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Citation for published version (APA):
Eijl, van, C. A. (1996). *A polyhedral approach to the discrete lot-sizing and scheduling problem*. [Phd Thesis 1 (Research TU/e / Graduation TU/e), Mathematics and Computer Science]. Technische Universiteit Eindhoven. https://doi.org/10.6100/IR461120

DOI:
10.6100/IR461120

Document status and date:
Published: 01/01/1996

Document Version:
Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

A Polyhedral Approach to the Discrete Lot-sizing and Scheduling Problem

Cleola van Eijl

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PROEFSCHRIFT

ter verkrijging van de graad van doctor aan de Technische Universiteit Eindhoven, op gezag van de Rector Magnificus, prof.dr. J.H. van Lint, voor een commissie aangewezen door het College van Dekanen in het openbaar te verdedigen op vrijdag 14 juni 1996 om 16.00 uur

door

CLEOLA ANGELINA VAN EIJL

geboren te Terneuzen

Dit proefschrift is goedgekeurd door de promotoren:

prof.dr. J.K. Lenstra
en
prof.dr. Y. Pochet

Copromotor: dr.ir. C.P.M. van Hoesel

Druk: Boek- en Offsetdrukkerij Letru, Helmond, (0492) 53 77 97

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1. Introduction

For several decades such divergent areas as production planning, transportation, telecommunication, and VLSI design have been an inexhaustible source of *combinatorial optimization problems*. These problems are usually characterized by a finite set of solutions from which the best solution with respect to a given objective function has to be found. The practical relevance of many combinatorial optimization problems stimulates the search for good solution methods. Unfortunately, most of these problems are *NP-hard* (cf. Garey and Johnson [16]). This implies that it is generally considered to be very unlikely that efficient solution methods for such problems will ever be developed. As a consequence, a vast amount of research in combinatorial optimization is devoted to the design of *approximation algorithms* that find reasonably good solutions in an acceptable amount of time. Nevertheless, algorithms that always provide the optimal solution but for which an efficient performance cannot be guaranteed, have also proven to be of great use in tackling hard combinatorial optimization problems. An optimization technique that has contributed to the successful solution of a large number of these problems, as for example listed by Jünger et al. [23] and Aardal and van Hoesel [1], is based on *polyhedral combinatorics*. In Section 1.1 we present basic concepts and results from polyhedral theory and discuss how polyhedral techniques can be applied in order to solve problems in the area of combinatorial optimization.

In this thesis we study the *discrete lot-sizing and scheduling problem* from a polyhedral point of view. This problem is one of the many lot-sizing models, which have been classified in a recent paper by Kuik et al. [24]. This paper also gives an extensive survey of the research in this area. The usual setting for a lot-sizing problem is a production facility that can produce several different items. One has to determine at what time and in what amount the items have to be produced in order to meet a given demand at minimum costs. Basic in lot-sizing is the trade-off between the costs related to inventory and the costs incurred by adjusting the facility for the production of a particular item. The typical features of the discrete lot-sizing and scheduling problem will be discussed in Section 1.2. This section is further devoted to the modeling of the problem and a review of earlier research on the subject.

An outline of the remainder of the thesis concludes this chapter.

1.1 Polyhedral combinatorics

In this section we will discuss how polyhedral methods can be applied to solve combinatorial optimization problems. First some basic concepts and results from polyhedral theory are introduced. For a detailed treatment of this subject, including proofs of the results stated here, the reader is referred to Nemhauser and Wolsey [30] and Schrijver [39].

For many combinatorial optimization problems, the solutions can be represented as n-dimensional integral vectors x that satisfy a set of linear constraints and for which the objective function is linear in the variables x_i, $1 \leq i \leq n$. Such a problem can thus be formulated as

$$\min\{cx : Ax \leq b, x \in \mathbb{Z}^n\}, \tag{1.1}$$

i.e., for given c, A, and b we have to find the integral vector x satisfying $Ax \leq b$ with minimum cost cx. Here c, A, and b are assumed to be integral. This formulation is called an *integer linear programming problem*, or IP for short. If only a subset of the variables has to be integral, then the problem is called a *mixed integer linear programming problem* (MIP). In the remainder we will only consider IPs, but the results hold for MIPs as well.

In this context an algorithm is said to be efficient if it solves every instance of a problem to optimality in a number of steps that is polynomial in the size of the input. The NP-hardness of most combinatorial optimization problems suggests that an efficient method for solving IPs is not likely to be ever developed. The integrality constraints form the complicating factor. If these constraints are omitted from the formulation, then the resulting problem is a *linear programming problem* (LP), called the *linear programming relaxation* or LP-relaxation of the IP under consideration. LPs are in general considerably easier than IPs, since they can be solved in polynomial time.

In order to obtain a solution to the IP one can apply *branch-and-bound*. This is a general implicit enumeration technique to solve combinatorial optimization problems. Here we restrict ourselves to the description of a linear programming based branch-and-bound procedure. In such a solution approach the original IP is partitioned into several subproblems by adding linear constraints. For example, in case of binary variables two subproblems can be created by fixing one variable to zero and one, respectively. For each subproblem we solve the corresponding LP-relaxation. If the linear program is infeasible or if its solution is integral, then the subproblem needs no further evaluation. Otherwise, a further refinement is made by splitting the subproblem into new subproblems. Note that the LP-relaxation provides a lower bound to the optimal value of the subproblem. If this lower bound is greater than or equal to the value of the currently best solution to the original problem, then the subproblem can also be discarded from further evaluation. Hence, in order to keep the number of subproblems that has to be evaluated as small as possible, it is important that strong lower bounds can be computed in an efficient way. Our efforts in applying polyhedral techniques are aimed at finding a strong LP-formulation of the problem at hand that serves as a good starting point for the branch-and-bound procedure.

Basic in polyhedral combinatorics is the existence of a linear program $\min\{cx : A'x \leq b'\}$ that solves the integer program (1.1). Before we state the two results from which this is an immediate consequence, we first introduce some terminology. Let $S = \{x^1, \dots, x^k\}$ be a set of vectors in $\mathbb{R}^n$. The *convex hull* of S, denoted by conv(S), is the set $\{\sum_{i=1}^k \lambda_i x^i : \sum_{i=1}^k \lambda_i = 1$ and $\lambda_i \geq 0$ for all $i\}$. The first result is that $\min\{cx : x \in S\} = \min\{cx : x \in \text{conv}(S)\}$ for any linear function c. Second, the convex hull of a finite number of points is a bounded *polyhedron*, i.e., there exists a finite set of linear inequalities $Ax \leq b$ such that $\text{conv}(S) = \{x \in \mathbb{R}^n : Ax \leq b\}$. We say that $Ax \leq b$ is a *linear description* of conv(S).

From the aforementioned it follows that the combinatorial optimization problem (1.1) can be solved by solving the linear program $\min\{cx : A'x \leq b'\}$, where $A'x \leq b'$ is a linear description of the convex hull of the set of feasible solutions $S = \{x : Ax \leq b, x \text{ integral}\}$. For most problems, however, it is very hard to find a complete linear description of conv(S). One therefore usually restricts oneself to the derivation of one or more classes of *valid* inequalities for the convex hull of S. These are inequalities that are satisfied by all elements of S. In particular, one is interested in the so-called *facet-defining* inequalities, which are necessary in the linear description of conv(S). Before we explain the latter term, some more definitions are introduced. Consider a polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ and let $\alpha x \leq \beta$ be a valid inequality for P. The set $F = \{x \in P : \alpha x = \beta\}$ is called a *face* of P and $\alpha x \leq \beta$ is said to *define* F. If F is neither empty nor equal to P, then it is said to be a *proper* face of P. A vector x is called a *direction* of P if there exist two different vectors x^1 and x^2 in P such that $x = x^1 - x^2$. Moreover, a set of vectors $x^1, \ldots, x^k$ is said to be *linearly independent* if $\sum_{i=1}^{k} \lambda_i x^i = 0$ implies that $\lambda_i = 0$ for all i. The *dimension* of a polyhedron P, denoted by dim(P), is defined as the maximum number of linearly independent directions in P. If $\dim(P) = n$, then P is said to be *full-dimensional*. This only occurs if there is no inequality that is satisfied at equality by all $x \in P$. Note that a face of a polyhedron is also a polyhedron. Now the *facets* of a polyhedron P are those faces F with $\dim(F) = \dim(P) - 1$. Together with the equalities that are satisfied by all $x \in P$, the facet-defining inequalities yield a linear description of P with as few constraints as possible. In order to prove that a valid inequality $\alpha x \leq \beta$ for a polyhedron P defines a facet, we proceed as follows. First, we have to show that there is a vector $x^0 \in P$ such that $\alpha x^0 < \beta$. This is usually trivial and therefore not explicitly mentioned. Then we provide $\dim(P) - 1$ linearly independent directions $x^1 - x^2$, where x^1 and x^2 are two different vectors in P that satisfy the inequality at equality.

Suppose that for the integer program (1.1) a partial linear description of the convex hull of the set of feasible solutions has been established. An explicit list of all inequalities, even when restricted to those that are facet-defining, usually yields a linear program that is too large to be handled by any LP-solver. However, in order to find the optimal solution to the IP, it suffices to include only a small number of inequalities, namely, the facet-defining inequalities that are satisfied at equality by this solution. This observation suggests the following procedure for solving IPs:

1. Take the LP-relaxation $\min\{cx : Ax \leq b\}$ as the first linear program. Go to 2.
2. Solve the current LP and denote the optimal solution by $\hat{x}$. If $\hat{x}$ is integral, then stop; otherwise go to 3.
3. Find a valid inequality that is violated by $\hat{x}$. If no such inequality can be found, then stop; otherwise add the inequality to the LP and go to 2.

This is called a *cutting plane* algorithm, since the inequalities found in Step 2 cut off the current LP-solution. The procedure of finding violated inequalities is called *separation*. A separation algorithm for a class of valid inequalities is called *exact* if it always finds a violated inequality in this class unless such an inequality does not exist. When no reasonably fast exact separation procedure is known, one usually applies a *heuristic* method. Then it is not guaranteed that there are no violated inequalities in the class at hand if the algorithm fails to find one.

The cutting plane algorithm terminates in one of the following two ways: either an integral solution is obtained in Step 2, or no violated inequality is identified in Step 3. In the first case, we have found an optimal solution to the IP. In the latter case, $\hat{x}$ is fractional and $c\hat{x}$ is a lower bound to the optimal value of the IP. Then an optimal integral solution can be obtained by applying branch-and-bound. In the branch-and-bound procedure one can try to improve the lower bounds by subjecting the subproblems to the cutting plane procedure. The combination of branch-and-bound and cutting planes is called *branch-and-cut*. Apart from good separation algorithms, the performance of a branch-and-cut algorithm depends on many other aspects such as the definition of the subproblems and the availability of strong upper bounds. Implementation issues in branch-and-cut algorithms are extensively discussed by Jünger et al. [23].

1.2 The discrete lot-sizing and scheduling problem

The main subject of our research is the discrete lot-sizing and scheduling problem, or DLSP for short. This production planning problem is concerned with a single machine that can produce a number of different items. The planning horizon is partitioned into small periods in each of which production occurs for at most one item. Furthermore, we assume an *all-or-nothing policy* with respect to the production level in one period, i.e., the production is either zero or at full capacity, which is defined as one unit of one item. This is often a reasonable assumption in short-term production planning, when the time periods are small. The demand for each item is dynamic and has to be satisfied without backlogging. In the first period of a production batch a so-called *startup* cost is incurred for setting up the machine for the item at hand. Since in multi-item problems startup costs usually arise when the machine switches from the production of one item to the production of another item, these costs are also known as changeover costs. Apart from startup costs, production and inventory costs have to be taken into account. Now DLSP is the problem of determining a production schedule that satisfies the given restrictions at minimum costs.

Startup costs are said to be *sequence-independent* if they only depend on the item for which the machine is set up. DLSP with sequence-independent startup costs is studied in Chapters 2 and 3. Moreover, a startup is assumed not to affect the production capacity. This assumption is reasonable, e.g., when the startups occur out of regular production hours. In Chapter 4 we consider an extension of DLSP in which startups take up an integral number of production periods.

In practice planning problems will usually have more complicating features than the ones captured by DLSP. Nevertheless, the study of these simplified models is a valuable aid in the solution of more realistic problems. Case-studies of production planning systems in which DLSP appears as a subproblem are presented by Van Wassenhove and Vanderhenst [40] and de Lange [25]. In the first paper only single-item problems have to be solved, whereas de Lange considers DLSP with sequence-dependent startup times. Moreover, Fleischmann [14] shows how DLSP can be used as an approximation for a capacitated lot-sizing problem with larger time periods in which production can occur for more than one item.

From the NP-hardness of DLSP (cf. Chapter 3) it follows that one should not strive for a polynomial-time solution procedure. Salomon [35] presents a dynamic programming algorithm for solving DLSP with sequence-independent costs. The running time of this algorithm is $O(M^2T\prod_{i=1}^{M} D_i)$, where M denotes the number of items, T the number of periods, and D_i the total demand for item i. Thus, when the number of items is fixed the problem can be solved in polynomial time. In particular, the single-item DLSP can be solved in $O(TD)$ time, where D denotes the total demand. For special cost functions faster algorithms are developed by van Hoesel [18] (see also [21]).

The dynamic programming algorithm for the single-item DLSP plays an important role in two other approaches to DLSP with sequence-independent startup costs, namely, by Lagrangean relaxation (Fleischmann [14]) and column generation (Cattrysse et al. [7]). An outline of both solution methods will be given in Chapter 3. The latter approach was in fact primarily developed for DLSP with sequence-independent startup times. In order to handle sequence-dependent startup costs Fleischmann [15] reformulates DLSP as a traveling salesman problem with additional time constraints and proposes a solution procedure in which lower bounds are obtained by Lagrangean relaxation. We return to this reformulation at the end of the section. Solution methods for DLSP with sequence-dependent startup times are discussed in recent papers by Salomon et al. [37] and Jordan and Drexl [22]. The former authors present an approach based on dynamic programming, whereas Jordan and Drexl transform the problem into a batch sequencing problem which is solved by a branch-and-bound algorithm.

The remainder of this section is mainly devoted to DLSP with sequence-independent startup costs. First an integer linear programming formulation of the problem is discussed. Then we will review polyhedral results that are obtained for DLSP and some generalizations. For a comprehensive survey of results for lot-sizing problems in the area of polyhedral combinatorics we refer to Pochet and Wolsey [33].

Let T denote the number of periods and let M denote the number of items. The demand for item i in period t is denoted by d_t^i. Since at most one unit of one item is produced per period, we assume without loss of generality that $d_t^i \in \{0, 1\}$ for all i and t. Furthermore, let p_t^i denote the cost for producing item i in period t, h_t^i the cost of holding one unit of item i in stock at the end of period t, and f_t^i the startup cost that is incurred when the machine is set up in period t for the production of item i.

The problem can be mathematically formulated using two types of binary variables: x_t^i, which indicates whether production occurs for item i in period t or not, and y_t^i, which takes the value one if a startup occurs for item i in period t and zero otherwise. For notational convenience we write x_{t_1,t_2}^i instead of $\sum_{t=t_1}^{t_2} x_t^i$, d_{t_1,t_2}^i instead of $\sum_{t=t_1}^{t_2} d_t^i$, etc. If $M = 1$, then the superscript i will be omitted.

We assume that initial inventories are zero. Consequently, the inventory of item i at the end of period t equals the total production of item i up to period t minus the total demand of item i up to this period. Hence, the inventory costs for item i can be expressed as $\sum_{t=1}^{T} h_t^i(x_{1,t}^i - d_{1,t}^i)$. Define $c_t^i = p_t^i + h_{t,T}^i$ and $H^i = -\sum_{t=1}^{T} h_t^i d_{1,t}^i$. Now DLSP can be modeled as

$$\text{(DLSP)} \qquad \min \sum_{i=1}^{M} \sum_{t=1}^{T} (c_t^i x_t^i + f_t^i y_t^i)$$

$$\text{s.t.} \qquad x_{1,t}^i \geq d_{1,t}^i \qquad (1 \leq i \leq M, 1 \leq t \leq T) \qquad (1.2)$$

$$x_t^i \leq x_{t-1}^i + y_t^i \qquad (1 \leq i \leq M, 1 \leq t \leq T, x_0^i = 0) \qquad (1.3)$$

$$\sum_{i=1}^{M} x_t^i \leq 1 \qquad (1 \leq t \leq T) \qquad (1.4)$$

$$x_t^i, y_t^i \in \{0, 1\} \qquad (1 \leq i \leq M, 1 \leq t \leq T) \qquad (1.5)$$

where the constant $\sum_{i=1}^{M} H^i$ is omitted from the objective function. Constraints (1.2) imply that for item i the total production up to period t is at least equal to the total demand up to this period. Inequalities (1.3) force that a startup for item i takes place in period t if production occurs for item i in this period, but not in the preceding one. Constraints (1.4) are the coupling constraints which state that in any period production can occur for only one item. Observe that when these constraints are not taken into account, the remaining problem consists of M single-item problems.

Note that a solution to (1.2) – (1.5) may have a positive inventory at the end of the planning horizon. Moreover, a startup is allowed to occur in a period without production, i.e., we may have $y_t^i = 1$ and $x_t^i = 0$ for some item i and period t. Also solutions in which two batches of the same item are scheduled contiguously, i.e., solutions with $x_{t-1}^i = y_t^i = x_t^i = 1$ for some i and t are allowed. One can easily add extra constraints to the model in order to exclude solutions having one of the aforementioned features. However, such a solution is never optimal when all costs are positive.

Polyhedral results for lot-sizing problems mainly concern the polyhedral structure of single-item models (cf. [33]). This also holds for the results described in this thesis. Obviously, valid inequalities for the single-item formulation remain valid for the multi-item problem.

For the single-item DLSP van Hoesel [18] characterizes one class of facet-defining inequalities, the so-called *hole-bucket inequalities*. A polynomial separation algorithm for these inequalities is given in [19]. Van Hoesel and Kolen present a *multicommodity flow* reformulation of the single-item DLSP in [20]. This model is obtained by splitting the original production variable x_t into variables x_{tk} which have value one if the production in period t satisfies the demand in the kth demand period and zero otherwise. For this formulation a complete linear description of the associated polyhedron is derived. Note that such a formulation has about $d_{1,T}T$ variables, whereas the original formulation has only $2T$ variables.

DLSP can be considered as a special case of the *capacitated lot-sizing problem with startup costs* or CLSS for short. We discuss here the single-item version of this problem. In CLSS the production in period t can attain any value between zero and the available capacity C_t. The amount of production in period t is denoted by X_t and incurs a cost c_tX_t. The binary setup variable Y_t indicates whether production can take place in period t or not. Obviously, Y_t must be one if $X_t > 0$. However, the machine might be set up for production, even if no production occurs. The setup cost f_t can be considered as the fixed cost incurred whenever the machine is able to produce. Moreover, if the machine is set up in a period t, but not in the preceding period, then a startup has to occur in period t. The startup variables and startup costs are here denoted by Z_t and g_t, respectively. Now CLSS can be formulated as

$$
\begin{array}{lll}
\text{(CLSS)} & \min \sum_{t=1}^{T} (c_tX_t + f_tY_t + g_tZ_t) & \\
\text{s.t.} & X_{1,t} \geq d_{1,t} & (1 \leq t \leq T) \\
 & Y_t \leq Y_{t-1} + Z_t & (1 \leq t \leq T, Y_0 = 0) \\
 & X_t \leq C_tY_t & (1 \leq t \leq T) \\
 & X_t \geq 0,\ Y_t, Z_t \in \{0,1\} & (1 \leq t \leq T)
\end{array}
$$

CLSS clearly reduces to DLSP if we set C_t equal to one and require $X_t = Y_t$ for every t. Hence, valid inequalities for CLSS can be easily turned into valid inequalities for DLSP. However, even if the original inequality is facet-defining for CLSS, the resulting inequality might be trivial or define a face of low dimension for DLSP.

Constantino [5] derives several classes of valid inequalities for CLSS. In relation to our work two classes are of great importance, namely, the *left* and *right supermodular inequalities*. The ideas behind these inequalities will be discussed in Chapter 2. For two subclasses, the *interval left* and *right supermodular inequalities*, Constantino establishes necessary and sufficient conditions for an inequality to be facet-defining and gives a polynomial-time separation algorithm. Moreover, he introduces valid inequalities for the single-item problem with more complicating features, such as lower bounds on the production and backlogging, and two classes of multi-item inequalities.

Magnanti and Vachani [28] and Sastry [38] study the special case of CLSS in which $C_t = 1$ for every t and the production variables are also binary. Hence, the only difference with DLSP is that the setup of the machine can be maintained during idle periods. In this way, one can avoid a more expensive startup. A class of facet-defining inequalities for this problem is discussed in [28]. This class forms the basis of the *skip inequalities* introduced by Sastry. He gives a complete characterization of facet-defining skip inequalities and presents a separation algorithm based on linear programming for one subclass.

For some problems it can be shown that a partial linear description of the related polyhedron always provides an optimal solution with respect to a certain class of objective functions. For lot-sizing problems this kind of result has been obtained for cost functions that satisfy the *Wagner-Whitin* property. For DLSP a cost function is said to be of the Wagner-Whitin type if the unit inventory cost h_t and the production cost p_t satisfy $h_t + p_t \geq p_{t+1}$ for every period t. Recalling that $c_t = p_t + h_{t,T}$, this is easily seen to be equivalent to $c_t \geq c_{t+1}$ for all t. The Wagner-Whitin property implies that it is always optimal to produce as late as possible. Hence, with Wagner-Whitin costs there always exists an optimal solution for which the inventory at the end of period $t-1$ is zero for any period t in which a production batch is started. Pochet and Wolsey [32] study four single-item lot-sizing problems and give for each of them a partial linear description of the convex hull that solves the problem in the presence of Wagner-Whitin costs. These polyhedra involve considerably fewer constraints than in the general cost case. In Section 2.3 we derive a similar result for the single-item DLSP.

We conclude this section with a brief discussion of the reformulation of DLSP as a traveling salesman problem with time windows (TSPTW). Consider a graph in which every unit of demand is represented by a node, i.e., if $d_t^i = 1$, then this unit of demand is represented by a node $v(i,t)$. With $v(i,t)$ a deadline t is associated. Now a production schedule can be considered as a tour that starts and finishes at a depot and visits every node before or at its deadline. The travel time between two nodes equals one if the nodes correspond to units of the same item, and one plus the required startup time otherwise. Startup and production costs are incorporated in the costs of using an arc, whereas inventory costs can be modeled as costs incurred when a node is visited strictly before its deadline.

Fleischmann [15] was the first to formulate DLSP as a TSPTW. He models DLSP with sequence-dependent startup costs and zero startup times as a TSPTW with time-dependent costs. A reformulation of DLSP with sequence-dependent startup times as a TSPTW is presented by Salomon et al. [37]. Both problems will not be considered in this thesis. However, in Chapter 5 we will study another variant of the TSP, the so-called *delivery man problem*, in which the objective is to find a tour starting from a given depot that minimizes the sum of the waiting times of the customers located at the nodes. We will present a MIP-formulation that can easily be turned into a formulation for the TSPTW. Furthermore, we will derive some additional valid inequalities and report computational results.

1.3 Outline of the thesis

The remainder of the thesis is organized as follows.

Chapters 2 and 3 are devoted to the discrete lot-sizing and scheduling problem with sequence-independent startup costs. In Chapter 2 we study the polyhedral structure of the single-item version of the formulation discussed in the previous section. We first investigate the general form of facet-defining inequalities for which all coefficients of the x-variables are either zero or one. Then three subclasses are discussed in more detail. In the last section we present a partial linear description of the convex hull of feasible solutions that solves the problem in the presence of Wagner-Whitin costs. This result already appeared in van Eijl and van Hoesel [11]. The multi-item problem is dealt with in Chapter 3. Its computational complexity is addressed first. Furthermore, we describe the implementation of a branch-and-cut algorithm and report computational results.

In Chapter 4 the problem with sequence-independent startup times is considered. We show that this problem can be modeled by a slight modification of the formulation for the problem that only involves startup costs. Valid inequalities for the latter formulation are hence easily turned into valid inequalities for the problem with startup times. We also present a multicommodity flow formulation for the single-item problem and show that its LP-relaxation solves the problem to optimality.

As mentioned before, Chapter 5 differs from the other chapters in that it focuses on the delivery man problem instead of the discrete lot-sizing and scheduling problem. We present a mixed integer programming formulation for this problem, derive classes of additional valid inequalities, and give some computational results. This chapter is based on van Eijl [10].

2. The single-item DLSP

In this chapter we study the polyhedral structure of the model discussed in the previous chapter when only one item is involved. Although the single-item DLSP is polynomially solvable, an explicit description of the convex hull of the set of feasible solutions to this formulation is not known. A partial linear description is given by van Hoesel [18]. Magnanti and Vachani [28] and Sastry [38] derive inequalities for a slightly more general problem in which also setup costs are involved. Furthermore, Constantino [5] derives several classes of valid inequalities for the capacitated lot-sizing problem with startup costs. This problem is a generalization of DLSP in which the production in period t can attain any value between zero and the available capacity in this period. Inequalities for these more general problems can easily be adapted to valid inequalities for DLSP. However, even if we start from a facet-defining inequality for the original problem, the resulting inequality for DLSP might be trivial or define a face of low dimension.

This chapter is organized as follows. In Section 2.1 the formulation from Chapter 1 is reviewed. In Section 2.2 we first investigate the general form of facet-defining inequalities for which all coefficients of the x-variables are either zero or one. Then three subclasses are discussed in more detail. The first subclass slightly extends the class of *right supermodular inequalities* of Constantino when adapted to DLSP. The inequalities of the first subclass are also related to the *skip inequalities* discussed in [38]. The second subclass generalizes the *hole-bucket inequalities* introduced by van Hoesel. The last subclass is again an extension of a class derived by Constantino, namely, the class of *interval left supermodular inequalities.* For all three subclasses we also address the separation problem. In the last section we present a partial linear description of the convex hull of feasible solutions that solves the problem in the presence of Wagner-Whitin costs.

2.1 Preliminaries

Throughout, the interval $\{t_1, \ldots, t_2\} \subseteq \{1, \ldots, T\}$ will be denoted by $[t_1, t_2]$. If $t_1 > t_2$, then $[t_1, t_2] = \emptyset$. Now the single-item version of the model discussed in the previous chapter reads as follows:

$$\text{(DLSP)} \qquad \min \sum_{t=1}^{T} (c_t x_t + f_t y_t)$$

$$\text{s.t.} \qquad x_{1,t} \geq d_{1,t} \qquad \text{for all } t \in [1, T] \qquad (2.1)$$

$$x_t \leq x_{t-1} + y_t \qquad \text{for all } t \in [1, T]\ (x_0 = 0) \qquad (2.2)$$

$$x_t, y_t \in \{0, 1\} \qquad \text{for all } t \in [1, T] \qquad (2.3)$$

Denote by $\mathcal{X}$ the set of feasible solutions to the above formulation, i.e., $\mathcal{X} = \{(x, y) \in \{0, 1\}^{2T} : (x, y)$ satisfies (2.1) and (2.2)$\}$. One of the main advantages of allowing a positive inventory at the end of the planning horizon and startups in periods without production, is that the convex hull of $\mathcal{X}$ is full-dimensional whenever $d_1 = 0$ (note that imposing $y_t \leq x_t$ for all t yields the equality $x_1 = y_1$). In the sequel $e(x_t)$ and $e(y_t)$ denote the unit vector of length $2T$ corresponding to the variable x_t and y_t, respectively.

Proposition 2.1.1 *The convex hull of $\mathcal{X}$ has dimension $2T$ if and only if $d_1 = 0$.*

PROOF. If $d_1 = 1$, then every solution satisfies $y_1 = x_1 = 1$, hence, in this case, the dimension of conv($\mathcal{X}$) is at most $2T - 2$.

In order to prove sufficiency, we show that the $2T$ unit vectors $e(x_t)$ and $e(y_t)$ for all t are directions in conv($\mathcal{X}$). First, let (x, y) be defined by $x_t = y_t = 1$ for all t and let $(\bar{x}, \bar{y})$ be the solution obtained from (x, y) by setting x_t to zero. Then $(x, y) - (\bar{x}, \bar{y}) = e(x_t)$. Furthermore, let $(\hat{x}, \hat{y})$ be obtained from $(\bar{x}, \bar{y})$ by setting y_t to zero. Then $(\bar{x}, \bar{y}) - (\hat{x}, \hat{y}) = e(y_t)$. □

Since it is usually easier to prove that a given inequality defines a facet when the associated polyhedron is full-dimensional, it is from now on always assumed that $d_1 = 0$. Consequently, an inequality $\alpha x + \beta y \geq \gamma$ can be shown to define a facet of conv($\mathcal{X}$) by exhibiting $2T - 1$ linearly independent directions $(x, y) - (\bar{x}, \bar{y})$, where (x, y) and $(\bar{x}, \bar{y})$ are feasible solutions to DLSP that satisfy the given inequality at equality. The following proposition gives necessary and sufficient conditions for the inequalities in the model and the trivial inequalities $x_t, y_t \geq 0$ and $x_t, y_t \leq 1$ to be facet-defining for conv($\mathcal{X}$). For the proof the reader is referred to van Hoesel [18] or van Hoesel and Kolen [19].

Proposition 2.1.2 (i) $x_t \geq 0$ *defines a facet of* conv($\mathcal{X}$) *if and only if* $(t = 1$ *and* $d_2 = 0)$ *or* $(t > 1$ *and* $d_{2,t} < t - 1)$;
(ii) $x_t \leq 1$ *defines a facet of* conv($\mathcal{X}$) *if and only if* $t > 1$;
(iii) $y_t \geq 0$ *defines a facet of* conv($\mathcal{X}$) *if and only if* $t > 1$ *and* $d_{2,t} < t - 1$;
(iv) $y_t \leq 1$ *defines a facet of* conv($\mathcal{X}$) *for all* t;
(v) $x_{1,t} \geq d_{1,t}$ *defines a facet of* conv($\mathcal{X}$) *if and only if* $d_t = 1$ *and either* $t = T$ *or* $d_{t+1} = 0$;
(vi) $x_t \leq x_{t-1} + y_t$ *defines a facet of* conv($\mathcal{X}$) *if and only if* $t = 1$ *or* $d_{2,t} < t - 1$. □

The LP-relaxation of the above formulation, i.e., $\min\{cx + fy : (x, y)$ satisfies (2.1), (2.2), and $0 \leq x_t, y_t \leq 1\}$, yields in general weak lower bounds.

Example 2.1.1 Let $T = 10$, $d_t = 1$ for $t \in \{3, 5, 7, 9, 10\}$, and $f_t = 10$, $c_t = 10 - t + 1$ (thus, $h_t = 1$, $p_t = 0$) for all t. Then

t	1	2	3	4	5	6	7	8	9	10
y_t		$\frac{1}{2}$				$\frac{1}{10}$				
x_t		$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{3}{5}$	$\frac{3}{5}$	$\frac{3}{5}$	$\frac{3}{5}$

with value 30 is the optimal solution to the LP-relaxation, whereas the optimal solution to DLSP has value 40 ($y_3 = x_3 = \ldots = x_7 = 1$). □

Now let t be a period with $d_t = 1$. Recall that $d_{1,t}$ denotes the total demand up to period t. Then in order to satisfy the demand up to t, we have to produce at least once in the interval $[d_{1,t}, t]$. This implies that at least one startup has to occur in the interval $[d_{1,t} + 1, t]$ if no production occurs in period $d_{1,t}$. This establishes the validity of the following inequality:

$$x_{d_{1,t}} + y_{d_{1,t}+1,t} \geq 1. \tag{2.4}$$

In the above example we have $d_7 = 1$ and $d_{1,7} = 3$. Thus, $x_3 + y_{4,7} \geq 1$ is a valid inequality for the instance of Example 2.1.1. For this instance the lower bound is substantially improved by adding the above constraints for all demand periods t to the LP-relaxation.

Example 2.1.1 (continued) The optimal solution to the LP-relaxation extended with inequalities (2.4) for all demand periods t is

t	1	2	3	4	5	6	7	8	9	10
y_t			1							
x_t			1	1	1	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{5}$

with value 37. □

From the general assumption that $d_1 = 0$ it follows that $t > d_{1,t}$, hence, the interval $[d_{1,t} + 1, t]$ is not empty. One easily checks that inequality (2.4) defines a facet of conv($\mathcal{X}$). The latter implies that for any other facet-defining inequality $\alpha x + \beta y \geq \gamma$, there exists a solution (x, y) satisfying $\alpha x + \beta y = \gamma$ and $x_{d_{1,t}} + y_{d_{1,t}+1,t} \geq 2$.

Note that in the above example the costs satisfy the Wagner-Whitin property, i.e., $c_t \geq c_{t+1}$ for all t. In Section 2.3 it will be shown that for such an instance the optimal solution to DLSP is yielded by the LP-relaxation extended with one of the classes of facet-defining inequalities of conv($\mathcal{X}$) discussed in the following section.

2.2 Facet-defining inequalities

In this section we derive additional classes of facet-defining inequalities of conv($\mathcal{X}$), where $\mathcal{X}$ denotes the set of solutions to (2.1) – (2.3). In particular, we study inequalities for which the coefficients of the x-variables are either zero or one. We first study the general form of such inequalities. Then three classes are discussed in more detail.

2.2.1 General form

Throughout this section, $\alpha x + \beta y \geq \gamma$ denotes a valid inequality for $\mathcal{X}$ other than one of the inequalities that define the LP-relaxation. Without loss of generality all coefficients are assumed to be integral. We first derive some restrictions on α and β when the inequality defines a facet of conv($\mathcal{X}$).

Lemma 2.2.1 *Let $\alpha x + \beta y \geq \gamma$ be a valid inequality for X other than one of the inequalities that define the LP-relaxation. If $\alpha x + \beta y \geq \gamma$ defines a facet of* conv($\mathcal{X}$), *then $\alpha_t \geq 0$ for all t, $\beta_1 = 0$, $\beta_t \geq 0$ for every $t > 1$, and $\alpha_t + \beta_t \geq \beta_{t+1}$ for every $t < T$. Moreover, if $\alpha_t \in \{0, 1\}$ for every t, then $\alpha_t + \beta_t \leq \beta_{t+1} + 1$ for every $t < T$.*

PROOF. Let $\bar{X}$ denote the set of feasible solutions to DLSP that satisfy $\alpha x + \beta y \geq \gamma$ at equality, i.e., $\bar{X} = \{(x, y) \in X : \alpha x + \beta y = \gamma\}$.

First, suppose that $\beta_t < 0$ for some t. Then $y_t = 1$ for every solution $(x, y) \in \bar{X}$. This implies that $\beta_t \geq 0$ must hold for every t if $\alpha x + \beta y \geq \gamma$ defines a facet of conv($\mathcal{X}$). Furthermore, β_1 must be zero, otherwise $x_1 = y_1$ for all solutions in $\bar{X}$.

Now suppose that $\alpha_t < 0$ for some t. If $\beta_t = 0$, then $x_t = 1$ for all $(x, y) \in \bar{X}$. Otherwise, if $\beta_t > 0$, then every $(x, y) \in \bar{X}$ satisfies $x_{t-1} + y_t \leq 1$ and if equality holds, then x_t equals one as well. Hence, if $\alpha_t < 0$ and $\beta_t > 0$, then $x_{t-1} + y_t = x_t$ for all $(x, y) \in \bar{X}$. This shows that $\alpha_t \geq 0$ for every period t if $\alpha x + \beta y \geq \gamma$ defines a facet of conv($\mathcal{X}$).

Next, suppose that $\alpha_t + \beta_t < \beta_{t+1}$ for some $t < T$. Let (x, y) be a solution in $\bar{X}$ satisfying $y_{t+1} = 1$. Since $\beta_{t+1} > 0$, we have $x_t = 0$ and $x_{t+1} = 1$. Moreover, $\beta_t \geq 0$ implies that either $y_t = 0$ or ($\beta_t = 0$ and $y_t = 1$), thus, $\beta_t y_t = 0$. Extend the production batch starting in $t + 1$ by producing in period t as well and denote the new solution by $(\bar{x}, \bar{y})$. Then $\bar{x}_t = \bar{y}_t = 1$, $\bar{y}_{t+1} = 0$, $\bar{x}_\tau = x_\tau$ for $\tau \neq t$, and $\bar{y}_\tau = y_\tau$ for $\tau \notin \{t, t+1\}$. Since $(\bar{x}, \bar{y})$ is obviously feasible, we have $\alpha x + \beta y = \alpha\bar{x} + \beta\bar{y} + \beta_{t+1} - \alpha_t - \beta_t(\bar{y}_t - y_t) > \alpha\bar{x} + \beta\bar{y} \geq \gamma$, contradicting our assumption that $(x, y) \in \bar{X}$. From this it follows that $y_{t+1} = 0$ for all solutions that satisfy $\alpha x + \beta y \geq \gamma$ at equality. Hence, if the inequality defines a facet of conv($\mathcal{X}$), then $\alpha_t + \beta_t \geq \beta_{t+1}$ for all $t < T$.

Finally, assume that $\alpha_t \in \{0, 1\}$ for all t. We claim that in this case $\alpha_t + \beta_t \leq \beta_{t+1} + 1$ for every $t < T$. Since $\beta_1 = 0$ and $\beta_2 \geq 0$, this obviously holds for $t = 1$. Hence, let $1 < t < T$ and suppose $\alpha_t + \beta_t > \beta_{t+1} + 1$. Then $\beta_t > 0$. Let (x, y) be a solution satisfying in $\bar{X}$ satisfying $y_t = 1$. Since $\alpha x + \beta y = \gamma$ and $\beta_t > 0$, we have $x_{t-1} = 0$ and $x_t = 1$. Let s be the first period that is not used for production. Then $s < t$ as $x_{t-1} = 0$. Now let $(\bar{x}, \bar{y})$ be the solution obtained from (x, y) by moving the production in period t to period s. Then $\bar{x}_s = 1, \bar{x}_t = 0, \bar{x}_\tau = x_\tau, \tau \notin \{s, t\}$, and $\bar{y}_t = 0$, $\bar{y}_{t+1} = 1$, $\bar{y}_s = 1$ if $s = 1$ and 0 otherwise (thus, $\beta_s\bar{y}_s = 0$), and $\bar{y}_\tau = y_\tau, \tau \notin \{s, t, t+1\}$. Since $\alpha_s \in \{0, 1\}$, we have $\alpha x + \beta y = \alpha\bar{x} + \beta\bar{y} + \alpha_t + \beta_t - \beta_{t+1} - \alpha_s - \beta_s\bar{y}_s > \alpha\bar{x} + \beta\bar{y} \geq \gamma$, a contradiction. Thus, $\alpha_t + \beta_t > \beta_{t+1} + 1$ implies that $y_t = 0$ for all solutions $(x, y) \in \bar{X}$. Hence, if $\alpha x + \beta y \geq \gamma$ defines a facet of conv($\mathcal{X}$), then $\alpha_t + \beta_t \leq \beta_{t+1} + 1$ must hold for every $t < T$. □

Before additional necessary conditions for an inequality to be facet-defining for conv($\mathcal{X}$) are discussed, let us first introduce some notation. Given an inequality $\alpha x + \beta y \geq \gamma$, we denote by t^* the last period t for which $\alpha_t + \beta_t > 0$. Notice that if $\alpha x + \beta y \geq \gamma$ satisfies the conditions stated in Lemma 2.2.1, then $\alpha_{t^*} + \beta_{t^*} = 1$. Furthermore, a period t is called a *hole* with respect to the inequality under consideration if $\alpha_t = \beta_t = 0$. In particular, all periods after t^* are holes.

Lemma 2.2.2 (cf. [18], [38]) *Let $\alpha x + \beta y \geq \gamma$ be a facet-defining inequality of* conv($\mathcal{X}$) *other than one of the inequalities that define the LP-relaxation, and assume that $\beta_t > 0$ for at least one period t. Let t^* be as defined above. Then this inequality satisfies the following properties:*

P1. *Period t^* is a demand period, thus, $d_{t^*} = 1$.*
P2. *For every period $t < t^*$ the number of holes in the interval $[t, t^*]$ is less than the total demand in this interval.*
P3. *If there is a hole before $t' = \min\{t : \beta_{t+1} > 0\}$, then for every $t \in [s, t' - 1]$, where s denotes the first hole, the number of holes in the interval $[s, t]$ exceeds the total demand in this interval.*

PROOF. Denote again by $\bar{\mathcal{X}}$ the set of feasible solutions to DLSP that satisfy $\alpha x + \beta y \geq \gamma$ at equality.

Ad P1. Suppose $d_{t^*} = 0$ and $\alpha_{t^*} = 1$. Let (x, y) be a feasible solution satisfying $x_{t^*} = 1$. We may assume that $y_t = x_t = 1$ for every $t > t^*$. Since $d_{t^*} = 0$, cancelling the production in period t^* yields another feasible solution, say, $(\bar{x}, \bar{y})$, and $\alpha x + \beta y \geq \alpha \bar{x} + \beta \bar{y} + \alpha_{t^*} > \gamma$. Hence, $x_{t^*} = 0$ for every $(x, y) \in \bar{\mathcal{X}}$. However, this contradicts the assumption that $\alpha x + \beta y \geq \gamma$ defines a facet of conv($\mathcal{X}$). Using similar arguments we find that $d_{t^*} = 0$ and $\beta_{t^*} = 1$ imply that $y_{t^*} = 0$ for every solution in $\bar{\mathcal{X}}$. Thus, $d_{t^*} = 1$ when $\alpha x + \beta y \geq \gamma$ defines a facet of conv($\mathcal{X}$).

Ad P2. Suppose $t < t^*$ is a period for which the number of holes in the interval $[t, t^*]$ is at least d_{t,t^*}. Let $(x, y) \in \bar{\mathcal{X}}$. Without loss of generality assume that $y_s = x_s = 1$ for every hole s. Then $x_{t,t^*-1} \geq d_{t,t^*}$ and $x_{t^*+1,T} \geq d_{t^*+1,T}$. Since $\alpha x + \beta y = \gamma$ and $\alpha_{t^*} + \beta_{t^*} > 0$, we must have $x_{t^*} = 0$ if $\alpha_{t^*} > 0$ and $y_{t^*} = 0$ if $\beta_{t^*} > 0$. This holds for every solution in $\bar{\mathcal{X}}$, which contradicts the assumption that $\alpha x + \beta y \geq \gamma$ defines a facet of conv($\mathcal{X}$).

Ad P3. Suppose there is a hole before t' and suppose that P3 is violated for some $t \in [s, t' - 1]$, where s denotes the first hole. That is, the number of holes in the interval $[s, t]$ is less than or equal to $d_{s,t}$. Since s is a hole, this implies $d_{s,t} \geq 1$. Also observe that $\alpha_\tau = 1$ and $\beta_\tau = 0$ for every period $\tau \leq t$ that is not a hole. Hence, in a solution (x, y) with $x_s = 0$, the demand of at least one period in $[s, t]$ is produced in a period τ satisfying $\alpha_\tau = 1$ and $\beta_\tau = 0$. This unit of production can be moved to period s while maintaining feasibility. From this we conclude that if (x, y) is a solution without production in period s, then $\alpha x + \beta y > \gamma$. Hence, $x_s = 1$ for every $(x, y) \in \bar{\mathcal{X}}$, which again contradicts the assumption that $\alpha x + \beta y \geq \gamma$ defines a facet of conv($\mathcal{X}$). □

Using similar arguments as in the proof of P2 and P3, one readily shows that if $\alpha x + \beta y \geq \gamma$ is a facet-defining inequality of conv($\mathcal{X}$) with $\beta_t = 0$ for all t, then the inequality is the production inequality $x_{1,t^*} \geq d_{1,t^*}$. In the sequel it is therefore always assumed that $\beta_t > 0$ for at least one period t.

As mentioned before, we will study facet-defining inequalities of conv($\mathcal{X}$) for which the coefficients of the x-variables are either zero or one. We start by investigating the general form of these inequalities.

In Section 1.2 we briefly discussed the capacitated lot-sizing problem with startup variables (CLSS). For this problem Constantino [5] derives several classes of valid inequalities. These inequalities are all of the form $I_t \geq LB(X, Y, Z)$, i.e., they yield a lower bound on the inventory I_t at the end of period t. By taking the capacity in every period equal to one and by identifying the setup and the production variables, valid inequalities for DLSP are obtained. One of the classes that yields nontrivial inequalities for DLSP is the class of *right supermodular inequalities* ([5], Section 2.4). These inequalities are obtained by establishing a lower bound on the inventory at the end of a certain period when, up to this period, the total production exceeds the total demand.

For DLSP these inequalities can be introduced as follows. Let t^* be a demand period and let $u, v \in [1, t^*]$, $u \leq v$, such that $v - u \geq d_{u,t^*}$. Then the inventory at the end of period t^* is at least one when all $v - u + 1$ periods in the interval $[u, v]$ are used for production. The latter occurs if there is production in v and no startup in the interval $[u+1, v]$, i.e., if $x_v - y_{u+1,v} = 1$. Observe that $x_v - y_{u+1,v}$ is integral and less than or equal to one for any solution $(x, y) \in \mathcal{X}$. Thus, $I_{t^*} \geq x_v - y_{u+1,v}$ is a valid inequality for $\mathcal{X}$.

Now let $V \subseteq [1, t^*]$ and let an interval $[u(v), v]$ satisfying $v - u(v) \geq d_{u(v),t^*}$ be associated with every $v \in V$. We claim that

$$I_{t^*} \geq \sum_{v \in V} (x_v - y_{u(v)+1,v}). \tag{2.5}$$

is valid for $\mathcal{X}$. In order to show this, let $(x, y) \in \mathcal{X}$ and let $j = \sum_{v \in V}(x_v - y_{u(v)+1,v})$. Assume that $j > 0$, otherwise (2.5) is obviously satisfied. Then $x_v - y_{u(v)+1,v}$ equals one for precisely j different elements of V, say, for $v \in \{v_1, \ldots, v_j\}$, where $v_1 < \ldots < v_j$. This implies that all intervals $[u(v_i), v_i]$, $1 \leq i \leq j$, are completely used for production in (x, y). Thus, the total production in the interval $[u(v_1), t^*]$ is at least the production in the periods $[u(v_1), v_1] \cup \{v_2, \ldots, v_j\}$, which is at least $d_{u(v_1),t^*} + j$. Hence, the overproduction in the interval $[u(v_1), t^*]$ amounts to at least j units, which proves our claim. Inequalities (2.5) are a direct adaptation of the right supermodular inequalities of Constantino to DLSP. In the sequel we will refer to them as *right stock-minimal inequalities*.

We can generalize the idea behind these inequalities in the following way. Suppose that for a demand period t^*, we are given a set of nonempty intervals $[u(v), v] \subseteq [1, t^*]$ and an integer $k \geq 1$ for which the following holds: the inventory at the end of period t^* is at least one in any solution to DLSP for which precisely k of these intervals are completely used for production. Then the following inequality is valid for $\mathcal{X}$:

$$I_{t^*} \geq \sum_{v \in V} (x_v - y_{u(v)+1,v}) - k + 1. \tag{2.6}$$

This can be shown using similar arguments as for (2.5), i.e., for inequality (2.6) with $k = 1$. Let $(x, y) \in \mathcal{X}$ and assume that $\sum_{v \in V}(x_v - y_{u(v)+1,v})$ equals $k + j - 1$ for some $j > 0$. Let $\{v_1, \ldots, v_{k+j-1}\}$, $v_1 < \ldots < v_{k+j-1}$, be the subset of elements in $v \in V$ that satisfy $x_v -$

$y_{u(v)+1,v} = 1$. Then all periods in $\cup_{i=1}^{k+j-1}[u(v_i), v_i]$ are used for production in (x, y). In particular, production occurs in every period in $\cup_{i=1}^{k}[u(v_i), v_i] \cup \{v_{k+1}, \ldots, v_{k+j-1}\}$. By assumption, using all periods in $\cup_{i=1}^{k}[u(v_i), v_i]$ for production yields an inventory of size at least one at the end of period t^*. This also implies that the demand in $[v_k + 1, t^*]$ can be satisfied from production in periods up to v_k. Thus, if all periods in $\cup_{i=1}^{k}[u(v_i), v_i] \cup \{v_{k+1}, \ldots, v_{k+j-1}\}$ are used for production, then the inventory at the end of period t^* is at least j. This establishes the validity of (2.6) for $\mathcal{X}$.

Example 2.2.1 Let $d_t = 1$ for $t \in \{2, 4, 6, 9, 10, 12, 14\}$.

First, let $t^* = 10$. We consider the following intervals: $[1, 6]$, $[4, 8]$, and $[8, 10]$. Thus, $V = \{6, 8, 10\}$, $u(6) = 1$, $u(8) = 4$, and $u(10) = 8$. One readily checks that $v - u(v) = d_{u(v),10}$ for each $v \in V$. Hence, by the above arguments, inequality (2.5) with $t^* = 10$ and the intervals $[u(v), v]$, $v \in V$, i.e.,

$$I_{10} \geq (x_6 - y_{2,6}) - (x_8 - y_{5,8}) - (x_{10} - y_{8,10}),$$

is valid for the instance at hand.

Next, let $t^* = 14$. We consider the following intervals: $[3, 8]$, $[6, 9]$, $[5, 10]$, $[12, 12]$, and $[13, 14]$. That is, $V = \{8, 9, 10, 12, 14\}$, $u(8) = 3$, $u(9) = 6$, $u(10) = 5$, $u(12) = 12$, and $u(14) = 13$. The demand periods and the intervals are depicted below.

t	1	2	3	4	5	6	7	8	9	10	11	12	13	14
d_t		1		1		1			1	1		1		1
			[· · ·	· · ·	· · ·	· · ·	· · ·	· · ·]				[··]		
						[· · ·	· · ·	· · ·	· · ·]				[· ·	· ·]
					[· · ·	· · ·	· · ·	· · ·	· · ·	· · ·]				

Now it is not difficult to check that for any feasible solution in which at least two of these intervals are completely used for production the inventory at the end of period 14 is at least one. Hence,

$$I_{14} \geq (x_8 - y_{4,8}) - (x_9 - y_{7,9}) - (x_{10} - y_{6,10}) - x_{12} - (x_{14} - y_{14}) - 1$$

is a valid inequality of the form (2.6) with $k = 2$ and t^*, V, and $[u(v), v]$, $v \in V$, as given. □

Using $I_{t^*} = x_{1,t^*} - d_{1,t^*}$, we can rewrite inequality (2.6) as

$$\sum_{t \in [1,t^*] \setminus V} x_t + \sum_{v \in V} y_{u(v)+1,v} \geq d_{1,t^*} - k + 1. \tag{2.7}$$

Note that the coefficient of x_t is either zero or one, whereas the coefficient of y_t is equal to $|\{v \in V : u(v) < t \leq v\}|$, which can obviously be larger than one.

We claim that every facet-defining inequality of conv($\mathcal{X}$) with x-coefficients in $\{0, 1\}$ is of the form (2.7) with $u(v) < u(v')$ for $v < v'$. In order to show this, consider an inequality

$\alpha x + \beta y \geq \gamma$ such that $\alpha_t \in \{0, 1\}$, $\beta_1 = 0$, and $\beta_t \geq 0$ for all other t. As usual, t^* denotes the last period t for which $\alpha_t + \beta_t > 0$. Define $V = \{t : t \leq t^* \text{ and } \alpha_t = 0\}$. Denote the elements of V by v_i, $i \in \{1, \dots, |V|\}$, where $v_i < v_{i+1}$. The following algorithm determines a period $u(v) \leq v$ for every $v \in V$ such that at termination $\bar{\beta}_t \geq |\{v \in V : u(v) < t \leq v\}|$ for all t.

begin DETERMINE_INTERVALS
  for $t = 1$ **to** t^* **do** $\bar{\beta}_t := \beta_t$;
  for $i = 1$ **to** $|V|$ **do begin**
    * Invariant: $\bar{\beta}_t = \beta_t - |\{j : 1 \leq j < i \text{ and } t \in [u(v_j) + 1, v_j]\}| \geq 0$ for all t *\
    $u(v_i) := \max\{t \leq v_i : \bar{\beta}_t = 0\}$;   * thus, $\bar{\beta}_t > 0$ for all $t \in [u(v_i) + 1, v_i]$ *\
    for $t = u(v_i) + 1$ **to** v_i **do** $\bar{\beta}_t := \bar{\beta}_t - 1$
  end
end.

From the assumption that $\beta_1 = 0$, it follows that $u(v_i)$ is well defined for every i. Furthermore, $u(v_i) \leq v_i$ and equality holds if and only if v_i is a hole. Observe also that $u(v_i) \leq u(v_{i+1})$.

Example 2.2.2 Let $\alpha x + \beta y \geq \gamma$ be the second inequality of Example 2.2.1 rewritten in the form (2.7), i.e.,

$$\begin{array}{l} \qquad\qquad + y_4 + y_5 + 2y_6 + 3y_7 + 3y_8 + 2y_9 + y_{10} \qquad\qquad\qquad + y_{14} \\ x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \qquad\qquad\qquad\qquad + x_{11} \quad + x_{13} \end{array} \geq 6.$$

Then $V = \{8, 9, 10, 12, 14\}$. Applying the above algorithm yields $u(8) = 3$, $u(9) = 5$, $u(10) = 6$, $u(12) = 12$, and $u(14) = 13$. Recall that the inequality was constructed from a different set of intervals, namely, with $[6, 9]$ and $[5, 10]$ instead of $[5, 9]$ and $[6, 10]$. However, this set cannot be obtained from DETERMINE_INTERVALS, since the algorithm yields periods $u(v)$, $v \in V$, with $u(v) \leq u(v')$ if $v \leq v'$. □

Lemma 2.2.3 *Let $\alpha x + \beta y \geq \gamma$ be a facet-defining inequality of* conv(X) *with $\alpha_t \in \{0, 1\}$ for all t, and let t^* and V be as defined before. Let $u(v)$, $v \in V$, be as provided by the above algorithm. Then $\beta_t = |\{v \in V : u(v) < t \leq v\}|$ for every $t \leq t^*$ and $u(v) < u(v')$ for $v < v'$.*

PROOF. Note that $\beta_t = |\{v \in V : u(v) < t \leq v\}|$ if and only if $\bar{\beta}_t = 0$ at termination of DETERMINE_INTERVALS. We first show that $\beta_{t^*} = |\{v \in V : u(v) < t^* \leq v\}|$. This obviously holds when $\beta_{t^*} = 0$, since in this case $\max_{v \in V} v < t^*$. If $t^* \in V$, i.e., if $\alpha_{t^*} = 0$, then in the last iteration of DETERMINE_INTERVALS we have $\bar{\beta}_{t^*} = \beta_{t^*} = 1$, hence, $u(t^*) < t^*$.

Now suppose there is a period $t < t^*$ such that $\beta_t > |\{v \in V : u(v) < t \leq v\}|$. Let s be the last period with this property. Observe that, as $\bar{\beta}_s > 0$ at termination of the algorithm, $u(v) \neq s$ for any $v \in V$. Furthermore, recall that $s \in V$ if and only if $\alpha_s = 0$. Then, by definition of s, we have

$$\begin{aligned} \beta_{s+1} &= |\{v \in V : u(v) < s + 1 \leq v\}| = |\{v \in V : u(v) < s \leq v\}| - (1 - \alpha_s) \\ &< \beta_s + \alpha_s - 1. \end{aligned}$$

However, by Lemma 2.2.1, $\alpha_t + \beta_t \leq \beta_{t+1} + 1$ for all t. Thus, $\beta_t = |\{v \in V : u(v) < t \leq v\}|$ for all t. It is readily checked that this, together with $\alpha_t + \beta_t \geq \beta_{t+1}$ for all t (cf. Lemma 2.2.1), implies that $u(v) < u(v')$ for $v < v'$. □

This result shows that any facet-defining inequality $\alpha x + \beta y \geq \gamma$ of conv($\mathcal{X}$) with $\alpha_t \in \{0, 1\}$ for all t can be written in the form (2.7) with $u(v) < u(v')$ for $v < v'$ and $k = d_{1,t^*} - \gamma + 1$. In order to show that $k \geq 1$, consider the solution (x, y) defined by $y_t = 1$ if and only if $t = 1$ or $t > t^*$, and $x_t = 1$ if and only if $y_t = 1$ or $t \in [2, d_{1,t^*}]$. Thus, production occurs in the first d_{1,t^*} periods and in every period after t^*. Recall from Lemma 2.2.1 that $\beta_1 = 0$ for any facet-defining inequality. Thus, $\alpha x + \beta y = \alpha x \leq d_{1,t^*}$. Hence, $\gamma \leq d_{1,t^*}$, which implies $k \geq 1$.

Now suppose we are given a demand period t^*, an integer $k \geq 1$, a subset $V \subseteq [1, t^*]$ such that $|V| \geq k$, and a set of intervals $[u(v), v]$, $v \in V$, satisfying $u(v) < u(v')$ for $v < v'$. An important question is how to establish the validity of the corresponding inequality (2.7) for conv($\mathcal{X}$). We already showed that the inequality is valid if for every subset W of V of size k the following holds: for every solution in $\mathcal{X}$ in which all periods in $\cup_{w \in W}[u(w), w]$ are used for production, the inventory at the end of period t^* is at least one. The lemma below asserts that this condition is also necessary for (2.7) to be valid. We first show that the aforementioned condition can be formally expressed as

$$\max_{v \in W}\left[\left|\bigcup_{w: w \in W, w \geq v} [u(w), w]\right| - d_{u(v),t^*}\right] \geq 1. \tag{2.8}$$

for every subset W of V of size k. In order to see the correctness of this expression, let $W \subset V$ and define $S = \cup_{w \in W}[u(w), w]$. Now suppose that for any solution in $\mathcal{X}$ for which production occurs in all periods in S the inventory at the end of period t^* is at least one. It is not hard to see that this holds if and only if there is a period $s \in S$ satisfying $|\{t \in S : t \geq s\}| \geq d_{s,t^*} + 1$. The latter condition is equivalent to (2.8) because of the following observation: if v is the maximum period in W satisfying $u(v) \leq s$, then $s \in [u(v), v]$, which yields $|\cup_{w: w \in W, w \geq v}[u(w), w]| = |\{t \in S : t \geq s\}| + s - u(v) \geq d_{s,t^*} + 1 + d_{u(v),s-1} = d_{u(v),t^*} + 1$.

Lemma 2.2.4 *Given a demand period* t^*, *an integer* $k \geq 1$, *a subset* $V \subseteq [1, t^*]$ *such that* $|V| \geq k$, *and a set of intervals* $[u(v), v]$, $v \in V$, *satisfying* $u(v) < u(v')$ *for* $v < v'$. *Then inequality* (2.7) *is valid for* X *if and only if*

$$\max_{v \in W}\left[\left|\bigcup_{w: w \in W, w \geq v} [u(w), w]\right| - d_{u(v),t^*}\right] \geq 1.$$

for every subset W *of* V *of size* k.

PROOF. Rewrite inequality (2.7) again in the form (2.6), thus, as

$$I_{t^*} \geq \sum_{v \in V}(x_v - y_{u(v)+1,v}) - k + 1.$$

We only have to prove the necessity of the condition. Therefore, suppose that there exists a subset W of V satisfying $|W| = k$ and $|\bigcup_{w:w\in W, w\geq v}[u(w), w]| \leq d_{u(v),t^*}$ for every $v \in W$. Then a feasible solution (x, y) that violates (2.7) can be constructed as follows. We can easily deal with the demand in periods $t > t^*$ by setting $y_t = x_t = 1$ for every period $t > t^*$ with positive demand. Hence, we only have to consider the production in the interval $[1, t^*]$. First, define $x_t = 1$ for every $t \in \bigcup_{v\in W}[u(v), v]$. Then, by assumption, the total production in the interval $[u(v), t^*]$ does not exceed the total demand in this interval for any $v \in W$. If the total production up to period t^* is strictly less than d_{1,t^*}, then let U consist of the first $d_{1,t^*} - |\bigcup_{v\in W}[u(v), v]|$ periods not used for production yet, and set $x_t = 1$ for every $t \in U$. Finally, set $y_1 = 1$ if $x_1 = 1$ and set $y_t = 1,\ t > 1$, if $x_t = 1$ and $x_{t-1} = 0$. Notice that if $y_t = 1$, then $t \notin \bigcup_{v\in W}[u(v) + 1, v]$. Then (x, y) is feasible and $x_{1,t^*} = d_{1,t^*}$, hence, $I_{t^*} = 0$. However, as $\{v \in V : x_v - y_{u(v)+1,v} = 1\} \supseteq W$, $\sum_{v\in V}(x_v - y_{u(v)+1,v})$ is at least k. Thus, (x, y) violates (2.7), which shows the necessity of the condition. □

In general, given a set of intervals $[u(v), v]$, there is no easy way to determine k. However, if some restrictions are imposed on the intervals or k, then we are able to describe (facet-defining) inequalities more explicitly. For example, if k equals the number of holes before t^* plus one, then the facet-defining inequalities of conv($\mathcal{X}$) of the form (2.7) can be completely characterized. This is the subject of the following subsection.

2.2.2 Hole-lifted right stock-minimal inequalities

In this subsection we study inequalities $\alpha x + \beta y \geq \gamma$ of the form (2.7) for which the right-hand side equals $d_{1,t^*} - |\{t \leq t^* : t \text{ is a hole}\}|$ for some period t^*. Then the inequality has the following form:

$$\sum_{t\in[1,t^*]\setminus V} x_t + \sum_{v\in V} y_{u(v)+1,v} \quad \geq \quad d_{1,t^*} - |V_0|, \tag{2.9}$$

where $V \subseteq [1, t^*]$, $u(v) \leq v$ for any $v \in V$, and V_0 is the set of holes in $[1, t^*]$. We always assume that $u(v) < v$ for any $v, v' \in V$ such that $v < v'$. This implies that if $u(v) = v$, then $v \notin [u(v'), v']$ for any $v' \in V\setminus\{v\}$. Since $V = \{t \leq t^* : \alpha_t = 0\}$ and $\beta_t = |\{v \in V : u(v) < t \leq v\}|$ for all t, we have $V_0 = \{v \in V : u(v) = v\}$. Furthermore, define $V_1 = V\setminus V_0$. Since the facet-defining production inequalities $x_{1,t} \geq d_{1,t}$ are the only facet-defining inequalities of the form (2.9) with $V_1 = \emptyset$, we assume throughout that $V_1 \neq \emptyset$.

With an inequality of the form (2.9), we associate three sets S_j, $0 \leq j \leq 2$, that partition the set of the demand periods up to period t^* as follows. First, S_0 is the set of demand periods before $\min_{v\in V} u(v)$. Observe that in any feasible solution the demand for a period in S_0 is produced in a period t with $\alpha_t = 1$ and $\beta_t = \beta_{t+1} = 0$. Next, let S_1 be the set of demand periods $t \leq t^*$ for which there is a period $t' \leq t$ such that the number of holes in the interval $[t', t]$ is at least $d_{t',t}$. Hence, if (x, y) is a solution that uses a hole before t^* for production, then this production can be assumed to satisfy the demand for one of the periods in S_1. This implies that the demand for a period $s \in S_2 = \{t \le t^* : d_t = 1 \text{ and } t \notin S_0 \cup S_1\}$ is always assumed to be produced in a period $t \le s$ satisfying $\alpha_t + \beta_t > 0$.

Throughout this subsection, the sets S_j, $0 \le j \le 2$, play an important role. We want to emphasize that these sets, just as V_0 and V_1, are always defined with respect to a certain inequality. For notational convenience, this is not indicated explicitly unless confusion can arise.

Example 2.2.3 Let $d_t = 1$ for $t \in \{2, 5, 7, 8, 9, 11, 12, 15\}$. Take $t^* = 15$, $V_0 = \{3, 4, 12\}$, $V_1 = \{9, 11, 15\}$, $u(9) = 5$, $u(11) = 7$, and $u(15) = 14$. The corresponding inequality (2.9) is

d_t :		1			1		1	1	1		1	1			1	
						$+\, y_6$	$+\, y_7$	$+\, 2y_8$	$+\, 2y_9$	$+\, y_{10}$	$+\, y_{11}$				$+\, y_{15}$	
	x_1	$+\, x_2$			$+\, x_5$	$+\, x_6$	$+\, x_7$	$+\, x_8$		$+\, x_{10}$			$+\, x_{13}$	$+\, x_{14}$		$\ge\ 5.$

For this inequality we have $S_0 = \{2\}$, $S_1 = \{5, 7, 12\}$, and $S_2 = \{8, 9, 11, 15\}$. □

In the sequel we derive necessary and sufficient conditions for an inequality (2.9) to be facet-defining for conv($\mathcal{X}$). In Lemma 2.2.2 we discussed three conditions that need to be satisfied by any facet-defining inequality with x-coefficients in $\{0, 1\}$. These conditions state, among other things, that t^* is a demand period (P1) and that for any period $t < t^*$ the number of holes in the interval $[t, t^*]$ is less than d_{t,t^*} (P2). Thus, if these conditions hold, then $t^* \notin S_1$ and, hence, $|S_1| = |V_0|$. Moreover, since we assume that $V_1 \neq \emptyset$, S_2 contains at least one element, namely t^*, whereas S_0 and S_1 might be empty.

Let us first determine the size of the intervals $[u(v), v]$, $v \in V_1$, for which (2.9) is valid.

Lemma 2.2.5 *Inequality* (2.9) *is valid for $\mathcal{X}$ if and only if $v - u(v) \ge |\{t \in S_2 : t \ge u(v)\}|$ for every $v \in V_1$.*

PROOF. We first prove the sufficiency of the condition. Therefore, assume that $v - u(v) \ge |\{t \in S_2 : t \ge u(v)\}|$ for every $v \in V_1$. We rewrite (2.9) in the form (2.6), i.e., we consider the inequality

$$I_{t^*} \ \ge\ \sum_{v \in V} (x_v - y_{u(v)+1,v}) - |V_0|, \tag{2.10}$$

where I_{t^*} denotes the stock at the end of period t^*.

Define $k^* = |V_0|$ and denote the elements of V_0 by v_k, $1 \le k \le k^*$. Let $\mathcal{X}_k$, $0 \le k \le k^*$, be the set of solutions to DLSP with production in v_j for $k < j \le k^*$. Thus, $\mathcal{X}_k = \{(x, y) \in \mathcal{X} : x_{v_j} = 1, k < j \le k^*\}$. We show by induction that

$$I_{t^*} \ \ge\ \sum_{v \in V_1} (x_v - y_{u(v)+1,v}) + \sum_{j=1}^{k} (x_{v_j} - 1) \tag{2.11}$$

is valid for $\mathcal{X}_k$, $0 \le k \le k^*$. Since (2.10) is (2.11) with $k = k^*$ and $\mathcal{X}_{k^*} = \mathcal{X}$, this establishes the sufficiency of the condition.

We first show that

$$I_{t^*} \geq \sum_{v \in V_1} (x_v - y_{u(v)+1,v}) \tag{2.12}$$

is valid for $\mathcal{X}_0$. Let (x, y) be a solution in $\mathcal{X}_0$. Then all holes before t^* are used for production. By definition of S_1, we can assume that the production in V_0 satisfies the demand in the periods in S_1. Thus, production in a period $t \in \bigcup_{v \in V_1}[u(v), v]$ satisfies the demand in a period in S_2 or contributes one unit to the inventory at the end of period t^*. Now let $\hat{\mathcal{X}}$ be the set of solutions to DLSP with respect to the demand function $\hat{d}$, where $\hat{d}_t = 0$ if $t \in S_1$ and $\hat{d}_t = d_t$ otherwise. Then (2.12) is valid for $\mathcal{X}_0$ if and only if the inequality is valid for $\hat{\mathcal{X}}$. Observe that (2.12) is a right-stock minimal inequality for $\hat{\mathcal{X}}$ (cf. Subsection 2.2.1), which is valid if $v - u(v) \geq \hat{d}_{u(v),t^*}$ for all $v \in V_1$. Hence, since $|\{t \in S_2 : t \geq u(v)\}| = \hat{d}_{u(v),t^*}$, (2.12) is valid for $\mathcal{X}_0$.

Now suppose that the validity of (2.11) for $\mathcal{X}_k$ has been established for $0 \leq k \leq k'$, where $k' < k^*$. We claim that

$$I_{t^*} \geq \sum_{v \in V_1} (x_v - y_{u(v)+1,v}) + \sum_{j=1}^{k'+1} (x_{v_j} - 1) \tag{2.13}$$

is satisfied by all $(x, y) \in \mathcal{X}_{k'+1}$. This obviously holds for $\{(x, y) \in \mathcal{X}_{k'+1} : x_{v_{k'+1}} = 1\} = \mathcal{X}_{k'}$. Therefore, let (x, y) be a solution in $\mathcal{X}_{k'+1}$ without production in period $v_{k'+1}$. Let $(\bar{x}, \bar{y})$ be the solution obtained from (x, y) by setting $x_{v_{k'+1}}$ and $y_{v_{k'+1}}$ to one. Note that $y_{v_{k'+1}}$ has coefficient zero in (2.13). Since (x, y) is a feasible solution to DLSP, the extra unit produced in period $v_{k'+1}$ increases the inventory at the end of period t^* by one, hence, $\bar{I}_{t^*} = I_{t^*} + 1$. Obviously, $(\bar{x}, \bar{y}) \in \mathcal{X}_{k'}$, thus, by the induction hypothesis,

$$\begin{aligned} I_{t^*} &= \bar{I}_{t^*} - 1 \geq \sum_{v \in V_1} (\bar{x}_v - \bar{y}_{u(v)+1,v}) + \sum_{j=1}^{k'} (\bar{x}_{v_j} - 1) - 1 \\ &= \sum_{v \in V_1} (x_v - y_{u(v)+1,v}) + \sum_{j=1}^{k'} (x_{v_j} - 1) - 1. \end{aligned}$$

This shows the validity of (2.13) for (x, y) and, hence, for $\mathcal{X}_{k'+1}$.

In order to prove the necessity of the condition, we define

$$M(W) = \max_{v \in W} [| \bigcup_{w: w \in W, w \geq v} [u(w), w]| - d_{u(v),t^*}].$$

Recall from Lemma 2.2.4 that $M(W) \geq 1$ means that the inventory at the end of period t^* has size at least one when all periods in $\bigcup_{v \in W}[u(v), v]$ are used for production. This lemma yields that (2.9) is only valid if $M(W) \geq 1$ for every subset W of V of size $|V_0| + 1$.

Suppose that $\bar{v} - u(\bar{v}) < |\{t \in S_2 : t \geq u(\bar{v})\}|$ for some $\bar{v} \in V_1$. Define $W = V_0 \cup \{\bar{v}\}$. If $v > \bar{v}$, then

$$| \bigcup_{w: w \in W, w \geq v} [u(w), w]| = |\{w \in V_0 : w \geq u(v)\}| \leq |\{t \in S_1 : t \geq u(v)\}| \leq d_{u(v),t^*}.$$

Otherwise,

$$
\begin{aligned}
|\bigcup_{w:w\in W, w\geq v} [u(w), w]| &= |\{w \in V_0 : w \geq u(v)\}| + \bar{v} - u(\bar{v}) + 1 \\
&\leq |\{t \in S_1 \cup S_2 : t \geq u(v)\}| = d_{u(v),t^*},
\end{aligned}
$$

since $v = u(v) < u(\bar{v})$ if $v < \bar{v}$. Thus, $\bar{v} - u(\bar{v}) < |\{t \in S_2 : t \geq u(\bar{v})\}|$ implies $M(W) \leq 0$. This concludes the proof of the lemma. □

In the proof we already noticed that the valid inequalities of the form (2.9) with $V_0 = \emptyset$ are the right stock-minimal (RSM) inequalities (2.5), which are the direct adaption of the right supermodular inequalities of Constantino [5] to DLSP. Roughly speaking, valid inequalities (2.9) are obtained from RSM inequalities by introducing holes in the interval $[1, t^*]$. In the proof of Lemma 2.2.5 the introduction of the holes occurs by lifting. Inequalities (2.9) are therefore called *hole-lifted right stock-minimal inequalities* or *HRSM inequalities* for short. Obviously, an RSM inequality is an HRSM inequality with $V_0 = \emptyset$.

In the sequel we derive necessary and sufficient conditions for an HRSM inequality to define a facet of conv($\mathcal{X}$). One necessary condition immediately follows from Lemma 2.2.5. Let $\alpha x + \beta y \geq \gamma$ be a valid HRSM inequality with $\bar{v} - u(\bar{v}) > |\{t \in S_2 : t \geq u(\bar{v})\}|$ for some $\bar{v} \in V_1$. Observe that $|\{t \in S_2 : t \geq u(\bar{v}) + 1\}| \leq |\{t \in S_2 : t \geq u(\bar{v})\}| \leq \bar{v} - (u(\bar{v}) + 1)$. Now the above lemma asserts that the inequality remains valid if $u(\bar{v})$ is replaced by $u(\bar{v}) + 1$. Recall that $\beta_t = |\{v \in V : u(v) < t \leq v\}$. Thus, substituting $u(\bar{v} + 1)$ for $u(\bar{v})$ decreases $\beta_{u(\bar{v})+1}$ by one. This implies the following condition.

Corollary 2.2.6 *If an HRSM inequality defines a facet of* conv($\mathcal{X}$), *then* $v - u(v) = |\{t \in S_2 : t \geq u(v)\}|$ *for every* $v \in V_1$. □

However, the following example shows that this condition is not sufficient.

Example 2.2.4 Let $d_t = 1$ for $t \in \{2, 3, 5\}$. Let $t^* = 5$ and $V = \{5\}$. If $u(5) < 5$, then $S_1 = \emptyset$ and $S_2 = \{t : d_t = 1 \text{ and } t \geq u(5)\}$, hence, $|S_2| = d_{u(5),t^*}$ for the corresponding HRSM inequality. One readily checks that if $u(5) \in \{3, 4, 5\}$, then $5 - u(5) = d_{u(5),t^*}$. Thus, by Lemma 2.2.5,

$$(1): x_{1,4} + y_5 \geq 3, \quad (2): x_{1,4} + y_{4,5} \geq 3, \quad \text{and} \quad (3): x_{1,4} + y_{3,5} \geq 3$$

are all valid HRSM inequalities. Obviously, neither (2) nor (3) can define a facet for the instance at hand. □

Lemma 2.2.7 *If an HRSM inequality defines a facet of* conv($\mathcal{X}$), *then* $u(v) \in S_1$ *for every* $v \in V_1$ *satisfying* $d_{u(v)} = 1$.

PROOF. Let $\alpha x + \beta y \geq \gamma$ be an HRSM inequality such that $v - u(v) = |\{t \in S_2 : t \geq u(v)\}|$ for all v. Moreover, let $\bar{v} \in V_1$ satisfy $d_{u(\bar{v})} = 1$ and $u(\bar{v}) \in S_2$. We will show that $\alpha x + \beta y \geq \gamma$, where $\bar{\beta}_{u(\bar{v})+1} = \beta_{u(\bar{v})+1} - 1$ and $\bar{\beta}_t = \beta_t$ for all $t \neq u(\bar{v}) + 1$, is also satisfied by all solutions $(x, y) \in \mathcal{X}$. This implies that $\alpha x + \beta y \geq \gamma$ does not define a facet of conv($\mathcal{X}$).

Write $\bar{\beta} y$ as $\sum_{v \in V_1} y_{\bar{u}(v)+1,v}$, then $\bar{u}(\bar{v}) = u(\bar{v}) + 1$ and $\bar{u}(v) = u(v)$ otherwise. For the new inequality we further have $\bar{S}_2 = S_2 \setminus \{u(\bar{v})\}$ (and $\bar{S}_0 = S_0 \cup \{u(\bar{v})\}$, cf. Example 2.2.4) if $u(\bar{v}) = \min_{v \in V} u(v)$, and $\bar{S}_2 = S_2$ otherwise. Then $v - \bar{u}(v) = |\{t \in \bar{S}_2 : t \geq \bar{u}(v)\}|$ for all $v \in V_1$. Hence, by Lemma 2.2.5, the new inequality is also valid for $\mathcal{X}$. □

Combining Corollary 2.2.6 and Lemma 2.2.7 yields the following necessary condition for an HRSM inequality to define a facet of conv($\mathcal{X}$):

C1. For every $v \in V_1$, $v - u(v) = |\{t \in S_2 : t \geq u(v)\}|$ and $u(v) \notin S_2$.

Recall that a facet-defining HRSM inequality must satisfy the conditions P1–P3 discussed in Lemma 2.2.2. In Theorem 2.2.9 it will be shown that, together with C1, these conditions are also sufficient for an HRSM inequality to define a facet of conv($\mathcal{X}$), provided that there exists a period t before $\min_{v \in V_1} u(v)$ that is not a hole. We already observed that this implies that t^* is a demand period and that $t^* \in S_2$. Furthermore, combining P3 and C1 yields the following:

Lemma 2.2.8 *If an HRSM inequality satisfies* P3 *and* C1, *then* $v_1 - u(v_1) = |S_2|$, *where* $v_1 = \min_{v \in V_1} v$.

PROOF. The validity of C1 yields that $v_1 - u(v_1) \leq |S_2|$. Now suppose that strict inequality holds. Then there is a period in S_2, say t, such that $t < u(v_1)$. By definition of S_2 and v_1, there must be a hole before t. Let s be the first hole. Since $t \notin S_1$, the number of holes in the interval $[s, t]$ must be strictly less than $d_{s,t}$. However, this contradicts the assumption that P3 is satisfied. □

In the proof of Theorem 2.2.9 we use the following notation with respect to an HRSM inequality $\alpha x + \beta y \geq \gamma$. For t satisfying $\beta_t > 0$ we define $v(t) = \max\{v \in V_1 : u(v) < t \leq v\}$. Since $u(v) < u(v')$ if $v, v' \in V$ and $v < v'$, and $u(v) = v$ for every $v \in V_0$, we have $V_0 \cap [t, v(t)] = \emptyset$. Hence, every period $\tau \in [t, v(t)]$ with $\alpha_\tau = 0$ belongs to V_1. This yields

$$\beta_t = |\{v \in V_1 : u(v) < t \leq v\}| = |\{v \in V_1 : t \leq v \leq v(t)\}| = \sum_{\tau=t}^{v(t)} (1 - \alpha_\tau). \qquad (2.14)$$

Furthermore, define $v_1 = \min_{v \in V_1} v$ and denote the elements of the set S_2 by $t_1, t_2, \ldots, t_{|S_2|}$, such that $t_1 < t_2 < \ldots < t_{|S_2|} = t^*$.

Theorem 2.2.9 *Let* $\alpha x + \beta y \geq \gamma$ *be an HRSM inequality satisfying* P1–P3 *and* C1, *and suppose that there is a period* $t < u(v_1)$ *that is not a hole. Then* $\alpha x + \beta y \geq \gamma$ *defines a facet of* conv($\mathcal{X}$).

PROOF. First, observe that Lemma 2.2.5 assures the validity of the inequality. We prove our claim by providing $2T-1$ linearly independent directions in the set $\bar{X} = \{(x, y) \in X : \alpha x + \beta y = \gamma\}$. We start by constructing two sets of solutions in $\bar{X}$: (x^v, y^v) for $v \in V_1$ and $(\bar{x}^t, \bar{y}^t)$ for t satisfying $\beta_t > 0$. In all these solutions production occurs in every period $t \in S_0 \cup V_0$ and in every demand period $t > t^*$. Note that $\beta_t = 0$ for all the aforementioned production periods, hence, for convenience, we set $y_t = 1$ for all these periods. Thus, for the partial solution defined up to now we have $\alpha x + \beta y = |S_0|$. The production in a period $t \in S_0$ or $t > t^*$ is assumed to satisfy the demand in that period. Furthermore, we can assume that the production in V_0 satisfies the demand for the periods in S_1. Hence, the solutions under construction only differ in the periods in which the demand for the periods in S_2 is produced.

First, for $v \in V_1$ we define (x^v, y^v) to be the solution in which the demand for S_2 is produced in the intervals $[u(v_1), v_1 - v + u(v) - 1]$ (empty if $v = v_1$) and $[u(v), v-1]$. Lemma 2.2.8 states that $v_1 - u(v_1) = |S_2|$ and $v - u(v) = |\{t \in S_2 : t > u(v)\}|$. This solution is clearly feasible. Using $\alpha_t = 1$ for $t \in [u(v_1), v_1 - 1]$, and $\beta_{u(v)} = |\{w \in V_1 : u(w) < u(v) \le w\}| = |\{w \in V_1 : u(v) \le w < v\}| = \sum_{t=u(v)}^{v-1}(1-\alpha_t)$, we get

$$\begin{aligned}\alpha x^v + \beta y^v &= |S_0| + \beta_{u(v_1)} y^v_{u(v_1)} + \alpha_{u(v_1), v_1 - v + u(v) - 1} + \beta_{u(v)} + \alpha_{u(v), v-1} \\ &= |S_0| + (v_1 - v + u(v) - u(v_1)) + v - u(v) \;=\; |S_0| + |S_2| \;=\; d_{1,t^*} - |V_0|,\end{aligned}$$

hence, $(x^v, y^v) \in \bar{X}$. In particular the solution (x^{v_1}, y^{v_1}) will be often used in the sequel as a starting point for the construction of directions in $\bar{X}$. This is the solution in which the demand for the periods in S_2 is produced in the interval $[u(v_1), v_1 - 1]$.

Next, for t satisfying $\beta_t > 0$ we denote by $(\bar{x}^t, \bar{y}^t)$ the solution in which the demand for S_2 is produced in the intervals $[u(v_1), v_1 - v(t) + t - 2]$ (again, this interval might be empty) and $[t, v(t)]$, where $v(t) = \max\{v \in V_1 : u(v) < t \le v\}$. By C1, $|\{\tau \in S_2 : \tau > u(v(t))\}| = v(t) - u(v(t))$, hence, $|\{\tau \in S_2 : \tau \ge t\}| = v(t) - u(v(t)) - (t - u(v(t)) - 1) = v(t) - t + 1$. Thus, $(\bar{x}^t, \bar{y}^t)$ is feasible. Using (2.14), which states that $\beta_t + \alpha_{t, v(t)} = v(t) - t + 1$, it is readily checked that $\alpha \bar{x}^t + \beta \bar{y}^t = \gamma$.

In the sequel we show that the following directions are in $\bar{X}$:

- (i) $e(x_t)$ for all t satisfying $\alpha_t = 0$;
- (ii) $e(y_t)$ for all t satisfying $\beta_t = 0$;
- (iiia) $e(x_t) + e(y_t) - e(x_{v_1 - 1})$ for all t satisfying $\beta_t > 0$ and ($t = T$ or $\beta_{t+1} = 0$);
- (iiib) $e(x_t) + e(y_t) - e(y_{t+1}) - e(x_{t'})$ for some $t' \ne t$ and all $t < T$ satisfying $\beta_t > 0$ and $\beta_{t+1} > 0$;
- (iv) $e(x_t) - e(x_{v_1 - 1})$ for all $t \ne v_1 - 1$ satisfying $\alpha_t = 1$.

Since $\alpha_{v_1 - 1} = 1$, (i) and (iv) yield $T-1$ different directions, whereas (ii), (iiia), and (iiib) yield T different directions. It is left to the reader to check that these $2T-1$ directions are linearly independent. This proves that $\alpha x + \beta y \ge \gamma$ defines a facet of conv(X).

Ad (i). Let t be a period such that $\alpha_t = 0$. Suppose there exists a solution $(x, y) \in \bar{X}$ satisfying $x_t = 0$ and $x_{t-1} = 1$ or $y_t = 1$. Then the solution $(\bar{x}, \bar{y})$ obtained from (x, y) by adding one unit of production in period t is also in $\bar{X}$, and $(\bar{x}, \bar{y}) - (x, y) = e(x_t)$. If $\beta_t > 0$, i.e., if $t = v$ for some $v \in V_1$, then (x^v, y^v) satisfies $x^v_{v-1} = 1$ and $x^v_v = 0$. In order to show that such a solution exists when $\beta_t = 0$, i.e., when t is a hole, it suffices to give a solution $(x, y) \in \bar{X}$ satisfying $x_s = 0$, where s is the first hole. Consider (x^{v_1}, y^{v_1}). If the solution obtained from (x^{v_1}, y^{v_1}) by moving the production in period s to period v_1 (and keeping y_s to one) is feasible, then we are done. Feasibility trivially holds if $s \geq u(v_1)$. Otherwise, we have that for every $t \in [s, u(v_1) - 1]$ the total production in $[s, t]$ equals the number of holes in this interval, which, by P3, is strictly larger than $d_{s,t}$.

Ad (ii). Let t be a period such that $\beta_t = 0$. Then the direction $e(y_t)$ is easily constructed from a solution $(x, y) \in \bar{X}$ satisfying $y_t = 0$. We have already seen that such a solution exists when t is a hole, i.e., when $\alpha_t = 0$. Consider the solution $(\bar{x}^{u(v_1)+1}, \bar{y}^{u(v_1)+1})$ defined above. In this solution the demand for the periods in S_2 is produced in the interval $[u(v_1) + 1, v_1]$, thus, in periods with positive y-coefficient. Hence, for every t satisfying $\beta_t = 0$ and $\alpha_t = 1$ we have $\bar{y}_t^{u(v_1)+1} = 0$, except when $t \in S_0$. Recall that we always assume that $d_1 = 0$, thus if $S_0 \neq \emptyset$, then $\alpha_1 = 1$ and $\bar{x}_1^{u(v_1)+1} = 0$. For $t \in S_0$ a solution $(x, y) \in \bar{X}$ with $\beta_t = 0$ is now readily constructed.

Ad (iiia). Recall that for t satisfying $\beta_t > 0$ we defined $v(t) = \max\{v \in V_1 : u(v) < t \leq v\}$. Now let t be a period satisfying $\beta_t > 0$ and either $t = T$ or $\beta_{t+1} = 0$. Using $\beta_t = |\{v \in V : u(v) < t \leq v\}|$, it is not difficult to see that in this case the following holds: $t = v(t)$ and $t \neq u(v)$ for any $v \in V_1$.

Consider the solution $(\bar{x}^t, \bar{y}^t)$. Since $t = v(t)$, the demand for the periods in S_2 is produced in the interval $[u(v_1), v_1 - 2]$ and in period t. Moving the production in period t to period $v_1 - 1$ yields the solution (x^{v_1}, y^{v_1}). Thus, $(\bar{x}^t, \bar{y}^t) - (x^{v_1}, y^{v_1}) = e(x_t) + e(y_t) - e(x_{v_1-1})$.

Ad (iiib). Let $t < T$ be a period satisfying $\beta_t > 0$ and $\beta_{t+1} > 0$. We will construct the direction $e(x_t) + e(y_t) - e(y_{t+1}) - e(x_{t'})$ for some $t' \neq t$. First, we show that if $t = u(v)$ for some $v \in V_1$, then the above direction with $t' = v$ is in $\bar{X}$. If $t \neq u(v)$ for any $v \in V_1$, then $t < v(t)$ (cf. (iiia)). In this case the direction $e(x_t) + e(y_t) - e(y_{t+1}) - e(x_{v_1-(v(t)-t+1)})$ will be established.

First, suppose that $t = u(v)$ for some $v \in V_1$. Consider the two solutions (x^v, y^v) and $(\bar{x}^{u(v)+1}, \bar{y}^{u(v)+1})$ defined previously. Observe that the second solution is obtained from the first by moving the production in period $u(v)$ to period v. These two solutions provide the direction $e(x_{u(v)}) + e(y_{u(v)}) - e(y_{u(v)+1}) - e(x_v)$.

Now assume that $t \neq u(v)$ for any $v \in V_1$. Then $\beta_{t+1} > 0$ implies $t < v(t)$. For convenience, define $i = v(t) - t + 1$. Note that $i \leq |S_2| \leq v_1 - u(v_1)$. Similar as in (iiia) we start from the solution $(\bar{x}^t, \bar{y}^t)$, in which the demand for the periods in S_2 is produced in the intervals $[u(v_1), v_1 - i - 1]$ and $[t, v(t)]$. Denote by (x, y) the solution obtained from $(\bar{x}^t, \bar{y}^t)$ by moving the production in period t to period $v_1 - i$. Since $t \neq u(v)$ for any $v \in V_1$, we have $\beta_{t+1} = |\{v \in V_1 : u(v) < t+1 \leq v\}| = |\{v \in V_1 : u(v) < t < v\}|$. Moreover, since $t \in V_1$ if and only if $\alpha_t = 0$, we have $\beta_t = |\{v \in V_1 : u(v) < t < v\}| + (1-\alpha_t) = \beta_{t+1} + (1-\alpha_t)$. Hence,

$$\alpha x + \beta y \;=\; \alpha \bar{x}^t + \beta \bar{y}^t + \beta_{t+1} + \alpha_{v_1-i} - \alpha_t - \beta_t \;=\; \gamma,$$

thus, $(x, y) \in \bar{X}$. Together with $(\bar{x}^t, \bar{y}^t)$, (x, y) yields the direction $e(x_t) + e(y_t) - e(y_{t+1}) - e(x_{v_1-i})$.

Ad (iv). Finally, we construct the direction $e(x_t) - e(x_{v_1-1})$ for every $t \neq v_1 - 1$ satisfying $\alpha_t = 1$. In most cases the direction is established by decomposing it into $(e(x_t) - e(x_s)) + (e(x_s) - e(x_{v_1-1}))$, where s is the first period that is not used for production in (x^{v_1}, y^{v_1}). Recall that (x^{v_1}, y^{v_1}) is the solution in which the demand for the periods in S_2 is produced in the interval $[u(v_1), v_1 - 1]$.

Let us first show that $s < u(v_1)$. If $S_0 \neq \emptyset$, then we already observed that $\alpha_1 = 1$ and $x_1^{v_1} = 0$. Hence, in this case, we have $s = 1$. Otherwise, if $S_0 = \emptyset$, then the only production periods in (x^{v_1}, y^{v_1}) up to t^* are the holes and the periods in the interval $[u(v_1), v_1 - 1]$. By assumption, there exists a period before $u(v_1)$ that is not a hole. From the above observation it follows that this period is not used for production in (x^{v_1}, y^{v_1}). Thus, also when $S_0 = \emptyset$ we have $s < u(v_1)$. Denote by (x, y) the solution obtained from (x^{v_1}, y^{v_1}) by setting y_s to one, and by $(\bar{x}, \bar{y})$ the solution obtained from (x, y) by moving the production in $v_1 - 1$ to s. Since $\alpha_s = 1$ and $\beta_s = 0$, both solutions are in $\bar{X}$. From these solutions the direction $e(x_s) - e(x_{v_1-1})$ is readily constructed.

From now on, let t be a period satisfying $\alpha_t = 1$ and $t \neq v_1 - 1$. First, we consider the case that $\beta_t = 0$ and $x_t^{v_1} = 0$. We may assume that in (x^{v_1}, y^{v_1}) the demand in t^* is satisfied by the production in $v_1 - 1$. Thus, if $x_t^{v_1} = 0$, then the solution obtained from (x^{v_1}, y^{v_1}) by moving the production in period $v_1 - 1$ to t is easily seen to be in $\bar{X}$. From these two solutions the direction $e(x_t) - e(x_{v_1} - 1)$ can be constructed.

For t satisfying $\beta_t = 0$, $x_t^{v_1} = 1$, and $t \neq u(v_1)$ we construct $e(x_t) - e(x_s)$. Note that $t \in S_0$ in this case, thus, $s = 1$. Hence, the direction $e(x_t) - e(x_s)$ is easily constructed from (x^{v_1}, y^{v_1}) and the solution obtained from the latter by moving the production in period t to period 1.

Next, let $\beta_t > 0$ and $t \in \bigcup_{v \in V_1 \setminus \{v_1\}} [u(v), v - 1]$. Define $\bar{v} = \min\{v \in V_1 : v > t\}$. Then $\alpha_\tau = 1$ for every $\tau \in [t, \bar{v} - 1]$ and $u(\bar{v}) < t$, since $\beta_t = |\{v \in V_1 : u(v) < t \leq v\}| = |\{v \in V_1 : v \geq \bar{v} \text{ and } t > u(v)\}| > 0$. Consider the solution $(x^{\bar{v}}, y^{\bar{v}})$ for which the demand for the periods in S_2 is produced in the intervals $[u(v_1), v_1 - \bar{v} + u(\bar{v}) - 1]$ and $[u(\bar{v}), \bar{v} - 1]$. Let (x, y) be the solution obtained from $(x^{\bar{v}}, y^{\bar{v}})$ by moving the production in the interval $[t+1, \bar{v} - 1]$ to $[v_1 - \bar{v} + u(\bar{v}), v_1 - t + u(\bar{v}) - 2]$. Then $(x, y) \in \bar{X}$ and $x_s = x_s^{\bar{v}} = 0$. Using similar arguments as before, the direction $e(x_t) - e(x_s)$ is now readily established.

We are left with the construction of the directions $e(x_t) - e(x_s)$ for $t \in [u(v_1), v_1 - 2]$. Assume that $v_1 - u(v_1) = |S_2| \geq 2$, otherwise we are done. Let $t \in [u(v_1), v_1 - 2]$ and define $i = v_1 - t$. We claim that there exists period t' such that $\beta_{t'} > 0$ and $v(t') - t' + 2 = i$. Then the solution $(\bar{x}^{t'}, \bar{y}^{t'})$ satisfies $\bar{x}^{t'}_t = 1$, and $\bar{x}^{t'}_{t+1} = \bar{x}^{t'}_s = 0$. Again, the solution obtained from $(x^{t'}, y^{t'})$ by moving the production in period t to period s is in $\mathcal{X}$ and yields, together with $(x^{t'}, y^{t'})$, the direction $e(x_t) - e(x_s)$. In order to prove our claim, consider the period $t_{|S_2|-i+2}$. Suppose first that $\beta_{t_{|S_2|-i+2}} = 0$. It is not hard to see that there exists a period $v \in V_1$ satisfying $v < t_{|S_2|-i+2}$, $\beta_v = 1$, and $\beta_{v+1} = 0$. In this case we take $t' = v - i + 2$. Then C1 implies $v - u(v) = |\{t \in S_2 : t \geq u(v)\}| \geq i - 1$, hence, $u(v) < t' \leq v$, thus, $\beta_{t'} > 0$, and $v(t') = v$. If $\beta_{t_{|S_2|-i+2}} > 0$, then take $t' = v(t_{|S_2|-i+2}) - i + 2$. Using the same arguments as before, we find that $\beta_{t'} > 0$ and $v(t') = v$. Hence, the direction $e(x_t) - e(x_s)$ can be constructed for every $t \in [u(v_1), v_1 - 2]$. This concludes the proof of the theorem. □

Example 2.2.5 Let $T = 12$ and $d_t = 1$ for $t \in \{3, 6, 8, 9, 11, 12\}$. Let $t^* = 12$, $V_0 = [4, 5]$, $V_1 = [9, 12]$, $u(9) = 6$, $u(10) = 7$, $u(11) = 8$, and $u(12) = 10$. The corresponding HRSM inequality is

$$\begin{array}{lll} & & + y_7 + 2y_8 + 3y_9 + 2y_{10} + 2y_{11} + y_{12} \\ x_1 + x_2 + x_3 & + x_6 + x_7 & + x_8 \end{array} \geq 4$$

with $S_0 = \{3\}$, $S_1 = \{6, 8\}$, and $S_2 = \{9, 11, 12\}$. By Theorem 2.2.9 the inequality defines a facet. This also holds for

$$\begin{array}{l} \quad\quad + y_3 + y_4 + y_5 + 2y_6 + y_7 + y_8 \\ x_1 + x_2 + x_3 + x_4 + x_5 \quad + x_7 \quad\quad + x_{10} \quad + x_{12} \end{array} \geq 4,$$

the HRSM inequality defined by $t^* = 12$, $V_0 = \{9, 11\}$, $V_1 = \{6, 8\}$, $u(6) = 2$, and $u(8) = 5$. Here $S_0 = \emptyset$, $S_1 = \{9, 11\}$, and $S_2 = \{3, 6, 8, 12\}$. □

If $u(v_1) = 1$ or all periods before $u(v_1)$ are holes, then C1 does not suffice for an HRSM inequality to define a facet of conv($\mathcal{X}$), as the following example shows.

Example 2.2.6 Let $d_t = 1$ for $t \in \{4, 6, 7\}$. Take $t^* = 7$, $V = V_1 = \{4, 6, 7\}$, $u(4) = 1$, $u(6) = 3$, and $u(7) = 5$. Then

$$\begin{array}{l} \quad\ + y_2 + y_3 + 2y_4 + y_5 + 2y_6 + y_7 \\ x_1 + x_2 + x_3 \quad\quad\quad + x_5 \end{array} \geq 3$$

is a valid HRSM inequality with $S_0 = S_1 = \emptyset$ and $S_2 = V$ One readily checks that P1–P3 and C1 are satisfied. However, every solution (x, y) that satisfies the above inequality at equality also satisfies the inequality $x_2 + y_{3,6} \geq 1$ at equality. The latter inequality is the facet-defining inequality $x_{d_{1,t}} + y_{d_{1,t}+1,t} \geq 1$ with $t = 6$ (cf. Section 2.1). □

In this case one extra condition is needed to guarantee that the inequality defines a facet of conv($\mathcal{X}$). The proofs of the necessity and sufficiency of this condition are rather technical and do not provide any further insight. Moreover, there is no easy way to test whether a given inequality satisfies this condition. Therefore, the following result is merely given for sake of completeness. The same notation as in Theorem 2.2.9 is used.

Theorem 2.2.10 *Suppose* $\alpha x + \beta y \geq \gamma$ *is an HRSM inequality satisfying* P1–P3, C1, *and either* $u(v_1) = 1$ *or all periods before* $u(v_1)$ *are holes. Then the inequality defines a facet of* conv(X) *if and only if for every* $i \in \{2, \ldots, |S_2|\}$ *one of the following holds:*
(i) *there exists a period* $t \leq t_{|S_2|-i+1}$ *such that* $\beta_t > 0$ *and* $v(t) \leq t + i - 2$;
(ii) *there exists a period* $v \in V_1$ *such that* $v_1 < v \leq t_{|S_2|+i-1}$ *and* $\alpha_{v-j} = 1$ *for every* $j \in \{1, \ldots, i\}$. □

Notice that the extra condition stated in the above theorem is always satisfied if $|S_2| = 1$. In this case, $S_2 = \{t^*\}$, where t^* is a demand period, and $S_1 = \{t < t^* : d_t = 1\}$. In order to satisfy C1, $u(v) = v - 1$ must hold for every $v \in V_1$ since $\{\tau \in S_2 : \tau \geq t\} = \{t^*\}$ for every $t \leq t^*$. Thus, in this case, $\alpha_t + \beta_t \leq 1$ for every t. If we take $V_0 = [1, d_{1,t^*} - 1]$ and $V_1 = [d_{1,t^*} + 1, t^*]$, then we obtain the inequality $x_{d_{1,t^*}} + y_{d_{1,t^*}+1,t^*} \geq 1$, hence, the inequalities (2.4) discussed at the end of Section 2.1 are a special case of the HRSM inequalities.

Before we address the separation problem for the HRSM inequalities, let us discuss how they relate to inequalities derived for generalizations of DLSP. We already mentioned that the RSM inequalities, i.e., the HRSM inequalities with $V_0 = \emptyset$, are a direct adaptation of the right supermodular inequalities of Constantino to DLSP. Sastry [38] derives a class of inequalities for the extension of DLSP with both startup and setup costs called *skip inequalities*. Denote by X_t, Y_t, and Z_t respectively the production variables, the setup variables, and the startup variables for this problem. If $\alpha x + \beta y \geq \gamma$ is a facet-defining HRSM inequality, then $\lambda X + \mu Y + \nu Z \geq \gamma$, with $\lambda_t = \alpha_t$ if $t \in [1, t^*] \setminus \bigcup_{v \in V}[u(v), v]$ and zero otherwise, $\mu_t = \alpha_t$ if $t \in [u(v), v]$ for some $v \in V$ and zero otherwise, and $\nu_t = \beta_t$ for all t, is a facet-defining skip inequality. Magnanti and Sastry [27] describe a linear programming based separation algorithm for a subset of the skip inequalities. With some slight modifications the separation algorithm described below applies to a much broader class of skip inequalities, which includes the subset for which separation is discussed in [27].

Let us now consider the separation problem for the HRSM inequalities. Y. Pochet (personal communication) proposes a dynamic programming algorithm for separating, not necessarily facet-defining, HRSM inequalities. This algorithm runs in $O(T^9)$ time, hence, the separation problem for the HRSM inequalities can be solved in polynomial time. However, it is obvious that such an algorithm is too time-consuming to be of practical use in a cutting plane method. Therefore we do not give any details. Instead we describe an $O((d_{1,T}T)^2)$ algorithm for the subclass of HRSM inequalities that, besides P1–P3 and C1, satisfy the following condition:

C2. If $V_0 \neq \emptyset$, then $\max_{v \in V_0} v < \min_{v \in V_1} v$,

i.e., if $t \leq t^*$ is the first period satisfying $\beta_t > 0$, then there are no holes in the interval $[t-1, t^*]$. This implies that $\max_{t \in S_1} t < \min_{t \in S_2} t$. In Example 2.2.5 only the first inequality satisfies this condition. Hence, the second inequality cannot be found by the separation algorithm described below.

Denote by s_i the ith demand period, i.e., $d_{s_i} = 1$ and $d_{1,s_i} = i$. Conditions P1 and P2 hold if and only if $t^* = s_{i^*}$ for some $i^* \in \{1, \ldots, d_{1,T}\}$ and $s_{i^*} \in S_2$. The validity of C2 implies the following. If $V_0 \neq \emptyset$, then $S_1 = \{s_{i_1+1}, \ldots, s_{i_2}\}$ and $S_2 = \{s_{i_2+1}, \ldots, s_{i^*}\}$ for some i_1 and i_2 satisfying $0 \leq i_1 < i_2 < i^*$. Moreover, $V_0 \subseteq [s_{i_1}+1, \bar{u}-1]$, where $\bar{u} = \min_{v \in V_1} u(v)$ and $|V_0| = i_2 - i_1$. Now in order to satisfy P3 we must have $s_{i_2} \geq \bar{u}$ or, equivalently, $i_2 \geq d_{1,\bar{u}-1}+1$. Furthermore, the number of holes in the interval $[s_{i_1}+1, s_i]$ must be at least $i - i_1 + 1$ for any $i \in \{i_1+1, \ldots, d_{1,\bar{u}-1}\}$, thus,

$$|[s_{i_1}+1, s_i] \cap V_0| \ \geq \ i - i_1 + 1 \ \text{ for } i \in \{i_1+1, \ldots, d_{1,\bar{u}-1}\}. \tag{2.15}$$

If $V_0 = \emptyset$, then $S_2 = \{s_{d_{1,\bar{u}-1}+1}, \ldots, s_{i^*}\}$. In this case define $i_2 = d_{1,\bar{u}-1}$. Condition C1 is satisfied if for all $v \in V_1$ the following holds: $v - u(v) = |\{t \in S_2 : t \geq u(v)\}|$ and if $d_{u(v)} = 1$, then $u(v) = s_i$ for some $i \leq i_2$. This yields

$$v - u(v) \ = \ i^* - \max(i_2, d_{1,u(v)}), \tag{2.16}$$

since $\min\{i : s_i \in S_2 \text{ and } s_i \geq u(v)\} = \max(i_2+1, d_{1,u(v)-1}+1) = \max(i_2, d_{1,u(v)} - d_{u(v)}) + 1$ and $i_2 \geq d_{1,u(v)}$ if $d_{u(v)} = 1$.

Let $(\hat{x}, \hat{y})$ be a solution to the LP-relaxation of DLSP. The question to be answered is: does there exist an HRSM inequality satisfying P1–P3, C1, and C2 that is violated by $(\hat{x}, \hat{y})$, thus, for which

$$(\hat{x}_{1,s_{i^*}} - i^*) + (|V_0| - \sum_{t \in V_0} \hat{x}_t) + \sum_{v \in V_1} (\hat{y}_{u(v)+1,v} - \hat{x}_v) \ < \ 0? \tag{2.17}$$

In our algorithm we first determine for each i^* and V_1 the set V_0 satisfying the aforementioned conditions for which the left-hand side of (2.17) is minimal. An important observation is that this set is the same for all i^* and V_1 with the same values of $u = \min_{v \in V_1} u(v)$ and $i_2 = i^* + u - \min_{v \in V_1} v$ $(= \min_{i \in S_2} i - 1)$; we will therefore refer to it as $F_u(i_2)$. Now for a given period u and a given index $i_2 < d_{1,T}$ satisfying $s_{i_2} \geq u$, $F_u(i_2)$ is determined as follows. For $i_1 \in \{0, \ldots, i_2 - 1\}$ let $F_u(i_1, i_2)$ be a subset of $i_2 - i_1$ periods in $[s_{i_1}+1, u-1]$ such that (2.15) is satisfied for $V_0 = F_u(i_1, i_2)$ and such that $\sum_{t \in F_u(i_1,i_2)} \hat{x}_t$ is maximal. Define $f_u(i_1, i_2) = |F_u(i_1, i_2)| - \sum_{t \in F_u(i_1,i_2)} \hat{x}_t$. Then $F_u(i_2) = F_u(i_1^*, i_2)$, where $i_1^* = \arg\min_{0 \leq i_1 < i_2} f_u(i_1, i_2)$. For $0 \leq i_1 < i_2$ we determine $F_u(i_1, i_2)$ and $f_u(i_1, i_2)$ in the following way:

determine $t \in [s_{i_1}+1, s_{i_1+1}-1]$ such that $\hat{x}_t$ is maximal;
$F_u(i_1, i_2) := \{t\}$; $f_u(i_1, i_2) := 1 - \hat{x}_t$;
for $i = i_1 + 1$ **to** $d_{1,u-1}$ **do begin**
    determine $t \in [s_{i_1}+1, s_i] \backslash F_u(i_1, i_2)$ such that $\hat{x}_t$ is maximal;
    $F_u(i_1, i_2) := F_u(i_1, i_2) \cup \{t\}$; $f_u(i_1, i_2) := f_u(i_1, i_2) + 1 - \hat{x}_t$
end

determine $F \subseteq [s_{i_1}+1, u-1] \backslash F_u(i_1, i_2)$ such that $|F| = i_2 - d_{1,u-1} - 1$
and $\sum_{t \in F} \hat{x}_t$ is maximal;
$F_u(i_1, i_2) := F_u(i_1, i_2) \cup F$; $f_u(i_1, i_2) := f_u(i_1, i_2) + |F| - \sum_{t \in F} \hat{x}_t$

Here it is assumed that $s_{i_1+1} > s_{i_1} + 1$ and $u - 1 - s_{i_1} \geq i_2 - i_1$, otherwise we set $F_u(i_1, i_2) = \emptyset$ and $f_u(i_1, i_2) = -\infty$. Determining $F_u(i_2)$ and $f_u(i_2) = |F_u(i_2)| - \sum_{t \in F_u(i_2)} \hat{x}_t$ for all periods $u \in [1, s_{d_{1,T}} - 1]$ and $i_2 \in \{d_{1,u-1}+1, \ldots, d_{1,T}-1\}$ can be done in $O((d_{1,T}T)^2)$ time.

Let u be a period in $[1, s_{d_{1,T}} - 1]$. If $u = u(v)$ for some $v \in V_1$, then we already observed that $v = u + i^* - \max(i_2, d_{1,u})$ (cf. (2.16)). Now define

$$g_u(i_2, i^*) = \hat{y}_{u+1, u+i^*-i_2} - \hat{x}_{u+i^*-i_2}$$

for $d_{1,u} \leq i_2 < i^* \leq d_{1,T}$. These values can be determined in $O(d_{1,T})$ time for u fixed, since $g_u(d_{1,u}, d_{1,u}+1) = \hat{y}_{u+1} - \hat{x}_{u+1}$,

$$g_u(d_{1,u}, i^*+1) = g_u(d_{1,u}, i^*) + \hat{x}_{u+i^*-d_{1,u}} + \hat{y}_{u+i^*-d_{1,u}+1} - \hat{x}_{u+i^*-d_{1,u}+1} \text{ for } i^* < d_{1,T},$$

and $g_u(i_2, i^*) = g_u(d_{1,u}, i^* - i_2 + d_{1,u})$ for $d_{1,u} < i_2 < i^*$. Now $(\hat{x}, \hat{y})$ violates an HRSM inequality satisfying P1–P3, C1, and C2 if and only if one of the following holds: there exist an index $i^* \in \{1, \ldots, d_{1,T}\}$ and a nonempty set $U \subseteq \{t \in [1, s_{i^*} - 1] : d_t = 0\}$ such that

$$(\hat{x}_{1,s_{i^*}} - i^*) + \sum_{u \in U} g_u(d_{1,u}, i^*) \ < \ 0, \tag{2.18}$$

or there exist two indices i_2 and i^* satisfying $1 \leq i_2 < i^* \leq d_{1,T}$, and a nonempty set $U \subseteq [i_2+1, s_{i^*}-1] \backslash \{s_{i_2}+1, \ldots, s_{i^*-1}\}$ with $u_{\min} = \min_{u \in U} u \leq s_{i_2}$ such that

$$(\hat{x}_{1,s_{i^*}} - i^*) + f_{u_{\min}}(i_2) + \sum_{u \in U} g_u(\max(i_2, d_{1,u}), i^*) \ < \ 0. \tag{2.19}$$

Since $(\hat{x}, \hat{y})$ satisfies $\hat{x}_{1,s_{i^*}} \geq i^*$, it is obvious that if (2.18) holds for i^* and U, then we can assume that $g_u(d_{1,u}, i^*) < 0$ for all $u \in U$. Suppose that (2.19) holds for i_2, i^*, and U, where $g_u(\max(i_2, d_{1,u}), i^*) \geq 0$ for some $u \in U$. If $u \neq u_{\min}$, then (2.19) also holds if U is replaced by $U \backslash \{u\}$. Therefore, suppose that $g_{u_{\min}}(i_2, i^*) \geq 0$. Since $f_u(i_2) \geq 0$ for any u and i_2, we must have that $U' = U \backslash \{u_{\min}\} \neq \emptyset$. It is not hard to see that $f_{u_1}(i_2) \geq f_{u_2}(i_2)$ if $u_1 < u_2$ and $s_{i_2} \geq u_2$. From this it follows that if $s_{i_2} \geq \min_{u \in U'} u$, then (2.19) also holds for i^*, i_2, and U'. Finally, if $s_{i_2} < \min_{u \in U'} u$, then $g_u(\max(i_2, d_{1,u}), i^*) = g_u(d_{1,u}, i^*)$ for all $u \in U'$. Then

$$\hat{x}_{1,s_{i^*}} - i^* + \sum_{u \in U'} g_u(d_{1,u}, i^*) \leq x_{1,s_{i^*}} - i^* + f_{u_{\min}}(i_2) + \sum_{u \in U} g_u(\max(i_2, d_{1,u}), i^*),$$

hence, the inequality with $V_0 = \emptyset$ corresponding to i^* and U is at least as much violated as the inequality with $V_0 = F_{u_{\min}}(i_2)$ corresponding to i^*, i_2, and U.

Now take $i^* \in \{1, \ldots, d_{1,T}\}$. The following algorithm provides the most violated HRSM inequality satisfying P1–P3, C1, and C2 with $t^* = s_{i^*}$, if one exists:

begin SEPARATION(i^*)
  $\Delta^{\text{opt}} := 0$; determine $g_u(i_2, i^*)$ for $u \in [1, s_{i^*} - 1]$ and $d_{1,u} \le i_2 < i^*$;
  $U := \{u \in [1, s_{i^*} - 1] : d_u = 0 \text{ and } g_u(d_{1,u}, i^*) < 0\}$;
  $\Delta := \hat{x}_{1,s_{i^*}} - i^* + \sum_{u \in U} g_u(d_{1,u}, i^*)$;
  if $\Delta < \Delta^{\text{opt}}$ **then begin** $U^{\text{opt}} := U$; $i_2^{\text{opt}} := 0$; $\Delta^{\text{opt}} := \Delta$ **end**
  for $i_2 = i^* - 1$ **downto** 1 **do begin**
    $U := \{u \in [i_2 + 1, s_{i^*} - 1] : (s_{i_2} \ge u \text{ or } d_u = 0) \text{ and } g_u(\max(i_2, d_{1,u}), i^*) < 0\}$;
    $\Delta := \hat{x}_{1,s_{i^*}} - i^* + \sum_{u \in U} g_u(\max(i_2, d_{1,u}), i^*)$; $u_{\min} := \min_{u \in U} u$;
    while $u_{\min} \le s_{i_2}$ **do begin**
      if $\Delta + f_{u_{\min}}(i_2) < \Delta^{\text{opt}}$ **then**
      begin $U^{\text{opt}} := U$; $i_2^{\text{opt}} := i_2$; $\Delta^{\text{opt}} := \Delta + f_{u_{\min}}(i_2)$ **end**
      $\Delta := \Delta - g_{u_{\min}}(i_2, i^*)$; $U := U \backslash \{u_{\min}\}$; $u_{\min} := \min_{u \in U} u$
    end
  end
end.

If $\Delta^{\text{opt}} < 0$, then $(\hat{x}, \hat{y})$ violates the inequality corresponding to i^*, i_2^{opt} (if $i_2^{\text{opt}} = 0$, then $V_0 = \emptyset$), and U^{opt}. If $f_u(i_2)$ and $F_u(i_2)$ are determined beforehand, then SEPARATION(i^*) runs in $O(d_{1,T}T)$ time.

Summarizing, the most violated HRSM inequality (2.9) for which P1–P3, C1, and C2 hold, if one exists, can be found as follows: first, determine $f_u(i_2)$ and $F_u(i_2)$ for $u \in [1, s_{d_{1,T}} - 1]$ and $d_{1,u-1} + 1 \le i_2 < i^* \le d_{1,T}$ and, second, run SEPARATION(i^*) for $i^* \in \{1, \ldots, d_{1,T}\}$. The most time consuming part is the determination of $f_u(i_2)$ and $F_u(i_2)$, hence, the separation algorithm runs in $O((d_{1,T}T)^2)$ time. Note that if we restrict ourselves to RSM inequalities, i.e., to HRSM inequalities with $V_0 = \emptyset$, then the separation can be performed in $O(d_{1,T}T)$ time.

2.2.3 Regular block inequalities

In this subsection we study another subclass of inequalities of the form (2.7). Let us first recall some definitions. If $\alpha x + \beta y \ge \gamma$ is an inequality of the form (2.7), then

$$\alpha x + \beta y = \sum_{t \in [1,t^*] \backslash V} x_t + \sum_{v \in V} y_{u(v)+1,v},$$

where t^* is a period with $\alpha_{t^*} + \beta_{t^*} = 1$, V a subset of $[1, t^*]$, and $u(v)$ a period associated with $v \in V$ such that $u(v) \le v$. We always assume that $u(v) < u(v')$ if $v < v'$. By definition, V is the set of periods $\{t : t \le t^* \text{ and } \alpha_t = 0\}$.

Similar as in the previous subsection, we denote by V_1 the subset of periods t in V with $\beta_t > 0$, hence, $V_1 = \{v \in V : u(v) < v\}$. In this subsection we study inequalities of the form

(2.7) for which the following holds:

$$\text{if } v, v' \in V_1 \text{ and } v < v', \text{ then } [v, v'] \subseteq V_1 \text{ or } [u(v), v] \cap [u(v'), v'] = \emptyset. \tag{2.20}$$

Suppose V_1 satisfies the above condition. Let $[\bar{v}, \bar{v}+q-1]$ be a maximal interval in V_1, i.e., $[\bar{v}, \bar{v}+q-1] \subseteq V_1$, if $\bar{v} > 1$, then $\bar{v}-1 \notin V_1$, and if $\bar{v}+q-1 < T$, then $\bar{v}+q \notin V_1$. Let $v \in V_1$. Condition (2.20) implies that $v < u(\bar{v})$ if $v < \bar{v}$, and $u(v) \geq \bar{v}+q$ if $v > \bar{v}+q$. Moreover, note that if $v \in V$ is a hole, i.e., if $u(v) = v$, then v is not contained in $[u(v'), v']$ for any $v' \in V_1$. This follows from the assumption that $u(v) < u(v')$ for any $v, v' \in V$ satisfying $v < v'$. Thus, in the interval $[u(\bar{v}), \bar{v}+q-1]$ we have $\alpha_t = 1$ for $t \in [u(\bar{v}), \bar{v}-1]$ and $\alpha_t = 0$ otherwise, and $\beta_t = 0$ if and only if $t = u(\bar{v})$. Furthermore, $\beta_{\bar{v}+q} = 0$ if $\bar{v}+q \leq T$. The interval $[u(\bar{v}), \bar{v}+q-1]$ is called a *(p, q)-block*, where $p = \bar{v} - u(\bar{v})$.

Example 2.2.7 Let $\alpha x + \beta y =$

$$\begin{array}{llll} + y_2 + y_3 & + y_6 + 2y_7 + 2y_8 + y_9 & + y_{11} + y_{12} & + y_{14} + y_{15} + y_{16} \\ x_1 \quad + x_4 + x_5 + x_6 & & + x_{10} + x_{11} & + x_{13} + x_{14} \end{array}$$

```
[·····]              [·········]              [·········]
     [·····]               [···········]                 [···········]
                                [···········]                       [·····]
```

In this example we have $V = V_1 = \{2, 3, 7, 8, 9, 12, 15, 16\}$. The intervals $[u(v), v]$, $v \in V_1$, are depicted above. There are four blocks: the $(1, 2)$-block $[1, 3]$, the $(2, 3)$-block $[5, 9]$, the $(2, 1)$-block $[10, 12]$, and the $(2, 2)$-block $[13, 16]$. □

An inequality $\alpha x + \beta y \geq \gamma$ of the form (2.7) satisfying (2.20) is called a *block inequality*. Observe that (2.7) is a block inequality if and only if there are no periods $t < T$ such that $\alpha_t = 0$, $\alpha_{t+1} = 1$, and $\beta_{t+1} > 0$.

Block inequalities generalize the *hole-bucket inequalities* discussed by van Hoesel in [18] (see also van Hoesel and Kolen [19]). These are inequalities of the form (2.7) with $\alpha_t + \beta_t \leq 1$ for every period t. Since $\beta_t = |\{v : u(v) < t \leq v\}|$, this implies that $u(v) = v - 1$ for every $v \in V_1$. Note that in this case (2.20) is always satisfied. Thus, a hole-bucket inequality is a block inequality with $p = 1$ for any (p, q)-block. Van Hoesel derives necessary and sufficient conditions for a hole-bucket inequality to define a facet of conv($\mathcal{X}$).

Here we will give sufficient conditions for a subclass of the block inequalities to define a facet of conv($\mathcal{X}$), namely, for block inequalities that only contain *regular* blocks. A (p, q)-block $[u(\bar{v}), \bar{v}+q-1]$ is called regular if $u(\bar{v}+i-1) = u(\bar{v}+i) - 1$, $1 < i \leq q$, i.e., if $|[u(\bar{v}+i), \bar{v}+i]| = |[u(\bar{v}), \bar{v}]| = p+1$ for all i. Otherwise, the block is called *nonregular*. In the above example the first three blocks are regular and the last block is nonregular. A block inequality that only contains regular blocks is called a *regular block inequality* or *R-block inequality* for short. Obviously, hole-bucket inequalities are contained in this subclass.

The following lemma states an important property of R-block inequalities. Let us first introduce some notation. In the sequel, the first and the last period of a block B are denoted by l_B and u_B, respectively. Moreover, we define $p_B = |\{t \in B : \alpha_t = 1\}|$ and $q_B = |\{t \in B : \alpha_t = 0\}|$, thus, $B = [l_B, u_B]$ is a (p_B, q_B)-block. When no confusion can arise, the subscript B is omitted.

Lemma 2.2.11 *Let $\alpha x + \beta y \geq \gamma$ be a valid R-block inequality of X that contains the (p, q)-block $B = [l, u]$. Let (x, y) be a solution satisfying $\alpha x + \beta y = \gamma$. Then $\sum_{t=l}^{u}(\alpha_t x_t + \beta_t y_t) = \min(x_{l,u}, p)$.*

PROOF. By definition, B is a regular block, hence, $u(v) = v - p$ for all $v \in B \cap V_1 = [l + p, u]$. Recall that $\alpha_t = 1$ for $t \in [l, l + p - 1]$ and $\alpha_t = 0$ for $t \in [l + p, u]$. Furthermore, let $t \in [l, u]$. If $v \in V_1$, then $t \in [u(v) + 1, v]$ if and only if $v \in [t, \min(t + p - 1, u)]$. Using this and the trivial observation that there are no holes in $[l, u]$, we get

$$\beta_t = |\{v \in V_1 : u(v) < t \leq v\}| = \sum_{s=t}^{\min(t+p-1,u)} (1 - \alpha_s). \tag{2.21}$$

In particular, we have $\beta_l = 0$. Recall further that $\beta_{u+1} = 0$ if $u < T$. Hence, without loss of generality, we may assume that $y_l = 1$ if $x_l = 1$ and $y_{u+1} = 1$ if $x_{u+1} = 1$. Thus, if $[t, t']$ is a production batch in (x, y) and $[t, t'] \cup B \neq \emptyset$, then $[t, t'] \subseteq B$.

Let t be a period in $[l, u]$ and $k \geq 1$ such that $[t, t + k - 1] \subseteq [l, u]$. Then

$$\beta_t + \sum_{s=t}^{t+k-1} \alpha_s \overset{(2.21)}{=} \left\{ \begin{array}{ll} k + \sum_{s=t+k}^{\min(t+p-1,u)}(1 - \alpha_s) & \text{if } k \leq p - 1 \\ p + \sum_{s=t+p}^{t+k-1} \alpha_s = p & \text{if } k \geq p \end{array} \right\} \geq \min(k, p). \tag{2.22}$$

This implies that $\sum_{t=l}^{u}(\alpha_t x_t + \beta_t y_t) \geq \min(x_{l,u}, p)$. In order to prove that equality holds, let us first consider the case that at least p periods in B are used for production in (x, y). Then $\sum_{t=l}^{u}(\alpha_t x_t + \beta_t y_t) \geq p$. Suppose that $\sum_{t=l}^{u}(\alpha_t x_t + \beta_t y_t) > p$. Denote by $(\bar{x}, \bar{y})$ the solution obtained from (x, y) by moving the production in B to the first $x_{l,u}$ periods in B. Then $\sum_{t=l}^{u}(\alpha_t \bar{x}_t + \beta_t \bar{y}_t) = \sum_{t=l}^{l+x_{l,u}-1} \alpha_t + \beta_l = p$, thus $\alpha \bar{x} + \beta \bar{y} < \alpha x + \beta y < \gamma$, which contradicts the validity of the inequality. Thus, if $x_{l,u} \geq p$, then $\sum_{t=l}^{u}(\alpha_t x_t + \beta_t y_t) = p$.

Next, suppose that the total production in B is less than p and suppose that $[t, t + k - 1] \subseteq [l, u]$ is a batch of length k in (x, y) satisfying $\sum_{t=l}^{u}(\alpha_t x_t + \beta_t y_t) > k$. If there is no production in $[l, t - 1]$, then define $t' = l$. Otherwise, let $t' - 1$ be the last period before t in which production occurs. Denote by $(\bar{x}, \bar{y})$ the solution obtained from (x, y) by moving the production in $[t, t + k - 1]$ to $[t', t' + k - 1]$. It is readily checked that $\beta_{t'} \bar{y}_{t'} + \sum_{s=t'}^{t'+k-1} \alpha_s \leq k$, hence, $\alpha \bar{x} + \beta \bar{y} < \alpha x + \beta y = \gamma$. Again, we find a contradiction. This concludes the proof of the lemma. □

Let $[t, t + k - 1]$ be contained in the regular block $[l, u]$. Using (2.22) it is not hard to see that $\beta_t + \sum_{s=t}^{t+k-1} \alpha_s = \min(k, p)$ if and only if $k \geq p$ or $t \in \{l, u - k + 1\}$. Together with the above lemma, this implies the following:

Corollary 2.2.12 *Let $\alpha x + \beta y \geq \gamma$ be a valid R-block inequality of X and let $B = [l, u]$ be a (p, q)-block. Suppose that (x, y) is a solution that satisfies the inequality at equality and for which k periods in B are used for production. If $k > p$, then the production in B occurs in one interval $[t, t+k-1]$ for some $t \in [l, u-k+1]$. Otherwise, the production occurs in $[l, l+k-1]$, or in $[u-k+1, u]$, or in two intervals $[l, l+k_1-1]$ and $[u-k_2+1, u]$, where $k_1, k_2 > 0$ and $k_1 + k_2 = k$.* □

In Lemma 2.2.2 we derived three properties, P1–P3, of facet-defining inequalities with x-coefficients in $\{0, 1\}$. The above result implies another property of facet-defining R-block inequalities.

Lemma 2.2.13 *Let $\alpha x + \beta y \geq \gamma$ be a facet-defining R-block inequality of* conv($\mathcal{X}$) *and let B be the first block. Then this inequality satisfies the following property:*

P4. *If all periods before l_B are holes and if s is a demand period satisfying $d_{1,s} = l_B$, then $s \geq u_B - p_B + 2$ or $\alpha x + \beta y \geq \gamma$ is equivalent to $x_{l_B} + y_{l_B+1,s} \geq 1$.*

PROOF. Suppose that $s \leq u_B - p_B + 1$ and $\alpha x + \beta y \geq \gamma$ is not equivalent to $x_{l_B} + y_{l_B+1,s} \geq 1$. Since $\alpha x + \beta y \geq \gamma$ defines a facet of conv($\mathcal{X}$), there must be a solution $(x, y) \in \mathcal{X}$ satisfying $\alpha x + \beta y = \gamma$ and $x_{l_B} + y_{l_B+1,s} \geq 2$. Then this solution contains two different batches with periods in B that both start before or in period s. However, this contradicts Corollary 2.2.12, which yields that the second batch does not start before period $u_1 - p_1 + 2$. □

In the sequel, we will give sufficient conditions for an R-block inequality to be facet-defining for conv($\mathcal{X}$). Therefore, we first have to deal with the following question: given an R-block structure $\alpha x + \beta y$, what is the maximal value of γ such that $\alpha x + \beta y \geq \gamma$ is valid? Van Hoesel shows that a greedy algorithm always yields the optimal value for a hole-bucket structure, i.e., when $p_B = 1$ for every block B. A generalization of this algorithm provides a solution $(\bar{x}, \bar{y})$ that, under certain conditions, is minimal with respect to $\alpha x + \beta y$, i.e., $\alpha\bar{x} + \beta\bar{y} = \min\{\alpha x + \beta y : (x, y) \in \mathcal{X}\}$. Then, obviously, $\alpha x + \beta y \geq \alpha\bar{x} + \beta\bar{y}$ is valid for $\mathcal{X}$.

FILL_BLOCKS

Consider the demand periods one by one in increasing order and determine for each demand period t a period to produce its demand according to the following rules:

1. If there is an empty hole $s \leq t$, then produce the demand for period t in s; otherwise go to 2.
2. If there is a partially filled block, then produce the demand for period t in the first empty period of this block; otherwise go to 3.
3. If there exists an empty block B with $l_B \leq t$, then produce the demand for period t in the first period of the first empty (p^*, q^*)-block, where $q^* = \max\{q_B : B$ empty and $l_B \leq t\}$ and $p^* = \min\{p_B : B$ empty, $l_B \leq t$, and $q_B = q^*\}$; otherwise go to 4.
4. Produce the demand for period t in t.

Denote the solution provided by this algorithm by $(\bar{x}, \bar{y})$, where $\bar{y}_t = 1$ if and only if $\bar{x}_t = 1$ and either $t = 1$ or $\bar{x}_{t-1} = 0$. Hence, startups only occur in periods t for which $\beta_t = 0$. Furthermore, for each demand period s we denote by $(\bar{x}^s, \bar{y}^s)$ the partial solution where $\bar{x}^s_t = 1$ for all periods t that are chosen by FILL_BLOCKS to produce the demand for some demand period in $[1, s]$. Observe that if block B' is chosen for production after B, then possibly $l_{B'} < l_B$. In that case, we have $q_{B'} \leq q_B$ and $p_{B'} > p_B$ when equality holds. The following example shows that $(\bar{x}, \bar{y})$ is not necessarily minimal with respect to the given inequality.

Example 2.2.8 Let $d_t = 1$ for $t \in \{4, 5, 7, 8, 10\}$ and let $\alpha x + \beta y =$

$$\begin{array}{llllllllll} & & +y_3 & +y_4 & & +y_6 & & +y_8 & +2y_9 & +y_{10} \\ x_1 & +x_2 & +x_3 & & +x_5 & & +x_7 & +x_8 & & \end{array}$$

The production periods determined by FILL_BLOCKS are $[2, 4] \cup [7, 8]$, hence, $\alpha\bar{x} = 4$. However, producing in the interval $[2, 6]$ yields a feasible solution (x, y) satisfying $\alpha x + \beta y = 3$. □

The following lemma gives a sufficient condition for $\alpha x + \beta y \geq \alpha\bar{x} + \beta\bar{y} = \alpha\bar{x}$ to be valid for $\mathcal{X}$.

Lemma 2.2.14 *Let $\alpha x + \beta y$ denote a regular block structure and let $(\bar{x}, \bar{y})$ denote the solution provided by* FILL_BLOCKS. *Then $\alpha x + \beta y \geq \alpha\bar{x}$ is valid for X if the following condition is satisfied for every demand period s: if period $l_B + j - 1$, $j \in \{1, \ldots, p\}$, of a (p, q)-block B is chosen for the production of the demand in s, then there is no empty (p', q')-block B' in $(\bar{x}^s, \bar{y}^s)$ with $l_{B'} \leq s$ such that $p' < p - j + 1$ or $q' > q$.*

PROOF. Suppose that the condition of the lemma is satisfied for all demand periods. Let $\bar{\mathcal{X}}$ be the set of minimal solutions with respect to $\alpha x + \beta y$. Thus, $(\hat{x}, \hat{y}) \in \bar{\mathcal{X}}$ if and only if $\alpha x + \beta y \geq \alpha\hat{x} + \beta\hat{y}$ is a valid inequality. We will show that for every demand period s there exists a solution $(\hat{x}, \hat{y}) \in \bar{\mathcal{X}}$ in which all periods that are determined by the algorithm to produce the demand for the demand periods up to s are used for production, i.e., if $\bar{x}^s_t = 1$, then $\hat{x}_t = 1$. This clearly proves the statement.

Let $(\hat{x}, \hat{y}) \in \mathcal{X}$. Without loss of generality we assume the following for $(\hat{x}, \hat{y})$: (i) every hole is used for production; (ii) if for a block B period $l_B + p_B - 1$ is used for production, then all periods in the interval $[l_B + p_B, u_B]$ are used for production, and (iii) if k periods of a block B are used for production, then these are the first k periods of B (cf. Corollary 2.2.12). Let s be the first demand period for which the algorithm determines a period to produce its demand, say t, that is not used for production in $(\hat{x}, \hat{y})$. Because of (i)–(iii), we have $\alpha_t = 1$. If $t \notin B$ for any block B, i.e., if $\beta_t = \beta_{t+1} = 0$, then all holes before s and all blocks B with $l_B \leq s$ are completely used for production both in $(\bar{x}^s, \bar{y}^s)$ and in $(\hat{x}, \hat{y})$. Hence, there is a period $t' \leq s$ such that $\alpha_{t'} = 1$, t' not in a block, and t' used for production in $(\hat{x}, \hat{y})$ but not in $(\bar{x}^s, \bar{y}^s)$. Moving the production in t' to t yields a solution in $\bar{\mathcal{X}}$ for which production occurs in every period t satisfying $\bar{x}^s_t = 1$.

Now assume that t is a period in a block B, say $t = l_B + j - 1$ for some $j \in \{1, \dots, p_B\}$. The definition of s implies that $\hat{x}_\tau = 1$ for $\tau \in [l_B, t-1]$. Then, by (iii), $\hat{x}_\tau = 0$ for every $\tau \in [t, u_B]$. Let t' be the first period that is used for production in $(\hat{x}, \hat{y})$ but not in $(\bar{x}^s, \bar{y}^s)$. Obviously, $t' \leq s$ and, by (i) and (iii), $\alpha_{t'} = 1$. If t' is a period that is not in a block, then the solution obtained from $(\hat{x}, \hat{y})$ by moving the production in t' to t is also in $\bar{X}$. Otherwise, t' is the first period of a block B' that is empty in $(\bar{x}^s, \bar{y}^s)$. Then, by assumption, $p_{B'} \geq p_B - j + 1$ and $q_{B'} \leq q_B$. Suppose that in $(\hat{x}, \hat{y})$ k periods of B' are used for production. Then, by (ii) and (iii), these are the first k periods of B' and $k = |B'|$ or $k < p_{B'}$. Define $k' = \min(k, u_B - t + 1)$ and let $(\hat{x}', \hat{y}')$ be the solution obtained by moving the production in the last k' periods of B' that are used for production to the first k' empty periods of B, i.e., to the interval $[t, t+k'-1]$. Using $p_{B'} \geq p_B - j + 1$ and $q_{B'} \leq q_B$, one readily checks that $\alpha\hat{x}' + \beta\hat{y}' \leq \alpha\hat{x} + \beta\hat{y}$. Since $(\hat{x}, \hat{y})$ is assumed to be minimal, equality holds. Hence, there exists a solution in $\bar{X}$ for which all periods in which production occurs in $(\bar{x}, \bar{y})$ are used for production. □

The above lemma gives a sufficient condition for FILL_BLOCKS to provide the correct right-hand side of an R-block inequality. However, this condition is not necessary, as the following example shows:

Example 2.2.9 Let $d_t = 1$ for $t \in \{5, 6, 7, 8, 9, 10\}$ and let $\alpha x + \beta y =$

$$\begin{array}{l} \quad\;\; + y_2 + 2y_3 + 2y_4 + y_5 \qquad + y_7 + y_8 \\ x_1 + x_2 + x_3 \qquad\qquad\qquad + x_6 \qquad\qquad\quad + x_9 + x_{10} \end{array}$$

The algorithm chooses period s to produce the demand in period $s + 4$, $1 \leq s \leq 6$, hence, $\alpha\bar{x} = 4$. It is readily checked that $\alpha x + \beta y \geq 4$ is indeed valid. However, when period 2, i.e., the second period of the $(3, 2)$-block $B = [1, 5]$, is chosen for the production of d_6, the $(1, 2)$-block $B' = [6, 8]$ is empty and $p_{B'} = 1 < 2 = p_B - j + 1$. □

If $p_B = 1$ for every block B, i.e., if $\alpha x + \beta y$ is a hole-bucket structure, then FILL_BLOCKS is equivalent to the greedy algorithm given in [18], which always provides the optimal value of the right-hand side of such an inequality. It is readily checked that in this case the condition of Lemma 2.2.14 is indeed always satisfied. This also holds when every block is a (p, q)-block for some p and q, i.e., $p_B = p$ and $q_B = q$ for every B.

In the following theorem we give sufficient conditions for an R-block inequality to be facet-defining for conv(X). Similar as before, let $(\bar{x}, \bar{y})$ be the solution provided by the algorithm for an R-block inequality $\alpha x + \beta y \geq \gamma$. Moreover, denote by $B_1, B_2, \dots, B_K$ the blocks that are used for production in $(\bar{x}, \bar{y})$, such that B_{k+1} is chosen for production after B_k. For notational convenience, we write l_k, u_k, etc., instead of l_{B_k}, u_{B_k}, etc. Recall that we may have $l_k > l_{k+1}$ for some k. In that case, either $q_k > q_{k+1}$ or $q_k = q_{k+1}$ and $p_k < p_{k+1}$ must hold.

Theorem 2.2.15 *Let $\alpha x + \beta y \geq \gamma$ be a valid R-block inequality satisfying* P1–P4 *and let $(\bar{x}, \bar{y})$ and B_k, $1 \leq k \leq K$, be as defined above. Suppose that $\alpha\bar{x} = \gamma$ and that the following conditions are satisfied:*

S1. FILL_BLOCKS *chooses period $l_K + p_K - 1$ for the production of the demand in t^*.*
S2. *If* FILL_BLOCKS *chooses period $l_k + p_k$ to produce the demand in period s, then there is an empty block B in $(\bar{x}^s, \bar{y}^s)$ with $l_B \leq s$, $p_B \in \{p_{k+1}, \ldots, p_K\}$, and $q_B \geq q_k$.*
S3. *If* FILL_BLOCKS *chooses period l_k for the production of the demand in t, then $t > u_k - p_k$ or $l_j < l_k$ for some $j > k$.*
S4. *If B is a block that is not used for production in $(\bar{x}, \bar{y})$ and $p_k < p_B$ for every block B_k with $l_k > l_B$, then there is a block B_k with $l_k < l_B$. Moreover, $p_{k'} \geq p_B$ and period $l_{k'} + p_{k'} - p_B$ is chosen by* FILL_BLOCKS *to produce the demand for some period $t > u_B - p_B$, where $k' = \max\{k : l_k < l_B\}$.*

Then $\alpha x + \beta y \geq \gamma$ defines a facet of conv($\mathcal{X}$).

PROOF. Denote by $\bar{\mathcal{X}}$ the set of feasible solutions to DLSP that satisfy $\alpha x + \beta y \geq \gamma$ at equality. Thus, in particular, $(\bar{x}, \bar{y}) \in \bar{\mathcal{X}}$. In order to prove that $\alpha x + \beta y \geq \gamma$ defines a facet of conv($\mathcal{X}$), we have to provide $2T - 1$ linearly independent directions. For every block B we construct solutions (x^{l_B}, y^{l_B}) and (x^{u_B}, y^{u_B}) in $\bar{\mathcal{X}}$ for which the only periods of this block that are used for production are the first p_B and the last p_B periods, respectively. Observe that, by S2, $q_1 \leq q_2 \leq \ldots \leq q_K$. Together with the earlier observation that $q_j \leq q_k$ if $j > k$ and $l_j < l_k$, this yields $q_j = q_k$ (and $p_j > p_k$) if $j > k$ and $l_j < l_k$. Together with P3, S2 furthermore implies that if there is a hole $t \leq s$ or a block B with $l_B \leq s$, s a demand period, then FILL_BLOCKS chooses either a hole or a period in a block for the production of the demand in s.

We first determine solutions (x^{l_k}, y^{l_k}), $1 \leq k \leq K$, that satisfy the following properties:

(a) all blocks B_j, $1 \leq j < k$, are completely used for production;
(b) only the first p_k periods of B_k are used for production;
(c) if B is used for production and $B \neq B_j$ for any $j < k$, then $q_B \geq q_k$ and $p_B \in \{p_{k+1}, \ldots, p_K\}$;
(d) $\alpha x^{l_k} + \beta y^{l_k} = \gamma$, thus, $(x^{l_k}, y^{l_k}) \in \bar{\mathcal{X}}$.

By S1, only the first p_K periods of B_K are used for production in $(\bar{x}, \bar{y})$, hence, we set $(x^{l_K}, y^{l_K}) = (\bar{x}, \bar{y})$. Now let $k < K$ and suppose that a solution (x^{l_j}, y^{l_j}) with the desired properties has been constructed for all $j > k$. Suppose period $l_k + p_k$ was chosen by FILL_BLOCKS to produce the demand for period s. If $s \geq l_j$ for some $j > k$, then let (x^{l_k}, y^{l_k}) be the solution obtained from (x^{l_j}, y^{l_j}) by moving the production in the last q_k periods of B_k to the first q_k empty periods of B_j, i.e., to the interval $[l_j + p_j, l_j + p_j + q_k - 1]$. Recall that $q_k \leq q_j$. It is easily seen that this solution satisfies (a)–(d). Otherwise, if $s < l_j$ for any $j > k$, then S2 yields that there is an empty block B in $(\bar{x}, \bar{y})^{d_{1,s}}$ such that $s \geq l_B$, $p_B = p_{k'}$ for some $k' \in \{k+1, \ldots, K\}$, and $q_B \geq q_k$. Thus, $B \neq B_j$ for any $j \leq k$. Moreover, since $l_B \leq s < l_j$ for any $j > k$, we conclude that B is empty in $(\bar{x}, \bar{y})$. Together with $l_B < l_{k'}$ and $p_B = p_{k'}$, this implies $q_B < q_{k'}$. Hence, by (c), B is empty in $(x^{l_{k'}}, y^{l_{k'}})$. Now let (x^{l_k}, y^{l_k}) be the solution obtained from $(x^{l_{k'}}, y^{l_{k'}})$ by moving the production in the last q_k periods of B_k and in (the first $p_{k'}$ periods of) $B_{k'}$ to block B. Again, one readily checks that this solution satisfies (a)–(d).

Second, denote by (x^{u_k}, y^{u_k}) the solution obtained from (x^{l_k}, y^{l_k}) by moving the production from the first p_k periods to the last p_k periods of the block B_k. This solution is feasible if $x^{u_k}_{1,l_k-1} \geq d_{1,u_k-p_k}$. This clearly holds if there is no demand period in the interval $[l_k, u_k - p_k]$. Therefore, suppose that there is at least one demand period in this interval. Define $s = \max\{t \in [l_k, u_k - p_k] : d_t = 1\}$. If FILL_BLOCKS chooses a period before l_k to produce the demand in s, then $x^{u_k}_{1,l_k-1} \geq \bar{x}^s_{1,l_k-1} = d_{1,s}$, since all holes before l_k and all blocks B_j, $j < k$, are completely used for production in (x^{u_k}, y^{u_k}). Otherwise, S3 asserts that there is a block B_j with $j > k$ and $l_j < l_k$, hence, $q_j = q_k$ and $p_j > p_k$. We may assume that (x^{l_k}, y^{l_k}) is obtained from (x^{l_j}, y^{l_j}) by moving the production in the last q_k periods of B_k to the last q_k $(= q_j)$ periods of B_j. Then B_j is completely used for production in (x^{u_k}, y^{u_k}), thus, $x^{u_k}_{1,l_k-1} \geq \bar{x}^s_{1,l_k-1} + p_j + q_j \geq d_{1,u_k}$. Hence, (x^{u_k}, y^{u_k}) is feasible. Since $\alpha x^{l_k} + \beta y^{l_K} = \gamma$, it follows from Corollary 2.2.12 that (x^{u_k}, y^{u_k}) is also in $\bar{\mathcal{X}}$.

Finally, let B be a block that is empty in $(\bar{x}, \bar{y})$. We first show that for any k, $1 \leq k \leq K$, we have $l_k \leq l_B$ or $p_k \leq p_B$. Suppose B_k satisfies $l_k > l_B$ and $p_k > p_B$. Since B_k was chosen for production by FILL_BLOCKS instead of B, we must have $q_B < q_k$. Then (c) implies that B is empty in (x^{l_k}, y^{l_k}). Let (x, y) be the solution obtained from (x^{l_k}, y^{l_k}) by moving the production in $[l_k + p_k - p_B - 2, l_k + p_k - 1]$ to the first $p_B + 1$ periods of B. Then $\alpha x + \beta y = \alpha x^{l_k} + \beta y^{l_k} - (p_B + 1) + p_B$, contradicting the validity of $\alpha x + \beta y \geq \gamma$. Hence, if $l_k > l_B$, then $p_k \leq p_B$. Let us now determine solutions (x^{l_B}, y^{l_B}) and (x^{u_B}, y^{u_B}) with the desired properties. We can easily deal with the case that $p_k = p_B$ for some block B_k with $l_k > l_B$. Then the solutions (x^{l_B}, y^{l_B}) and (x^{u_B}, y^{u_B}) that are obtained from (x^{l_k}, y^{l_k}) by moving the production in $[l_k, l_k + p_k - 1]$ to respectively the first p_B and the last p_B periods of B are clearly in $\bar{\mathcal{X}}$. Therefore, assume that $p_k < p_B$ for all blocks B_k with $l_k > l_B$. By S4, $k' = \max\{k : l_k < l_B\}$ is well defined and $p_{k'} \geq p_B$. By (c), B is empty in $(x^{l_{k'}}, y^{l_{k'}})$. Moreover, all blocks B_j, $j < k'$, are completely used for production in this solution, hence, by S4, the production in the interval $[l_{k'} + p_{k'} - p_B - 1, l_{k'} + p_{k'} - 1]$ can be moved to the last p_B periods of B while feasibility is maintained. Denote this solution by (x^{u_B}, y^{u_B}). Then (x^{u_B}, y^{u_B}) satisfies $\alpha x + \beta y \geq \gamma$ at equality, just as (x^{l_B}, y^{l_B}), the solution obtained from (x^{u_B}, y^{u_B}) by moving the production in the last p_B periods of B to the first p_B periods of this block.

Now in order to prove that $\alpha x + \beta y \geq \gamma$ defines a facet of conv($\mathcal{X}$), we will show that the following directions are in $\bar{\mathcal{X}}$:

- (i) $e(x_t)$ for all t satisfying $\alpha_t = 0$;
- (ii) $e(y_t)$ for all t satisfying $\beta_t = 0$;
- (iiia) $e(x_{t+p-1}) + e(y_t) - e(x_{t-1}) - e(y_{t-1})$ for all $t \in \bigcup_B [l_B + 1, u_B - p_B + 1]$;
- (iiib) $e(x_{t-q-1}) + e(y_t) - e(x_{t-1}) - e(y_{t-1})$ for all $t \in \bigcup_B [u_B - p_B + 2, u_B]$;
- (iv) $e(x_t) - e(x_{l_K + p_K - 1})$ for all $t \neq l_K + p_K - 1$ satisfying $\alpha_t = 1$.

Note that $\beta_t > 0$ if and only if $t \in [l_B + 1, u_B]$ for some block B. Thus, (ii), (iiia), and (iiib) yield T different directions. Since $\alpha_{l_K+p_K-1} = 1$, (i) and (iv) yield $T - 1$ different directions. One readily checks that these $2T - 1$ directions are linearly independent. This shows that $\alpha x + \beta y \geq \gamma$ defines a facet of conv(X).

Ad (i) and (ii). The directions $e(x_t)$ and $e(y_t)$ for t satisfying $\alpha_t = 0$ and $\beta_t = 0$, respectively, can be constructed in a similar way as in the proof of Theorem 2.2.9. Therefore, we omit the details.

Ad (iii). Let t be a period satisfying $\beta_t > 0$. Then $t \in B$ for some (p, q)-block $B = [l, u]$ and $t > l$. Consider the previously defined solution (x^u, y^u) in which only the last p periods of B are used for production. For $t \in [l + 1, u - p + 1]$ the direction $e(x_{t+p-1}) + e(y_t) - e(x_{t-1}) - e(y_{t-1})$ is constructed from the solutions obtained from (x^u, y^u) by moving the production in $[u - p + 1, u]$ to $[t, t + p - 1]$ and $[t - 1, t + p - 2]$, respectively. From Corollary 2.2.12 and $\alpha x^u + \beta y^u = \gamma$, it follows that these solutions are also in $\bar{X}$. Furthermore, for $t \in [u - p + 2, u]$ the direction $e(x_{t-q-1}) + e(y_t) - e(x_{t-1}) - e(y_{t-1})$ is constructed from the solutions obtained from (x^u, y^u) by moving the production in the interval $[u - p + 1, s - 1]$ to the interval $[u - p - q + 1, s - q - 1] = [l, s - q - 1]$ for $s \in \{t - 1, t\}$. Again, by Corollary 2.2.12, these solutions satisfy $\alpha x + \beta y \geq \gamma$ at equality.

Ad (iv). We show that the direction $e(x_t) - e(l_K + p_K - 1)$ can be constructed for every period $t \neq l_K + p_K - 1$ with $\alpha_t = 1$. In the proof we often restrict ourselves to the construction of a direction $e(x_t) - e(x_{t'})$, where t' is a period for which the direction $e(x_{t'}) - e(x_{l_K+p_K-1})$ has already been constructed.

First, let t be a period such that $\alpha_t = 1$, $\beta_t = 0$, and $\bar{x}_t = 0$. Since period $l_K + p_K - 1$ was chosen by FILL_BLOCKS to produce the demand in period t^*, the solution obtained from $(\bar{x}, \bar{y})$ by moving the production in period $l_K + p_K - 1$ to t is in $\bar{X}$. Together with $(\bar{x}, \bar{y})$ and the direction $e(y_t)$ constructed previously, this solution provides the desired direction. If t is a period with $\alpha_t = 1$, $\bar{x}_t = 1$, and t not in a block, then t is a demand period before the first hole and before $\min_B l_B$. In this case we have $\alpha_1 = 1$ and $\bar{x}_1 = 0$, as it is always assumed that $d_1 = 0$. Thus, the directions $e(x_t) - e(x_1)$ and $e(x_1) - e(x_{l_K+p_K-1})$ are easily constructed.

Next, let $t \in [l, l + p - 1]$, where $B = [l, u]$ is a (p, q)-block that is not used for production in $(\bar{x}, \bar{y})$. Consider the solution (x^l, y^l) in which the first p periods of B are used for production. Recall that (x^l, y^l) was obtained from (x^{l_k}, y^{l_k}) for some $k \in \{1, \ldots, K\}$, say k, by moving the production in $[l_k + p_k - p, l_k + p_k - 1]$ to $[l, l + p - 1]$. Denote by (x, y) the solution obtained from (x^l, y^l) by moving the production in $[t + 1, l + p - 1]$ to the last $l + p - t - 1$ periods of B. We already observed that $\alpha x + \beta y = \gamma$. Furthermore, $x_{l_k+p_k-p} = 0$ and moving the production in period t to $l_k + p_k - p$ yields another solution in $\bar{X}$. Hence, the direction $e(x_t) - e(x_{l_k+p_k-p})$ can be constructed for every $t \in [l, l + p - 1]$. Together with the direction $e(x_l) - e(x_{l_K+p_K-1})$, which has already been established because $\alpha_l = 1$, $\beta_l = 0$, and $\bar{x}_l = 0$, this yields the desired direction for $t \in [l, l + p - 1]$.

What is left to show is that the direction $e(x_t) - e(x_{l_K+p_K-1})$ can be constructed for every $t \in \bigcup_{k=1}^{K}[l_k, l_k + p_k - 1]$, $t \neq l_K + p_K - 1$. We will first establish this direction for the periods in $[l_K, l_K + p_K - 2]$. Then it is shown how $e(x_t) - e(x_{l_K+p_K-1})$ can be constructed for $t \in [l_k, l_k + p_k - 1]$, $k < K$ if the direction has already been established for all periods in $\bigcup_{j=k+1}^{K}[l_j, l_j + p_j - 1]$. However, we start by showing that there exists a period t such that $\alpha_t = 1$ and $\bar{x}_t = 0$. Note that if there exist periods that are neither a hole nor in a block, then at least one of these periods is empty in $(\bar{x}, \bar{y})$. Therefore, suppose that there is no period t with $\alpha_t = 1$ that is not in a block. Furthermore, suppose that all blocks are (partially) used for production in $(\bar{x}, \bar{y})$. Then there are no blocks but B_k, $1 \leq k \leq K$, hence, by S1, $\alpha x + \beta y = \alpha\bar{x} = \sum_{k=1}^{K} p_k$. Let k' satisfy $l_{k'} = \min_k l_k$. By P4, the total demand up to period $u_{k'} - p_{k'} + 1$ is less than $l_{k'}$, hence, the solution $(\hat{x}, \hat{y})$ in which all periods but $[l_{k'}, u_{k'} - p_{k'} + 1]$ are used for production is feasible. However, $\alpha\hat{x} + \beta\hat{y} = \sum_{k=1}^{K} p_k - p_{k'} + (p_{k'} - 1)$, which contradicts the validity of $\alpha x + \beta y \geq \gamma$. Hence, there always exists a period t with $\alpha_t = 1$ for which no production occurs in $(\bar{x}, \bar{y})$.

We will first construct the direction $e(x_{t_K}) - e(x_t)$ for every $t \in [l_K, l_K + p_K - 1]$, where $t_K = \min\{t : \alpha_t = 1 \text{ and } \bar{x}_t = 0\}$. From the above arguments it follows that t_K is well defined and, by S1, $\beta_{t_K} = 0$. Note that the direction $e(x_{t_K}) - e(x_{l_K+p_K-1})$ has already been established. Thus, let $t \in [l_K, l_K + p_K - 2]$ and let (x, y) be the solution obtained from $(\bar{x}, \bar{y})$ by moving the production in the interval $[t + 1, l_K + p_K - 1]$ to the last $l_K + p_K - t - 1$ periods in the block B_K, i.e., to the interval $[t + q_K + 1, u_K]$. Then $(x, y) \in \bar{X}$. From this solution the direction $e(x_{t_K}) - e(x_{l_K+j-1})$ can be easily constructed, provided that moving the production in period t to period t_K yields a feasible solution. This trivially holds if $t_K < l_K$. Hence, suppose that $t_K > u_K$. Note that this implies that each period before t_K is either a hole or in a block. Then the production in t can be moved to t_K if the total production in (x, y) up to period $t + q_K$, which equals $t - 1$, is not less than the total demand up to this period. Similar as before, let $l_{k'} = \min_k l_k$. We already observed that the solution for which production occurs in all periods except in the interval $[l_{k'}, u_{k'} - p_{k'} + 1]$ is feasible. Thus, the total demand up to period $t + q_K$ is at most $t + q_K - (q_k + 1)$. The proof is concluded by showing that $q_k = q_K$. Therefore, consider again the solution $(\bar{x}, \bar{y})$. If $q_k < q_K$, then we can move the production in the last $q_k + 1$ periods of B_k to to the first $q_k + 1$ empty periods of B_K while maintaining feasibility. However, this new solution (x', y') satisfies $\alpha x' + \beta y' = \alpha\bar{x} + \beta\bar{y} - 1$, contradicting again the validity of $\alpha x + \beta y \geq \alpha\bar{x}$. Thus, $q_k = q_K$, since we already observed that $q_k \leq q_K$.

Now let $k < K$ and suppose that the direction $e(x_t) - e(l_K + p_K - 1)$ has been established for every $t \in \bigcup_{j=k+1}^{K}[l_j, l_j + p_j - 1]$. Consider the solution (x^{l_k}, y^{l_k}), in which in B_k only production occurs in the first p_k periods. Moreover, all blocks B_j, $j < k$, are completely used for production. Using similar arguments as before, one easily shows that $t_k = \min\{t : \alpha_t = 1$ and $x_t^{l_k} = 0\}$ is well defined. Then either t_k is empty in $(\bar{x}, \bar{y})$ or t_k is the first period of a block B_j for some $j > k$. In both cases the direction $e(x_{t_k}) - e(x_{l_K+p_K-1})$ has already been established. Now the direction $e(x_{t_k}) - e(x_t)$, $t \in [l_k, l_k + p_k - 1]$, can be constructed in a similar way as for $t \in [l_K, l_K + p_K - 2]$. This concludes the proof of the theorem. □

Example 2.2.10 Let $T = 15$ and $d_t = 1$ for $t \in \{4, 6, 7, 9, 11, 13, 15\}$. Consider the following R-block inequalities:

$$\begin{array}{l} \quad\;\; + y_2 + 2y_3 + y_4 \qquad + y_6 + 2y_7 + 2y_8 + y_9 \qquad + y_{11} + y_{12} + y_{13} \qquad + y_{15} \\ x_1 + x_2 \qquad\qquad + x_5 + x_6 \; + x_7 \qquad\qquad + x_{10} + x_{11} + x_{12} \qquad + x_{14} \end{array} \geq 5$$

and

$$\begin{array}{l} \quad\;\; + y_2 + y_3 \qquad + y_5 \qquad + y_7 + y_8 \qquad\qquad + y_{11} + 2y_{12} + 2y_{13} + y_{14} \\ x_1 + x_2 \qquad + x_4 \qquad + x_6 + x_7 \qquad + x_9 + x_{10} + x_{11} \qquad\qquad\qquad + x_{15} \end{array} \geq 5.$$

Given the left-hand side of the first inequality, FILL_BLOCKS chooses period i, $1 \leq i \leq 7$, to produce the demand for the ith demand period. With respect to the left-hand side of the second inequality FILL_BLOCKS proceeds as follows: first, the $(1, 1)$-block $[4, 5]$ is completely filled, then the $(2, 1)$-block $[1, 3]$ is filled, and finally periods 10 and 11 are chosen to produce the demand for period 13 and 15, respectively. In both cases $(\bar{x}, \bar{y})$ is minimal with respect to the R-block structure at hand because of Lemma 2.2.14. It is left to the reader to check that P1–P4 and S1–S4 hold in either case. Hence, both inequalities are facet-defining for the instance at hand. □

We already observed that $\alpha x + \beta y \geq \alpha\bar{x}$ is always valid for $\mathcal{X}$ if $p_B = 1$ for all B or if every block is a (p, q)-block for some p and q. Van Hoesel [18] proves that conditions P1–P4 and S1–S3 are both necessary and sufficient for a hole-bucket inequality to be facet-defining for conv($\mathcal{X}$). Note that S1 implies S4 in this case. Let us now consider the subclass of R-block inequalities with $p_B = p$ and $q_B = q$ for all B. Using similar arguments as in [18], we can show that such an inequality must satisfy S1–S4 in order to define a facet of conv($\mathcal{X}$). Together with Theorem 2.2.15, this yields a complete characterization of the facet-defining inequalities of in this subclass.

With respect to separation we mention the following results. Van Hoesel and Kolen [19] present a separation algorithm for the hole-bucket inequalities that is based on dynamic programming. They define an acyclic network such that each path corresponds to a facet-defining hole-bucket inequality and vice versa. The running time of this algorithm is $O(T^5)$. A similar approach can be used for the separation of R-block inequalities with $p_B = p$ and $q_B = q$ for all blocks B. Because of their large running times, these separation algorithms will not be used in our computational experiments in Chapter 3. Therefore, we will not describe them in more detail.

2.2.4 Hole-lifted left stock-minimal inequalities

The last subclass of facet-defining inequalities with x-coefficients in $\{0, 1\}$ that we discuss is also based on a class of inequalities derived by Constantino for the capacitated lot-sizing problem with startup costs, namely, the class of *interval left supermodular inequalities* (cf. [5], Section 2.2). In the sequel, we will not make use of the general form (2.7) of facet-defining inequalities with x-coefficients in $\{0, 1\}$. Although the inequalities discussed here can also be stated in this form, there is a much more insightful way to introduce them. Similar as before, s_i denotes the ith demand period, thus, $d_{s_i} = 1$ and $d_{1,s_i} = i$.

Let t be a period in $[0, T]$ and let s be the jth demand period after t, thus, $s = s_{d_{1,t}+j}$. If no production occurs in the interval $[t + j, s]$, then the inventory at the end of period t must be at least one, or, equivalently, the total production up to period t must be at least $d_{1,t} + 1$. Consider the following inequality:

$$x_{1,t} + x_{t+j} + y_{t+j+1,s} \geq d_{1,t} + 1. \tag{2.23}$$

Observe that $x_{t+j} + y_{t+j+1,s}$ is nonnegative and integral for any feasible solution to DLSP. Moreover, $x_{t+j} + y_{t+j+1,s} = 0$ implies that there is no production in $[t + j, s]$. Hence, (2.23) forces the inventory at the end of period t to be at least one if no production occurs in the interval $[t + j, s]$.

Proposition 2.2.16 *Let $t \in [0, T]$ and let $J \subseteq \{1, \ldots, d_{t+1,T}\}$. Then*

$$x_{1,t} + \sum_{j \in J} (x_{t+j} + y_{t+j+1,s_{d_{1,t}+j}}) \quad \geq \quad d_{1,t} + |J| \tag{2.24}$$

is valid for X.

Before we prove the above proposition, we give an example.

Example 2.2.11 Let $T = 12$ and $d_t = 1$ for $t \in \{2, 4, 5, 7, 8, 9, 10, 12\}$. Take $t = 3$ and $J = \{1, 3, 4, 6, 7\}$. Then $\{s_{d_{1,t}+j} : j \in J\} = \{4, 7, 8, 10, 12\}$. By Proposition 2.2.16,

$$\begin{array}{ccccc} & + y_4 & + y_6 + 2y_7 + y_8 + y_9 + 2y_{10} + y_{11} + y_{12} & & \geq 6 \\ x_1 + x_2 + x_3 & + x_5 + x_6 & + x_8 + x_9 & & \end{array}$$

is a valid inequality for the given instance. □

PROOF OF PROPOSITION 2.2.16. The proof is by induction on $|J|$. First, note that for $J = \emptyset$ the above inequality is the production inequality $x_{1,t} \geq d_{1,t}$, hence, (2.24) is valid for $|J| = 0$. Now let $J \subseteq \{1, \ldots, d_{t+1,T}\}$, $|J| \geq 1$, and suppose (2.24) is valid for all subsets J' of size $|J'| < |J|$. Let (x, y) be a solution to DLSP violating (2.24) for the given choice of J. Let $j^* = \max_{j \in J} j$ and define $J' = J \backslash \{j^*\}$. By the induction hypothesis, (2.24) is valid for J', hence,

$$\begin{aligned} d_{1,t} + |J| \quad &> \quad x_{1,t} + \sum_{j \in J} (x_{t+j} + y_{t+j+1,s_{d_{1,t}+j}}) \\ &= \quad x_{1,t} + \sum_{j \in J'} (x_{t+j} + y_{t+j+1,s_{d_{1,t}+j}}) + x_{t+j^*} + y_{t+j^*+1,s_{d_{1,t}+j^*}} \\ &\geq \quad d_{1,t} + |J| - 1 + x_{t+j^*} + y_{t+j^*+1,s_{d_{1,t}+j^*}}. \end{aligned}$$

Since (x, y) is integral, we have $x_{t+j^*} + y_{t+j^*+1,s_{d_{1,t}+j^*}} = 0$, which implies that no production occurs in the interval $[t + j^*, s_{d_{1,t}+j^*}]$. Moreover, (x, y) satisfies the production requirements, thus, $x_{1,t+j^*-1} = x_{1,s_{d_{1,t}+j^*}} \geq d_{1,t} + j^*$. This yields

$$\begin{aligned} x_{1,t} + \sum_{j\in J}(x_{t+j} + y_{t+j+1,s_{d_{1,t}+j}}) &\geq x_{1,t} + \sum_{j\in J'} x_{t+j} = x_{1,t+j^*-1} - \sum_{j\in\{1,\dots,j^*-1\}\setminus J'} x_{t+j} \\ &\geq d_{1,t} + j^* - (j^* - 1 - |J'|) = d_{1,t} + |J|, \end{aligned}$$

which contradicts the assumption that (x, y) violates (2.24) for the given choice of J. □

An inequality of the form (2.24) is called a *left stock-minimal inequality* or *LSM inequality* for short. LSM inequalities are a direct adaptation of the interval left submodular inequalities of Constantino to DLSP. In Subsection 2.2.2 we discussed HRSM inequalities, which generalize the RSM inequalities, which in their turn are a direct adaption of the interval right submodular inequalities of Constantino. A similar generalization of the LSM inequalities will be discussed later.

We first investigate under which conditions an LSM inequality defines a facet of conv($\mathcal{X}$). If $J = \emptyset$, then (2.24) is the production inequality $x_{1,t} \geq d_{1,t}$, which defines a facet of conv($\mathcal{X}$) if and only if $d_t = 1$ and either $t = T$ or $d_{t+1} = 0$ (Proposition 2.1.2). If $t = 0$, then (2.24) is the sum of inequalities of the form $x_j + y_{j+1,s_j} \geq 1$. These inequalities are facet-defining, hence, if $t = 0$, then (2.24) does not define a facet unless $|J| = 1$. Therefore, assume that $t > 0$ and $J \neq \emptyset$. If (2.24) defines a facet of conv($\mathcal{X}$), then at least the general properties P1–P3 stated in Lemma 2.2.2 have to be satisfied. Note that for $J \neq \emptyset$ the demand period $s_{d_{1,t}+j^*}$, where $j^* = \max_{j\in J} j$, is the last period τ with $\alpha_\tau + \beta_\tau > 0$, hence, P1 is obviously satisfied. The only holes before $s_{d_{1,t}+j^*}$ are the periods $t + j$, $1 \leq j < \min_{j\in J} j$. Hence, for every $\tau < s_{d_{1,t}+j^*}$ the number of holes in $[\tau, s_{d_{1,t}+j^*}]$ is strictly less than the demand in this interval, thus, P2 holds. Finally, P3 states that the number of holes in the interval $[t + 1, t + j]$, $1 \leq j < \min_{j\in J} j$, must exceed $d_{t+1,t+j}$. If $\min_{j\in J} j > 1$, then this condition is satisfied if and only if $d_{t+1} = 0$. Moreover, if $1 \in J$ and $d_{t+1} = 1$, then (2.24) is equivalent to the inequality with t replaced by $t + 1$ and J replaced by $J' = \{j - 1 : j \in J \text{ and } j > 1\}$. Hence, we only have to consider the case that $t \in [1, T - 1]$, $d_{t+1} = 0$, and $J \neq \emptyset$.

Theorem 2.2.17 *Let* $t \in [1, T - 1]$ *such that* $d_{t+1} = 0$ *and let* $\emptyset \neq J \subseteq \{1, \dots, d_{t+1,T}\}$. *Then*

$$x_{1,t} + \sum_{j\in J}(x_{t+j} + y_{t+j+1,s_{d_{1,t}+j}}) \geq d_{1,t} + |J|$$

defines a facet of conv($\mathcal{X}$).

PROOF. Let $\alpha x + \beta y \geq \gamma$ be an inequality of the form (2.24) such that t and J satisfy the required conditions. Let $\bar{X}$ be the set of solutions in X that satisfy the inequality at equality. Let $j^* = \max_{j\in J} j$. For $\tau \in [t + 1, s_{d_{1,t}+j^*}]$ we define $j_\tau = \min\{j : 1 \leq j \leq j^* \text{ and } s_{d_{1,t}+j} \geq \tau\}$.

Thus, $s_{d_{1,t}+j_\tau}$ is the first demand period after $\tau - 1$. We claim that the following $2T-1$ linearly independent directions are in $\bar{X}$:

(i) $e(x_\tau)$ for all periods τ with $\alpha_\tau = 0$;
(ii) $e(y_\tau)$ for all periods τ with $\beta_\tau = 0$;
(iii) $e(x_\tau) - e(x_{d_{1,t}+1})$ for all periods $\tau \neq d_{1,t}+1$ with $\alpha_\tau = 1$;
(iva) $e(y_\tau) + e(x_{\tau+j^*-j_\tau}) - e(y_{\tau-1}) - e(x_{\tau-1})$ for all periods τ with $\beta_\tau > 0$ and $j_\tau = j_{\tau-1}$;
(ivb) $e(y_\tau) + e(x_{t+j_\tau-1}) - e(y_{\tau-1}) - e(x_{\tau-1})$ for all periods τ with $\beta_\tau > 0$ and $j_\tau > j_{\tau-1}$.

These directions can be constructed using similar arguments as in the proofs of Theorems 2.2.9 and 2.2.15. Therefore, we restrict ourselves to providing solutions $(x^\tau, y^\tau) \in \bar{X}$ for all $\tau \in [t+1, s_{d_{1,t}+j^*}]$ that can be used for the construction of the aforementioned directions.

Let $\tau \in [t+1, s_{d_{1,t}+j^*}]$ and define (x^τ, y^τ) as follows. First, set $y_{s_j} = x_{s_j} = 1$ for $j \le d_{1,t}$ and for $j > d_{1,t} + j^*$. Furthermore, produce the demand for $s_{d_{1,t}+j}$, $1 \le jj^*$, in the intervals $[t+1, t+j_\tau - 1]$ and $[\tau, \tau + j^* - j_\tau]$. The feasibility of (x^τ, y^τ) follows from the definition of j_τ. Hence, it remains to show that $\alpha x^\tau + \beta y^\tau = \gamma = d_{1,t} + |J|$. Observe that $d_{t+1} = 0$ implies that $j_\tau < \tau - t$ for all $\tau > t+1$. Thus, we have $x^\tau_{\tau-1} = 0$ and, hence, $y^\tau_\tau = 1$. Then

$$\begin{aligned} \alpha x^\tau + \beta y^\tau &= d_{1,t} + \beta_{t+1} + \alpha_{t+1,t+j_\tau-1} + \beta_\tau + \alpha_{\tau,\tau+j^*-j_\tau} \\ &= d_{1,t} + |\{j \in J : j < j_\tau\}| + \beta_\tau + \alpha_{\tau,\tau+j^*-j_\tau}. \end{aligned}$$

If $\tau > t + j^*$, then $\beta_\tau = |\{j \in J : s_{d_{1,t}+j} \ge \tau\}| = |\{j \in J : j \ge j_\tau\}|$ and $\alpha_{\tau,\tau+j^*-j_\tau} = 0$. Otherwise, if $\tau = t + j$ for some $j \in \{1, \dots, j^*\}$, then $\beta_\tau = |\{j \in J : s_{d_{1,t}+j} \ge \tau$ and $j < j\}|$ and $\alpha_{\tau,\tau+j^*-j_\tau} = \alpha_{t+j,t+j^*} = |\{j \in J : j \ge j\}|$. Thus in both cases we have $\alpha x^\tau + \beta y^\tau = d_{1,t} + |J|$.
□

Example 2.2.12 Let $T = 12$ and $d_t = 1$ for $t \in \{3, 6, 8, 9, 11, 12\}$. Let $t = 3$ and $J = \{3, 4, 5\}$, thus, $\{s_{d_{1,t}+j} : j \in J\} = \{9, 11, 12\}$. By Theorem 2.2.17,

$$\begin{array}{l} \qquad\qquad\qquad\quad + y_7 + 2y_8 + 3y_9 + 2y_{10} + 2y_{11} + y_{12} \\ x_1 + x_2 + x_3 \quad + x_6 + x_7 \ + x_8 \end{array} \ge 4 \tag{2.25}$$

is a facet-defining LSM inequality for this instance. This also holds for

$$\begin{array}{l} \qquad\qquad\qquad\qquad\quad + y_6 \qquad + y_8 + y_9 + y_{10} + y_{11} + y_{12} \\ x_1 + x_2 + x_3 + x_4 + x_5 \qquad + x_7 \qquad + x_9 \end{array} \ge 4, \tag{2.26}$$

where $t = 4$, $J = \{1, 3, 5\}$, and, hence, $\{s_{d_{1,t}+j} : j \in J\} = \{6, 9, 12\}$. □

Consider the first inequality in the above example. Let $V_0 = \{t : \alpha_t = \beta_t = 0\} = [4, 5]$ and let $V_1 = \{t : \alpha_t = 0 < \beta_t\} = [9, 12]$. Then the right-hand side equals $d_{1,12} - |V_0|$. Recall that all facet-defining inequalities of the general form (2.7) with right-hand side $d_{1,t^*} - |V_0|$ are contained in the class of HRSM inequalities discussed in Subsection 2.2.2. Here t^* denotes as usual the last period t with $\alpha_t + \beta_t > 0$ and V_0 is the set of holes before t^*. Thus, (2.25) also belongs to the class of facet-defining HRSM inequalities (cf. Example 2.2.5). It is not hard to see that this holds for every inequality (2.24) for which J is an interval $\{j', \dots, j^*\}$. Observe that in this case the inequality is also a block inequality containing one block, namely, the $(|J|, s_{d_{1,t}+j^*} - t - j^*)$-block $[t + j', s_{d_{1,t}+j^*}]$. Recall that a block is said to be regular if $u(v) = u(v+1) - 1$ for all periods v and $v + 1$ in the intersection of V_1 and the block. Thus, the $(3, 4)$-block $[6, 12]$ in (2.24) is nonregular. In fact, the block is regular if and only if $s_{d_{1,t}+j^*} = s_{d_{1,t}+j'} + j^* - j'$.

In Section 2.3 we will give a partial linear description of conv($\mathcal{X}$) that solves the problem when the costs satisfy the Wagner-Whitin property. This linear description will consist of the inequalities in the LP-relaxation of DLSP and the LSM inequalities with $J = \{1, \dots, j^*\}$, $1 \le j^* \le d_{t+1,T}$.

We next discuss an extension of the LSM inequalities by introducing holes in the interval $[1, t]$. In the sequel, we use the following notation: if t is a period, and j and k are two nonnegative integers, then $s(t, j, k)$ denotes the $(j + k)$th demand period after t, i.e., $s(t, j, k) = s_{d_{1,t}+j+k}$.

Proposition 2.2.18 *Let $t \in [0, T]$, $k \ge 0$, and $H \subseteq [1, t]$ such that the inventory at the end of period t is at least k when all periods in H are used for production. Furthermore, let $J \subseteq \{1, \dots, d_{t+1,T} - k\}$. Then*

$$\sum_{\tau \in [1,t] \setminus H} x_\tau + \sum_{j \in J} (x_{t+j} + y_{t+j+1,s(t,j,k)}) \ge d_{1,t} + k + |J| - |H|. \tag{2.27}$$

is valid for $\mathcal{X}$.

PROOF. The proof is similar to the proof of Lemma 2.2.5, in which we gave a necessary and sufficient condition for the HRSM inequalities to be valid for $\mathcal{X}$. Rewrite (2.27) as follows:

$$I_t \ge k + \sum_{j \in J} (1 - x_{t+j} - y_{t+j+1,s(t,j,k)}) + \sum_{\tau \in H} (x_\tau - 1), \tag{2.28}$$

where I_t denotes as usual the stock at the end of period t, thus, $I_t = x_{1,t} - d_{1,t}$. Define $h^* = |H|$ and denote the periods in H by t_h, $1 \le h \le h^*$. Furthermore, let $\mathcal{X}_h$, $0 \le h \le h^*$, be the set of solutions to DLSP in which the periods t_i, $h < i \le h^*$, are used for production. Hence, $\mathcal{X}_h = \{(x, y) \in \mathcal{X} : x_{t_i} = 1, h < i \le h^*\}$. We show by induction that

$$I_t \ge k + \sum_{j \in J} (1 - x_{t+j} - y_{t+j+1,s(t,j,k)}) + \sum_{i=1}^{h} (x_{t_i} - 1) \tag{2.29}$$

is valid for $\mathcal{X}_h$, $0 \le h \le h^*$. Since $\mathcal{X}_{h^*} = \mathcal{X}$, this obviously proves the statement.

We first show that

$$I_t \ge k + \sum_{j \in J} (1 - x_{t+j} - y_{t+j+1,s(t,j,k)}) \tag{2.30}$$

is valid for $\mathcal{X}_0$. Similar as in the proof of Proposition 2.23, we use induction on $|J|$. First, let $J = \emptyset$ and let (x, y) be a solution in $\mathcal{X}_0$. Then $x_\tau = 1$ for all $\tau \in H$. Hence, by assumption, $I_t \geq k$. The rest of the proof is analogous to the proof of Proposition 2.23, to which the reader is referred for more details.

Now suppose that the validity of (2.30) for $\mathcal{X}_h$ has been established for $0 \leq h \leq h'$, where $h' < h^*$. We claim that

$$I_t \geq k + \sum_{j \in J}(1 - x_{t+j} - y_{t+j+1,s(t,j,k)}) + \sum_{h=1}^{h'+1}(x_{t_h} - 1) \qquad (2.31)$$

is satisfied by all $(x, y) \in \mathcal{X}_{h'+1}$. This obviously holds for $\{(x, y) \in \mathcal{X}_{h'+1} : x_{t_{h'+1}} = 1\} = \mathcal{X}_{h'}$. Therefore, let (x, y) be a solution in $\mathcal{X}_{h'+1}$ without production in period $t_{h'+1}$. Let $(\bar{x}, \bar{y})$ be the solution obtained from (x, y) by setting $x_{t_{h'+1}}$ and $y_{t_{h'+1}}$ to one. Since (x, y) is a feasible solution to DLSP, the extra unit produced in period $t_{h'+1}$ increases the inventory at the end of period t by one, hence, $\bar{I}_t = I_t + 1$. Obviously, $(\bar{x}, \bar{y}) \in \mathcal{X}_{h'}$, hence the induction hypothesis yields that

$$\begin{aligned} I_t &= \bar{I}_t - 1 \geq k + \sum_{j \in J}(1 - \bar{x}_{t+j} - \bar{y}_{t+j+1,s(t,j,k)}) + \sum_{h=1}^{h'}(\bar{x}_{t_h} - 1) - 1 \\ &= k + \sum_{j \in J}(1 - x_{t+j} - y_{t+j+1,s(t,j,k)}) + \sum_{h=1}^{h'}(x_{t_h} - 1) - 1. \end{aligned}$$

This shows the validity of (2.31) for (x, y) and, hence, for $\mathcal{X}_{h'+1}$. □

An inequality of the form (2.27) is called a *hole-lifted left stock-minimal inequality* or HLSM inequality for short. Obviously, an HLSM inequality with $H = \emptyset$ is an LSM inequality. We already characterized the facet-defining LSM inequalities of conv($\mathcal{X}$). Therefore, consider an HLSM inequality with $|H| \geq 1$. In order for (2.27) to define a facet of conv($\mathcal{X}$), the conditions P1–P3 of Lemma 2.2.2 must be satisfied. This imposes the following restrictions on the set H: $|H| < d_{h,T}$, where $h = \min_{\tau \in H} \tau$, and

$$|H \cap [h, \tau]| > d_{h,\tau} \text{ for every } \tau \in [h, t]. \qquad (2.32)$$

Thus, in particular, $|H| > d_{h,t}$. Observe that if H satisfies the aforementioned restrictions, then the stock at the end of period t is at least $|H| - d_{h,t}$ when all periods in H are used for production. Thus, $k = |H| - d_{h,t}$ in (2.27). Furthermore, $J \neq \emptyset$, since there must be at least one period with positive y-coefficient. Note that from $J \subseteq \{1, \ldots, d_{t+1,T} - k\}$, $J \neq \emptyset$, and $k = |H| - d_{h,t}$ it follows that $|H| < d_{h,T}$. The following theorem implies that P1–P3 are also sufficient for an HLSM inequality with $H \neq \emptyset$ to be facet-defining of conv($\mathcal{X}$). The result can be proven in a similar way as Theorem 2.2.17.

Theorem 2.2.19 *Let $t \in [1, T]$ and let H be a nonempty subset of $[1, t]$ satisfying (2.32). Furthermore, let $J \subseteq \{1, \dots, d_{h,T} - |H|\}$, $J \neq \emptyset$, where $h = \min_{\tau \in H} \tau$. Then*

$$\sum_{\tau \in [1,t] \setminus H} x_\tau + \sum_{j \in J} (x_{t+j} + y_{t+j+1, s(t,j,|H|-d_{h,t})}) \geq d_{1,h-1} + |J| \tag{2.33}$$

defines a facet of conv(X). □

It is not hard to show that every HLSM inequality (2.33) for which J is an interval, can also be considered as an HRSM inequality with $V_0 = H \cup \{t + j : 1 \leq j \leq j_{\min} - 1\}$ and $V_1 = [t + j_{\max} + 1, s(t, j_{\max}, |H| - d_{h,t})]$, where $j_{\min} = \min_{j \in J} j$ and $j_{\max} = \max_{j \in J} j$. See for example the first inequality below.

Example 2.2.13 Let $T = 12$ and $d_t = 1$ for $t \in \{2, 4, 8, 10, 11, 12\}$. Take $t = 7$ and $H = \{1, 2, 4, 5\}$. Furthermore, take $J = \{1, \dots, |H| - d_{1,t}\} = \{1, 2\}$. Then $s(t, j, 2) = 10 + j$, $j \in J$. By Theorem 2.2.19

$$\begin{array}{l} \qquad\qquad\qquad + y_9 + 2y_{10} + 2y_{11} + y_{12} \\ x_3 \quad + x_6 + x_7 + x_8 + x_9 \end{array} \geq 2,$$

is a facet-defining inequality for this instance. This also holds for

$$\begin{array}{l} \qquad\qquad\qquad\qquad\qquad\qquad + y_9 + y_{10} + y_{11} + y_{12} \\ x_1 + x_2 + x_3 + x_4 + x_5 \quad + x_7 + x_8 \qquad + x_{10} \end{array} \geq 4,$$

where $t = 7$, $H = \{6\}$, and $J = \{1, 3\}$. □

To conclude this subsection, we describe a polynomial algorithm to separate HLSM inequalities. Let $(\hat{x}, \hat{y})$ denote a solution to the LP-relaxation of DLSP. The algorithm provides the most violated inequality of the form

$$\sum_{\tau \in [1,t] \setminus H} x_\tau + \sum_{j \in J} (x_{t+j} + y_{t+j+1, s(t,j,|H|-d_{h,t})}) \geq d_{1,h-1} + |J|, \tag{2.34}$$

where $t \in [1, T]$ and either $H = \emptyset$ and $J \subseteq \{1, \dots, d_{h,T} - |H|\}$ or H and J satisfy the conditions of Proposition 2.2.19, or concludes that such an inequality does not exist.

In the sequel, we assume that t is fixed. We first make the following observation with respect to the set H. Let $H \subseteq [1, t]$, $H \neq \emptyset$, and define $h = \min_{\tau \in H} \tau$. Define $k_1 = d_{1,h-1}$ and $k_2 = |H| + k_1$. Then s_{k_1} is the last demand period before h, hence, $H \subseteq [s_{k_1} + 1, t]$. Now (2.32) is equivalent to

$$|H \cap [s_{k_1} + 1, s_k]| > k - k_1 \text{ for all } k \in \{k_1 + 1, \dots, d_{1,t}\}. \tag{2.35}$$

Note that the above condition can only be satisfied if $k_2 > d_{1,t}$.

We first deal with the separation problem for inequalities (2.34) with $H \subseteq [s_{k_1} + 1, t]$ and $|H| = k_2 - k_1$ for given integers k_1 and k_2 satisfying either $0 \leq k_1 < d_{1,t} < k_2 < d_{1,T}$ or $k_1 =$

$d_{1,t} \leq k_2 < d_{1,T}$. The question to be answered is: do there exist a set $H \subseteq [s_{k_1}+1, t]$ of size $k_2 - k_1$ satisfying (2.35), and a set $J \subseteq \{1, \ldots, d_{1,T} - k_2\}$ such that

$$\sum_{\tau \in [1,t] \setminus H} \hat{x}_\tau + \sum_{j \in J} (\hat{x}_{t+j} + \hat{y}_{t+j+1, s_{k_2+j}}) < k_1 + |J|? \tag{2.36}$$

Note that $H = \emptyset$ if and only if $k_1 = k_2 = d_{1,t}$, hence, in this case (2.35) is satisfied. Furthermore, if $H \neq \emptyset$ and $h = \min_{\tau \in H} \tau$, then $|H| - d_{h,t} = k_2 - d_{1,t}$, $d_{t+1,T} - (|H| - d_{h,t}) = d_{1,t} - k_2$, and $s(t, j, k_2 - d_{1,t}) = s_{k_2+j}$. For $1 \leq j \leq d_{1,T} - k_2$ we define

$$f_j(t, k_2) = \hat{x}_{t+j+1} + \hat{y}_{t+j+1, s_{k_2+j}}.$$

Then the separation problem boils down to finding sets H and J such that

$$\sum_{\tau \in [1,t] \setminus H} \hat{x}_\tau - k_1 + \sum_{j \in J} f_j(t, k_2) - |J| \; < \; 0.$$

Since $\sum_{\tau \in [1,t] \setminus H} \hat{x}_\tau \geq \hat{x}_{1, s_{k_1}} \geq k_1$ for any $H \subseteq [s_{k_1}+1, t]$, a most violated HLSM inequality, if one exists, will satisfy $f_j(t, k_2) \leq 1$ for all $j \in J$. Hence, take $J = \{j : 1 \leq j \leq d_{1,T} - k_2$ and $f_j(t, k_2) < 1\}$. From all possible sets H we take the one that maximizes $\sum_{\tau \in H} \hat{x}_\tau$. Thus, if $k_1 = d_{1,t}$, then we take the $k_2 - k_1$ periods τ in $[s_{k_1}+1, t]$ with largest value of $\hat{x}_\tau$. If $k_1 < d_{1,t}$, then H must be chosen in such a way that $|H \cap [s_{k_1}+1, s_k]| > k - k_1$ for all $k \in \{k_1+1, \ldots, d_{1,t}\}$. This is achieved in the same way as in the separation algorithm for the HRSM inequalities. First, we determine a set $H(t, k_1) \subseteq [s_{k_1}+1, t]$ of size $d_{1,t} - k_1$ that satisfies $|H(t, k_1) \cap [s_{k_1}+1, s_j]| > k - k_1$ for all $k \in \{k_1+1, \ldots, d_{1,t}\}$, and $\sum_{\tau \in H(t,k_1)} \hat{x}_\tau$ is maximal:

determine $\tau \in [s_{k_1}+1, s_{k_1+1}-1]$ such that $\hat{x}_\tau$ is maximal; $H(t, k_1) := \{\tau\}$;
for $k = k_1 + 1$ **to** $d_{1,t}$ **do begin**
  determine $\tau \in [s_{k_1}+1, s_j] \setminus H(t, k_1)$ such that $\hat{x}_\tau$ is maximal; $H(t, k_1) := H(t, k_1) \cup \{\tau\}$
end

Let H consists of the set $H(t, k_1)$ and the $k_2 - d_{1,t} - 1$ (≥ 0) periods τ in $[s_{k_1}+1, t] \setminus H(t, k_1)$ with largest value of $\hat{x}_\tau$. It is readily checked that H satisfies the required conditions. Observe that $(\hat{x}, \hat{y})$ violates an inequality of the form (2.36) for t, k_1, and k_2 if and only if (2.36) is violated for the given choices of H and J.

The following algorithm provides the most violated HLSM inequality, if one exists, for a given period t. Note that $f_j(t, k_2) = f_j(t-1, k_2) + \hat{x}_{t+j} - \hat{x}_{t+j-1} - \hat{y}_{t+j}$ and $f_j(t, k_2) = f_{j+1}(t-1, k_2-1)$ provided that $j < d_{1,T} - k_2$ and $k_2 > d_{1,t}$. Hence, the value of $f_j(t, k_2)$ can be evaluated in an efficient way.

begin SEPARATION(t)
  $\Delta^{\text{opt}} = 0$; determine $f_j(t, k_2)$ for $d_{1,t} \le k_2 < d_{1,T}$ and $1 \le j \le d_{1,T} - k_2$;
  for $k_2 = d_{1,t}$ **to** $d_{1,T} - 1$ **do begin**
    $J_{k_2} := \{j : 1 \le j \le d_{1,T} - k_2 \text{ and } f_j(t, k_2) < 1\}$; $\Delta_{k_2} := \sum_{j \in J_{k_2}} f_j(t, k_2) - |J_{k_2}|$
  end
  for $k_1 = 0$ **to** $d_{1,t}$ **do begin**
    if $k_1 = d_{1,t}$ **then begin** $H := \emptyset$; $k^* := d_{1,t}$ **end**
    else begin
      determine $H(t, k_1)$ as described previously; $H := H(t, k_1)$; $k^* := d_{1,t} + 1$
    end
    $\Delta := \hat{x}_{1,t} - \sum_{\tau \in H} \hat{x}_\tau - k_1$;
    for $k_2 = k^*$ **to** $d_{1,T} - 1$ **do begin**
      if $\Delta + \Delta_{k_2} < \Delta^{\text{opt}}$ **then begin** $J^{\text{opt}} := J_{k_2}$; $H^{\text{opt}} := H$; $\Delta^{\text{opt}} := \Delta + \Delta_{k_2}$ **end**
      determine $\tau \in [s_{k_1} + 1, t] \backslash H$ such that $\hat{x}_\tau$ is maximal; $H := H \cup \{\tau\}$; $\Delta := \Delta - \hat{x}_\tau$
    end
  end
end.

If $\Delta^{\text{opt}} < 0$ at termination of SEPARATION(t), then (2.27) is violated for t, H^{opt}, and J^{opt}. Applying the above algorithm for every $t \in [1, T-1]$ gives an $O((d_{1,T}T)^2)$ algorithm to find the most violated HLSM inequality. Note that the separation can be performed in $O(d_{1,T}T)$ time when we restrict ourselves to HLSM inequalities with $H = \emptyset$, i.e., to the LSM inequalities (2.24).

2.2.5 Further remarks

In this section we studied facet-defining inequalities of conv($\mathcal{X}$) with x-coefficients in $\{0, 1\}$. We derived some general properties of these inequalities and discussed three subclasses in more detail. We already observed that the intersection of these three subclasses is not empty. The HLSM inequalities for which K is an interval also belong to the class of HRSM inequalities. In fact, it is not hard to show that these are the only inequalities that are contained in both classes. Moreover, these inequalities are block inequalities with one, not necessarily regular, block. The inequalities $x_{d_{1,t}} + y_{d_{1,t}+1,t} \ge 1$, where t is a demand period, are contained in the intersection of the three subclasses.

Example 2.2.14 Let $T = 12$ and $d_t = 1$ for $t \in \{3, 6, 8, 9, 11, 12\}$. Take $V = \{8, 10, 11, 12\}$, $u(8) = 2$, $u(10) = 5$, $u(11) = 7$, and $u(12) = 10$. Then

$$\begin{array}{l} \quad\quad\quad\;\; + y_3 + y_4 + y_5 + 2y_6 + 2y_7 + 3y_8 + 2y_9 + 2y_{10} + 2y_{11} + y_{12} \\ x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \quad\quad\quad + x_9 \end{array} \ge 6$$

is a facet-defining HRSM inequality for this instance, since $S_2 = \{3, 6, 8, 9, 11, 12\}$, $d_{u(v)} = 0$, and $v - u(v) = |\{t \in S_2 : t \ge u(v)\}|$ for every $v \in V_1$. Obviously, the inequality is neither a block inequality nor an HLSM inequality. An example of a facet-defining R-block inequality that is not contained in one of the other two subclasses is

$$\begin{array}{l} \qquad + y_2 + y_3 \qquad + y_5 + y_6 + y_7 \qquad + y_9 + y_{10} + y_{11} \\ x_1 + x_2 \qquad + x_4 + x_5 + x_6 \qquad + x_8 + x_9 + x_{10} \qquad + x_{12} \end{array} \geq 5.$$

Finally, inequality (2.26) in Example 2.2.12 is an HLSM inequality that defines a facet for the given instance and belongs to neither of the other two subclasses. □

For most instances the three subclasses, together with the inequalities in the model, only yield a partial description of conv($\mathcal{X}$).

Example 2.2.15 Let $T = 11$ and $d_t = 1$ for $t \in \{7, 9, 10, 11\}$. For this instance

$$\begin{array}{l} \qquad + y_3 + y_4 + y_5 + 2y_6 + 2y_7 + y_8 + 2y_9 + y_{10} + y_{11} \\ x_1 + x_2 \qquad + x_4 + x_5 \qquad\qquad + x_8 \end{array} \geq 3$$

defines a facet. □

It is readily seen that the above inequality does not belong to any of the three subclasses discussed in the previous subsections. In fact, even a complete characterization of the facet-defining inequalities with x-coefficients in $\{0, 1\}$ would not give a complete linear description of conv($\mathcal{X}$), since for most instances there exist facet-defining inequalities with x-coefficients larger than one.

Example 2.2.16 Let $T = 8$ and $d_t = 1$ for $t \in \{4, 6, 7, 8\}$. For this instance

$$\begin{array}{l} \qquad + 3y_2 \qquad + 2y_4 + y_5 \qquad + 2y_7 + 2y_8 \\ 3x_1 \qquad + 3x_3 \quad + x_4 + x_5 + 2x_6 + 2x_7 \end{array} \geq 7$$

and

$$\begin{array}{l} \qquad + y_2 + y_3 \qquad + 2y_5 \qquad + 2y_7 + y_8 \\ x_1 + x_2 \qquad + 2x_4 \qquad + 2x_6 \qquad + x_8 \end{array} \geq 4$$

are facet-defining. □

2.3 Wagner-Whitin costs

In this section we give a partial linear description of the convex hull of feasible solutions to DLSP that solves the problem in the presence of Wagner-Whitin costs. Recall that the costs are said to satisfy the Wagner-Whitin property if $c_t \geq c_{t+1}$ for all t. This implies that there exists an optimal solution that satisfies the zero-inventory property, which means that the inventory at the end of period $t-1$ is zero for any period t in which a production batch is started.

Pochet and Wolsey [32] derive similar results for four other single-item lot-sizing models. For each problem they give an extended formulation for which the LP-relaxation always yields a solution that satisfies the zero-inventory property. By projection they obtain an LP-formulation in the original variables that solves the problem in the presence of Wagner-Whitin costs. We will prove our result in a different way, without the introduction of additional variables.

We start from the formulation of DLSP presented in Section 2.1 with the extra restriction that overproduction is not allowed, i.e., we require $x_{1,T} = d_{1,T}$. Note that if c_t is nonnegative for all t, then there always exists an optimal solution to the more general formulation that satisfies this constraint. Denote by RDLSP the LP-relaxation of DLSP extended with the $O(d_{1,t}T)$ inequalities

$$x_{1,t} + \sum_{i=1}^{j}(x_{t+i} + y_{t+i+1,s_{d_{1,t}+i}}) \geq d_{1,t} + j \tag{2.37}$$

for $t \in [0, T-1]$ and $j \in \{1, \ldots, d_{t+1,T}\}$. As usual, s_i denotes the ith demand period in $[1, T]$. Inequalities (2.37) are a special case of the LSM inequalities for which K is an interval (cf. Subsection 2.2.4). It was already observed that these inequalities also belong to the class of RSM inequalities discussed in Subsection 2.2.2. We will prove the following result:

Theorem 2.3.1 *If the cost function satisfies the Wagner-Whitin property, then the objective value of RDLSP equals the objective value of DLSP.*

When c_t strictly decreases in t, an even stronger result can be proven, namely, that RDLSP solves DLSP.

Theorem 2.3.2 *If $c_t > c_{t+1}$ for every period t, then any optimal solution of RDLSP is a convex combination of feasible solutions of DLSP, i.e., the set of optimal solutions of RDLSP has integral extreme points.*

The remainder of this section is mainly devoted to the proof of the second theorem. Afterwards we will prove Theorem 2.3.1 as a corollary to Theorem 2.3.2. From now on it is therefore assumed that $c_t > c_{t+1}$ for all t. The proof uses a partitioning of a solution (x, y) of RDLSP into a set of batches $\mathcal{B}$, where a batch $B = [p^B, q^B]$ is identified with the partial solution (x^B, y^B) defined by $y^B_{p^B} = 1$, $x^B_t = 1$ for $t \in [p^B, q^B]$, and all other variables equal zero. Furthermore, a *value* b^B, $0 < b^B \leq 1$, is attached to every batch B such that $(x, y) = \sum_{B \in \mathcal{B}} b^B (x^B, y^B)$. We say that $\mathcal{B}$ satisfies the *partitioning condition* if

$$\forall_{i \in \{1,\ldots,d_{1,T}\}} \sum_{B \in \mathcal{B}: s_i \in I^B} b^B = 1, \tag{2.38}$$

where I^B consists of the first $|B|$ demand periods in $[p^B, T]$.

The proof of Theorem 2.3.2 consists of the following two steps. First, we prove that the partitioning condition is a sufficient condition for (x, y) to be a convex combination of solutions of DLSP (Lemma 2.3.3). Second, we present a greedy algorithm that partitions any optimal solution (x^*, y^*) of RDLSP into a set of batches $\mathcal{B}$ with values b^B, $B \in \mathcal{B}$, such that $(x^*, y^*) = \sum_{B\in\mathcal{B}} b^B(x^B, y^B)$ and the partitioning condition is satisfied. Combining these results yields that all extreme points of the set of optimal solutions of RDLSP are integral.

Lemma 2.3.3 *Given a set of batches $\mathcal{B}$ with values b^B, $0 < b^B \le 1$, $B \in \mathcal{B}$, such that* (2.38) *is satisfied. Then $(x, y) := \sum_{B\in\mathcal{B}} b^B(x^B, y^B)$ is a convex combination of solutions of DLSP.*

PROOF. The lemma is proven by induction on the number of batch-pairs (B, D) in $\mathcal{B}$ with intersecting demand sets I^B and I^D, which is denoted by ν. Thus, $\nu = |\{(B, D) : B, D \in \mathcal{B}, B \ne D, \text{ and } I^B \cap I^D \ne \emptyset\}|$.

If $\nu = 0$, i.e., if no two batches have an intersecting demand set, then, by (2.38), each batch B in $\mathcal{B}$ has value $b^B = 1$, and the lemma follows immediately.

Now let $\nu > 0$ and suppose that the result has been established for sets of batches that satisfy the partitioning condition and for which at most $\nu - 1$ batch-pairs have intersecting demand sets. In order to show that (x, y) can be written as a convex combination of solutions of DLSP, we introduce the following definition: a subset $\mathcal{D}$ of $\mathcal{B}$ is said to yield a partition of the set of demand periods $\{s_1, \ldots, s_j\}$, $1 \le j \le d_{1,T}$, if $\cup_{B\in\mathcal{D}:i\in I^B} s_i = \{s_1, \ldots, s_j\}$ and no two batch-pairs in $\mathcal{D}$ have intersecting demand set. We will construct a subset $\mathcal{D}$ of $\mathcal{B}$ that yields a partition of the set $\{s_1, \ldots, s_{d_{1,T}}\}$. First, we take a batch B whose demand set contains the first demand period s_1 and set $\mathcal{D} = \{B\}$. Suppose that we have found a set of batches $\mathcal{D}$ that yields a partition of the first $i < d_{1,T}$ demand periods. Then there exists a batch $D \in \mathcal{B}\backslash\mathcal{D}$ such that the demand set I^D contains s_{i+1} but not s_i. This follows from

$$\sum_{B\in\mathcal{B}\backslash\mathcal{D}:s_{i+1}\in I^B} b^B = \sum_{B\in\mathcal{B}:s_{i+1}\in I^B} b^B = 1 = \sum_{B\in\mathcal{B}:s_i\in I^B} b^B > \sum_{B\in\mathcal{B}\backslash\mathcal{D}:s_i\in I^B} b^B.$$

The demand set of D is $\{s_{i+1}, \ldots, s_{i'}\}$ for some $i' \in \{i + 1, \ldots, d_{1,T}\}$. Adding D to $\mathcal{D}$ gives a partition of the demand periods $\{s_1, \ldots, s_{i'}\}$. We proceed in this way until $\mathcal{D}$ yields a partition of $\{s_1, \ldots, s_{d_{1,T}}\}$. By construction, the integral vector $(x', y') := \sum_{B\in\mathcal{D}}(x^B, y^B)$ is a feasible solution of DLSP.

Set $\bar{b} = \min\{b^B : B \in \mathcal{D}\}$ and define $\overline{\mathcal{B}} = \mathcal{B}\backslash\{B \in \mathcal{D} : b^B = \bar{b}\}$. Note that, by (2.38) and the assumption that $\nu > 0$, we have $\bar{b} < 1$. Set $\bar{b}^B = (b^B - \bar{b})/(1 - \bar{b})$ for $B \in \overline{\mathcal{B}} \cap \mathcal{D}$ and $\bar{b}^B = b^B/(1 - \bar{b})$ for $B \in \overline{\mathcal{B}}\backslash\mathcal{D}$. Let $i \in \{1, \ldots, d_{1,T}\}$. Since there is exactly one batch $B \in \mathcal{D}$ such that $s_i \in I^B$, we have

$$\sum_{B\in\overline{\mathcal{B}}:s_i\in I^B} \bar{b}^B = \sum_{B\in\mathcal{D}:s_i\in I^B} \frac{b^B - \bar{b}}{1 - \bar{b}} + \sum_{B\in\mathcal{B}\backslash\mathcal{D}:s_i\in I^B} \frac{b^B}{1 - \bar{b}} = \frac{\sum_{B\in\mathcal{B}:s_i\in I^B} b^B - \bar{b}}{1 - \bar{b}} = 1.$$

Hence, $\overline{\mathcal{B}}$ satisfies the partitioning condition. Since $\bar{b} < 1$, there is at least one batch-pair (B, D) with $B \in \overline{\mathcal{B}}$ and $D \in \mathcal{B}\backslash\overline{\mathcal{B}}$ such that $I^B \cap I^D \ne \emptyset$. This implies that the number of pairwise intersecting demand sets in $\overline{\mathcal{B}}$ is less than ν, the number of pairwise intersecting demand sets in $\mathcal{B}$. Now the induction hypothesis yields that $(x'', y'') := \sum_{B\in\overline{\mathcal{B}}} \bar{b}^B(x^B, y^B)$ is a convex combination of integral solutions. Thus, so is $(x, y) = \bar{b}(x', y') + (1-\bar{b})(x'', y'')$. □

Using the above lemma and the observation that one can always add extra startups to a solution, it is not hard to show the following:

Corollary 2.3.4 *If (x, y) is a feasible solution of RDLSP and $\mathcal{B}$ a set of batches B with values b^B, $B \in \mathcal{B}$, such that $x = \sum_{B\in\mathcal{B}} b^B x^B$, $y \geq \sum_{B\in\mathcal{B}} b^B y^B$, and the partitioning condition is satisfied, then (x, y) is a convex combination of solutions of DLSP.* □

From the above results it follows that, in order to prove Theorem 2.3.2, it suffices to show that any optimal solution (x^*, y^*) of RDLSP can be partitioned into a set of batches $\mathcal{B}$ with values b^B, $B \in \mathcal{B}$, such that $x^* = \sum_{B\in\mathcal{B}} b^B x^B$, $y^* \geq \sum_{B\in\mathcal{B}} b^B y^B$, and the partitioning condition is satisfied. In the sequel, (x^*, y^*) denotes an optimal solution of RDLSP. We claim that the following algorithm provides a set of batches $\mathcal{D}$ with the desired properties.

```
begin CONSTRUCT_BATCHES
    for t = 1 to T do begin x̄_t := x*_t; ȳ_t := y*_t; d̄_t := d_t end
        \* x̄_t is called the residual production, etc. *\
    𝒟 := ∅;
    while x̄_{1,T} > 0 do begin
        q^D := last period with positive residual production;
        p^D := last period in [1, q^D] with positive residual startup;
        D := [p^D, q^D];
        J^D := set of demand periods with positive residual demand in [p^D, T];
        b^D := min{ȳ_{p^D}, min_{t∈D} x̄_t, min_{t∈J^D} d̄_t};
        ȳ_{p^D} := ȳ_{p^D} − b^D;
        for t ∈ D do x̄_t := x̄_t − b^D;
        for t ∈ J^D do d̄_t := d̄_t − b^D;
        𝒟 := 𝒟 ∪ {D}
    end
end.
```

Observe that $\bar{x}_t$, $\bar{y}_t$, and $\bar{d}_t$ are non-increasing and nonnegative during the execution of the algorithm. Moreover, the residual demands $\bar{d}_{s_i}$ are non-increasing in i. It is also easily seen that $\bar{x}_t \leq \bar{y}_t + \bar{x}_{t-1}$ holds for all t. Therefore, $\bar{x}_{q^D} = \min_{t\in D} \bar{x}_t > 0$, and if $J^D \neq \emptyset$, then $\bar{d}_{\bar{s}} = \min_{t\in J^D} \bar{d}_t$, where $\bar{s}$ denotes the last period with positive residual demand. We will prove that during the execution of the algorithm the following invariant holds:

(I$_1$) $\forall_{t\in[1,T]}\ x^*_t = \bar{x}_t + \sum_{B\in\mathcal{D}:t\in B} b^B;$

(I$_2$) $\forall_{t\in[1,T]}\ y^*_t = \bar{y}_t + \sum_{B\in\mathcal{D}:t=p^B} b^B;$

(I$_3$) $\forall_{i\in\{1,\ldots,\bar{d}_{1,T}\}}\ 1 = \bar{d}_{s_i} + \sum_{B\in\mathcal{D}:s_i\in J^B} b^B;$

(I$_4$) $\forall_{B\in\mathcal{D}}\ |J^B| = |B|;$

(I$_5$) $\forall_{t\in[1,T-1]}\ \bar{x}_{1,t} \geq \bar{d}_{1,t}$ and $\bar{x}_{1,T} = \bar{d}_{1,T}$.

Note that for $t < p^B$, the residual values are equal to the original values, i.e., $\bar{x}_t = x^*_t$, $\bar{y}_t = y^*_t$, and $\bar{d}_t = d_t$.

Suppose that (I$_1$)$-$(I$_5$) hold during the execution of the algorithm. At termination of the algorithm we have $\bar{x}_t = 0$, $\bar{y}_t \geq 0$, and, by (I$_5$), $\bar{d}_t = 0$ for all t. Hence, by (I$_1$) and (I$_2$), the set of batches $\mathcal{D}$ provided by CONSTRUCT_BATCHES satisfies $x^* = \sum_{B\in\mathcal{D}} b^B x^B$ and $y^* \geq \sum_{B\in\mathcal{D}} b^B y^B$. Moreover, from (I$_4$) it follows that J^B, $B \in \mathcal{B}$, can be identified with I^B, the set of the first $|B|$ demand periods in $[p^B, T]$. Together with (I$_3$), this implies that the set $\mathcal{D}$ satisfies the partitioning condition. Now Corollary 2.3.4 yields that (x^*, y^*) is a convex combination of feasible solutions of DLSP. Thus, the validity of the invariant during the execution of the algorithm implies the validity of Theorem 2.3.2.

The invariant is easily checked to hold initially. We will prove that if the invariant holds at the beginning of an iteration, then it also holds at the end of that iteration. In the sequel the *current* iteration is the one for which validity of the invariant is proven. We denote the batch defined in the current iteration by D. The set of batches that are constructed in previous iterations is denoted by $\mathcal{D}$. Now (I$_1$)$-$(I$_3$) are easily checked to hold at the end of the current iteration, and (I$_5$) follows from (I$_4$). The latter holds at the end of the current iteration if $|J^D| = |D|$. Hence, we are left with the proof of $|J^D| = |D|$.

PROOF OF $|J^D| = |D|$.
We first show that $|J^D| > |D|$ implies that (x^*, y^*) is not optimal. Next, we show that if $|J^D| < |D|$, then (x^*, y^*) violates a constraint of type (2.37). Both results contradict the assumption that (x^*, y^*) is an optimal solution of RDLSP, which leads to the conclusion that $|J^D| = |D|$.

PART 1: $|J^D| \leq |D|$.
Assume that $|J^D| > |D|$. We claim that in this case we can move an amount $\epsilon > 0$ from the production in period q^D to period $q^D + 1$ while maintaining feasibility. Since $c_{q^D} > c_{q^D+1}$, this yields a cheaper solution than (x^*, y^*), which contradicts the optimality of (x^*, y^*). In order to prove our claim, it suffices to show that the following constraints have positive slack, i.e., they are not satisfied at equality:

(i) $x^*_{q^D} \geq 0;$

(ii) $x^*_{q^D+1} \leq 1;$

(iii) $x^*_{q^D+1} \leq y^*_{q^D+1} + x^*_{q^D};$

(iv) $\forall_{t,j:t+j=q^D}\ x^*_{1,t} + \sum_{i=1}^{j}(x^*_{t+i} + y^*_{t+i+1,s_{\bar{d}_{1,t+i}}}) \geq d_{1,t} + j.$

By definition of q^D, we have $x^*_{q^D} \geq \bar{x}_{q^D} > 0$. For the proof of $x^*_{q^D+1} < 1$, we use the following important observation: if period s has positive residual demand in the current iteration, then $s \in J^B$ for every batch $B \in \mathcal{D}$ with $p^B \leq s$. Now let s' be the first demand period after q^D. Then $\bar{d}_{s'} > 0$, since $|J^D| > |D|$. Hence, if $B \in \mathcal{D}$ satisfies $q^D + 1 \in B$, then $s' \in J^B$. Together with $\bar{x}_{q^D+1} = 0$, this yields

$$x^*_{q^D+1} = x^*_{q^D+1} - \bar{x}_{q^D+1} \stackrel{(I_1)}{=} \sum_{B:q^D+1 \in B} b^B \leq \sum_{B:s' \in J^B} b^B = 1 - \bar{d}_{s'} < 1.$$

In order to show that (iii) is not satisfied at equality, notice that whenever $\bar{x}_{q^D+1}$ decreases in an iteration, one of the variables $\bar{x}_{q^D}$ or $\bar{y}_{q^D+1}$ decreases by the same amount. At the beginning of the current iteration, strict inequality holds since $0 = \bar{x}_{q^D+1} < \bar{x}_{q^D}$.

Finally, consider a constraint (2.37) such that $t + j = q^D$. Now

$$\begin{aligned}
& x^*_{1,t} + \sum_{i=1}^{j} (x^*_{t+i} + y^*_{t+i+1,s_{d_{1,t}+i}}) \\
\stackrel{(I_1),(I_2)}{=} \quad & x^*_{1,t} + \sum_{i=1}^{j} (\bar{x}_{t+i} + \bar{y}_{t+i+1,s_{d_{1,t}+i}}) + \sum_{i=1}^{j} \sum_{B \in \mathcal{D}: q^B \geq t+i,\, p^B \leq s_{d_{1,t}+i}} b^B \\
\geq \quad & x^*_{1,t} + \bar{x}_{t+1,t+j} + \bar{y}_{t+2,p^D} + \sum_{i=1}^{j} \sum_{B \in \mathcal{D}: q^B \geq t+i,\, p^B \leq s_{d_{1,t}+i}} b^B \\
\stackrel{(*)}{\geq} \quad & x^*_{1,t} + \bar{x}_{t+1,t+j} + \bar{y}_{t+2,p^D} + \sum_{i=1}^{j} \sum_{B \in \mathcal{D}: s_{d_{1,t}+i} \in J^B} b^B \\
\stackrel{(I_3)}{=} \quad & x^*_{1,t} + \bar{x}_{t+1,q^D} + \bar{y}_{t+2,p^D} + \sum_{i=1}^{j} (1 - \bar{d}_{s_{d_{1,t}+i}}),
\end{aligned}$$

where $(*)$ holds because $s_{d_{1,t}+i} \in J^B$ implies $p^B \leq s_{d_{1,t}+i}$, and $t + i \leq t + j = q^D \leq q^B$. In order to show that strict inequality holds for the constraint under consideration, we distinguish two cases. First, suppose that $p^D > t + 1$. Then, since $\bar{y}_{p^D} > 0$, $\bar{x}_{1,q^D} = \bar{x}_{1,T}$, and $\bar{d}_{1,t} = d_{1,t}$, we have

$$\begin{aligned}
& x^*_{1,t} + \bar{x}_{t+1,q^D} + \bar{y}_{t+2,p^D} + \sum_{i=1}^{j} (1 - \bar{d}_{s_{d_{1,t}+i}}) \\
> \ & \bar{x}_{1,T} + \sum_{i=1}^{j} (1 - \bar{d}_{s_{d_{1,t}+i}}) \stackrel{(I_5)}{\geq} \bar{d}_{1,s_{d_{1,t}+j}} + \sum_{i=1}^{j} (1 - \bar{d}_{s_{d_{1,t}+i}}) \ = \ \bar{d}_{1,t} + j \ = \ d_{1,t} + j.
\end{aligned}$$

If $p^D \leq t + 1$, then the assumption that $|J^D| > |D|$ implies that $|J^D| > t + j - p^D + 1 \geq j$. Hence, if $\bar{s}$ denotes the last period with positive residual demand, then $\bar{s} > s_{d_{1,t}+j}$. Then, by

(I$_5$), $\bar{x}_{1,q^D} = \bar{x}_{1,T} = \bar{d}_{1,\bar{s}} > \bar{d}_{1,s_{d_{1,t}+j}}$. We have

$$x^*_{1,t} + \bar{x}_{t+1,q^D} + \bar{y}_{t+2,p^D} + \sum_{i=1}^{j}(1 - \bar{d}_{s_{d_{1,t}+i}}) \overset{(*)}{\geq} \bar{x}_{1,q^D} + \sum_{i=d_{1,p^D-1}+1}^{d_{1,t}+j} (1 - \bar{d}_{s_i})$$
$$> \bar{d}_{1,s_{d_{1,t}+j}} + \sum_{i=d_{1,p^D-1}+1}^{d_{1,t}+j} (1 - \bar{d}_{s_i}) = d_{1,p^D-1} + d_{1,t} + j - d_{1,p^D-1} = d_{1,t} + j,$$

where $(*)$ follows from the validity of (I$_1$) for every period in $[p^D, t]$, the validity of (I$_3$) for $i \in \{d_{1,p^D-1}+1, \ldots, d_{1,t}\}$, and the observations that for any $B \in \mathcal{D}$ we have $q^B \geq q^D > t$ and at most $t - p^B + 1$ periods with positive residual demand in $[p^B, t]$.

We conclude that none of the constraints (i)–(iv) is satisfied at equality, which establishes the validity of $|J^D| \leq |D|$.

PART 2: $|J^D| \geq |D|$.
Suppose that $|J^D| < |D|$. We claim that in this case constraint (2.37) with $t = p^D - 1$ and $j = |J^D|$ is violated by (x^*, y^*), i.e.,

$$x^*_{1,p^D-1} + \sum_{i=1}^{|J^D|}(x^*_{p^D+i-1} + y^*_{p^D+i,s_{d_{1,p^D-1}+i}}) < d_{1,p^D-1} + |J^D|.$$

First, suppose that $|J^D| = 0$. Then $x^*_{1,p^D-1} = \bar{x}_{1,p^D-1} < \bar{x}_{1,p^D} \leq \bar{d}_{1,T} = \bar{d}_{1,p^D-1} = d_{1,p^D-1}$, which establishes our claim. In the sequel, we therefore assume that $|J^D| > 0$. In the proof we use the following observation:

$$\forall_{t\in[p^D+1,\bar{s}]}\ \bar{y}_t = 0, \tag{2.39}$$

where $\bar{s}$ denotes the last period with positive residual demand. Note that for $t \in [p^D+1, q^D]$ this holds by choice of p^D. Therefore, suppose that $\bar{y}_\tau > 0$ for some $\tau \in [q^D+1, \bar{s}]$. Similar as in Part 1, we claim that in this case we can obtain a cheaper solution than (x^*, y^*) by moving an amount $\epsilon > 0$ from the production in period q^D to period τ. In order to prove our claim, it again suffices to show that the following constraints are not satisfied at equality: $x^*_{q^D} \geq 0$, $x^*_\tau \leq 1$, $x^*_\tau \leq y^*_\tau + x^*_{\tau-1}$, and $x^*_{1,t} + \sum_{i=1}^{j}(x^*_{t+i} + y^*_{t+i+1,s_{d_{1,t}+i}}) \geq d_{1,t} + j$ for all t and j such that $t + j = q^D$. For most cases the same arguments as in Part 1 can be used. Therefore, we only show that the last inequality is not satisfied at equality when $t + j = q^D$, $p^D \leq t+1$, and $\bar{s} \leq d_{1,t} + j$:

$$x^*_{1,t} + \sum_{i=1}^{j}(x^*_{t+i} + y^*_{t+i+1,s_{d_{1,t}+i}}) \geq x^*_{1,t} + \bar{x}_{t+1,q^D} + \bar{y}_\tau + \sum_{i=1}^{j}(1 - \bar{d}_{s_{d_{1,t}+i}})$$
$$> \bar{x}_{1,q^D} + \sum_{i=d_{1,p^D-1}+1}^{d_{1,t}+j}(1 - \bar{d}_{s_{d_{1,t}+i}}) \geq \bar{d}_{1,s_{d_{1,t}+j}} + \sum_{i=d_{1,p^D-1}+1}^{d_{1,t}+j}(1 - \bar{d}_{s_i}) = d_{1,t} + j.$$

For more details we refer to Part 1.

Note that $s_{d_{1,p^D-1}+|J^D|}$ is the last period with positive residual demand, hence, the right-hand side of (2.37) with $t = p^D - 1$ and $j = |J^D|$ equals $d_{1,\bar{s}}$. We have

$$
\begin{aligned}
& x^*_{1,p^D-1} + \sum_{i=1}^{|J^D|} (x^*_{p^D-1+i} + y^*_{p^D-1+i+1, s_{d_{1,p^D-1}+i}}) \\
\overset{(I_1),(I_2)}{=} \quad & \bar{x}_{1,p^D-1+|J^D|} + \sum_{i=1}^{|J^D|} \bar{y}_{p^D-1+i+1, s_{d_{1,p^D-1}+i}} + \sum_{i=1}^{|J^D|} \sum_{B \in \mathcal{D}: q^B \geq p^D-1+i,\, p^B \leq s_{d_{1,p^D-1}+i}} b^B \\
\overset{(2.39)}{=} \quad & \bar{x}_{1,p^D-1+|J^D|} + \sum_{i=1}^{|J^D|} \sum_{B \in \mathcal{D}: q^B \geq p^D-1+i,\, p^B \leq s_{d_{1,p^D-1}+i}} b^B \\
\overset{(*)}{\leq} \quad & \bar{x}_{1,p^D-1+|J^D|} + \sum_{i=1}^{|J^D|} \sum_{B \in \mathcal{D}: s_{d_{1,p^D-1}+i} \in J^B} b^B \\
\overset{(I_3)}{=} \quad & \bar{x}_{1,p^D-1+|J^D|} + \sum_{i=1}^{|J^D|} (1 - \bar{d}_{s_{d_{1,p^D-1}+i}}) \overset{(\dagger)}{<} d_{1,\bar{s}}.
\end{aligned}
$$

Note that in the current iteration all demand periods in $[p^D, \bar{s}]$ have positive residual demand. Thus, for each $B \in \mathcal{B}$ with $p^B \leq s_{d_{1,p^D-1}+i}$, $i \leq |J^D|$, we have $s_{d_{1,p^D-1}+i} \in J^B$. This shows the validity of $(*)$. Moreover, the assumption that $|J^D| < |D|$ yields that $p^D - 1 + |J^D| < q^D$, hence, by definition of q^D and (I_5), we have $\bar{x}_{1,p^D-1+|J^D|} < \bar{x}_{1,q^D} = \bar{d}_{1,\bar{s}}$. From this the validity of ($\dagger$) immediately follows.

This concludes the proof of $|J^D| = |D|$ and, hence, the proof of Theorem 2.3.2. □

As a corollary we can prove Theorem 2.3.1 as follows. For arbitrary $\epsilon > 0$ the cost function $c'_t := c_t + (T - t)\epsilon$ satisfies the requirements of Theorem 2.3.2. Therefore, for every $\epsilon > 0$ there exists an optimal solution of RDLSP that is an integral extreme point. Since the objective function is continuous in ϵ, there must be an integer optimal solution of RDLSP for $\epsilon = 0$. However, we do not necessarily find that for $\epsilon = 0$ all extreme points of the set of optimal solutions of RDLSP are integral.

In the following chapter we develop a branch-and-cut algorithm in order to solve multi-item problems. As only problems with Wagner-Whitin costs will be considered, the $O(d_{1,T}T)$ constraints of type (2.37) are expected to yield strong cutting planes.

We conclude this section by presenting some computational results for single-item problems with costs that do not satisfy the Wagner-Whitin property. In general, such a problem will not be solved by the addition of inequalities (2.37) only. Our only purpose here is to investigate the quality of the lower bounds obtained when some of the inequalities discussed in the previous section are added to the LP-relaxation. For more details on the implementation the reader is referred to the following chapter.

We consider the following lower bounds:

z_0 : optimal value of the LP-relaxation of DLSP;
z_1 : optimal value of the LP-relaxation of DLSP + inequalities (2.37);
z_2 : optimal value of the LP-relaxation of DLSP + LSM inequalities;
z_3 : optimal value of the LP-relaxation of DLSP + LSM + RSM inequalities.

Recall that the LSM inequalities can be separated in $O(d_{1,T}T)$ time. Also the separation algorithm for the RSM inequalities runs in $O(d_{1,T}T)$ time. We already observed that inequalities (2.37) are contained in the intersection of these two classes.

Our test set consists of twelve instances I.$x.y$, where $x \in \{1, 2, 3\}$ and $y \in \{a, b, c, d\}$. Here x refers to one of three randomly generated demand patterns with $T = 96$ and total demand $d_{1,T} = 65$, and y denotes the type of production cost p_t:

a : $p_t = 10$ if $t \bmod 7 \in \{0, 6\}$ and 0 otherwise;
b : $p_t = 20$ if $t \bmod 7 \in \{0, 6\}$ and 0 otherwise;
c : p_t is randomly chosen in $\{1, \ldots, 10\}$ for all t;
d : p_t is randomly chosen in $\{1, \ldots, 20\}$ for all t.

Cases a and b can be interpreted as follows: one period represents one day and producing during the weekend incurs an extra cost of 10 and 20, respectively. We always take $f_t = 10$ and $h_t = 1$ for all t. The objective is to minimize $\sum_{t=1}^{T}(p_t x_t + f_t y_t + h_t I_t)$, where $I_t = x_{1,t} - d_{1,t}$.

inst	z^*	z_0	g_0	z_1	g_1	z_2	g_2	z_3	g_3
I.1.a	224	202.3	9.7	224	0				
I.2.a	215	197.4	8.2	215	0				
I.3.a	237	208.9	11.9	237	0				
I.1.b	284	256.0	9.9	281.5	0.9	281.5	0.9	281.5	0.9
I.2.b	227	216.9	4.5	227	0				
I.3.b	292	271.5	7.0	292	0				
I.1.c	456	414.9	9.0	453.6	0.5	454.5	0.3	456	0
I.2.c	459	405.8	11.6	456.5	0.5	458.0	0.2	459	0
I.3.c	496	453.5	8.6	494.5	0.3	494.5	0.3	496	0
I.1.d	696	670.7	3.6	690.7	0.8	692.7	0.5	694.5	0.2
I.2.d	747	719.9	3.6	743.5	0.5	745.5	0.2	746.5	0.1
I.3.d	722	708.0	1.9	720.0	0.3	720.0	0.3	722	0

Table 2.1: Lower bounds and gaps for instances with non-Wagner-Whitin costs

For each instance Table 2.1 shows the optimal value z^*, the lower bounds z_i, $0 \leq i \leq 3$, and the corresponding integrality gaps $g_i = 100\% \times (z^* - z_i)/z^*$. The addition of inequalities (2.37) already yields lower bounds that differ less than 1% from the optimal value. The gap is further reduced by the addition of inequalities from the two larger classes, but is not closed for three out of twelve instances.

3. The multi-item DLSP

The multi-item DLSP is an NP-hard problem, as will be shown in Section 3.1. In order to solve multi-item problems to optimality we have developed a branch-and-cut algorithm based on the following integer programming formulation:

$$\text{(DLSP)} \qquad \min \sum_{i=1}^{M} \sum_{t=1}^{T} \left((p_t^i + h_{t,T}^i) x_t^i + f_t^i y_t^i - h_{t,T}^i d_t^i \right)$$

$$\text{s.t.} \qquad x_{1,t}^i \ \geq \ d_{1,t}^i \qquad \text{for all } i \text{ and } t \tag{3.1}$$

$$x_t^i \ \leq \ x_{t-1}^i + y_t^i \qquad \text{for all } i \text{ and } t \ (x_0^i = 0) \tag{3.2}$$

$$\sum_{i=1}^{M} x_t^i \ \leq \ 1 \qquad \text{for all } t \tag{3.3}$$

$$x_t^i, y_t^i \in \{0, 1\} \qquad \text{for all } i \text{ and } t \tag{3.4}$$

As far as the objective function is concerned, recall that p_t^i denotes the cost of producing item i in period t and f_t^i the cost of setting up the machine for item i in period t. Moreover, a cost $h_t^i(x_{1,t}^i - d_{1,t}^i)$ is incurred for the inventory of item i at the end of period t. For a more detailed discussion of the above formulation the reader is referred to Chapter 1.

The cutting plane procedure incorporates separation routines for some of the inequalities discussed in the previous chapter. In addition we introduce valid inequalities for the multi-item problem in Section 3.2. Section 3.3 describes the branch-and-cut algorithm in some detail and reports on its computational performance. The last section discusses other solution methods for DLSP as proposed in the literature and compares their performance to the performance of the branch-and-cut algorithm.

3.1 Complexity

This section discusses complexity results for DLSP. For a general introduction to the theory of computational complexity the reader is referred to Garey and Johnson [16].

First observe that there exists at least one feasible solution to DLSP if and only if for each period t the total demand up to t does not exceed the available capacity up to t, i.e., if and only if

$$\sum_{i=1}^{M} d_{1,t}^i \ \leq \ t \ \text{ for all } t. \tag{3.5}$$

Thus, the feasibility of an instance can be checked in $O(MT)$ time. Moreover, the single-item DLSP can be solved in $O(d_{1,T}T)$ time by a straightforward dynamic programming algorithm (see, e.g., van Hoesel [18]). Complexity results for the multi-item problem are discussed by Salomon et al. in [36]. They show that DLSP is solvable in polynomial time when the startup costs are zero. They further claim that DLSP with zero production costs and constant inventory and startup costs per item is NP-hard, but their proof is not correct since the proposed reduction from PARTITION is not polynomial. However, as pointed out by G. Woeginger, a similar reduction from 3-PARTITION shows the correctness of their claim.

For ease of presentation, the instances for DLSP considered in this section do not necessarily satisfy our general assumption that the demand function is binary. It is not hard to see that for such an instance there exists an equivalent instance with binary demand and a cost function that differs only by a constant value from the original one.

Proposition 3.1.1 (Woeginger, personal communication) *DLSP with $p_t^i = 0$, $h_t^i = h_1^i$, and $f_t^i = f_1^i$ for all i and t is NP-hard.*

PROOF. As mentioned before, the reduction is from 3-PARTITION. This problem is NP-complete in the strong sense ([16], p. 224).

3-PARTITION
Given an integer B and a multiset A consisting of $3n$ positive integers a_i, $1 \leq i \leq 3n$, with $B/4 < a_i < B/2$ and $\sum_{i=1}^{3n} a_i = nB$, does there exist a partition of A into n pairwise disjoint subsets A_j, $1 \leq j \leq n$, such that the elements in A_j add up to B?

Let I_1 be an instance of 3-PARTITION encoded in unary. Let I_2 be an instance of DLSP with $T = n(B+1)$ periods, $3n+1$ items, and

$$
\begin{aligned}
d_t^i &= \begin{cases} a_i, & t = T \\ 0, & \text{otherwise} \end{cases} && 1 \leq i \leq 3n, \\
d_t^{3n+1} &= \begin{cases} 1, & t = j(B+1),\ 1 \leq j \leq n \\ 0, & \text{otherwise} \end{cases} \\
p_t^i &= 0,\ h_t^i = 0,\ f_t^i = 1, && 1 \leq i \leq 3n,\ 1 \leq t \leq T, \\
p_t^{3n+1} &= 0,\ h_t^{3n+1} = 1,\ f_t^{3n+1} = 0, && 1 \leq t \leq T.
\end{aligned}
$$

The cost of a schedule for I_2 is at least $3n$ and equality holds if and only if production for item $3n+1$ occurs in period $j(B+1)$, $1 \leq j \leq n$, and the demand for item i, $1 \leq i \leq 3n$, is produced in exactly one batch. From this it immediately follows that I_1 is a yes-instance if and only if there exists a schedule for I_2 with cost $3n$. □

The instances used in our computational experiments have the cost structure considered in the above proposition. The following result is concerned with the complexity of DLSP when only startup costs are taken into account.

Proposition 3.1.2 *DLSP with $p_t^i = h_t^i = 0$ for all i and t is NP-hard.*

PROOF. The reduction is from a special case of the following problem:

SCHEDULING JOBS OF EQUAL LENGTH (SEL)
Given an integer C, a planning horizon of T periods and n jobs of length p, $p \in \mathbb{Z}^+$, which have to be scheduled on one machine without preemption. Starting job j in period t incurs a cost c_{jt}. Does there exist a feasible schedule with cost less than or equal to C?

Crama and Spieksma [6] prove that SEL is NP-complete, even for $p = 2$ and $c_{jt} \in \{0, 1\}$ for all j and t. The proof is based on a reduction from 3-DIMENSIONAL MATCHING. We use this result to prove our claim.

Let I_1 be an instance of SEL with T periods, n jobs of length 2, and processing costs $c_{jt} \in \{0, 1\}$ for all jobs j and periods t. Let I_2 be an instance of DLSP with T periods, n items, and

$$\begin{aligned} d_t^i &= 0,\ t \neq T, \ \text{and}\, d_T^i = 2, && 1 \leq i \leq n, \\ h_t^i &= p_t^i = 0,\ f_t^i = n + c_{it} + 1, && 1 \leq i \leq n,\ 1 \leq t \leq T. \end{aligned}$$

Assume without loss of generality that $C \leq n$. We prove our claim by showing that there exists a schedule for I_1 with cost C if and only if there exists a schedule for I_2 with cost $n(n+1) + C$. Obviously, a schedule for I_1 with cost C corresponds to a schedule for I_2 with cost $n(n+1) + C$. Observe that a schedule for I_2 is a feasible schedule for I_1 if and only if all batches have length 2, i.e., if and only if exactly n startups occur. Such a schedule has cost less than $(n+1)^2$, whereas a schedule with more than n startups has cost at least $(n+1)^2$. Thus, a schedule for I_2 with cost $n(n+1) + C$ corresponds to a schedule for I_1 with cost C. □

It is still an open problem whether DLSP with zero production and inventory costs and constant startup costs per item ($f_t^i = f_1^i$ for all i) is NP-hard. In particular, we are left with the question whether the minimum number of startups of any feasible solution can be determined in polynomial time ($f_t^i = 1$ for all i and t). The complexity of this problem is closely related to the complexity of the following problem:

BATCH SCHEDULING WITH UNIT CHANGEOVER COSTS (BS-UCC)
Suppose we are given J jobs and an integer C. Each job has a processing time $\hat{p}_j$ and a deadline $\hat{d}_j$, and belongs to a job class $\gamma_j \in \{1, \ldots, n\}$. These jobs have to be scheduled without preemption on one machine such that no job finishes after its deadline. If a job is not scheduled immediately after another job from the same class, then a unit changeover cost is incurred. The question to be answered is whether there exists a feasible schedule with total changeover cost less than or equal to C.

Bruno and Downey [4] prove that BS-UCC is NP-complete by a reduction from PARTITION. However, they also show that the problem is polynomially solvable when the number of deadlines is part of the problem description. To our knowledge, it is still an open problem whether BS-UCC with an arbitrary number of deadlines is NP-complete in the strong sense (cf. Garey and Johnson [16], p. 238). The following proposition shows that an affirmative answer to the latter question will make it very unlikely that the minimum number of startups can be determined in an efficient way.

Proposition 3.1.3 *If BS-UCC with an arbitrary number of deadlines is strongly NP-complete, then DLSP with $p_t^i = h_t^i = 0$ and $f_t^i = 1$ for all i and t is NP-hard.*

Proof. Given an instance I_1 of BS-UCC encoded in unary. Without loss of generality we assume that $\max_j \hat{d}_j = \sum_j \hat{p}_j$. Let I_2 be an instance of DLSP with $T = \max_j \hat{d}_j$, n (= number of classes) items, and

$$d_t^i = \sum_{j:\hat{d}_j=t,\gamma_j=i} \hat{p}_j \qquad \text{for all } i \text{ and } t,$$
$$p_t^i = h_t^i = 0, \; f_t^i = 1 \qquad \text{for all } i \text{ and } t.$$

We show that there exists a schedule for I_1 with cost C if and only if there exists a schedule for I_2 with cost C. Obviously, a schedule for I_1 with cost C corresponds to a solution for I_2 with cost C, i.e., a solution with C startups. However, a solution for I_2 might yield a preemptive schedule for I_1. Let k_i, $1 \leq i \leq n$, denote the number of jobs in job class i and denote the jobs by J_{ik}, $i \in \{1, \ldots, n\}, k \in \{1, \ldots, k_i\}$, such that job J_{ik} belongs to job class i and $\hat{d}_{J_{ik}} \geq \hat{d}_{J_{il}}$ for $l < k$. Then a solution for I_2 can be considered as a feasible schedule for I_1 if for every period t that is the first period of a production batch the following holds: if production occurs for item i in period t, then the total production for i from period t up to period T must equal $\sum_{l=k}^{k_i} \hat{p}_{J_{il}}$ for some $k \in \{1, \ldots, k_i\}$. Such a solution is called *nonpreemptive*.

We claim that any feasible solution for I_2 can be transformed in polynomial time into a nonpreemptive solution without increasing the number of startups. Therefore, consider a preemptive schedule for I_2 with C startups. Denote by X_t^i the total production for item i from period t up to period T. Let t be the first period of a batch for some item, say i, such that $X_t^i \neq \sum_{l=k}^{k_i} \hat{p}_{J_{il}}$ for any $k \in \{1, \ldots, k_i\}$. Without loss of generality, assume that t is maximal. Let k^* satisfy $\sum_{l=k^*+1}^{k_i} \hat{p}_{J_{il}} < X_t^i < \sum_{l=k^*}^{k_i} \hat{p}_{J_{il}}$. Note that from the feasibility of the schedule under consideration it follows that the deadline of job J_{ik^*} is at least t. Define $\Delta = \sum_{l=k^*}^{k_i} \hat{p}_{J_{il}} - X_t^i$. Now there is at least one batch for item i which ends before t, say in period t'. For convenience, assume that the length of this batch is at least Δ. Move the production for i in the interval $[t' - \Delta + 1, t']$ to $[t - \Delta, t - 1]$ and move the production in period τ, $\tau \in [t' + 1, t - 1]$, in the original schedule to period $\tau - \Delta$. One readily sees that the new schedule is still feasible and has at most C startups. Repeating this argument finally results in a nonpreemptive schedule for I_2 with at most C startups.

From this we conclude that if there exists a solution for I_2 with cost C, then there exists a feasible schedule for I_1 with cost $\leq C$. This completes the proof of the proposition. □

3.2 Valid inequalities

In the following section we discuss a branch-and-cut algorithm for the multi-item DLSP that is based on the formulation presented in the beginning of this chapter. Let $\mathcal{X}$ denote the set of solutions to DLSP, i.e., $\mathcal{X} = \{(x, y) : (x, y) \text{ satisfies } (3.1) - (3.4)\}$. An effective use of the cutting plane procedure requires additional valid inequalities for $\mathcal{X}$ that can serve as cutting planes. We already observed that when the coupling constraints $\sum_i x_t^i \le 1$, $1 \le t \le T$, are omitted, the remaining problem consists of M single-item problems, which are denoted by DLSP$_i$, $1 \le i \le M$:

$$
\begin{array}{llll}
(\text{DLSP}_i) & \min \sum_{t=1}^{T} \left((p_t^i + h_{t,T}^i)x_t^i + f_t^i y_t^i - h_{t,T}^i d_t^i\right) & & \\
\text{s.t.} & x_{1,t}^i \ge d_{1,t}^i & \text{for all } t & (3.6) \\
 & x_t^i \le x_{t-1}^i + y_t^i & \text{for all } t \ (x_0^i = 0) & (3.7) \\
 & x_t^i, y_t^i \in \{0, 1\} & \text{for all } t & (3.8)
\end{array}
$$

Denote by $\mathcal{X}_i$, $1 \le i \le M$, the set of solutions to DLSP$_i$. Obviously, all inequalities that are valid for $\mathcal{X}_i$ are also valid for $\mathcal{X}$. This holds in particular for the inequalities discussed in the previous chapter. The latter are called *single-item* inequalities for DLSP, since all variables with nonzero coefficient have the same index i. By taking the demand for other items into account we can derive single-item inequalities for item i that are valid for $\mathcal{X}$, but not for $\mathcal{X}_i$. This is the subject of the following subsection. In Subsection 3.2.2 we discuss some *multi-item* inequalities.

3.2.1 Single-item inequalities

In multi-item problems the number of periods up to period t that can be used for the production of item i is usually strictly less than t, since demand for other items has to be satisfied as well. In the sequel we always assume that there exists at least one feasible solution to DLSP, thus, $\sum_i d_{1,t}^i \le t$ for all t.

Let U_t^i denote the maximum number of production periods for item i up to period t in any feasible solution. It is not difficult to see that U_t^i has the following value:

$$U_T^i = T - \sum_{j \ne i} d_{1,T}^j \quad \text{and} \quad U_t^i = \min(U_{t+1}^i, t - \sum_{j \ne i} d_{1,t}^j) \quad \text{for } t < T. \tag{3.9}$$

Since DLSP is assumed to be feasible, we have $U_t^i \ge d_t^i$ for all i and t. Moreover, $U_{t+1}^i - U_t^i \in \{0, 1\}$ for $t < T$.

Example 3.2.1 Let $T = 12$ and $M = 3$. Consider the following demand function.

t	1	2	3	4	5	6	7	8	9	10	11	12
d_t^1	0	0	0	1	0	1	0	0	1	0	0	1
d_t^2	0	0	1	0	0	0	1	0	0	0	0	1
d_t^3	0	0	0	1	0	0	1	0	0	1	0	0

Using (3.9), the values of U_t^i can be easily determined.

t	1	2	3	4	5	6	7	8	9	10	11	12
U_t^1	1	2	2	2	3	3	3	4	5	5	6	6
U_t^2	1	2	2	2	3	3	3	3	4	4	5	5
U_t^3	1	2	2	2	3	3	3	4	4	5	5	5

Given i and t, one readily constructs a solution (x, y) satisfying $x_{1,t}^i = U_t^i$. For example, let $i = 1$ and $t = 6$. Suppose item 1 is produced in periods 4, 5, and 6. Then the periods 1, 2, 3, and 7 must be used for the production of items 2 and 3, say, $x_1^2 = x_2^2 = 1$ and $x_3^3 = x_7^3 = 1$. Together with $x_{10}^3 = x_{11}^2 = x_{12}^1 = 1$, this yields a feasible solution satisfying $x_{1,6}^1 = U_6^1 = 3$. From this it also immediately follows that a solution satisfying $x_{1,6}^1 \geq 4$ does not exist, since in that case the total production up to period 7 must be at least $4 + d_{1,7}^2 + d_{1,7}^3 = 8$. □

Now let $i \in \{1, \ldots, M\}$ and $k \in \{1, \ldots, d_{1,T}^i\}$. Denote by s_k^i the kth demand period of item i. Recall from Section 2.1 that

$$x_k^i + y_{k+1,s_k^i}^i \geq 1$$

defines a facet of conv($\mathcal{X}_i$). The validity of this inequality follows from the observation that at least one unit of item i must be produced in the interval $[k, s_k^i]$.

The inequality $x_3^1 + y_{4,9}^1 \geq 1$ is therefore valid for the instance in the above example. However, from $U_4^1 = 2$ and $U_5^1 = 3$ it follows that the third production period for item 1 cannot occur before period 5. Thus, in the interval [5, 9] at least one period must be used for the production of item 1. This establishes the validity of the inequality $x_5^1 + y_{6,9}^1 \geq 1$ for the instance in Example 3.2.1. Note that the face defined by $x_3^1 + y_{4,9}^1 \geq 1$ is strictly contained in the face defined by $x_5^1 + y_{6,9}^1$, since $x_3^1 + y_{4,9}^1 = \sum_{t=4}^{5}(x_{t-1}^1 + y_t^1 - x_t^1) + x_5^1 + y_{6,9}^1 \geq x_5^1 + y_{6,9}^1$ for all feasible solutions.

Generalizing the above argument leads to the following class of valid inequalities for $\mathcal{X}$:

$$x_{e_k^i}^i + y_{e_k^i+1,s_k^i}^i \geq 1, \tag{3.10}$$

where e_k^i denotes the first period in which the kth unit of item i can be produced, i.e., $e_k^i = \min\{t : U_t^i = k\}$. If $k < e_k^i$, then (3.10) yields a stronger cut than $x_k^i + y_{k+1,s_k^i}^i \geq 1$.

In Example 3.2.1 we observed that a production batch for item 1 starting in period 4 cannot consist of more than three periods. Thus, in multi-item problems the demand for other items imposes a restriction on the length of a production batch for item i starting in a certain period t. Suppose that there exists a feasible solution for which a production batch for item i starts in period t. If this batch has length l, then the total production for item i up to period $t+l-1$ is at least $d^i_{1,t-1}+l$. By definition of U^i_t, we must have $d^i_{1,t-1}+l \leq U^i_{t+l-1}$. On the other hand, if $d^i_{1,t-1}+l = U^i_{t+l-1}$ for some i, t, and l, then there exists a feasible solution to DLSP in which all periods in $[t, t+l-1]$ are used for the production of item i. Thus, if we define

$$l^i_t = \max\{l : t+l-1 \leq T \text{ and } d^i_{1,t-1}+l \leq U^i_{t+l-1}\},$$

then l^i_t denotes the maximum length of a production batch for item i starting in period t in any feasible solution. Observe that $l^i_t = l^i_{t+1}+1$ if $d^i_t = 1$ and $t < T$.

Now consider an item i and a period t and suppose that $t + l^i_t \leq T$. By definition of l^i_t, the interval $[t, t+l^i_t]$ cannot be completely used for the production of item i. Thus, if item i is produced in period $t+l^i_t$, then the machine has to be set up for item i in one of the periods in $[t+1, t+l^i_t]$. This is implied by the following inequality:

$$x^i_{t+l^i_t} \leq y^i_{t+1,t+l^i_t}. \tag{3.11}$$

Since $t+l^i_t = t+1+l^i_{t+1}$ when $d^i_t = 1$, the strongest cuts of the above form are obtained for $d^i_t = 0$.

3.2.2 Multi-item inequalities

The class of multi-item inequalities discussed first is adapted from the class of *uncapacitated multi-item inequalities* introduced by Constantino ([5], Section 4.3). The basic idea behind these inequalities is the same as for the LSM inequalities: if there is no production for item i in the interval $[t, t']$, then the inventory for item i at the end of period $t-1$ must be at least $d^i_{t,t'}$.

Let $i \in \{1, \ldots, M\}$, $J \subseteq \{1, \ldots, M\} \setminus \{i\}$, and let t_1, t_2, and t_3 be three periods satisfying $1 \leq t_1 \leq t_2 \leq t_3 \leq T$ and $d^i_{t_1,t_3} \geq 1$. Suppose that all periods in $[t_1, t_2]$ are used for production of item $j \in J$ and that there is no startup for item i in the interval $[t_2+1, t']$, where $t' \in [t_2+1, t_3]$. Then, clearly, the demand for item i in the interval $[t_1, t']$ must be produced before period t_1. This is forced by the following inequality:

$$x^i_{1,t_1-1} \geq d^i_{1,t_1-1} + d^i_{t_1,t_3} \sum_{j \in J} (x^j_{t_2} - y^j_{t_1+1,t_2}) - \sum_{t=t_2+1}^{t_3} d^i_{t,t_3} y^i_t. \tag{3.12}$$

Its validity for $\mathcal{X}$ is shown as follows. Since we can produce at most one item per period, $\sum_{j \in J}(x^j_{t_2} - y^j_{t_1+1,t_2})$ either equals one or is nonpositive. In the latter case, the inequality is clearly valid. Hence, suppose that the expression equals one. Then there exists an item $j \in J$ for which production occurs in every period $t \in [t_1, t_2]$. If there is no startup in the interval $[t_2+1, t']$ for some $t' \in [t_2+1, t_3]$, then there is no production for item i in $[t_1, t']$. In this case, all demand for item i in $[t_1, t']$ has to be satisfied from stock. In other words, the total production for item i up to period t_1-1 must be at least $d^i_{1,t'}$. Note that if $y^i_t=0$ for every $t\in[t_2+1,t_3]$, then the right-hand side of (3.12) equals d^i_{1,t_3}. Otherwise, the right-hand side is at most $d^i_{1,t'}$, where $t'=\min\{t\in[t_2,t_3-1]: y^i_{t+1}=1\}$. This establishes the validity of (3.12). Note that the inequality remains valid when the term $\sum_{t=t_2+1}^{t_3} d^i_{t,t_3} y^i_t$ is replaced by $x^i_{t_2+1,t_3}$.

The separation problem for inequalities (3.12) can be solved in $O((\sum_i \log d^i_{1,T})T^2)$ time. In order to show this, rewrite the right-hand side of (3.12) as

$$d^i_{1,t_1-1}+d^i_{t_1,t_2}\sum_{j\in J}(x^j_{t_2}-y^j_{t_1+1,t_2})+\sum_{t=t_2+1}^{t_3} d^i_t\big(\sum_{j\in J}(x^j_{t_2}-y^j_{t_1+1,t_2})-y^i_{t_2+1,t}\big). \quad (3.13)$$

Let $(\hat{x},\hat{y})$ denote the current LP-solution and assume that i, t_1, and t_2 are fixed. Then, as $\hat{y}^i_{t_2+1,t}$ is nonnegative and nondecreasing in t, (3.13) is maximal for $J^*=\{j\neq i: \hat{x}^j_{t_2}>\hat{y}^j_{t_1+1,t_2}\}$ and

$$t^*_3=\min(t_2,\max\{t\in[t_2+1,T]: d^i_t=1 \text{ and } \sum_{j\in J^*}(\hat{x}^j_{t_2}-\hat{y}^j_{t_1+1,t_2})>\hat{y}^i_{t_2+1,t}\}).$$

Hence, in order to find out whether there exists an inequality of the form (3.12) that is violated by $(\hat{x},\hat{y})$ for i, t_1, and t_2, it suffices to check violation for J^* and t^*_3. From the trivial observation that $\hat{y}^i_{t_2+1,t}=\hat{y}^i_{1,t}-\hat{y}^i_{1,t_2}$ it follows that t^*_3 can be determined in $O(\log d^i_{1,T})$ time by applying binary search on the demand periods s^i_k, provided that the values $\hat{y}^i_{1,t}$ are determined beforehand. Moreover, if for all t the value $\hat{Y}^i_t:=\sum_{\tau=1}^{t} d^i_\tau \hat{y}^i_{1,\tau}$ is known as well, then $\sum_{t=t_2+1}^{t^*_3} d^i_t \hat{y}^i_{t_2+1,t}$ can be efficiently computed as $\hat{Y}^i_{t^*_3}-\hat{Y}^i_{t_2}-d^i_{t_2+1,t^*_3}\hat{y}^i_{1,t_2}$. Both $\hat{y}^i_t$ and $\hat{Y}^i_t$ can be calculated in $O(MT)$ time. Now for $0\le t_1\le t_2\le T$ we proceed as follows:

begin SEPARATION(t_1,t_2)
  $Z:=\sum_i \max(0,(\hat{x}^i_{t_2}-\hat{y}^i_{t_1+1,t_2}))$;
  for $i=1$ **to** M **do begin**
    $Z^i:=Z-\max(0,\hat{x}^i_{t_2}-\hat{y}^i_{t_1+1,t_2})$;
    determine $t_3=\max(t_2,\max\{t\in[t_2+1,T]: d^i_t=1 \text{ and } Z^i>\hat{y}^i_{t_2+1,t}\})$;
    check whether $\hat{x}^i_{1,t_1-1}<d^i_{1,t_1-1}+d^i_{t_1,t_3}Z^i-\sum_{t=t_2+1}^{t_3} d^i_t\hat{y}^i_{t_2+1,t}$
  end
end.

Obviously, $\hat{x}^i_{1,t_1-1}$ and $\hat{x}^i_{t_2}-\hat{y}^i_{t_1+1,t_2}$ can be updated in constant time in the iteration corresponding to (t_1,t_2). Hence, the separation procedure runs in $O((\sum_i \log d^i_{1,T})T^2)$ time.

We conclude this section by showing two examples of multi-item inequalities that are strongly related to inequalities (3.11). Consider again Example 3.2.1. If item 1 is produced in periods 3, 4, and 5, then one of the periods 6 and 7 must be used for the production of item 2 and the other for the production of item 3. This observation yields the following valid multi-item inequalities for the problem at hand:

$$x^1_5\le y^1_{4,5}+y^2_t+y^3_t,\ t=6,7, \text{ and } x^1_5\le y^1_{4,5}+x^j_6+y^j_7,\ j=2,3.$$

It is not hard to deduce valid multi-item inequalities for the general problem from this example. However, computational tests showed that these inequalities rarely improve the lower bound. Therefore, we do not discuss them in more detail.

3.3 Branch-and-cut

In order to solve multi-item problems to optimality we have developed a branch-and-cut algorithm for DLSP based on the integer programming formulation presented in the beginning of this chapter. The main steps of such an algorithm have already been explained in Chapter 1. In Subsection 3.3.1 we discuss some specific features of our implementation, in particular, the ones concerning the cutting plane procedure. Subsection 3.3.2 reports on the computational performance of the cutting plane procedure and different variants of the branch-and-cut algorithm. Throughout, $(\hat{x}, \hat{y})$ denotes the current LP-solution.

3.3.1 Implementation issues

This subsection gives a detailed description of the cutting plane procedure. Furthermore, we discuss the computation of upper bounds and branching strategies.

The cutting plane procedure

The current procedure incorporates separation routines for the RSM and LSM inequalities that are facet-defining for DLSP$_i$, $1 \leq i \leq M$. These routines, that will be discussed in some more detail hereafter, also identify violated inequalities of the form (3.10) and (3.11). Furthermore, cuts are generated from the class of multi-item inequalities (3.12).

In Subsection 2.2.4 we presented an $O((d^i_{1,T}T)^2)$ separation algorithm for the HLSM inequalities that are facet-defining for DLSP$_i$. Preliminary experiments showed that for the instances in our test set all cuts generated by this algorithm belonged to the subclass of LSM inequalities, i.e., the HLSM inequalities with $H = \emptyset$. These inequalities have the following form:

$$x^i_{1,t} + \sum_{j \in J}(x^i_{t+j} + y^i_{t+j+1,s^i_{d(t,j)}}) \geq d^i_{1,t} + |J|, \tag{3.14}$$

where $t \in [0, T-1]$, $d^i_{t+1} = 0$, $J \subseteq \{1, \ldots, d^i_{t+1,T}\}$, and where $s^i_{d(t,j)}$ denotes the jth demand period for item i after t. Recall from Subsection 2.2.4 that the separation problem for inequalities (3.14) can be solved in $O(d^i_{1,T}T)$ time. Because of the aforementioned observation the cutting plane procedure only incorporates the less time-consuming separation routine for inequalities (3.14). This routine also solves the separation problem for inequalities (3.10). In the cutting plane algorithm the separation routine for inequalities (3.10) and (3.14) is successively called for $i = 1, \ldots, M$. In the sequel we refer to this procedure as SEP1.

For the HRSM inequalities we implemented the version of the separation algorithm discussed in Subsection 2.2.2 that only identifies violated RSM inequalities, i.e., HRSM inequalities with $V_0 = \emptyset$. For item i these inequalities can be written in the following form:

$$x^i_{1,t} + \sum_{u \in U} (y^i_{u+1,u+d^i_{u,t}} - x^i_{u+d^i_{u,t}}) \geq d^i_{1,t}, \tag{3.15}$$

where t satisfies $d^i_t = 1$ and $U \subseteq \{\tau : \tau < t \text{ and } d^i_\tau = 0\}$. The separation for inequalities (3.11) is easily incorporated in the $O(d^i_{1,T}T)$ separation procedure for inequalities (3.15). The successive calls of this routine for all items are denoted by SEP2.

Violated inequalities of the form (3.12) are identified by the separation algorithm outlined in the previous section.

Rather than running all available separation routines in each iteration of the cutting plane procedure, it is often preferable to call certain separation routines only when other algorithms have failed to identify violated inequalities. Here both the effectiveness of the cuts and the computational effort needed to generate them are taken into account. Moreover, one should decide on the maximum number of inequalities added in one iteration, as the size of the formulation may considerably influence the time needed to solve the linear programs.

After some preliminary experiments we decided to use the following separation strategy in our cutting plane algorithm. First, SEP1 is called, i.e., for each item i it is checked whether there are violated inequalities of the form (3.10) or (3.14). For each item all violated inequalities of the first type and only the most violated inequality of the second type, if any, are added to the formulation. If at least $M/2$ cuts are generated by SEP1, then the separation phase is left and the new linear program is solved. Otherwise, SEP2 is called. Similar as for SEP1, this separation procedure generates for each item all violated inequalities of the form (3.11) and only the most violated inequality of the form (3.15). The multi-item inequalities (3.12) are only checked on violation if no cuts have been generated by the other two separation procedures. Also in this case we only add the most violated inequality per item. This means that for each item i, there is at most one combination of t_1, t_2, t_3, and $J \subseteq \{1, \ldots, M\} \backslash \{i\}$ for which (3.12) is added to the formulation.

The size of the formulation can be reduced by eliminating previously added inequalities that do not seem to play a role anymore. In our algorithm every ten iterations all inequalities with slack larger than 0.1 are deleted.

Computation of upper bounds

In the branch-and-bound procedure a subproblem can be discarded from further evaluation if its lower bound is greater than or equal to the value of the best known feasible solution. Thus, the quality of the available upper bounds may have a considerable influence on the size of the search tree. Good feasible solutions can often be constructed from the LP-solutions occurring during the branch-and-cut algorithm. We have implemented two LP-based heuristics that are called each time a linear program has been solved.

Both algorithms work as follows. First, a set of pairs (i_k, l_k), where $i_k \in \{1, \ldots, M\}$ and $l_k \in \mathbb{Z}^+$ such that $\sum_{k:i_k=i} l_k = d^i_{1,T}$ for all i, is determined. The two heuristics differ in the way these pairs are constructed from the LP-solution. This will be discussed in more detail later. By associating start times t_k to the pairs (i_k, l_k), we obtain production batches $B_k = [t_k, t_k + l_k - 1]$, where B_k is a production batch for item i_k. The periods t_k are determined such that $t_k + l_k \leq t_{k+1}$ and the production schedule consisting of the batches B_k has minimum cost. Since all instances in our test set have Wagner-Whitin costs and constant startup costs per item, the production schedule is optimal with respect to the given restrictions if t_k is chosen as late as possible. Let $\bar{s}_k$ be the demand period for which the demand is produced in the first period of B_k. Here it is assumed that the demand for the jth demand period of item i is produced in the jth production period for this item. Let K be the number of pairs (i_k, l_k). Then $t_K = \bar{s}_K$ and $t_k = \min(\bar{s}_k, t_{k+1} - l_k)$ for $k < K$. Obviously, the schedule is feasible if and only if $t_1 \geq 1$.

If this construction provides us with a feasible solution, then we apply an improvement heuristic that boils down to joining two batches of the same item that are not far apart in the current solution. Loosely speaking, for each pair of indices k_1 and k_2 satisfying $k_1 < k_2$, $i_{k_1} = i_{k_2}$, and $i_j \neq i_{k_1}$ for $k_1 < j < k_2$, it is checked whether the current solution is improved by moving B_{k_2} forwards and appending it to B_{k_1}, or by moving B_{k_1} backwards and appending it to B_{k_2}. Note that such a move involves a recalculation of the optimal periods t_k, which can be performed in time linear in the number of batches. For instances with constant inventory and production costs per item, recalculating the costs can also be done in linear time. If a better solution is found in this way, then we take the one that gives the largest improvement and repeat the above procedure.

The heuristics differ in the way the initial sequence is constructed from the LP-solution. The first heuristic determines for every i and j, $1 \leq j \leq d^i_{1,T}$, the first period t for which the total production for item i up to this period exceeds j. This period is denoted by t^i_j. Let (i_h, t_h), $1 \leq h \leq H := \sum_{i=1}^{M} d^i_{1,T}$, denote the sequence of pairs (i, t^i_j) sorted in order of nondecreasing t^i_j. Set $h_1 = 1$ and determine K and indices h_k, $1 < k \leq K$, such that h_k is the first index h after h_{k-1} for which i_h differs from $i_{h_{k-1}}$, and $i_{h_K} = i_h$ for $h_K \leq h \leq H$. Then the pairs $(i_k, l_k) = (i_{h_k}, h_{k+1} - h_k)$, $1 \leq k \leq K$ and $h_{K+1} = H + 1$, form the initial set.

In the second heuristic we first determine for each item i the set $\{t : \hat{x}^i_t > 0 \text{ and } (t = 1 \text{ or } \hat{x}^i_{t-1} = 0)\}$. Let j_i denote the cardinality of this set and denote its elements by $t^i_1, \ldots, t^i_{j_i}$, where $t^i_j < t^i_{j+1}$ for all j. Furthermore, set $t^i_{j_i+1} = T + 1$. For notational convenience, let $\hat{X}^i_j$ denote the total production for item i up to period $t^i_{j+1} - 1$ in the current LP-solution. Determine l^i_j successively for $j = 1, \ldots, j_i$, where $l^i_j = \lceil \hat{X}^i_j - \sum_{j'<j} l^i_{j'} \rceil$ if $\hat{X}^i_j - \lfloor \hat{X}^i_j \rfloor$ exceeds a given threshold value (0.7 in our implementation), and $l^i_j = \lfloor \hat{X}^i_j - \sum_{j'<j} l^i_{j'} \rfloor$ otherwise. The initial set (i_k, l_k), $1 \leq k \leq K := \sum_i j_i$, is now formed by the pairs (i, l^i_j) sorted such that the corresponding periods t^i_j are nondecreasing.

Branching strategies

If the current subproblem cannot be discarded from further evaluation after the termination of the cutting plane procedure, then new subproblems will be created according to a prespecified branching strategy. The simplest strategy in the presence of binary variables is to create two subproblems by fixing one fractional variable to zero and one, respectively.

Two well-known branching strategies for general 0−1 problems are the one in which a fractional variable closest to $\frac{1}{2}$ is chosen as branching variable and the one in which the fractional variable closest to 1 is selected. We considered both strategies in our computational experiments, with the restriction that only fractional x-variables are selected.

Moreover, we tested the performance of branching strategies that construct a schedule by fixing the periods one by one either in increasing or in decreasing order. For both orders two variants were implemented. In the first variant we simply branch on the fractional x-variable with smallest respectively largest index t. We do not report results for this variant, since it was outperformed by the second variant, in which the branching variable is selected from the set of fractional x-variables of which the value is in a certain interval $[\frac{1}{2} - \mu_1, \frac{1}{2} + \mu_2]$. The idea behind this is to avoid branching on a variable that is close to zero or one in the current LP-solution $(\hat{x}, \hat{y})$. We take $\mu_1 = \frac{1}{2} - \frac{1}{2}\max\{\hat{x}_t^i : \hat{x}_t^i \leq \frac{1}{2}\}$ and $\mu_2 = \frac{1}{2}\min\{\hat{x}_t^i : \hat{x}_t^i \geq \frac{1}{2}\}$. Thus, results are reported for the strategies in which from all fractional x-variables with value in $[\frac{1}{2} - \mu_1, \frac{1}{2} + \mu_2]$ the one with smallest respectively largest index t is selected for branching.

Apart from defining a branching strategy, one also has to specify in which order the subproblems are to be evaluated. We examine two well-known search strategies: *best-bound* and *depth-first*. In the first strategy the subproblem with the smallest lower bound is selected, whereas depth-first search selects the subproblem that is created last. Since the subproblem with the smallest lower bound must be evaluated anyway in order to prove optimality, one may expect that the size of the search tree is smaller in a best-bound search. However, in a best-bound search two subproblems that are subsequently evaluated can be far apart in the search tree, whereas in a depth-first search they often differ only in the fixing of one variable. As a consequence, the computational effort per node is usually higher in a best-bound search.

It may happen that, despite the addition of violated inequalities, the LP-value does not increase significantly during a number of consecutive iterations. This effect is called *tailing off*. In order to prevent this, a branching step is forced at any node but the root node when the LP-value has not been improved by more than 0.1% during the last three iterations. At the root node cutting planes are added as long as violated inequalities are identified.

3.3.2 Computational results

The implementation of the branch-and-cut algorithm is based on MINTO ([29], version 2.0), which is a software system for solving mixed integer linear programs by means of a linear programming based branch-and-bound algorithm. Within the general framework the user can embed problem specific functions such as separation routines, primal heuristics, and branching strategies. MINTO also provides the optional use of preprocessing techniques, construction of feasible solutions, and generation of generic inequalities such as knapsack cover inequalities. However, in our experiments none of these system functions were used. All computational results were obtained on a SUN Sparcstation 5 using CPLEX 2.1 as LP-solver.

Instances

In order to test our branch-and-cut algorithm a set of 486 instances was generated. These instances have either 60 or 100 periods and either 2, 4, or 6 items. Moreover, we consider variations in the total demand $D = \rho \times T$, where the capacity utilization ρ equals respectively 0.65, 0.80, and 0.90. This yields 18 different (T, M, ρ) combinations. As far as the inequalities $x^i_{1,t} \geq d^i_{1,t}$ in the IP-formulation are concerned, it clearly suffices to include only those for i and t satisfying $d^i_t = 1$. Hence, the size of the initial LP-formulation is larger for larger values of each of the three parameters T, M, and ρ. This also holds for the size of the classes of inequalities from which cuts are generated.

The instances are further characterized by two parameters δ and f, that both relate to the cost function. Let us mention first that the production cost p^i_t equals zero for all i and t. Moreover, both the inventory costs and the startup costs are constant per item, i.e., $h^i_t = h^i_1$ and $f^i_t = f^i_1$ for all i and t. The parameter $\delta \in \{0, 1, 2\}$ indicates whether or not h^i_1 and f^i_1 have the same value for different items i. If $\delta \in \{0, 1\}$, then $h^i_1 = 2$ for all items i; otherwise, h^i_1 is randomly generated from the interval $[1, 3]$. Furthermore, f^i_1 equals f for all i if $\delta = 0$, but is randomly generated from the interval $[\frac{1}{2}f, \frac{3}{2}f]$ for $\delta \in \{1, 2\}$. We consider instances with f equal to 10, 20, and 40, respectively. Three instances were generated for each (T, M, ρ, δ, f) combination. This resulted in a test set of $18 \times 9 \times 3 = 486$ instances.

Given T, M, and ρ, the demand periods are generated as follows. For each item i we generate $d^i_{1,T}$ demand periods, where $d^i_{1,T}$ is randomly taken from the interval $[\lfloor\frac{1}{2}D/M\rfloor, \lfloor\frac{3}{2}D/M\rfloor]$ such that $\sum_i d^i_{1,T} = D$. The demand periods are randomly generated in such a way that the feasibility condition $\sum_i d^i_{1,t} \leq t$ is satisfied for all t. Moreover, all instances satisfy $d^i_1 = 0$ for all items i and $d^i_T = 1$ for at least i.

Recall that an instance is said to have Wagner-Whitin costs if $h^i_t + p^i_t \geq p^i_{t+1}$ for all items i and periods t. This obviously holds for all instances in our test set.

Lower bounds

We first investigate the effectiveness of the cutting plane procedure in improving the bounds obtained from the LP-relaxation. Recall that the cutting plane procedure incorporates two separation routines for single-item inequalities and one for multi-item inequalities. We will also consider the lower bounds obtained when only single-item inequalities are added to the formulation.

The quality of a lower bound z is often expressed in terms of the integrality gap $g = 100\% \times (z^* - z)/z^*$, where z^* denotes the value of the optimal integral solution. Table 3.1(a) reports integrality gaps with respect to the following lower bounds:

- z_0 : the optimal value of the LP-relaxation of DLSP;
- z_1 : the value of the LP-solution after the addition of single-item inequalities only;
- z_2 : the value of the LP-solution after the addition of both single-item and multi-item inequalities.

Column $\bar{g}_j$, $0 \le j \le 2$, shows the average value of the integrality gap with respect to z_j over the 27 instances within one (T, M, ρ) combination. Note that these entries represent percentages. The number between brackets denotes the number of problems (out of 27) for which an integral solution was obtained. Table 3.1(b) provides information on the average number of linear programs solved ($\bar{l}_j$), the average number of cuts added ($\bar{c}_j$), and the average computation time ($\bar{t}_j$, in seconds) needed to obtain the lower bound z_j, $j = 1, 2$.

		$T = 60$			$T = 100$		
M	ρ	$\bar{g}_0$	$\bar{g}_1$	$\bar{g}_2$	$\bar{g}_0$	$\bar{g}_1$	$\bar{g}_2$
	0.65	53.1 (0)	0.16 (16)	0.00 (25)	52.0 (0)	0.22 (14)	0.04 (20)
2	0.80	51.6 (0)	0.46 (18)	0.17 (21)	51.8 (0)	0.68 (5)	0.22 (13)
	0.90	46.7 (0)	0.87 (14)	0.43 (18)	44.9 (0)	0.86 (5)	0.34 (7)
	0.65	54.4 (0)	0.28 (12)	0.11 (20)	54.9 (0)	0.14 (11)	0.03 (21)
4	0.80	52.5 (0)	0.37 (4)	0.16 (12)	52.6 (0)	0.35 (4)	0.10 (13)
	0.90	49.9 (0)	0.55 (7)	0.24 (15)	48.1 (0)	0.65 (1)	0.38 (4)
	0.65	54.1 (0)	0.04 (15)	0.00 (25)	55.0 (0)	0.14 (8)	0.05 (16)
6	0.80	53.7 (0)	0.37 (8)	0.10 (12)	54.1 (0)	0.35 (5)	0.16 (12)
	0.90	52.1 (0)	0.43 (4)	0.21 (15)	49.9 (0)	0.59 (0)	0.23 (4)

Table 3.1(a): Quality of the lower bounds at the root node

		$T = 60$						$T = 100$					
M	ρ	$\bar{l}_1$	$\bar{c}_1$	$\bar{t}_1$	$\bar{l}_2$	$\bar{c}_2$	$\bar{t}_2$	$\bar{l}_1$	$\bar{c}_1$	$\bar{t}_1$	$\bar{l}_2$	$\bar{c}_2$	$\bar{t}_2$
	0.65	25	79	5	26	81	5	48	129	15	51	135	16
2	0.80	30	101	6	34	109	7	56	176	22	67	197	26
	0.90	31	117	7	41	133	9	57	205	26	79	246	36
	0.65	22	114	7	24	119	7	42	199	25	45	206	26
4	0.80	27	143	12	33	158	12	48	244	38	55	265	42
	0.90	28	161	15	38	188	17	53	293	51	66	326	59
	0.65	19	129	9	20	132	9	36	233	27	38	238	27
6	0.80	24	166	14	28	178	14	45	309	53	53	336	57
	0.90	26	191	18	33	214	21	49	357	73	67	415	87

Table 3.1(b): Average number of LPs, average number of cuts, and average computation time at the root node

First observe that the LP-relaxation is rather weak in general; the average gap is about 50%. However, the addition of single-item inequalities already yields lower bounds that, on average, differ less than 1% from the optimal value. Furthermore, about two-third of the remaining gap is closed when also multi-item inequalities are added. The cutting plane procedure yields an integral solution for 273 problems, which amounts to 56% of the total test set.

Table 3.1(a) suggests that the problems with highest capacity utilization ρ are the most difficult. In general, the integrality gaps $\bar{g}_1$ and $\bar{g}_2$ increase and the number of problems solved to optimality decreases for increasing values of ρ. From the presented results we cannot draw solid conclusions about the effect of the number of periods T or the number of items M on the quality of the lower bounds. However, the number of problems for which an integral solution is found is significantly higher for $T = 60$ than for $T = 100$.

Table 3.1(b) shows that computation times increase for increasing values of T, M, and ρ. This is not surprising, considering the larger size of the initial formulation and the larger number of generated cuts. Recall from our discussion of the cutting plane procedure that the more items, the more cuts can be added in one iteration. This may explain why the number of linear programs that are solved decreases when M increases (and T and ρ are kept constant). However, this decrease in the number of linear programs is outweighed by the increase in the computational effort spent on one linear program.

In order to obtain more insight into the effect of increasing either T, M, or ρ on the quality of the lower bounds, we performed some additional tests. Starting from the combination $(T, M, \rho) = (100, 4, 0.80)$, we varied, one at a time, the values of T, M, and ρ. Moreover, we report results for different values of T while M and ρ are kept to 6 and 0.80, respectively. For each new combination (T, M, ρ) 27 instances were generated in the same way as described previously. Tables 3.2(a)–(d) provide information on the integrality gap after the addition of both single-item and multi-item inequalities ($\bar{g}$), and the computation time ($\bar{t}$, in seconds). Again, the entries are averages over the 27 instances within one (T, M, ρ) combination.

T	60	100	150	200
$\bar{g}$	0.16	0.10	0.14	0.24
$\bar{t}$	12	42	132	322

Table 3.2(a): Average gaps and computation times for different values of T $(M = 4, \rho = 0.80)$

T	60	100	150	200
$\bar{g}$	0.10	0.16	0.20	0.21
$\bar{t}$	17	59	159	370

Table 3.2(b): Average gaps and computation times for different values of T $(M = 6, \rho = 0.80)$

M	2	4	6	8	10
$\bar{g}$	0.22	0.10	0.16	0.11	0.16
$\bar{t}$	26	42	57	60	67

Table 3.2(c): Average gaps and computation times for different values of M $(T = 100, \rho = 0.80)$

ρ	0.50	0.65	0.80	0.90	0.95	0.99
$\bar{g}$	0.00	0.03	0.10	0.38	0.68	0.59
$\bar{t}$	13	26	42	57	88	100

Table 3.2(d): Average gaps and computation times for different values of ρ $(T = 100, M = 4)$

One may conclude from Tables 3.2(a) and (b) that the lower bounds become weaker when the number of periods becomes larger. A more solid conclusion can be reached about the computational effort required for the cutting plane procedure: computation times grow rapidly for increasing T. The number of items does not seem to have much influence on the size of the gap. Finally, Table 3.2(d) confirms the previously observed relation between the value of ρ and the quality of the lower bounds. For larger values of ρ we expect that a fair number of variables can be fixed beforehand. This may explain the decrease in $\bar{g}$ when ρ increases from 0.95% to 0.99%.

Recall that, besides T, M, and ρ, an instance is characterized by two other parameters that refer to the cost function. The results did not suggest any relation between the values of these parameters and the size of the integrality gap.

Upper bounds

Table 3.1(a) reports that 67% of the 60-period problems (163 out of 243) and 45% of the 100-period problems (110 out of 243) were solved to optimality without branching. The use of the primal heuristics increased these percentages to 76% and 57%, which corresponds to 184 and 139 problems, respectively. Note that in the presence of integral cost coefficients the optimality of a feasible solution is established as soon as its value and the value of the current LP-solution differ less than the greatest common divisor of the cost coefficients. This feature is incorporated in MINTO.

For the instances that were not solved to optimality at the root node, the gap between the value of the best available solution at termination of the cutting plane procedure and the value of the optimal solution was on average 0.65%.

Branching

For 163 out of 486 problems of our original test set branching was required in order to prove optimality. Here we report on the performance of different variants of the branch-and-cut algorithm with respect to these problems. Results are presented for the following branching strategies, which were discussed in more detail in the previous subsection:

- FRAC: selects the fractional x-variable closest to $\frac{1}{2}$;
- MAX: selects the fractional x-variable closest to 1;
- FIRST: selects the fractional x-variable with smallest index t and value in $[\frac{1}{2} - \mu_1, \frac{1}{2} + \mu_2]$, where $\mu_1 = \frac{1}{2} - \frac{1}{2}\max\{\hat{x}_t^i : \hat{x}_t^i \leq \frac{1}{2}\}$ and $\mu_2 = \frac{1}{2}\min\{\hat{x}_t^i : \hat{x}_t^i \geq \frac{1}{2}\}$;
- LAST: selects the fractional x-variable with largest index t and value in $[\frac{1}{2} - \mu_1, \frac{1}{2} + \mu_2]$, where μ_1 and μ_2 are as defined for FIRST.

Each branching strategy was tested in combination with both best-bound search and depth-first search. For each variant of the branch-and-cut algorithm a limit of two hours was imposed on the computation time spent on one instance. Three instances could not be solved within this limit for at least one of the eight variants.

Table 3.3 gives a first impression of the performance of the different branch-and-cut procedures. The entries are averages over the 160 instances that were solved to optimality by all variants within the imposed time limit of two hours. Besides the average number of nodes in the branch-and-bound tree ($\bar{n}$) and the average computation time ($\bar{t}$, in seconds), we report the average ratio between the time needed to solve the problem to optimality and the time spent at the root node ($\bar{r}_t$).

	HALF			MAX			FIRST			LAST		
	$\bar{n}$	$\bar{t}$	$\bar{r}_t$	$\bar{n}$	$\bar{t}$	$\bar{r}_t$	$\bar{n}$	$\bar{t}$	$\bar{r}_t$	$\bar{n}$	$\bar{t}$	$\bar{r}_t$
best-bound	21	76	1.55	43	105	1.97	38	114	2.04	31	90	1.78
depth-first	19	63	1.39	39	81	1.63	37	98	1.83	32	81	1.64

Table 3.3: Average performance of the different branching and search strategies for 160 instances

For both search strategies HALF yields the best results on average. For all branching strategies the problems are solved faster on average by depth-first search than by best-bound search. In the previous subsection we remarked that in a depth-first strategy usually more nodes have to be evaluated than when best-bound is applied. Apparently, this does not hold in our case. Because of the above results, depth-first is the default search strategy in the sequel.

We observed that for the majority of the instances the size of the branch-and-bound tree was fairly small for all branching strategies. Therefore, the differences in the average results reported in Table 3.3 are mainly due to a relatively small set of problems. In the remainder we investigate the performance of the different branch-and-cut procedures separately for so-called easy and hard instances. An instance is said to be easy if the average computation time required by the four branch-and-cut algorithms is at most twice the time spent at the root node. This definition labels 142 out of 163 instances (87%) as easy.

We first provide some statistics related to the easy problems. Table 3.4 presents average results for each (T, ρ) combination. Column #inst reports the number of instances over which the average is taken. Furthermore, $\bar{g}_{\text{root}}$ gives the average gap at the root node and $\bar{n}$, $\bar{t}$, and $\bar{r}_t$ have the same meaning as in Table 3.3.

It appears that strategy HALF also yields the best results when only the easy instances are taken into account. Except for the problems with $(T, \rho) = (100, 0.80)$ HALF performs at least as good as any of the other strategies. However, there is not much difference between the four variants. Note that $\bar{r}_t$ almost always increases when ρ increases. This is not surprising, since the average gap at the root node is larger for larger values of ρ, hence, we may expect that more nodes have to be evaluated in order to prove optimality. For $\rho = 0.65$ about 10% of the total computation time is taken up by the branching phase, whereas for $\rho = 0.90$ this is more than 20% on average.

				HALF			MAX			FIRST			LAST		
T	ρ	#inst	$\bar{g}_{\text{root}}$	$\bar{n}$	$\bar{t}$	$\bar{r}_t$	$\bar{n}$	$\bar{t}$	$\bar{r}_t$	$\bar{n}$	$\bar{t}$	$\bar{r}_t$	$\bar{n}$	$\bar{t}$	$\bar{r}_t$
60	0.65	7	0.40	5	9	1.09	5	9	1.09	7	10	1.13	11	11	1.21
	0.80	21	0.44	6	16	1.10	8	17	1.13	8	17	1.15	8	17	1.14
	0.90	25	0.57	9	24	1.22	15	26	1.31	10	25	1.26	11	25	1.28
100	0.65	16	0.20	6	29	1.09	8	29	1.11	6	29	1.09	8	29	1.09
	0.80	31	0.30	16	54	1.24	17	53	1.22	15	55	1.25	11	50	1.15
	0.90	42	0.40	12	68	1.21	25	77	1.36	14	71	1.27	19	74	1.33

Table 3.4: Average performance of the different branching strategies for the easy instances

The last table presents results for the 21 hard instances. For each instance we give the number of nodes (n) and the computation time (t) required for each of the four different branching strategies. An instance is identified as I.T.M.D.k, where D denotes the total demand, i.e., $D = \rho \times T$, and $k \in \{1, \ldots, 27\}$. Not surprisingly, the majority of these instances has capacity utilization $\rho = 0.90$. Column g_{root} shows the integrality gap at the root node. The time spent before entering the branching phase is reported in column t_{root}. The last row presents average results over the 19 instances for which no entry '—' appears in the corresponding row. This entry indicates that the problem could not be solved within two hours.

For the 19 instances for which average results are reported in the last row, strategy HALF yields considerably better results than the other three strategies. Note however that LAST performs as least as good as HALF for a fair number of these problems. Moreover, for the two instances that are not considered in the average results, LAST outperforms the other strategies by far.

3.4 Related research

In this section we give an outline of two other solution methods for DLSP proposed in the literature. The performance of these methods is compared to the performance of the branch-and-cut algorithm.

Fleischmann [14] proposes a branch-and-bound algorithm using Lagrangean relaxation for the determination of both the lower and upper bounds. This approach is based on the same formulation as our branch-and-cut procedure. Relaxing the coupling constraints $\sum_i x_t^i \leq 1$ for all t decomposes the problem into M single-item problems, which are solved by means of dynamic programming (DP). The Lagrangean multipliers are updated iteratively by a standard subgradient optimization technique. Feasible solutions are obtained by successively solving the subproblems for item $i = 1, \ldots, M$ by a modification of the DP-algorithm in which the periods already used for the production of item $j < i$ are skipped. The branching strategy constructs a schedule by fixing the periods one by one in decreasing order. That is, from a

			HALF		MAX		FIRST		LAST	
instance	g_{root}	t_{root}	n	t	n	t	n	t	n	t
I.60.2.54.13	4.66	27	317	119	533	170	675	295	895	294
I.60.4.54.18	1.50	22	23	41	29	36	21	37	187	153
I.60.4.54.24	1.48	32	31	89	113	137	51	125	25	74
I.60.4.54.25	0.64	33	9	46	105	100	73	129	3	40
I.60.6.48.6	1.49	24	97	116	191	156	189	212	41	78
I.60.6.54.23	1.23	29	35	96	95	130	39	113	29	85
I.100.4.80.24	1.02	84	43	198	449	659	103	418	41	211
I.100.4.90.12	0.99	94	115	369	207	439	141	439	295	835
I.100.4.90.15	0.65	103	235	803	215	494	245	912	457	1594
I.100.4.90.18	0.64	66	77	211	197	288	499	924	33	101
I.100.4.90.26	1.43	74	119	299	193	317	365	923	39	145
I.100.4.90.27	0.94	148	29	286	103	484	75	569	45	360
I.100.6.80.19	1.59	100	335	1630	665	1860	355	1591	737	3865
I.100.6.80.22	0.44	111	17	169	419	825	89	316	31	182
I.100.6.90.3	0.71	179	—	—	—	—	—	—	559	4186
I.100.6.90.10	1.16	177	665	3083	—	—	1631	6540	147	907
I.100.6.90.15	0.48	94	75	277	473	733	697	1253	103	262
I.100.6.90.16	0.53	151	21	227	103	396	389	896	21	208
I.100.6.90.19	1.07	85	33	129	353	671	227	435	1073	1764
I.100.6.90.21	0.12	127	231	463	53	211	259	439	5	137
I.100.6.90.25	0.26	139	3	156	33	259	159	1001	15	190
Average (19)	1.10	81	97	301	238	440	245	580	214	557

Table 3.5: Performance of the different branching strategies for the hard instances

subproblem in which periods $t+1$ up to T are fixed, $M+1$ new subproblems are created by either assigning production of item i, $1 \leq i \leq M$, to period t or leaving t idle.

Cattrysse et al. [7] present a solution method for DLSP based on column generation. This algorithm has been primarily developed for the problem with sequence-independent startup times. The columns in the master problem represent production schedules for the different items. The method is used as a heuristic as it only solves the LP-relaxation of the master problem. The procedure starts with a restricted set of columns and generates new production schedules by solving single-item problems by dynamic programming. These are added to the formulation as long as they price out. A dual ascent procedure followed by subgradient optimization provides approximately optimal dual variables of the LP-relaxation of the master problem. Feasible solutions are obtained from the upper bounding procedure developed by Fleischmann and from an enumeration algorithm that tries to construct a feasible schedule from the single-item production schedules generated so far.

Fleischmann tested his algorithm on data sets from the literature on a capacitated lot-sizing problem with larger time periods during which production can occur for more than one item. This problem is transformed in a discrete lot-sizing and scheduling problem by a discretization of the production periods. Unlike Fleischmann, we can solve most of the problems TV (T varies between 63 and 96, 8 items, total demand 61) without branching, but his algorithm requires much less computation time. Furthermore, neither of his problems G1 and G2 (250 periods, 3 items, $\rho \in \{0.84, 0.90\}$) was solved by our algorithm within one hour. In general, we conclude that the approach based on Lagrangean relaxation can handle much larger instances than the branch-and-cut procedure within the same amount of time.

Cattrysse et al. present computational results for a test set of 45 instances with $T = 60$, $M \in \{2, 4, 6\}$, and ρ either ≤ 0.55, between 0.55 and 0.75, or > 0.75. The cost structure is somewhat different from ours; for more details we refer to [7]. The authors compare the performance of their algorithm to Fleischmann's procedure, where the latter is applied without branching. The column generation approach yields smaller gaps and a larger number of problems that are solved to optimality, but requires a larger computational effort. We may expect that, when branching is applied, Fleischmann's algorithm needs less time than the column generation procedure in order to reach gaps of the same size.

Finally, we compare the results reported by Cattrysse et al. to the results that we obtained for our set of 60-period problems. The cutting plane procedure yields slightly better gaps when ρ is (about) 0.65. For larger values of ρ column generation performs better. This also holds with respect to the computation times, although we have no insight into the extra time that will be needed by this approach in order to solve all instances to optimality. Moreover, we cannot say anything about how the two approaches will compare for larger values of T.

4. Sequence-independent startup times

In this chapter we study the extension of DLSP in which startups take up an integral number of periods in which no production can occur. We restrict ourselves to the problem with sequence-independent startup times, i.e., the startup times only depend on the item for which the machine is set up. In Section 4.1 we present an integer programming formulation of the problem that is similar to the formulation of DLSP studied in the previous chapters. Valid inequalities for the latter formulation are therefore easily turned into valid inequalities for the problem with startup times. Section 4.2 deals with this subject in some more detail. In Section 4.3 we discuss a multicommodity flow formulation of the single-item problem. By reformulating the problem as a shortest path problem we derive a complete linear description in the new variables.

4.1 Formulation

The setting for the problem with sequence-independent startup times, denoted by DLSS, is the same as for DLSP, except that a startup for item i takes up σ_i periods, $\sigma_i \in \mathbb{Z}^+$, in which no production can occur. As before, T denotes the number of periods of the planning horizon, M the number of items, and $d_t^i \in \{0, 1\}$ the demand for item i in period t. We assume that the machine is turned off at the beginning of the planning horizon, hence, $\sigma_i + 1$ is the first period in which item i can be produced. As a consequence, it is always assumed that there is no demand for item i before period $\sigma_i + 1$.

As for DLSP, the binary variable x_t^i indicates whether production occurs for item i in period t or not. If item i is produced in period t, then this incurs a cost c_t^i. Furthermore, we introduce variables z_t^i that equal one if the interval $[t - \sigma_i, t - 1]$ is used for setting up the machine for item i and zero otherwise. Thus, $z_t^i = 1$ indicates that a production batch of item i can be started in period t. Associated with z_t^i is the startup cost g_t^i. Obviously, x_t^i and z_t^i are only defined for $t \in [\sigma_i + 1, T]$.

If startups do not affect the production capacity, then there exists a production schedule that satisfies all demand in time if and only if $\sum_i d_{1,t}^i \leq t$ for all t (cf. Section 3.1). In the presence of startup times, however, the existence of such a production schedule is not easily established. In order to prevent infeasibility of an instance, we therefore assume that the demand can always be satisfied by producing out of the regular production periods. This incurs a penalty cost c_0^i per unit of item i. The nonnegative integer variable X_0^i represents the total number of units of item i produced in this way. For convenience, this production is said to occur in period 0. In practice, X_0^i may correspond to the demand quantity of item i that cannot be satisfied, in which case $c_0^i X_0^i$ represents the lost purchases.

Now DLSS can be formulated in the following way:

$$\min \sum_{i=1}^{M}\left[c_0^i X_0^i + \sum_{t=\sigma_i+1}^{T}\left(c_t^i x_t^i + g_t^i z_t^i\right)\right]$$

$$\text{s.t.} \quad X_0^i + x^i_{\sigma_i+1,t} \geq d^i_{\sigma_i+1,t} \qquad (\sigma_i < t \leq T, 1 \leq i \leq M) \tag{4.1}$$

$$x_t^i \leq x_{t-1}^i + z_t^i \qquad (\sigma_i < t \leq T, 1 \leq i \leq M, x^i_{\sigma_i} = 0) \tag{4.2}$$

$$\sum_{i:\sigma_i<t}\left(x_t^i + \sum_{\tau=\max(\sigma_i+1,t+1)}^{\min(t+\sigma_i,T)} z_\tau^i\right) \leq 1 \qquad (1 \leq t \leq T) \tag{4.3}$$

$$X_0^i \in \mathbb{N},\ x_t^i, z_t^i \in \{0,1\} \qquad (\sigma_i < t \leq T, 1 \leq i \leq M) \tag{4.4}$$

Constraints (4.1) imply that the demand up to period t is always satisfied, either from production in the regular production periods up to t or from production in period 0. Constraints (4.2) force the machine to be set up for item i in $[t-\sigma_i, t-1]$ if this item is produced in period t but not in period $t-1$. Constraints (4.3) assure that no production occurs when the machine is being set up. Moreover, they assure that in one period at most one item is produced.

As in the case of DLSP, we can add extra constraints to the formulation in order to exclude solutions with a positive inventory at the end of the planning horizon or with $z_t^i = 1$ and $x_t^i = 0$ for some i and t. Note that in the latter case, constraints (4.3) imply that for any item j period $t+\sigma_j+1$ is the first period after t in which this item can be produced. Clearly, such a solution is never optimal in the presence of positive startup costs.

Recall that $z_t^i = 1$ indicates that the machine is being set up for item i in $[t-\sigma_i, t-1]$. Thus, $z_t^i = 1$ implies that we can start a production batch of item i in period t. Observe that the z-variables have the same meaning as the y-variables in the formulation of DLSP studied in the previous chapters. Hence, if a startup for some item i only incurs a cost but does not affect the production capacity, then this can be easily incorporated in the above formulation by taking $\sigma_i = 0$. Moreover, most of the inequalities derived for DLSP can therefore easily be turned into valid inequalities for the set of solutions to (4.1) – (4.4), as will be shown in the following section.

We conclude this section by comparing our model of DLSS to a slightly different formulation proposed by Cattrysse et al. in [7]. Instead of z_t^i they use binary variables v_t^i that indicate whether the machine is being set up for item i in period t. For ease of explanation, assume that $\sigma_i > 0$ for all items i. Thus, if a production batch for item i starts in period t, then v_τ^i must equal one for every $\tau \in [t-\sigma_i, t-1]$. This incurs a cost $\sum_{\tau=t-\sigma_i}^{t-1} \bar{g}_\tau^i$, where $\bar{g}_\tau^i$ is the cost associated with v_τ^i. The correctness of the formulation below is now readily seen.

$$\min \sum_{i=1}^{M}\left[c_0^i X_0^i + \sum_{t=\sigma_i+1}^{T} c_t^i x_t^i + \sum_{t=1}^{T} \bar{g}_t^i v_t^i\right] \quad \text{s.t. (4.1),}$$

$$x_t^i \leq x_{t-1}^i + v_{t-k}^i \qquad (1 \leq k \leq \sigma_i < t \leq T, 1 \leq i \leq M, x^i_{\sigma_i} = 0)$$

$$\sum_{i:\sigma_i<t} x_t^i + \sum_i v_t^i \leq 1 \qquad (1 \leq t \leq T)$$

$$X_0^i \in \mathbb{N},\ x_t^i \in \{0,1\} \qquad (\sigma_i < t \leq T, 1 \leq i \leq M)$$

$$v_t^i \in \{0,1\} \qquad (1 \leq t \leq T, 1 \leq i \leq M)$$

Observe that for $\sigma_i > 1$ there exist solutions (X_0, x, v) with $v_t^i = 1$ for k subsequent periods, where k is not a multiple of σ_i. Such a solution does not correspond to any feasible solution to (4.1) – (4.4). However, one easily verifies that if (X_0, x, z) satisfies (4.1) – (4.4), then (X_0, x, v) with

$$v_t^i = \sum_{\tau=\max(\sigma_i+1,t+1)}^{\min(t+\sigma_i,T)} z_\tau^i$$

for all i and t is a solution to the above formulation. In particular, if $g_t^i = \sum_{\tau=t-\sigma_i}^{t-1} \bar{g}_\tau^i$ and $\bar{g}_t^i > 0$ for all i and t, then both formulations yield the same optimal solution to DLSS. Furthermore, it is not difficult to show that if $g_t^i = \sum_{\tau=t-\sigma_i}^{t-1} \bar{g}_\tau^i$ for all i and t, then the LP-relaxation of our model yields lower bounds that are as least as good as the lower bounds obtained from the LP-relaxation of the model proposed in [7].

4.2 Valid inequalities

In this section we give some results on valid inequalities for DLSS. We mainly consider valid inequalities for the single-item problem, i.e., inequalities that are valid for all (X_0, x, z) satisfying

$$X_0 + x_{\sigma+1,t} \geq \bar{d}_{\sigma+1,t} \qquad (\sigma < t \leq \bar{T}) \qquad (4.5)$$

$$x_t \leq x_{t-1} + z_t \qquad (\sigma < t \leq \bar{T}, x_\sigma = 0) \qquad (4.6)$$

$$x_t + \sum_{\tau=t+1}^{\min(t+\sigma,\bar{T})} z_t \leq 1 \qquad (\sigma \leq t \leq \bar{T}, x_\sigma = 0) \qquad (4.7)$$

$$X_0 \in \mathbb{N},\ x_t, z_t \in \{0, 1\} \qquad (\sigma < t \leq \bar{T}) \qquad (4.8)$$

where $\bar{T}$ denotes the number of periods and $\bar{d}_t \in \{0, 1\}$ for all t ($\bar{d}_{1,\sigma} = 0$). For ease of presentation, we renumber the periods such that period 1 is the first period in which production can occur. Let T denote the last period of the new planning horizon, i.e., $T = \bar{T} - \sigma$. Moreover, define $d_t = \bar{d}_{t+\sigma}$, $1 \leq t \leq T$. Then the above formulation can be restated as

$$X_0 + x_{1,t} \geq d_{1,t} \qquad (1 \leq t \leq T) \qquad (4.9)$$

$$x_t \leq x_{t-1} + z_t \qquad (1 \leq t \leq T, x_0 = 0) \qquad (4.10)$$

$$x_t + \sum_{\tau=t+1}^{\min(t+\sigma,T)} z_\tau \leq 1 \qquad (0 \leq t \leq T, x_0 = 0) \qquad (4.11)$$

$$X_0 \in \mathbb{N},\ x_t, z_t \in \{0, 1\} \qquad (1 \leq t \leq T) \qquad (4.12)$$

Let $\mathcal{X}_\sigma$ denote the set of feasible solutions to (4.9) – (4.12). Every valid inequality for $\mathcal{X}_\sigma$ clearly yields a valid inequality for the set of feasible solutions to (4.5) – (4.8) by substituting $x_{t+\sigma}$ and $z_{t+\sigma}$ for x_t and z_t, respectively, for every $t \in [1, T]$.

Let $\sigma > 0$. Because of (4.11), we have $\mathcal{X}_\sigma \subset \mathcal{X}_{\sigma-1}$ and $\text{conv}(\mathcal{X}_\sigma) \subset \text{conv}(\mathcal{X}_{\sigma-1})$. Thus, all valid inequalities for $\mathcal{X}_{\sigma-1}$ are valid for $\mathcal{X}_\sigma$. In particular, all valid inequalities for $\mathcal{X}_0$ are valid for $\mathcal{X}_\sigma$. Obviously, the subset of solutions in $\mathcal{X}_0$ with $X_0 = 0$ is the set of solutions to the formulation of the single-item DLSP studied in Chapter 2 with y_t replaced by z_t. For the latter formulation we derived several classes of facet-defining inequalities $\alpha x + \beta z \geq \gamma$ with $\alpha_t \in \{0, 1\}$ for all t. Recall that these inequalities can be written as $I_t \geq LB(x, z)$, i.e., they yield a lower bound on the inventory I_t at the end of period t. If production may also occur in period 0, then $I_t = X_0 + x_{1,t} - d_{1,t}$. Thus, the inequalities from Chapter 2 become valid for $\mathcal{X}_0$ by adding the term X_0 to the left-hand side. It is not hard to see these inequalities are also facet-defining for conv($\mathcal{X}_0$).

For two subclasses we will give conditions for the inequalities to be facet-defining for conv($\mathcal{X}_\sigma$). However, we will not prove in detail that an inequality defines a facet since these proofs are similar to the proofs in Chapter 2. As a matter of fact, they are usually facilitated by the possibility to produce in period 0. The following lemma implies that, in order to prove that an inequality defines a facet of conv($\mathcal{X}_\sigma$), one has to construct $2T$ linearly independent directions in the subset of solutions in $\mathcal{X}_\sigma$ that satisfy the inequality at equality.

Lemma 4.2.1 Conv($\mathcal{X}_\sigma$) *has dimension* $2T + 1$ *for all* $\sigma \geq 0$. □

Since demand can always be produced in period 0, one readily constructs the $2T + 1$ unit vectors as directions in $\mathcal{X}_\sigma$. It is obvious that there always exists a feasible solution with $X_0 = 0$ for the single-item DLSS. However, if the variable X_0 is omitted from the formulation, then the dimension of the convex hull of $\mathcal{X}_\sigma$ depends on the demand function. For example, if $\sigma = 2$ and $d_2 = 1$, then all solutions in $\mathcal{X}_2$ with $X_0 = 0$ satisfy $x_1 + z_2 = 0$ and $z_3 = 0$.

The following lemma gives a necessary condition for an inequality to be facet-defining for conv($\mathcal{X}_\sigma$).

Lemma 4.2.2 *Let* $\sigma > 0$. *Let* $\alpha_0 X_0 + \alpha x + \beta z \geq \gamma$ *be a facet-defining inequality of* conv($\mathcal{X}_\sigma$) *that is not of the form* (4.11). *Then for any* $t \leq T - \sigma$ *the following holds:*

C1. *There exists a solution in* $\mathcal{X}_\sigma$ *without production in the interval* $[t, t + \sigma]$ *that satisfies the inequality at equality.*

PROOF. Define $\bar{\mathcal{X}}_\sigma = \{(X_0, x, z) \in \mathcal{X}_\sigma : \alpha_0 X_0 + \alpha x + \beta z = \gamma\}$. From the assumption that $\alpha x + \beta y \geq \gamma$ defines a facet of conv($\mathcal{X}_\sigma$) and is not of the form (4.11), it follows that for any $t \leq T - \sigma$ there must be at least one solution $(X_0, x, z) \in \bar{\mathcal{X}}_\sigma$ satisfying $x_t + z_{t+1,t+\sigma} = 0$. Since $x_t + z_{t+1,t+\sigma} = 0$ implies that no production occurs in $[t, t + \sigma]$, this proves the statement. □

We discussed before how facet-defining inequalities derived in Chapter 2 can easily be turned into valid inequalities of $\mathcal{X}_\sigma$. In the sequel, we will give conditions for the inequalities of two subclasses to be facet-defining of conv($\mathcal{X}_\sigma$). For all inequalities $X_0 + \alpha x + \beta z \geq \gamma$ in these subclasses the following holds: for any $t \in [1, T - \sigma]$ there exists a solution in $\mathcal{X}_\sigma$ satisfying $X_0 + \alpha x + \beta z = \gamma + 1$ and $x_t + z_{t+1,t+\sigma} = 0$. The following lemma shows how such an inequality can be strengthened with respect to conv($\mathcal{X}_\sigma$) when condition C1 is violated for some period t.

Lemma 4.2.3 *Let $\sigma > 0$ and $t \in [1, T-\sigma]$. Let $X_0 + \alpha x + \beta z \geq \gamma$ be as described above. If there is no solution in $\mathcal{X}_\sigma$ without production in $[t, t+\sigma]$ that satisfies the inequality at equality, then*

$$X_0 + \alpha x + \beta z + x_t + z_{t+1,t+\sigma} \geq \gamma + 1 \tag{4.13}$$

is valid for $\mathcal{X}_\sigma$ and defines a higher-dimensional face for $\operatorname{conv}(\mathcal{X}_\sigma)$ *than the original inequality.* □

It is not hard to see that if C1 is also violated for t and $\sigma' < \sigma$, then (4.13) is valid for $\mathcal{X}_{\sigma'}$ as well. However, the new inequality does not necessarily define a higher-dimensional face with respect to $\operatorname{conv}(\mathcal{X}_{\sigma'})$.

Example 4.2.1 Let $T = 6$ and $d_5 = d_6 = 1$. Then

$$\begin{array}{ll} \qquad\qquad\qquad\qquad\quad + z_5 & \\ X_0 + x_1 + x_2 + x_3 + x_4 & \geq 1 \end{array}$$

defines a facet of $\operatorname{conv}(\mathcal{X}_0)$ but not for $\operatorname{conv}(\mathcal{X}_1)$, since for any solution in $\mathcal{X}_0$ and, hence, in $\mathcal{X}_1$ that satisfies the inequality at equality production occurs in at least one of the periods 5 and 6 (see also Proposition 4.2.4 below). By Lemma 4.2.3

$$\begin{array}{ll} \qquad\qquad\qquad\qquad\quad + z_5 + z_6 & \\ X_0 + x_1 + x_2 + x_3 + x_4 + x_5 & \geq 2 \end{array}$$

is valid for $\mathcal{X}_0$ and $\mathcal{X}_1$. Moreover, the inequality defines a facet of both $\operatorname{conv}(\mathcal{X}_0)$ and $\operatorname{conv}(\mathcal{X}_1)$. □

We first consider the LSM inequalities (cf. Subsection 2.2.4), which have the following form:

$$X_0 + x_{1,t} + \sum_{j \in J} (x_{t+j} + z_{t+j+1, s_{d(t,j)}}) \ \geq \ d_{1,t} + |J|, \tag{4.14}$$

where $t \in [0, T-1]$, $d_{t+1} = 0$, $J \subseteq \{1, \ldots, d_{t+1,T}\}$, and where $s_{d(t,j)}$ denotes the jth demand period after t. These inequalities are facet-defining for $\operatorname{conv}(\mathcal{X}_0)$.

Proposition 4.2.4 *Let $\sigma > 0$. Inequality* (4.14) *defines a facet of* $\operatorname{conv}(\mathcal{X}_\sigma)$ *if and only if for every $j \in \{1, \ldots, d_{t+1,T}\}$ the following holds: if $s_{d(t,j)} \leq t + j + \sigma$, then $j \in J$.* □

The necessity of the condition follows from the observation that for $j \in \{1, \ldots, d_{t+1,T}\} \setminus J$ there is no solution in $\mathcal{X}_0$, and, hence, none in $\mathcal{X}_\sigma$, without production in $[t + j, s_{d(t,j)}]$ that satisfies (4.14) at equality. By Lemma 4.2.3, we can add $x_{t+j} + z_{t+j+1, s_{d(t,j)}} \geq 1$ to (4.14). This obviously yields another inequality of the form (4.14) with $J' = J \cup \{j\}$.

We next consider the RSM inequalities (cf. Subsection 2.2.2), which can be written in the form

$$X_0 + x_{1,t^*} + \sum_{u \in U} (z_{u+1,u+d_{u,t^*}} - x_{u+d_{u,t^*}}) \geq d_{1,t^*}, \tag{4.15}$$

where t^* is a demand period and $U \subseteq \{t : t < t^* \text{ and } d_t = 0\}$. For convenience, define $V = \{u + d_{u,t^*} : u \in U\}$. Moreover, denote by $u(v)$, $v \in V$, the element in U satisfying $u + d_{u,t^*} = v$. Again, these inequalities are facet-defining for $\text{conv}(\mathcal{X}_0)$.

Unlike in the case of constraints (4.14), we can only give sufficient conditions for (4.15) to be facet-defining for $\text{conv}(\mathcal{X}_\sigma)$, $\sigma > 0$. We use the following notation with respect to an inequality of the form (4.15): if t^* is not the last demand period, then t' denotes the first demand period after t^*, otherwise we set $t' = T + 1$. Furthermore, define $v_{\min} = \min_{v \in V} v$.

Proposition 4.2.5 *Let* $\sigma > 0$*. If* $t' > \min(T, t^* + \sigma + 1)$ *or if* $t' = t^* + \sigma + 1$ *and* ($v_{\min} + \sigma \leq t^*$ *or* $t^* \in V$)*, then inequality* (4.15) *defines a facet of* $\text{conv}(\mathcal{X}_\sigma)$ □

With respect to $\text{conv}(\mathcal{X}_1)$ we can give both necessary and sufficient conditions for (4.15) to define a facet. These are easily obtained from the following lemma, which provides a strong relation between facet-defining inequalities of $\text{conv}(\mathcal{X}_0)$ and $\text{conv}(\mathcal{X}_1)$.

Lemma 4.2.6 *A facet-defining inequality of* $\text{conv}(\mathcal{X}_0)$ *defines a facet of* $\text{conv}(\mathcal{X}_1)$ *if and only if for every* $t < T$ *there exists a solution in* $\mathcal{X}_1$ *without production in* $[t, t+1]$ *that satisfies the inequality at equality.*

PROOF. We already proved the necessity of the condition. Hence, let $\alpha_0 X_0 + \alpha x + \beta z \geq \gamma$ be a facet-defining inequality of $\text{conv}(\mathcal{X}_0)$, and denote by $\bar{\mathcal{X}}_\sigma$ the set of solutions in $\mathcal{X}_\sigma$, $\sigma \in \{0, 1\}$, that satisfy the inequality at equality. Moreover, suppose that for every $t < T$ there exists a solution in $\bar{\mathcal{X}}_1$ without production in $[t, t+1]$. Since the inequality defines a facet of $\text{conv}(\mathcal{X}_0)$, there exist $2T$ linearly independent directions in $\bar{\mathcal{X}}_0$ (cf. Lemma 4.2.1). The direction $e(z_t)$ is easily established for all $t > 1$ with $\beta_t = 0$, where $e(z_t)$ denotes the unit vector of length $2T + 1$ corresponding to the variable z_t. We may assume that the other directions are obtained from solutions in $\bar{\mathcal{X}}_0$ that satisfy $x_t + z_{t+1} \leq 1$ for all t, thus, these are directions in $\bar{\mathcal{X}}_1$ as well. Hence, it suffices to show that $e(z_t)$ is also a direction in $\bar{\mathcal{X}}_1$ for all $t > 1$ with $\beta_t = 0$. To that end, let $t > 1$ satisfy $\beta_t = 0$. By assumption, there exists a solution $(X_0, x, z) \in \bar{\mathcal{X}}_1$ satisfying $x_{t-1} + z_t = 0$. Then $(X_0, x, \bar{z})$ with $\bar{z}_t = 1$ and $\bar{z}_\tau = z_\tau$ for $\tau \neq t$ is also in $\bar{\mathcal{X}}_1$ and $(X_0, x, \bar{z}) - (X_0, x, z) = e(z_t)$. □

It is not likely that a simple relation between the facet-defining inequalities of $\text{conv}(\mathcal{X}_{\sigma-1})$ and $\text{conv}(\mathcal{X}_\sigma)$ can be established for $\sigma > 1$. In any case, the following example shows that the above result for $\sigma = 1$ does not hold for general σ.

Example 4.2.2 Let $T = 10$ and $d_t = 1$ for $t \in \{7, 8, 9, 10\}$. Then

$$\begin{array}{l} \quad\quad\quad\;\; + z_3 + 2z_4 + z_5 \quad\quad + z_7 + z_8 \\ X_0 + x_1 + x_2 + x_3 \quad\quad\quad\; + x_6 + x_7 \end{array} \geq 2 \tag{4.16}$$

is an inequality of the form (4.15) with $t^* = 8$ and $U = \{2, 3, 6\}$ that defines a facet of conv($\mathcal{X}_1$) (cf. Proposition 4.2.7 below). Furthermore, for each $t \in [1, 8]$ there exists a solution in $\mathcal{X}_2$ without production in $[t, t+2]$ that satisfies (4.16) at equality: take $X_0 = 2$, $z_9 = x_9 = x_{10} = 1$ for $t \leq 6$, and $X_0 = 0$, $z_2 = x_2 = \ldots = x_5 = 1$ for $t = 7, 8$. However, (4.16) does not define a facet of conv($\mathcal{X}_2$), since $x_7 + z_8 = x_8$ for all $(X_0, x, z) \in \mathcal{X}_2$ that satisfy the inequality at equality. □

Let us consider inequalities (4.15) again. Throughout, we use the following notation with respect to such an inequality: $\hat{t} = \max\{t \in [t^*, T] : d_\tau = 1 \text{ for all } \tau \in [t^*, t]\}$. For every inequality of the form (4.15) condition C1 is satisfied with respect to $\mathcal{X}_1$ for $t < t^*$ and for $t \geq \hat{t}$, i.e., for these periods t there exists a solution in $\mathcal{X}_1$ without production in $[t, t+1]$ that satisfies (4.15) at equality. For $t \in [t^*, \hat{t} - 1]$ the condition is satisfied if and only if one of the following holds: (i) $\hat{t} = t^*$, (ii) $\hat{t} = t^* + 1$ and $v_{\min} < t^*$, or (iii) $\hat{t} > t^* + 1$ and $\{v, v+1\} \subseteq V$ for some $v < t^*$. Hence, by Lemma 4.2.6, we have the following result.

Proposition 4.2.7 *Inequality* (4.15) *defines a facet of* conv($\mathcal{X}_1$) *if and only if one of the following holds:* (i) $\hat{t} = t^*$, (ii) $\hat{t} = t^* + 1$ *and* $v_{\min} < t^*$, *or* (iii) $\hat{t} > t^* + 1$ *and* $\{v, v+1\} \subseteq V$ *for some* $v < t^*$. □

The above result implies that (4.15) does not define a facet of conv($\mathcal{X}_1$) if either $\hat{t} \geq t^* + 1$ and $v_{\min} = t^*$ or $\hat{t} \geq t^* + 2$, $v_{\min} < t^*$, and $v + 1 \notin V$ for any $v \in V$. Denote by R the set of periods t that violate condition C1 with respect to $\mathcal{X}_1$. Then $R = [t^*, \hat{t} - 1]$ if $\hat{t} \geq t^* + 1$ and $v_{\min} = t^*$, and $R = [t^* + 1, \hat{t} - 1]$ otherwise. Inequality (4.15) can now be strengthened with respect to conv($\mathcal{X}_1$) by applying Lemma 4.2.3 for at least one $t \in R$. The question is: for which $R' \subseteq R$ is

$$X_0 + x_{1,t^*} + \sum_{u \in U}(z_{u+1,u+d_{u,t^*}} - x_{u+d_{u,t^*}}) + \sum_{\tau \in R'}(x_\tau + z_{\tau+1}) \quad \geq \quad d_{1,t^*} + |R'| \tag{4.17}$$

facet-defining for conv($\mathcal{X}_1$)?

First, notice that if $v_{\min} = t^*$, then (4.15) is also of the form (4.14) with $t = u(t^*) - 1$ and $J = \{1, \ldots, t^* - u(t^*) + 1\}$. Then, by Proposition 4.2.4, (4.17) defines a facet of conv($\mathcal{X}_1$) if and only if $R' = R = [t^*, \hat{t} - 1]$. It is also not difficult to check that if $\hat{t} \geq t^* + 2$, $|V| = \{v\}$, and $v < t^*$, then (4.17) defines a facet of conv($\mathcal{X}_1$) if and only if $R' = R = [t^* + 1, \hat{t} - 1]$. For the remaining cases we have the following result:

Proposition 4.2.8 *Let* (4.17) *satisfy* $\hat{t} \geq t^* + 2$, $|V| \geq 2$, *and* $v + 1 \notin V$ *for any* $v \in V$. *Furthermore, let* $R' = \{t^* + 2j - 1 : j = 1, \ldots, \lfloor(\hat{t} - t^*)/2\rfloor\}$. *Then* (4.17) *is valid for* $\mathcal{X}_1$. *The inequality defines a facet of* conv($\mathcal{X}_1$) *if and only if one of the following holds:* (i) $\hat{t} = t^* + 2$,

(ii) *there exists* $v \in V$ *such that* $v+2 \in V$ *or* $u(v) = v-1$, *or* (iii) $\hat{t} - t^*$ *is even and there exists* $v \in V$ *such that* $v+3 \in V$ *or* $u(v) = v-2$. □

In all cases, the validity of (4.17) for $\mathcal{X}_1$ can be shown by sequentially applying Lemma 4.2.3 to the elements of R' in increasing order. Obviously, if $(X_0, x, z) \in \mathcal{X}_0 \backslash \mathcal{X}_1$, then the solution obtained from (X_0, x, z) by setting z_t to zero for all t satisfying $x_{t-1} = z_t = 1$ yields a solution in $\mathcal{X}_1$. Hence, a valid inequality for $\mathcal{X}_1$ with nonnegative z-coefficients is also valid for $\mathcal{X}_0$. In particular, (4.17) is valid for $\mathcal{X}_0$ if it is valid for $\mathcal{X}_1$. Moreover, it is not difficult to show that (4.17) defines a facet of conv$(\mathcal{X}_0)$ whenever the inequality is valid for $\mathcal{X}_0$. Then, by Lemma 4.2.6, the result that (4.17) defines a facet of conv$(\mathcal{X}_1)$ for the given set R' is established by checking that for any $t \in R$ there exists a solution in $\mathcal{X}_1$ without production in $[t, t+1]$ that satisfies (4.17) at equality.

Example 4.2.3 Let $T = 10$ and $d_t = 1$ for $t \in [5, 10]$. Then

$$\begin{array}{l} \qquad\qquad\qquad\quad + z_3 + z_4 + z_5 + z_6 \\ X_0 + x_1 + x_2 + x_3 \qquad\qquad\quad + x_5 \end{array} \geq 2$$

defines a facet of conv$(\mathcal{X}_0)$ but not for conv$(\mathcal{X}_1)$, since $x_{t-1} + z_t = 1$, $t \in [8, 10]$, for any solution $(X_0, x, z) \in \mathcal{X}_1$ that satisfies the inequality at equality. Now the above proposition yields that

$$\begin{array}{l} \qquad\qquad\qquad\quad + z_3 + z_4 + z_5 + z_6 \qquad + z_8 \qquad + z_{10} \\ X_0 + x_1 + x_2 + x_3 \qquad\qquad\quad + x_5 \qquad + x_7 \qquad + x_9 \end{array} \geq 4$$

defines a facet of both conv$(\mathcal{X}_0)$ and conv$(\mathcal{X}_1)$. Notice that this inequality does not belong to any of the three subclasses of facet-defining inequalities of conv$(\mathcal{X}_0)$ that were discussed in Subsections 2.2.2 through 2.2.4. □

Let us finally show how the multi-item inequalities (3.12) (cf. Section 3.2) can be strengthened in the presence of startup times. Consider the multi-item formulation presented at the beginning of this chapter. For convenience, we add $x_t^i = z_t^i = 0$ for all i and $t \leq \sigma_i$. Then the following modification of (3.12) is valid for the set of solutions to (4.1) – (4.4):

$$X_0^i + x_{1,t_1-1}^i \;\geq\; d_{1,t_1-1}^i + d_{t_1,t_3}^i \sum_{j \in J} (x_{t_2}^j - z_{t_1+1,t_2}^j) - \sum_{t'=t_2+1}^{t_3} d_{t',t_3}^i z_{t'}^i, \tag{4.18}$$

where $i \in \{1, \ldots, M\}$, $J \subseteq \{1, \ldots, M\} \backslash \{i\}$, and t_1, t_2, and t_3 are three periods that satisfy $1 \leq t_1 \leq t_2 \leq t_3 \leq T$ and $d_{t_1,t_3}^i \geq 1$. These inequalities were derived from the following observation: if there is no production for item i in the interval $[t_1, t']$, where $t' \in [t_2+1, t_3]$, then the inventory for item i at the end of period $t_1 - 1$ must be at least $d_{t_1,t'}^i$. If startup times are zero, then there is no production for item i in $[t_1, t_2]$ if the whole interval is used for production of item j. The latter is implied by $x_{t_2}^j - z_{t_1+1,t_2}^j = 1$. In the presence of positive startup times, however, we cannot produce item i in the interval $[t_1, t_2]$ if all periods in $[t_1 + \sigma_j, t_2]$ are used for production of item j, which is implied by $x_{t_2}^j - z_{t_1+\sigma_j+1,t_2}^j = 1$. Moreover, if item

j is produced in period t_2, then $t_2 + \sigma_i + 1$ is the first period after t_2 in which production for item i can occur. The above inequality can therefore be strengthened to

$$X_0^i + x_{1,t_1-1}^i \geq d_{1,t_1-1}^i + d_{t_1,t_3}^i \sum_{j\in J}(x_{t_2}^j - z_{t_1+\sigma_j+1,t_2}^j) - \sum_{t=t_2+\sigma_i+1}^{t_3} d_{t,t_3}^i z_t^i. \quad (4.19)$$

Recall that inequalities (4.14), (4.15), and (4.18) (with $X_0^i = 0$ for all i) served as cutting planes in the branch-and-cut algorithm for DLSP discussed in the preceding chapter. Obviously, the separation routines can be used in a cutting plane algorithm for DLSS as well. The strengthenings of inequalities (4.15) or (4.18) presented above can easily be incorporated. Computational experiments will have to show whether the addition of these inequalities is as effective in reducing the integrality gap for DLSS as for DLSP.

4.3 A multicommodity flow formulation

A natural formulation for single-item lot-sizing problems is the one in which the variables are only indexed by t. However, the introduction of new variables often yields a tighter formulation with fewer constraints. Various types of extended formulations are discussed in the literature; for a review we refer to Pochet and Wolsey [33]. In many cases a complete linear description in the new variables is derived.

One such extended formulation is the *multicommodity flow* formulation, in which the production variables x_t are split into variables x_{tk} that specify how much is produced in period t to satisfy the demand in the kth demand period. Van Hoesel and Kolen [20] present a multicommodity flow formulation for DLSP in which not only the production variables x_t but also the startup variables y_t are split. By modeling DLSP as a shortest path problem they derive an LP-formulation in these variables that always yields an integral solution.

Here we give a similar result for the single-item problem with startup times. We consider DLSS with the additional restrictions that we always produce when the machine has been set up for production ($z_t \leq x_t$) and that overproduction is not allowed. As in Section 4.2, the periods are numbered such that period 1 is the first period in which production can occur. Thus, we consider the following formulation of DLSS:

$$\min \quad c_0 X_0 + \sum_{t=1}^{T}(c_t x_t + g_t z_t)$$

$$\text{s.t.} \quad X_0 + x_{1,t} \geq d_{1,t} \qquad (1 \leq t < T) \qquad (4.20)$$

$$X_0 + x_{1,T} = d_{1,T} \qquad (4.21)$$

$$z_t \leq x_t \leq x_{t-1} + z_t \qquad (1 \leq t \leq T, x_0 = 0) \qquad (4.22)$$

$$x_t + \sum_{\tau=t+1}^{\min(t+\sigma,T)} z_\tau \leq 1 \qquad (0 \leq t \leq T, x_0 = 0) \qquad (4.23)$$

$$X_0 \in \mathbb{N},\ x_t, z_t \in \{0,1\} \qquad (1 \leq t \leq T) \qquad (4.24)$$

In a multicommodity flow formulation of DLSS the variables x_t and z_t are split into binary variables x_{tk} and z_{tk}, respectively. As usual, s_k denotes the kth demand period. Then $x_{tk} = 1$ if and only if production occurs in period t to satisfy the demand in s_k. Similarly, $z_{tk} = 1$ if and only if period t is used to produce for period s_k after a startup in $[t - \sigma, t - 1]$. The integer variable X_0 is split into binary variables x_{0k} that indicate whether the demand for period s_k is produced in period 0. Without loss of generality we assume that the demand for s_k, $k > 1$, is not produced before the demand for s_{k-1}, i.e., we only consider solutions to DLSS that satisfy

$$\text{if } x_{t_1,k-1} = x_{t_2,k} = 1, \text{ then } t_1 < t_2 \text{ or } t_1 = t_2 = 0. \tag{4.25}$$

In the sequel a complete linear description of the convex hull of the set of these solutions will be established. The result is obtained by modeling DLSS as a shortest path problem on an acyclic network such that the variables x_{tk} and z_{tk} form a subset of the flow variables in the corresponding linear program. Elimination of the other variables yields a linear program in the x- and z-variables that always has an optimal integral solution.

We first present a dynamic programming algorithm that solves DLSS. This algorithm will be transformed into the shortest path formulation mentioned above. For convenience, define $K = \{1, \ldots, d_{1,T}\}$.

Now let $\phi(t, k)$, $t \in [1, s_k]$ and $k \in K$, denote the minimum cost of a solution for periods 0 up to t in which period t is used for production and the total production up to t equals k. If $t < k$, then the production in period 0 is at least $k - t$, which incurs a cost of c_0 per unit. Also observe that if $t \in [2, \sigma]$ is used for production and the total production up to t equals $k > 1$, then either t is the first period in $[1, t]$ in which production occurs or period $t - 1$ used for production as well.

Furthermore, let $\psi(t, k)$, $t \in [2, s_k - \sigma]$ and $k \in K$, denote the minimum cost of a solution for periods 0 up to t for which the following holds: period t is not used for production, period $t - 1$ is not used for setting up the machine, and the total production up to t equals $k - 1$. In this case the kth unit cannot be produced before $t + \sigma$. Hence, $\psi(t, k)$ is only defined for $t \leq s_k - \sigma$. Note that if the $(k - 1)$st unit is produced in some period $t \in [s_k - \sigma + 1, s_{k-1} + 1]$, then the kth unit must be produced in period $t + 1$. Moreover, if the kth unit is produced in $t > s_{k-1} + 1$, then a startup occurs in the interval $[t - \sigma, t - 1]$.

Using the above observations we obtain the following forward recursion is easily verified:

$$\phi(t, 1) = g_t + c_t \text{ for } t \in [1, s_1], \text{ and}$$

$$\phi(t, k) = c_t + \begin{cases} c_0(k-1) + g_1 & \text{for } t = 1 \\ \min(c_0(k-1) + g_t, \phi(t-1, k-1)) & \text{for } t \in [2, \sigma + 1] \\ \min(\phi(t-1, k-1), \psi(t-\sigma, k) + g_t) & \text{for } t \in [\sigma + 2, \min(s_{k-1} + 1, s_k - \sigma)] \\ \psi(t-\sigma, k) + g_t & \text{for } t \in [s_{k-1} + 2, s_k - \sigma] \\ \phi(t-1, k-1) & \text{for } t \in [s_k - \sigma + 1, s_{k-1} + 1] \end{cases}$$

$$\psi(t,k) = \begin{cases} \min(c_0(k-1), \phi(1,k-1)) & \text{for } t=2 \\ \min(\psi(t-1,k), \phi(t-1,k-1)) & \text{for } t \in [3, \min(s_{k-1}+1, s_k-\sigma)] \\ \psi(t-1,k) & \text{for } t \in [s_{k-1}+2, s_k-\sigma] \end{cases}$$

for $k \in K\backslash\{1\}$. Calculation of $\min\left(c_0 d_{1,T}\,, \min_{t\in[1,s_{d_{1,T}}]} \phi(t, d_{1,T})\right)$ yields the minimum cost of any feasible solution to DLSS.

The dynamic programming algorithm gives rise to a shortest path formulation in the following way. For convenience, define $K_1 = K\backslash\{1\}$. We define a network that, apart from a source S and a sink S', consists of the following nodes:

- u_{tk} for all $t \in [0, s_k]$ and $k \in K$, except for $(t,k) = (0, d_{1,T})$;
- v_{tk} for all $t \in [2, s_{k-1}+1]$ and $k \in K_1$;
- w_{tk} for all $t \in [2, s_k - \sigma]$ and $k \in K_1$.

Every solution to DLSS will correspond to a path from S to S'. Node u_{tk} will be on the path if and only if the kth unit of demand is produced in period t. Node v_{tk} will be on the path if and only if the demand for s_{k-1} is produced in period $t-1$. Finally, node w_{tk} will be on the path if the following holds: there is no production in period t, the total production up to period t equals $k-1$, and the machine is not being set up in $t-1$.

Table 4.1 lists the arcs in the network together with the corresponding flow variable and cost. Figure 4.1 shows the network corresponding to the instance with $T=6$, $\sigma=2$, and demand periods $s_1=2$, $s_2=5$, and $s_3=6$.

arc	variable	cost	defined for
(S, u_{01})	x_{01}	c_0	
(S, u_{t1})	z_{t1}	g_t	$t \in [1, s_1]$
$(u_{0,k-1}, u_{0k})$	x_{0k}	c_0	$k \in K_1\backslash\{d_{1,T}\}$
$(u_{0,k-1}, u_{tk})$	z_{tk}	g_t	$t \in [1, \sigma+1]$ and $k \in K_1$
$(u_{0,k-1}, w_{2k})$	γ_{1k}	0	$k \in K_1$
$(u_{t-1,k-1}, v_{tk})$	$x_{t-1,k-1}$	c_{t-1}	$t \in [2, s_{k-1}+1]$ and $k \in K_1$
(v_{tk}, u_{tk})	α_{tk}	0	$t \in [2, s_{k-1}+1]$ and $k \in K_1$
(v_{tk}, w_{tk})	β_{tk}	0	$t \in [2, \min(s_{k-1}+1, s_k-\sigma)]$ and $k \in K_1$
$(w_{tk}, w_{t+1,k})$	γ_{tk}	0	$t \in [2, s_k-\sigma-1]$ and $k \in K_1$
$(w_{tk}, u_{t+\sigma,k})$	$z_{t+\sigma,k}$	$g_{t+\sigma}$	$t \in [2, s_k-\sigma]$ and $k \in K_1$
$(u_{0,d_{1,T}-1}, S')$	$x_{0,d_{1,T}}$	c_0	
$(u_{t,d_{1,T}}, S')$	$x_{t,d_{1,T}}$	c_t	$t \in [1, s_{d_{1,T}}]$

Table 4.1: Arcs in the network

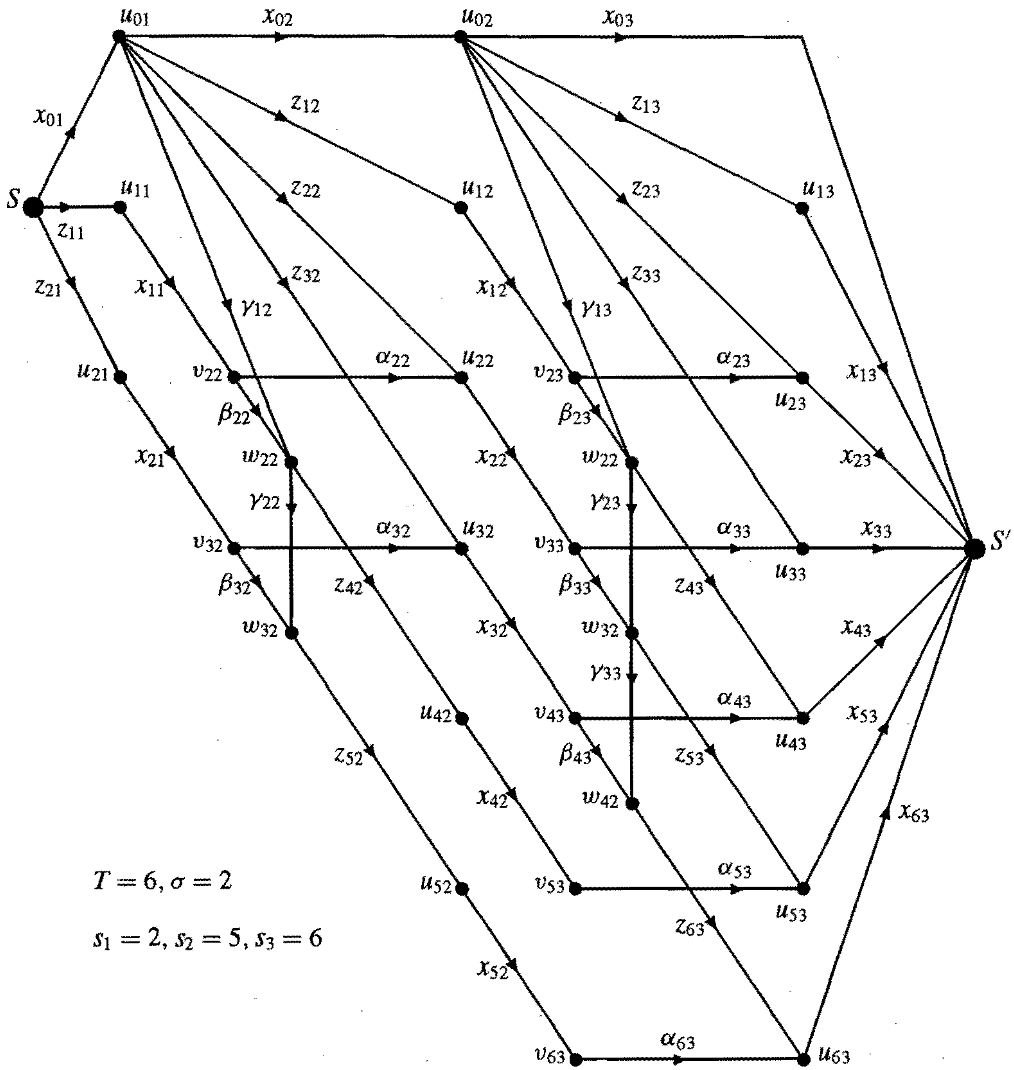

Figure 4.1: Shortest path formulation of DLSS

It is not hard to see that for $t \in [1, s_k]$ and $k \in K$ the value of $\phi(t, k) - c_t$ equals the length of the shortest path from S to u_{tk} and, hence, that $\phi(t, k)$ corresponds to the length of the shortest path from S to $v_{t+1,k+1}$ for $k < d_{1,T}$. Furthermore, $\psi(t, k)$ corresponds to the length of the shortest path from S to w_{tk}. Note that the unique shortest path from S to u_{0k}, $k < d_{1,T}$, has length c_0k. Thus, the length of the shortest path from S to S' is $\min(c_0 d_{1,T}, \min_{t\in[1,s_{d_{1,T}}]} \phi(t, d_{1,T}))$. Hence, DLSS can be solved by solving the shortest path problem on this network.

Consider a path from S to S'. Observe that the intermediate nodes are nondecreasing in both the index t and the index k. Thus, for every $k < d_{1,T}$ there is precisely one node u_{tk}, $t \in [0, s_k]$, on the path, hence, $x_{tk} = 1$ for precisely one $t \in [0, s_k]$. The latter also holds for $k = d_{1,T}$. Furthermore, if $x_{0k} = 1$, $k > 1$, then $x_{0,k-1} = 1$, and if $x_{tk} = 1$ for $t > 0$ and $k > 1$, then $x_{\tau,k-1} = 1$ for some $\tau < t$. Thus, (4.25) is satisfied. Note further that $z_{tk} = 1$ implies $x_{tk} = 1$. Finally, if $z_{tk} = 1$ for $k > 1$, then $x_{\tau,k-1} = 1$ for some $\tau \le \max(0, t - \sigma - 1)$.

The other variables can be interpreted as follows. If $\alpha_{tk} = 1$, then the demand for s_{k-1} is produced in $t - 1$ and the demand for s_k is produced in period t. If $\beta_{tk} = 1$, then the demand for s_{k-1} is produced in $t - 1$ but there is no production in period t. Finally, if $\gamma_{tk} = 1$, then there is no production in period t and the total production up to t equals $k - 1$. Moreover, period t is not used for setting up the machine.

The shortest path problem can be solved by the linear program formed by the flow conservation constraints and the nonnegativity constraints. The flow conservation constraints yield explicit expressions for the α-, β-, and γ-variables in terms of the x- and z-variables. Using these and the nonnegativity constraints, we can obtain an equivalent linear program in the x- and z-variables only. It is a tedious but straightforward procedure to check that elimination of the α-, β-, and γ-variables yields the linear program

$$\min \sum_{k=1}^{d_{1,T}} \left[c_0 x_{0k} + \sum_{t=1}^{s_k} (c_t x_{tk} + g_t z_{tk}) \right] \tag{4.26}$$

$$\text{s.t.} \sum_{t=0}^{s_k} x_{tk} = 1 \quad \text{for } k \in K \tag{4.27}$$

$$z_{t1} = x_{t1} \quad \text{for } t \in [1, s_1] \tag{4.28}$$

$$z_{tk} = x_{tk} \quad \text{for } t \in \{1\} \cup [s_{k-1} + 2, s_k] \text{ and } k \in K_1 \tag{4.29}$$

$$z_{tk} \le x_{tk} \le x_{t-1,k-1} + z_{tk} \quad \text{for } t \in [2, \min(s_k - \sigma, s_{k-1} + 1)] \text{ and } k \in K_1 \tag{4.30}$$

$$z_{tk} \le x_{tk} = x_{t-1,k-1} + z_{tk} \quad \text{for } t \in [s_k - \sigma + 1, s_{k-1} + 1] \text{ and } k \in K_1 \tag{4.31}$$

$$\sum_{\tau=t}^{s_{k-1}} x_{\tau,k-1} + \sum_{\tau=t+1}^{t+\sigma} z_{\tau k} \le \sum_{\tau=t+1}^{s_k} x_{\tau k} \quad \text{for } t \in [1, s_k - \sigma - 1] \text{ and } k \in K_1 \tag{4.32}$$

$$x_{0k}, x_{tk}, z_{tk} \ge 0 \quad \text{for } t \in [1, s_k] \text{ and } k \in K \tag{4.33}$$

From the above discussion it follows that the linear program (4.26) – (4.33) always yields an integral solution. Hence, we have proven the following result:

Theorem 4.3.1 *The linear program* (4.26) – (4.33) *solves DLSS.* □

A formulation for the multi-item DLSS in the variables x^i_{tk} and z^i_{tk} is obtained by taking the single-item formulations for the different items together with the linking constraints

$$\sum_{i\in M}\left(\sum_{k:s^i_k\geq t} x^i_{tk} + \sum_{\tau=t+1}^{t+\sigma_i}\sum_{k:s^i_k\geq\tau} z^i_{\tau k}\right) \leq 1 \text{ for all } t. \tag{4.34}$$

Obviously, its LP-relaxation will not yield an integral solution in general. However, it is expected to give stronger lower bounds than the LP-relaxation of the formulation in the natural variables. A disadvantage of the multicommodity flow formulation is its large number of variables, which is about $d_{1,T}T$. If the formulation is used in a branch-and-cut algorithm, then it may be computationally advantageous to start with an LP-formulation in which only a subset of the variables is included and to add nonactive variables only when they price out. This is an interesting topic for further research.

5. The delivery man problem

This chapter deals with the delivery man problem or DMP for short. This problem is a variant of the well-known traveling salesman problem (TSP) in which the objective is to find a tour starting from a given depot that minimizes the sum of the waiting times of the customers. The DMP can also be interpreted as a single-machine scheduling problem with sequence-dependent processing times in which the total flow time of the jobs has to be minimized.

Polyhedral methods have been proven to be very successful for the TSP and many of its extensions. Most of these extensions deal with extra conditions on the graph structure, e.g. precedence constraints, such that the problem can still be formulated as a $0-1$ model. However, we will formulate the DMP as a mixed integer programming problem by introducing time variables (Section 5.1). This formulation can easily be adapted to the problem with time windows, i.e., when each customer has to be visited within a specified interval. Only a few papers deal with this kind of extension. Ascheuer [2] has developed a branch-and-cut code for the TSP with time windows. Escudero and Sciomachen [12] and Escudero, Guignard, and Malik [13] study the sequential ordering problem with time windows, where the sequential ordering problem is the problem of finding a minimum weight Hamiltonian path subject to precedence constraints. For this problem Maffioli and Sciomachen [26] propose a formulation that is similar to our formulation of the DMP.

In Section 5.2 we will derive additional classes of valid inequalities in order to strengthen the linear programming relaxation. The quality of the lower bounds obtained from the LP-relaxation and the effectiveness of the new inequalities in reducing the gap are studied computationally in Section 5.3.

This chapter differs from the previous chapters in that it is primarily devoted to the DMP instead of the DLSP. Nevertheless, the formulation presented here may also serve as a basis for a polyhedral approach to the DLSP with sequence-dependent startup times. To that end, we use the reformulation of the latter problem as a TSP with time windows (cf. Section 1.2). The DLSP then only differs from the DMP with time windows as far as the objective function is concerned.

5.1 Formulation

The delivery man problem is formally stated as follows. We consider the complete directed graph $G = (V \cup \{0\}, A)$, where $V = \{1, \dots, n\}$. With each arc (i, j) we associate a nonnegative integer travel time p_{ij}. It is assumed that visiting time is included in the travel time, hence, the arrival time at node i equals the departure time at node i. Node 0 is the depot, i.e., every tour starts and finishes in node 0. Furthermore, we assume that every tour starts at time 0.

Then the waiting time of the customer located at node i is equal to the departure time at node i. Now the problem is to find a tour that minimizes the sum of the departure times at the nodes.

The DMP can be modeled using the following two types of variables. For every arc (i, j) there is a binary variable x_{ij} that indicates whether the arc (i, j) is included in the tour or not, and a time variable t_{ij} defined as follows:

$$t_{ij} = \begin{cases} \text{departure time at node } i, & \text{if } x_{ij} = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Since we assume that every tour starts at time 0, the variables t_{0j} can be omitted from the model. This gives rise to the following formulation of DMP, in which C denotes a large constant:

$$\min \sum_{i=1}^{n} \sum_{j=0, j\neq i}^{n} t_{ij}$$

$$\text{s.t.} \quad \sum_{j=0, j\neq i}^{n} x_{ij} = 1 \qquad (0 \le i \le n) \tag{5.1}$$

$$\sum_{i=0, i\neq j}^{n} x_{ij} = 1 \qquad (0 \le j \le n) \tag{5.2}$$

$$\sum_{i=1, i\neq j}^{n} t_{ij} + \sum_{i=0, i\neq j}^{n} p_{ij}x_{ij} = \sum_{k=0, k\neq j}^{n} t_{jk} \qquad (1 \le j \le n) \tag{5.3}$$

$$0 \le t_{ij} \le Cx_{ij} \qquad (0 \le i, j \le n, i \neq j, i \neq 0) \tag{5.4}$$

$$x_{ij} \in \{0, 1\} \qquad (0 \le i, j \le n, i \neq j) \tag{5.5}$$

Constraints (5.1) and (5.2) ensure that every node, including the depot, is visited exactly once. Constraints (5.3) guarantee that if $x_{ij} = 1$, then the departure time at node j equals the departure time at node i plus the travel time p_{ij}. These constraints also prevent subtours, unless there exists a set of subtours such that every node is in precisely one subtour and $p_{ij} = 0$ for all arcs (i, j) that are involved in these subtours. If C is an upper bound on the departure time at node i, e.g. $C = n \cdot \max_{i,j} p_{ij}$, then (5.4) is valid when $x_{ij} = 1$. Furthermore, these constraints force that $t_{ij} = 0$ if $x_{ij} = 0$.

This model can easily be adapted to the problem with time windows (DMPTW), i.e., when each node i has to be visited within a specified interval $[e_i, l_i]$. The delivery man may arrive at node i before e_i, but cannot deliver before the opening of the time window. In this case, the departure time at node i is strictly larger than the arrival time at node i. To model this, equalities (5.3) must be replaced by

$$\sum_{i=1, i\neq j}^{n} t_{ij} + \sum_{i=0, i\neq j}^{n} p_{ij}x_{ij} \le \sum_{k=0, k\neq j}^{n} t_{jk} \qquad (1 \le j \le n). \tag{5.6}$$

Furthermore, substituting

$$e_i x_{ij} \leq t_{ij} \leq l_i x_{ij} \qquad (0 \leq i, j \leq n, i \neq j, i \neq 0) \qquad (5.7)$$

for (5.4) yields that $t_{ij} = 0$ if $x_{ij} = 0$ and that the departure time at node i is in the time window $[e_i, l_i]$ otherwise. The sum of the waiting times is now $\sum_i (\sum_j t_{ij} - e_i)$.

It is evident that, apart from the objective function, the formulation for the DMPTW models the TSP with time windows as well. A well-known model for the TSPTW is the one with variables x_{ij} and t_i, where t_i denotes the departure time at node i (cf. Desrochers et al. [8]). Then the set of feasible tours is the set of solutions to (5.1), (5.2), (5.5), and

$$t_i + p_{ij} + M_{ij}(x_{ij} - 1) \leq t_j \qquad (0 \leq i, j \leq n, i \neq j, j \neq 0, t_0 = 0)$$
$$e_i \leq t_i \leq l_i \qquad (1 \leq i \leq n)$$

where $M_{ij} \geq l_i + p_{ij} - e_j$. Compared to this model an important feature of the formulation with variables x_{ij} and t_{ij} is that it does not involve big-M coefficients in the constraints.

As mentioned before, the DMP can also be considered as a model for the single-machine scheduling problem with sequence-dependent processing times. In this case we have n jobs, a dummy job 0 that is scheduled twice, once at the beginning and once at the end of the schedule, and sequence-dependent processing times p_{ij}. Furthermore, in the presence of time windows, every job i is released at time e_i and has to be started not later than l_i.

A frequently used objective in scheduling problems is minimizing the completion time of the last job in the schedule. In our formulation this can easily be expressed as $\sum_i (t_{i0} + p_{i0}x_{i0})$. For this problem Maffioli and Sciomachen [26] have proposed a similar formulation in which, in addition to the t_{ij}, time variables y_{ij} and u_{ij} are introduced to denote the departure time at node j and the time the delivery man has to wait at node j, respectively, when i is visited before j. Variables u_{ij} are only defined for arcs (i, j) satisfying $e_i + p_{ij} < e_j$. Now a feasible tour is a vector (x, t, u, y) satisfying (5.1), (5.2), (5.5) – (5.7), and

$$\sum_{i=0, i\neq j}^{n} y_{ij} = \sum_{k=0, k\neq j}^{n} t_{jk} \qquad (1 \leq j \leq n)$$
$$u_{ij} = y_{ij} - p_{ij}x_{ij} - t_{ij} \qquad (0 \leq i, j \leq n, i \neq j, j \neq 0, t_{0j} = 0)$$
$$e_j x_{ij} \leq y_{ij} \leq l_j x_{ij} \qquad (0 \leq i, j \leq n, i \neq j, j \neq 0)$$
$$(e_j - p_{ij} - l_i)^+ x_{ij} \leq u_{ij} \leq (e_j - p_{ij} - e_i)^+ x_{ij} \qquad (0 \leq i, j \leq n, i \neq j, j \neq 0, e_0 = 0)$$

We will also consider this model and the one with variables x_{ij} and t_i in our computational experiments.

5.2 Valid inequalities

In the sequel, let $\mathcal{X}$ and $\mathcal{X}_{tw}$ denote the set of feasible solutions to DMP and DMPTW, respectively. That is, $\mathcal{X} = \{(x, t) : (x, t)$ satisfies (5.1) – (5.5)$\}$ and $\mathcal{X}_{tw} = \{(x, t) : (x, t)$ satisfies

(5.1), (5.2), and (5.5) – (5.7)}. In this section we derive valid inequalities for $\mathcal{X}$ and $\mathcal{X}_{\text{tw}}$. These inequalities contain both x- and t-variables, which may lead to a stronger connection between the two types of variables.

5.2.1 Inequalities for the delivery man problem

The inequalities presented in this section can be considered as a generalization of the inequalities derived by Queyranne [34] for the single-machine scheduling problem with sequence-independent processing times. We will briefly discuss the latter, using the terminology of the DMP. Recall that for every solution to this problem the departure time equals the arrival time at any node but the depot.

Sequence-independent processing times correspond to travel times p_{ij} that only depend on node i, i.e., $p_{ij} = p_i$. Furthermore, $p_0 = 0$. For the model that only involves departure times t_i, $i \in V$, Queyranne derived the following inequalities:

$$\sum_{i \in S} p_i t_i \quad \geq \quad \sum_{i,j \in S, i<j} p_i p_j, \tag{5.8}$$

where S is a nonempty subset of V. He proves that the set of all inequalities of the form (5.8) defines the convex hull of the set of all feasible tours.

The validity of (5.8) is easily shown. Let $S = \{\pi(1), \ldots, \pi(s)\}$ and suppose without loss of generality that the nodes in S appear in the tour in the order $\pi(1), \ldots, \pi(s)$. Thus, $t_{\pi(i)} \leq t_{\pi(i+1)}$ for $1 \leq i < s$. Then $t_{\pi(i)} \geq \sum_{j=1}^{i-1} p_{\pi(j)}$, and equality holds if every node in S is visited before any node in $V \backslash S$. Hence,

$$\sum_{i \in S} p_i t_i = \sum_{i=1}^{s} p_{\pi(i)} t_{\pi(i)} \geq \sum_{i=1}^{s} \sum_{j=1}^{i-1} p_{\pi(i)} p_{\pi(j)} = \sum_{i,j \in S, i<j} p_i p_j,$$

which establishes the validity of (5.8) for the model with variables t_i only.

Let us first indicate how the above inequalities can be transformed into valid inequalities for our model. The terms $p_i t_i$ in (5.8) will be split into terms $p_{ij} t_{ij}$. At the right-hand side p_i will be replaced by $\sum_j p_{ij} x_{ij}$, since a travel time p_{ij} occurs only if $x_{ij} = 1$. Unfortunately, this yields quadratic terms at the right-hand side. These have to be linearized, as we can only deal with *linear* inequalities. However, we will first present the quadratic inequalities obtained in this way and show their validity.

In the sequel, a tour is identified with a solution $(x, t) \in \mathcal{X}$, thus (x, t) satisfies the constraints (5.1) – (5.5). Furthermore, the following notation is used. As usual, $x(S) = \sum_{i,j \in S} x_{ij}$ and $x(S_1, S_2) = \sum_{i \in S_1, j \in S_2} x_{ij}$. If S is a subset of V, then $S_0 = S \cup \{0\}$. We abbreviate $\sum_{j \neq i} t_{ij}$ by t_i.

Proposition 5.2.1 *For all $S \subseteq V$, $S \neq \emptyset$,*

$$\sum_{i,j \in S} p_{ij} t_{ij} \geq \frac{1}{2}\left(\sum_{i \in S_0, j \in S} p_{ij} x_{ij}\right)^2 - \frac{1}{2}\sum_{i \in S_0, j \in S} p_{ij}^2 x_{ij} \tag{5.9}$$

is a valid (quadratic) inequality for X.

PROOF. Let $S = \{\pi(1), \pi(2), \ldots, \pi(s)\} \subseteq V$ and define $\bar{S} = V \backslash S$. The validity of (5.9) is first established for tours for which every node in S is visited before any node in $\bar{S}$, i.e., $t_i \leq t_j$ for all $i \in S$, $j \in \bar{S}$. Let (x, t) be a tour satisfying this restriction and assume that the elements of S are visited in the order $\pi(1), \pi(2), \ldots, \pi(s)$. Thus $x_{\pi(i),\pi(i+1)} = 1$, $0 \leq i < s$, where $\pi(0) = 0$. Since we assumed that (x, t) satisfies (5.3), we have

$$t_{\pi(1),\pi(2)} = p_{\pi(0),\pi(1)}, \; t_{\pi(i),\pi(i+1)} = t_{\pi(i-1),\pi(i)} + p_{\pi(i-1),\pi(i)} = \sum_{j=0}^{i-1} p_{\pi(j),\pi(j+1)}, \; 2 \leq i < s.$$

Obviously, $t_{\pi(i),j} = x_{\pi(i),j} = 0$ for $j \neq \pi(i+1)$, $0 \leq i < s$, and $t_{\pi(s),j} = x_{\pi(s),j} = 0$ for $j \in S$. Hence,

$$\begin{aligned}\sum_{i,j \in S} p_{ij} t_{ij} &= \sum_{i=1}^{s-1} p_{\pi(i),\pi(i+1)} t_{\pi(i),\pi(i+1)} = \sum_{i=1}^{s-1} p_{\pi(i),\pi(i+1)} \cdot \left(\sum_{j=0}^{i-1} p_{\pi(j),\pi(j+1)}\right) \\ &= \tfrac{1}{2}\left(\sum_{i=0}^{s-1} p_{\pi(i),\pi(i+1)}\right)^2 - \tfrac{1}{2}\sum_{i=0}^{s-1} p_{\pi(i),\pi(i+1)}^2 = \frac{1}{2}\left(\sum_{i \in S_0, j \in S} p_{ij} x_{ij}\right)^2 - \frac{1}{2}\sum_{i \in S_0, j \in S} p_{ij}^2 x_{ij}.\end{aligned}$$

Thus (5.9) is satisfied at equality by all tours for which $t_i \leq t_j$ for all $i \in S$, $j \in \bar{S}$ holds.

To establish the validity of (5.9) for all tours, we introduce the notion of a *block* S', which is defined as a maximal set of nodes of S that are visited successively in the tour, i.e., if $|S'| > 1$, then $x(S') = |S'| - 1$, and $x(S \backslash S', S') = x(S', S \backslash S') = 0$. Obviously, if $x(S) = |S| - k^*$ for some $k^* \geq 1$, then the set S is partitioned into k^* blocks S_k of size $s_k \geq 1$, $k = 1, \ldots, k^*$. Denote the elements of S_k by $\pi_k(1), \ldots, \pi_k(s_k)$, and assume that the nodes in S are visited in the order $\pi_1(1), \ldots, \pi_1(s_1), \pi_2(1), \ldots, \pi_2(s_2), \ldots, \pi_{k^*}(1), \ldots, \pi_{k^*}(s_{k^*})$. Since (x, t) satisfies (5.3), we have

$$t_{\pi_k(i),\pi_k(i+1)} = t_{\pi_k(i-1),\pi_k(i)} + p_{\pi_k(i-1),\pi_k(i)}, \qquad 2 \leq i \leq s_k - 1, \; 1 \leq k \leq k^*.$$

Furthermore, if node $\pi_1(1)$ is visited first in the tour, then $x_{0,\pi_1(1)} = 1$ and $t_{\pi_1(1)} = p_{0,\pi_1(1)}$. Otherwise, $\sum_{i \in S} p_{0i} x_{0i} = 0$. Also notice that if i is the last node of a block, then the terms t_{ij} and x_{ij} in (5.9) are all equal to zero. Combining the above observations yields

$$t_{\pi_k(i),\pi_k(i+1)} \geq p_{0,\pi_1(1)} x_{0,\pi_1(1)} + \sum_{l=1}^{k-1} \sum_{j=1}^{s_l - 1} p_{\pi_l(j),\pi_l(j+1)} + \sum_{j=1}^{i-1} p_{\pi_k(j),\pi_k(j+1)} \tag{5.10}$$

for $1 \leq i \leq s_k - 1,\ 1 \leq k \leq k^*$, and

$$\begin{aligned}
\sum_{i,j\in S} p_{ij}t_{ij} &= \sum_{k=1}^{k^*}\sum_{i=1}^{s_k-1} p_{\pi_k(i),\pi_k(i+1)}t_{\pi_k(i),\pi_k(i+1)} \\
&\overset{(5.10)}{\geq} \sum_{k=1}^{k^*}\sum_{i=1}^{s_k-1} p_{\pi_k(i),\pi_k(i+1)}\cdot\left(p_{0,\pi_1(1)}x_{0,\pi_1(1)} + \sum_{l=1}^{k-1}\sum_{j=1}^{s_l-1} p_{\pi_l(j),\pi_l(j+1)} + \sum_{j=1}^{i-1} p_{\pi_k(j),\pi_k(j+1)}\right) \\
&= \frac{1}{2}\left(p_{0,\pi_1(1)}x_{0,\pi_1(1)} + \sum_{k=1}^{k^*}\sum_{i=1}^{s_k-1} p_{\pi_k(i),\pi_k(i+1)}\right)^2 - \frac{1}{2}\left(p^2_{0,\pi_1(1)}x_{0,\pi_1(1)} + \sum_{k=1}^{k^*}\sum_{i=1}^{s_k-1} p^2_{\pi_k(i),\pi_k(i+1)}\right) \\
&= \frac{1}{2}\left(\sum_{i\in S_0,\, j\in S} p_{ij}x_{ij}\right)^2 - \frac{1}{2}\sum_{i\in S_0,\, j\in S} p^2_{ij}x_{ij}. \qquad (5.11)
\end{aligned}$$

This concludes the proof of the validity of inequality (5.9) for all tours. □

If all travel times are strictly positive, then (5.9) is satisfied at equality by all tours (x, t) for which $x(0, S) = 1$, $x(S) = |S| - k^*$ for some $k^* \geq 1$, and $|S_k| = 1$ for $1 < k \leq k^*$. This can be seen as follows. If $p_{ij} > 0$ for all i, j, then $t_{\pi_k(1)} > t_{\pi_{k-1}(s_{k-1})}$ for $1 < k \leq k^*$ and $t_{\pi_1(1)} \geq \sum_i p_{0i}x_{0i}$. Hence, equality holds in (5.10), and, consequently, in (5.11) if $x(0, S) = 1$, $k = 1$, and $1 \leq i \leq s_1 - 1$. These restrictions are equivalent to the ones mentioned before.

Observe that when we set $p_{ij} = p_i$ in the above inequality, we do not get inequality (5.8), unless $S = V$. In order to obtain (5.8), we would need for every $i \in S$ the terms $p_{ij}t_{ij}$ and $p_{ij}x_{ij}$ for every $j \in V$. However, these terms appear in (5.9) only for $j \in S$.

As mentioned before, we can only deal with linear inequalities. In the sequel, we discuss a linearization of the right-hand side of (5.9) that yields valid linear inequalities for $\mathcal{X}$.

Let $S \subseteq V$ and define $\Gamma(S)$ to be the set of all values that $\sum_{i\in S_0, j\in S} p_{ij}x_{ij}$ can attain, thus, $\Gamma(S) = \{\sum_{i\in S_0, j\in S} p_{ij}x_{ij} : (x, t) \text{ is a tour}\}$. Note that the assumption that the travel times are integral implies that all elements of $\Gamma(S)$ are integral. Let γ_1 and γ_2, $\gamma_1 < \gamma_2$, be two consecutive elements of $\Gamma(S)$, i.e., every $\gamma \in \Gamma(S)$ satisfies $\gamma \leq \gamma_1$ or $\gamma \geq \gamma_2$. Then

$$\Big(\sum_{i\in S_0,\, j\in S} p_{ij}x_{ij} - \gamma_1\Big)\Big(\sum_{i\in S_0,\, j\in S} p_{ij}x_{ij} - \gamma_2\Big) \geq 0$$

and, hence,

$$\begin{aligned}
\left(\sum_{i\in S_0,\, j\in S} p_{ij}x_{ij}\right)^2 &= \Big(\sum_{i\in S_0,\, j\in S} p_{ij}x_{ij} - \gamma_1\Big)\Big(\sum_{i\in S_0,\, j\in S} p_{ij}x_{ij} - \gamma_2\Big) + (\gamma_1 + \gamma_2)\cdot\sum_{i\in S_0,\, j\in S} p_{ij}x_{ij} - \gamma_1\gamma_2 \\
&\geq (\gamma_1 + \gamma_2)\cdot\sum_{i\in S_0,\, j\in S} p_{ij}x_{ij} - \gamma_1\gamma_2
\end{aligned}$$

for every tour (x, t). Thus, substituting $(\gamma_1 + \gamma_2)\cdot\sum_{i\in S_0, j\in S} p_{ij}x_{ij} - \gamma_1\gamma_2$ for the quadratic term in the right-hand side of (5.9) yields a valid linear inequality of $\mathcal{X}$. This proves the following statement.

Proposition 5.2.2 *Let $S \subseteq V$, $S \neq \emptyset$. Then for every pair (γ_1, γ_2) of consecutive elements of $\Gamma(S)$*

$$\sum_{i,j \in S} p_{ij} t_{ij} \geq \frac{1}{2} \sum_{i \in S_0, j \in S} p_{ij}(\gamma_1 + \gamma_2 - p_{ij}) x_{ij} - \frac{1}{2}\gamma_1 \gamma_2 \tag{5.12}$$

is valid for X. □

In this way we obtain for every S a set of valid linear inequalities. Notice that these inequalities differ in the coefficients of the x-variables and the constant term, but *not* in the coefficients of the t-variables. In a separation algorithm for (5.12) we only have to check violation for at most one inequality yielded by S. For every triple $(\gamma_1, \gamma_2, \gamma_3)$ of consecutive elements of $\Gamma(S)$, where $\gamma_1 < \gamma_2 < \gamma_3$, and $y \in \mathbb{R}$, we have

$$(\gamma_2 + \gamma_3)y - \gamma_2\gamma_3 - [(\gamma_1 + \gamma_2)y - \gamma_1\gamma_2] = (\gamma_3 - \gamma_1)(y - \gamma_2) > 0 \text{ if and only if } y > \gamma_2.$$

Let $(\hat{x}, \hat{t})$ be a solution to the LP-relaxation of our formulation for DMP and let $S \subseteq V$. Define $\hat{y} = \sum_{i \in S_0, j \in S} p_{ij}\hat{x}_{ij}$. Then it is easily seen from the above observation that $(\gamma_1 + \gamma_2)\hat{y} - \gamma_1\gamma_2$ is maximal for $\gamma_1 = \max\{\gamma \in \Gamma(S) \mid \gamma \leq \hat{y}\}$ and $\gamma_2 = \min\{\gamma \in \Gamma(S) \mid \gamma > \hat{y}\}$. From this it follows that when for a given subset S some inequalities of the form (5.12) are violated by $(\hat{x}, \hat{t})$, the inequality with γ_1 and γ_2 as defined above is the most violated one. Thus, in a separation routine it suffices to consider at most one inequality for every subset S.

However, we expect the inequalities (5.12) to be rather weak in general. Let us therefore consider the tours that satisfy (5.12) at equality. Clearly, such a tour also satisfies (5.9) at equality. If we restrict ourselves to the case that all travel times are strictly positive, then it was observed before that for every tour (x, t) satisfying (5.9) at equality there is a subset $S' \subseteq S$ such that $x(S') = x(S) = |S'| - 1$ and $x(0, S') = 1$. In general, we cannot say much about the number of such tours for which $\sum_{i \in S_0, j \in S} p_{ij}x_{ij}$ has a particular value, but we expect it to be small. Notice that even if there would exist a linear inequality that is satisfied at equality by all tours that satisfy (5.9) at equality, then this would not define a facet. This follows from the observation that every tour (x, t) satisfying (5.9) at equality also satisfies $x(0, S) = 1$, hence $x_{0i} = 0$ for every $i \notin S$.

Furthermore, there will not be an efficient way in general to determine $\Gamma(S)$. In a separation routine that uses the ideas described previously, it will usually be too time-consuming to determine $\gamma_1 = \max\{\gamma \in \Gamma(S) \mid \gamma \leq \hat{y}\}$ and $\gamma_2 = \min\{\gamma \in \Gamma(S) \mid \gamma > \hat{y}\}$. Hence, checking violation will have to be restricted to the inequality with $\gamma_1 = \lfloor \hat{y} \rfloor$ and $\gamma_2 = \lfloor \hat{y} + 1 \rfloor$. From the assumption that all travel times are integral, it is easily seen that these inequalities are always valid for X. However, if neither γ nor $\gamma + 1$ is an element of $\Gamma(S)$, then there is no feasible solution that satisfies such an inequality at equality.

A second class of valid linear inequalities can be derived in a similar way. In this case we start from the following class of quadratic inequalities. For $S \subseteq V$, define $\bar{S} = V \backslash S$ and let T_S denote the set of arcs $(i, j) \in V \times V$ for which at most one of i and j is in $\bar{S}$, i.e., $T_S = (V \times V) \backslash (\bar{S} \times \bar{S})$.

Proposition 5.2.3 *Let $S \subseteq V$, $S \neq \emptyset$, and let T_S be as defined above. Then*

$$\sum_{(i,j)\in T_S} p_{ij}t_{ij} \;\geq\; \frac{1}{2}\left(\sum_{j\in V} p_{0j}x_{0j} + \sum_{(i,j)\in T_S} p_{ij}x_{ij}\right)^2 - \frac{1}{2}\sum_{j\in V} p_{0j}^2 x_{0j} - \frac{1}{2}\sum_{(i,j)\in T_S} p_{ij}^2 x_{ij} \quad (5.13)$$

is a valid (quadratic) inequality for $\mathcal{X}$.

PROOF. We first restrict ourselves to tours in which no two nodes in $\bar{S}$ are visited successively before the last node of S. Let (x, t) be such a tour. Define $\bar{S}' = \{j \in \bar{S} \mid x(S \cup \{0\}, j) = 1\}$. Furthermore, let $S' = S \cup \bar{S}' = \{\pi(1), \ldots, \pi(s')\}$ and suppose $x_{\pi(i),\pi(i+1)} = 1, 1 \leq j \leq s' - 1$. Note that $\pi(s') \in \bar{S}$ when $S' \neq V$. By definition of $\bar{S}'$, we have

$$\sum_{i,j\in\bar{S}'} p_{ij}t_{ij} = \sum_{i\in S,\, j\in\bar{S}\setminus\bar{S}'} p_{ij}t_{ij} = \sum_{i\in\bar{S}\setminus\bar{S}',\, j\in S} p_{ij}t_{ij} = 0. \quad (5.14)$$

Hence,

$$\begin{aligned}\sum_{i,j\in S'} p_{ij}t_{ij} &= \sum_{i,j\in S} p_{ij}t_{ij} + \sum_{i,j\in\bar{S}'} p_{ij}t_{ij} + \sum_{i\in S,\, j\in\bar{S}'} p_{ij}t_{ij} + \sum_{i\in\bar{S}',\, j\in S} p_{ij}t_{ij} \\ &= \sum_{i,j\in S} p_{ij}t_{ij} + \sum_{i\in S,\, j\in\bar{S}} p_{ij}t_{ij} + \sum_{i\in\bar{S},\, j\in S} p_{ij}t_{ij} = \sum_{(i,j)\in T_S} p_{ij}t_{ij}.\end{aligned}$$

Clearly, (5.14) and the above equality also hold with vaiables x_{ij} instead of t_{ij}. Furthermore, $\sum_{j\in S'} p_{0j}x_{0j} = \sum_{j\in V} p_{0j}x_{0j}$. Combining these results with the observation that (5.9) is satisfied at equality for the subset S', we get

$$\begin{aligned}\sum_{(i,j)\in T_S} p_{ij}t_{ij} &= \sum_{i,j\in S'} p_{ij}t_{ij} = \frac{1}{2}\left(\sum_{i\in S_0',\, j\in S'} p_{ij}x_{ij}\right)^2 - \frac{1}{2}\sum_{i\in S_0',\, j\in S'} p_{ij}^2 x_{ij} \\ &= \frac{1}{2}\left(\sum_{j\in S'} p_{0j}x_{0j} + \sum_{i,j\in S'} p_{ij}x_{ij}\right)^2 - \frac{1}{2}\sum_{j\in S'} p_{0j}^2 x_{0j} - \frac{1}{2}\sum_{i,j\in S'} p_{ij}^2 x_{ij} \\ &= \frac{1}{2}\left(\sum_{j\in V} p_{0j}x_{0j} + \sum_{i,j\in T_S} p_{ij}x_{ij}\right)^2 - \frac{1}{2}\sum_{j\in V} p_{0j}^2 x_{0j} - \frac{1}{2}\sum_{i,j\in T_S} p_{ij}^2 x_{ij}.\end{aligned}$$

The proof that (5.13) is also valid for all other tours is analogous to the corresponding proof in Proposition 5.2.1 and is therefore omitted. □

From the above proof it follows that (5.13) is satisfied at equality by all tours in which no two nodes in $\bar{S}$ are visited successively before the last node of S. Hence, unlike in the case of (5.9), there exist tours (x, t) such that $x(0, \bar{S}) = 1$ which satisfy (5.13) at equality.

To these quadratic inequalities we can apply a linearization that is similar to the one described for (5.9). Therefore, we will not discuss it in detail. Define $\Gamma(T_S) = \{\sum_{j\in V} p_{0j}x_{0j} + \sum_{(i,j)\in T_S} p_{ij}x_{ij} : (x, t) \in \mathcal{X}\}$.

Proposition 5.2.4 *Let $S \subseteq V$, $S \neq \emptyset$. Then for every pair (γ_1, γ_2) of consecutive elements of $\Gamma(T_S)$*

$$\sum_{(i,j)\in T_S} p_{ij}t_{ij} \geq \frac{1}{2}\sum_{j\in V} p_{0j}(\gamma_1+\gamma_2-p_{0j})x_{0j} + \frac{1}{2}\sum_{(i,j)\in T_S} p_{ij}(\gamma_1+\gamma_2-p_{ij})x_{ij} - \frac{1}{2}\gamma_1\gamma_2 \quad (5.15)$$

is valid for X. □

From the proof of Proposition 5.2.3 it follows that (5.13) is satisfied at equality by all tours in which no two nodes in $\bar{S}$ are visited successively before the last node of S. Hence, unlike in the case of (5.9), there exist tours (x, t) such that $x(0, \bar{S}) = 1$ which satisfy (5.13) at equality. In general, (5.13) will be satisfied at equality by more tours than (5.9). However, since $\Gamma(T_S)$ will usually be larger than $\Gamma(S)$, the linearization of (5.13) will yield a larger set of linear inequalities than the linearization of (5.9). We therefore do not expect inequalities (5.15) to be much stronger than inequalities (5.12).

5.2.2 Inequalities for the delivery man problem with time windows

Let us now consider tours (x, t) that satisfy constraints (5.1), (5.2), and (5.5)– (5.7), thus, $(x, t) \in X_{\text{tw}}$. Constraint (5.6) yields that the departure time at node i may be strictly larger than the arrival time. In this case we say that waiting time occurs at node i. Note that waiting affects the left-hand side of the inequalities derived in the previous subsection, but not the right-hand side. Hence, inequalities (5.12) and (5.15) are also valid for the problem with time windows.

In the remainder of this section we show how time windows can be incorporated in the inequalities derived previously. We first present two classes of valid quadratic inequalities for X_{tw} that involve earliest and latest departure times, respectively. Detailed proofs of their validity are omitted, since these are similar to the proof of Proposition 5.2.1. The inequalities will be linearized in the same way as the quadratic inequalities derived for the problem without time windows.

The first class of inequalities can be considered as a generalization of (5.9). Let S be a set of nodes and let $e_S = \min_{i\in S} e_i$. Introduce an extra node $0'$ that has to be left at time e_S and for which $p_{0'i} = p_{i0'} = 0$ for every $i \in V$. This extra node can be considered as a depot for the set S. Let (x, t) be a tour satisfying $x(S) = s - 1$, where $s = |S|$. Subtracting e_S from all departure times yields a vector for which (5.9) holds, if 0 is replaced by $0'$. Recalling that $p_{0'i} = 0$ for all $i \in V$, this shows the validity of the following inequality for tours satisfying $x(S) = s - 1$.

Proposition 5.2.5 *Let $S \subseteq V$, $S \neq \emptyset$, and define $e_S = \min_{i\in S} e_i$. Then*

$$\sum_{i,j\in S} p_{ij}t_{ij} \geq \frac{1}{2}\left(\sum_{i,j\in S} p_{ij}x_{ij}\right)^2 - \frac{1}{2}\sum_{i,j\in S} p_{ij}^2 x_{ij} + e_S \sum_{i,j\in S} p_{ij}x_{ij} \quad (5.16)$$

is a valid (quadratic) inequality for X_{tw}. □

The validity of the above inequality for all other tours can be easily proven, using similar arguments as in Proposition 5.2.1. From the above interpretation it immediately follows that (5.16) is satisfied at equality by all tours for which the departure time at the first node of S equals e_S, $x(S) = s - 1$, and no waiting time occurs at any node of S, except possibly at the one visited first. Thus, in this case, $t_{\pi(i),\pi(i+1)} = e_S + \sum_{j=1}^{i-1} p_{\pi(j),\pi(j+1)}$ holds for $i = 1, \ldots, s-1$, where it is assumed that $S = \{\pi(1), \ldots, \pi(s)\}$ and the nodes are visited in this order. Note that such a tour might not exist.

Until now, we discussed inequalities that involved a certain moment *from* which the tours were considered. For the second class of quadratic inequalities, tours will be considered *until* a moment l_S, defined as the maximum latest departure time of the nodes in a set S.

Proposition 5.2.6 *Let $S \subseteq V$, $S \neq \emptyset$, and define $l_S = \max_{i \in S} l_i$. Then*

$$\sum_{i,j \in S} p_{ij} t_{ij} \leq l_S \sum_{i,j \in S} p_{ij} x_{ij} - \frac{1}{2} \left(\sum_{i,j \in S} p_{ij} x_{ij} \right)^2 - \frac{1}{2} \sum_{i,j \in S} p_{ij}^2 x_{ij} \tag{5.17}$$

is a valid (quadratic) inequality for X_{tw}.

PROOF. We show that equality holds for all tours for which the departure time at the last node of S equals l_S, $x(S) = s - 1$, where $s = |S|$, and no waiting time occurs at any node in S, except possibly in the one visited first. As in the previous case, such a tour might not exist. Let $S = \{\pi(1), \pi(2), \ldots, \pi(s)\} \subseteq V$ and suppose (x, t) satisfies $x_{\pi(i),\pi(i+1)} = 1$, $1 \leq j \leq s - 1$. Furthermore, let $t_{\pi(s)} = l_S$ and

$$t_{\pi(i),\pi(i+1)} = t_{\pi(i+1)} - p_{\pi(i),\pi(i+1)} = l_S - \sum_{j=i}^{s-1} p_{\pi(j),\pi(j+1)} \tag{5.18}$$

for $i = 1, \ldots, s - 1$. Then

$$\begin{aligned} \sum_{i,j \in S} p_{ij} t_{ij} &= \sum_{i=1}^{s-1} p_{\pi(i),\pi(i+1)} t_{\pi(i),\pi(i+1)} = \sum_{i=1}^{s-1} p_{\pi(i),\pi(i+1)} \left(l_S - \sum_{j=i}^{s-1} p_{\pi(j),\pi(j+1)} \right) \\ &= l_S \sum_{i=1}^{s-1} p_{\pi(i),\pi(i+1)} - \frac{1}{2} \left(\sum_{i=1}^{s-1} p_{\pi(i),\pi(i+1)} \right)^2 - \frac{1}{2} \sum_{i=1}^{s-1} p_{\pi(i),\pi(i+1)}^2 \\ &= l_S \sum_{i,j \in S} p_{ij} x_{ij} - \frac{1}{2} \left(\sum_{i,j \in S} p_{ij} x_{ij} \right)^2 - \frac{1}{2} \sum_{i,j \in S} p_{ij}^2 x_{ij}. \end{aligned}$$

Notice that for a tour (x, t) such that $x_{\pi(i),\pi(i+1)} = 1$, $1 \leq i < s$, (5.18) is the latest possible departure time for every $i \in S$. Hence, if $t_{\pi(i-1),\pi(i)} + p_{\pi(i-1),\pi(i)} < t_{\pi(i),\pi(i+1)}$ for some $i, 2 < i < s$, then the left-hand side of (5.17) is *less* than when equality holds. As the right-hand side is the same in both cases, this shows the validity of (5.17) for all tours (x, t) with

$x(S) = s - 1$. For tours (x, t) satisfying $x(S) < s - 1$ validity can be proven in a similar way as in Proposition 5.2.1. □

To (5.16) and (5.17) almost the same linearization as the one described for (5.9) can be applied. Define $\Gamma(S) = \{\sum_{i,j \in S} p_{ij} x_{ij} : (x, t) \in \mathcal{X}_{tw}\}$.

Proposition 5.2.7 *Let $S \subseteq V$, $S \neq \emptyset$, and let e_S and l_S be as defined before. Then for every pair (γ_1, γ_2) of consecutive elements of $\Gamma(S)$*

$$\sum_{i,j \in S} p_{ij} t_{ij} \geq \frac{1}{2} \sum_{i,j \in S} p_{ij}(\gamma_1 + \gamma_2 - p_{ij}) x_{ij} - e_S \sum_{i,j \in S} p_{ij} x_{ij} - \frac{1}{2} \gamma_1 \gamma_2 \tag{5.19}$$

and

$$\sum_{i,j \in S} p_{ij} t_{ij} \leq l_S \sum_{i,j \in S} p_{ij} x_{ij} - \frac{1}{2} \sum_{i,j \in S} p_{ij}(\gamma_1 + \gamma_2 + p_{ij}) x_{ij} + \frac{1}{2} \gamma_1 \gamma_2 \tag{5.20}$$

are valid for $\mathcal{X}_{tw}$. □

For the quadratic inequalities (5.16) and (5.17) we already noticed that the number of tours satisfying the inequality at equality will be small in general, hence, this will certainly hold for (5.19) and (5.20).

5.3 Computational experiments

The main purpose of our computational experiments is to study the quality of the lower bounds for the DMP and DMPTW given by the LP-relaxation and the improvement of these lower bounds by adding cuts from some specific classes. We compare these bounds for the three models discussed in Section 5.1, i.e., the model introduced in Section 5.1 with variables x_{ij} and t_{ij} (model 1), the model proposed in [26] with variables x_{ij}, t_{ij}, y_{ij}, and u_{ij} (model 2), and the model with variables x_{ij} and t_i (model 3). The first subsection briefly describes the steps of our cutting plane procedure. Computational results are reported in Subsection 5.3.2. All experiments have been performed using MINTO (cf. Section 3.3).

5.3.1 Implementation issues

Before solving the initial LP one usually tries to reduce the size of the problem or improve the formulation by preprocessing techniques such as fixing variables and improving bounds. We restrict ourselves to fixing variables x_{ij} in the presence of time windows. Since for all our instances the travel times p_{ij} will satisfy the triangle inequality, variable fixing can be done in the following way: if $e_i + p_{ij} > l_j$ ($e_i > \min_j l_j$, $l_i < \max_j e_j$), then x_{ij} (x_{0i}, x_{i0}) is set to zero. After a variable has been fixed, it is eliminated from the formulation. Observe that $x_{ij} = 0$ implies $t_{ij} = 0$ (model 1, 2) and $y_{ij} = 0$, $u_{ij} = 0$ (model 2). Therefore, preprocessing may considerably reduce the large number of variables in our models (e.g., $2n^2 + n$ in model 1), especially when the time windows are tight.

It is a trivial observation that all inequalities derived for the TSP, such as subtour elimination constraints (SECs), 2-matching constraints, comb inequalities, etc. (cf. Grötschel and Padberg [17]), are also valid for the three models considered here and, hence, can be used as cutting planes to improve the lower bounds obtained from the LP-relaxations of these formulations. We will restrict ourselves to the addition of SECs, which have the following form:

$$x(S) \leq |S| - 1, \; S \subset \{0, \ldots, n\}, \; 2 \leq |S| \leq n.$$

Our cutting plane procedure works as follows. First, the exact separation algorithm described by Padberg and Rinaldi [31] is applied to check whether the LP-solution satisfies all SECs. Violated SECs found in this way are added to the formulation and the LP is solved again. This step is repeated until the LP-solution satisfies all SECs.

Next, we check whether violated inequalities of the form (5.12) or (5.15) can be identified. Notice that this step can only be applied to the formulations with variables t_{ij}, i.e., to model 1 and 2. Our (not very sophisticated) separation algorithms are inspired by the $O(n\log n)$ exact algorithm for inequalities (5.8), i.e., $\sum_{i\in S} p_i t_i \geq \sum_{i,j\in S, i<j} p_i p_j$ for $S \subseteq V$. Given an LP-solution $\hat{t}$, the separation problem for these inequalities amounts to checking whether (5.8) is violated for $S = \{\pi(1), \ldots, \pi(i)\}, 1 \leq i \leq n$, where the permutation $\pi : V \to V$ satisfies $\hat{t}_{\pi(i)} \leq \hat{t}_{\pi(i+1)}$ (cf. Queyranne [34]).

Let us now give an outline of the separation algorithm for inequalities (5.12). Let $(\hat{x}, \hat{t})$ be the LP-solution and let the permutation $\pi : V \to V$ be such that $\hat{t}_{\pi(i)} \leq \hat{t}_{\pi(i+1)}$. We check whether $(\hat{x}, \hat{t})$ violates (5.12) for $S = \{\pi(1), \ldots, \pi(i)\}, 1 \leq i \leq n$, $\gamma_1 = \lfloor \hat{y} \rfloor$ and $\gamma_2 = \lfloor \hat{y} + 1 \rfloor$, where $\hat{y} = \sum_{i\in S_0, j\in S} p_{ij}\hat{x}_{ij}$ (cf. Subsection 5.2.1). This procedure is repeated a fixed number of times with a permutation π that is obtained from the permutation $\bar{\pi}$ in the previous iteration by putting $\bar{\pi}(i) = \pi(i+1)$ and $\bar{\pi}(i+1) = \pi(i)$ for $i \in V'$, where V' is a randomly chosen subset of V of size at most $|V|/2$.

A similar separation heuristic is used to identify violated inequalities (5.15). The heuristics for (5.19) and (5.20) differ slightly from the one described above, but we will not discuss them since we never found violated inequalities of these types. This is possibly due to the fact that these separation algorithms were only called when neither violated SECs nor violated inequalities of type (5.12) or (5.15) were identified.

5.3.2 Results

We report results for twelve sets of five randomly generated instances with $n = 15$. These sets were constructed from two sets of five matrices (p_{ij}), which are denoted by *grid* and *sched*, respectively. For *grid* the 15 nodes and the depot are randomly generated lattice points of a 20×20 grid. The travel time p_{ij} equals the Manhattan distance between i and j, i.e., $p_{ij} = |a_i - a_j| + |b_i - b_j|$, where (a_i, b_i) denotes the pair of coordinates of node i. The second set of matrices, *sched*, results from the interpretation of the DMP as a machine scheduling problem with sequence-dependent processing times. Here p_{ij} can be considered as the sum of a fixed processing time p_i for job i and a changeover time s_{ij}. The (integer) processing times p_i, $i \neq 0$, and changeover times s_{ij}, $j \neq 0$, are randomly chosen from the interval $[5, 15]$ and $[0, 4]$, respectively. Furthermore, $p_0 = 0$ and $s_{i0} = 0$. Observe that only in *grid* the matrices are symmetric. However, in both sets the travel times p_{ij} satisfy the triangle inequality. Furthermore, for fixed i the value of p_{ij} in the first set is in the interval $[1, 38]$, whereas in the second set this value only ranges from p_i to $p_i + 4$.

Each of these two sets of matrices gave rise to six sets of five instances by the addition of time windows. These sets are denoted by *grid_k* and *sched_k*, where $k \in \{0, \ldots, 5\}$.

- For $k = 0$ the instances are instances for the ordinary DMP, i.e., no time windows are involved.
- For $k = 1$ the nodes are partitioned into three clusters of size five. Every node in cluster c, $1 \leq c \leq 3$, has a time window $[(c-1)W_{inst}, cW_{inst}]$, where $W_{grid} = 80$ and $W_{sched} = 100$.
- For $k = 2$ the nodes are partitioned into five clusters of size three. Every node in cluster c, $1 \leq c \leq 5$, has a time window $[(c-1)W_{inst}, cW_{inst}]$, where $W_{grid} = 50$ and $W_{sched} = 70$.
- The instances in *grid_k* and *sched_k*, $k \in \{3, 4, 5\}$, have the following form. First, a random solution to the DMP is generated. The departure time at node i in this solution is denoted by t_i^*. Then for each node an earliest departure time e_i is randomly generated such that t_i^* is in the interval $[e_i, e_i + W_k]$, where $W_3 = 60$, $W_4 = 40$, and $W_5 = 20$.

All instances were tested with respect to the objective function $\sum_{i=1}^{n} t_i$.

Table 5.1 shows integrality gaps with respect to a lower bound z and the best known upper bound $\bar{z}$ to the value of the optimal solution to the DMPTW, where the integrality gap is defined as $100\% \times (\bar{z} - z)/\bar{z}$. The value $\bar{z}$ was found by a branch-and-cut algorithm in which in every node a feasible solution to the DMPTW was constructed from the LP-solution (max. 2000 nodes). The last column of Table 5.1 lists the number of problems (out of five) for which the upper bound $\bar{z}$ was proven to be optimal.

For all three models column $\bar{g}_0$ shows the average gap over five instances, where z is the value of the LP-relaxation of the model. The average gap after SECs have been added to the formulation is reported in column $\bar{g}_1$. Finally, for model 1 and 2 column $\bar{g}_2$ shows the average gap after both SECs and inequalities (5.12) and (5.15) have been added.

Since every solution to the LP-relaxation of model 2 yields a feasible solution to the LP-relaxation of model 1, the lower bounds obtained from the latter cannot exceed the lower bounds obtained from the first. Table 5.1 shows that the bounds obtained from model 2 can be considerably better than the bounds obtained from model 1. The LP-relaxation of the third formulation yields bounds that are inferior to the corresponding bounds obtained from model

	model 1			model 2			model 3		
inst	$\bar{g}_0$	$\bar{g}_1$	$\bar{g}_2$	$\bar{g}_0$	$\bar{g}_1$	$\bar{g}_2$	$\bar{g}_0$	$\bar{g}_1$	opt
grid_0	56.5	53.5	33.2	56.5	53.5	30.4	64.9	64.9	0
grid_1	25.6	23.3	19.6	23.1	21.0	17.5	35.1	35.1	1
grid_2	17.2	15.0	14.2	12.9	11.2	10.1	24.3	24.3	1
grid_3	11.1	9.5	7.2	8.7	7.4	4.5	16.9	16.9	4
grid_4	6.1	5.4	5.3	3.3	2.8	2.7	8.4	8.4	4
grid_5	6.3	2.8	2.3	4.8	2.5	1.9	10.2	9.9	5
sched_0	81.0	81.0	5.9	81.0	81.0	5.0	97.1	97.1	0
sched_1	27.3	26.3	7.0	26.1	25.0	6.1	39.2	39.2	0
sched_2	11.9	11.0	5.8	10.4	9.6	3.6	21.5	21.5	2
sched_3	16.6	16.0	4.8	16.3	15.6	3.5	32.0	32.0	2
sched_4	4.4	4.2	3.4	3.2	3.0	1.8	11.0	11.0	5
sched_5	1.7	1.5	1.2	1.2	1.2	0.7	12.8	12.7	5

Table 5.1: Quality of the lower bounds

1 and which are rarely improved by the addition of SECs. Also for model 1 and 2 we conclude that the reduction of the gap by the addition of SECs is rather limited. This especially holds for the second set of problem instances. However, the addition of inequalities (5.12) and (5.15) substantially improves the lower bounds for these instances. For the instances of the sets *grid_k*, the gaps are also reduced by the addition of inequalities (5.12) and (5.15), but clearly not as much as for the instances of the sets *sched_k*.

For model 1 the instances of *grid_k* and *sched_k*, $k \in \{1, 2, 4\}$, were also tested with respect to the ordinary TSP objective, i.e., minimizing $\sum_{i,j} p_{ij}x_{ij}$. Table 5.2 shows the results for both objective functions. As in the previous table, $\bar{g}_0$ shows the average gap obtained from the LP-relaxation of model 1, $\bar{g}_1$ gives the average gap after SECs have been added, and $\bar{g}_2$ shows the average gap after the addition of SECs and inequalities (5.12) and (5.15). As far as the last column is concerned, we mention that all instances of the TSPTW were solved to optimality by N. Ascheuer (personal communication).

We also compare the effect of preprocessing as described in the previous section for the two problems. Column $\hat{g}_0$ reports the average gap for the value of the LP-relaxation of model 1 without preprocessing. We observe that variable fixing hardly improves the value of the LP-relaxation with respect to the DMP objective. Only for the instances with the smallest time windows ($k = 5$, not reported in Table 5.2 we found that preprocessing increased the value of the LP-relaxation by more than 0.5%. With respect to the TSP objective, however, preprocessing actually leads to an improved formulation for all instances.

Furthermore, we conclude from Table 5.2 that the addition of SECs may be much more ef-

obj	inst	$\hat{g}_0$	$\bar{g}_0$	$\bar{g}_1$	$\bar{g}_2$	opt
$\sum_i t_i$	*grid_1*	25.6	25.6	23.3	19.6	1
	grid_2	17.5	17.2	15.0	14.2	1
	grid_4	6.4	6.1	5.4	5.3	4
$\sum_{i,j} p_{ij}x_{ij}$	*grid_1*	32.6	27.3	12.9	12.9	5
	grid_2	36.3	27.8	18.1	18.1	5
	grid_4	41.2	22.8	11.6	11.6	5
$\sum_i t_i$	*sched_1*	27.3	27.3	26.3	7.0	0
	sched_2	11.9	11.9	11.0	5.8	2
	sched_4	4.5	4.4	4.2	3.4	5
$\sum_{i,j} p_{ij}x_{ij}$	*sched_1*	2.1	1.6	1.4	1.4	5
	sched_2	3.2	2.4	2.3	2.3	5
	sched_4	4.7	3.0	2.8	2.8	5

Table 5.2: Comparison of DMPTW and TSPTW

fective for the TSPTW than for the DMPTW. However, this does not hold for the inequalities derived in Section 5.2. Although violated inequalities of this kind were identified for the instances of the TSPTW, the value of the objective function was not improved by adding these inequalities. Apparently, the addition of violated inequalities of the form (5.12) and (5.15) hardly influences the value of the x-variables, but only changes the value of the t-variables.

5.3.3 Further remarks

The aim of our computational experiments was to get some idea of the quality of the lower bounds obtained from the LP-relaxation of the proposed formulation and the effectiveness of the new inequalities in reducing the gap. The results show that the addition of the inequalities derived in Section 5.2 can substantially improve the lower bound. Nevertheless, further study is necessary to reach more solid conclusions about the possibility to solve the DMP and DMPTW efficiently by means of polyhedral methods and the value of our model in this approach.

The performance of the cutting plane procedure will undoubtedly be improved when more extensive preprocessing is applied and other classes of inequalities are incorporated.

In the preprocessing phase we only applied a simple rule to eliminate arcs. As mentioned before, the formulation can also be improved by reducing the time windows. This may also lead to a strengthening of inequalities (5.19) and (5.20). Time windows can be tightened by applying the rules described by Desrosiers et al. [9].

Furthermore, the structure of the time windows may yield precedences between nodes.

If node i has to be visited before node j, then the following constraint can be added to our formulation of the DMPTW:

$$\sum_{k \neq i,j} t_{ik} + \sum_{k \neq i,j} p_{ik} x_{ik} \leq \sum_{k \neq i,j} t_{kj}. \tag{5.21}$$

Balas et al. [3] discuss a strengthening of the SECs that takes precedences into account. These inequalities can also be added in order to obtain better lower bounds.

Because of the time windows some paths will not occur as a subpath in any feasible tour. For example, if

$$e_{\pi(1)} + \sum_{i=1}^{k-1} p_{\pi(i),\pi(i+1)} > l_{\pi(k)}$$

for some subset $\{\pi(1), \dots, \pi(k)\}$ of V, then the path $(\pi(1), \dots, \pi(k))$ will not be contained in any feasible tour. In this case

$$\sum_{i=1}^{k-1} x_{\pi(i),\pi(i+1)} \leq k-2$$

is a valid inequality for all three models considered in our experiments. For obvious reasons, this is called an infeasible path inequality. Ascheuer [2] discusses several generalizations of these inequalities. His branch-and-cut code for the TSPTW is based on a formulation that only involves x-variables. The time constraints are modeled implicitly by the infeasible path constraints. Ascheuer concludes from his computational experiments that this model is superior to the one with variables x_{ij} and t_i (model 3 in our experiments). However, this formulation can only be used with respect to the TSP objective.

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Samenvatting

Dit proefschrift is gebaseerd op onderzoek op het gebied van de combinatorische optimalisering. Combinatorische optimaliseringsproblemen kunnen gekarakteriseerd worden door een eindige, maar mogelijk zeer grote, verzameling oplossingen waaruit de oplossing met de laagste kosten moet worden gevonden. Het bekendste voorbeeld van een dergelijk probleem is ongetwijfeld het handelsreizigersprobleem, waarbij de kortste route langs een gegeven verzameling steden moet worden gevonden. Veel van de in de combinatorische optimalisering bestudeerde problemen zijn geïnspireerd door praktische problemen uit uiteenlopende gebieden als produktieplanning, telecommunicatie, transport en VLSI-ontwerp.

Een groot deel van de combinatorische optimaliseringsproblemen is NP-lastig, hetgeen betekent dat het probleem hoogstwaarschijnlijk niet efficiënt, d.w.z. in polynomiale tijd, op te lossen is. Hoewel er daarom meestal gebruik wordt gemaakt van algoritmen die binnen een redelijke tijd een oplossing van acceptabele kwaliteit geven, hebben methoden die gegarandeerd de optimale oplossing vinden inmiddels ook hun waarde bewezen. Voor een groot aantal problemen zijn goede resultaten geboekt met optimaliseringsalgoritmen waarbij gebruik wordt gemaakt van polyhedrale technieken. Hierbij wordt een probleem geformuleerd als een (gemengd) geheeltallig lineair programmeringsprobleem en vervolgens opgelost met een branch-and-cut algoritme. Dit is een op lineaire programmering (LP) gebaseerd branch-and-bound algoritme dat is uitgebreid met een zogenaamde snedemethode. Hiermee worden toegelaten ongelijkheden, d.w.z. ongelijkheden waaraan alle oplossingen van het oorspronkelijke probleem voldoen, aan het LP-probleem toegevoegd. Het doel hiervan is het verkrijgen van een betere LP-formulering die hopelijk zal leiden tot een kleinere zoekboom.

De bestudering van de polyhedrale struktuur van een probleem leidt meestal tot één of meer klassen toegelaten ongelijkheden. We zijn hierbij met name geïnteresseerd in facet-definiërende ongelijkheden. Dit zijn de ongelijkheden die noodzakelijk zijn in een volledige lineaire beschrijving van de verzameling toegelaten oplossingen van het probleem. Het toevoegen van alle gevonden ongelijkheden aan de initiële formulering levert meestal een LP-probleem op dat te groot is om rechtstreeks te kunnen worden opgelost. We beginnen daarom met een LP-probleem met een beperkt aantal ongelijkheden en lossen dit op. Indien de LP-oplossing ook een oplossing voor het oorspronkelijke probleem is, dan is deze oplossing optimaal en is het oorspronkelijke probleem dus opgelost. Zo niet, dan wordt geprobeerd één of meer ongelijkheden te vinden waaraan de LP-oplossing niet voldoet. Als dit lukt, dan worden de gevonden ongelijkheden aan het LP-probleem toegevoegd en wordt dit probleem opnieuw opgelost. Omdat de oude LP-oplossing niet meer aan de huidige LP-formulering voldoet, wordt een nieuwe oplossing gevonden die minstens zo goed is als de vorige. De hierboven beschreven stappen kunnen nu worden herhaald. Als er geen geschonden ongelijkheden gevonden worden, dan kan men branch-and-bound toepassen om een optimale oplossing te vinden. Het vinden van geschonden ongelijkheden wordt separatie genoemd. Efficiënte separatiemethoden zijn van groot belang voor de kwaliteit van een branch-and-cut algoritme. Daarnaast dient bij de ontwikkeling van een dergelijke methode ook aandacht te worden besteed aan diverse andere aspecten, zoals het vertakkingsschema en de zoekstrategie.

Een groot deel van dit proefschrift is gewijd aan de ontwikkeling van een branch-and-cut algoritme voor het discrete serie-grootte probleem (DSP). In dit produktieplanningsprobleem beschouwen we een machine waarmee verschillende goederen geproduceerd kunnen worden. De planningshorizon bestaat uit een aantal korte perioden waarin ten hoogste één goed geproduceerd kan worden. Karakteristiek voor het discrete serie-grootte probleem is de 'alles-of-niets' produktie: als er geproduceerd wordt in een periode, dan wordt de machinecapaciteit volledig benut. Gegeven is in welke perioden en in welke hoeveelheden de goederen afgeleverd moeten worden. Er dient nu een produktieschema ontworpen te worden waarmee aan deze vraag voldaan wordt en waarvan de kosten zo laag mogelijk zijn. De belangrijkste kosten zijn de voorraadkosten van de goederen die niet direct afgeleverd kunnen worden en de opstartkosten. Dit zijn de kosten die in rekening worden gebracht wanneer er een nieuwe produktieserie wordt gestart, bijvoorbeeld voor het instellen van de machine voor het te produceren goed. We nemen aan dat de opstartkosten volgorde-onafhankelijk zijn, hetgeen betekent dat de kosten alleen afhangen van het goed waarvoor wordt opgestart.

Het vinden van een optimaal produktieschema komt in feite neer op het vinden van een optimaal schema voor ieder goed afzonderlijk met daarbij als extra eis dat de verschillende schema's elkaar niet mogen overlappen. In hoofdstuk 2 bestuderen we daarom de polyhedrale struktuur van het discrete serie-grootte waarin slechts één goed wordt geproduceerd (1-DSP). Dit probleem is weliswaar polynomiaal oplosbaar, maar de verkregen ongelijkheden kunnen gebruikt worden in een snedemethode voor het probleem met meerdere goederen, dat NP-lastig is. Er worden verschillende klassen facet-definiërende ongelijkheden afgeleid. Voor twee klassen worden efficiënte separatie-algoritmen ontwikkeld. Verder wordt aangetoond dat het toevoegen van ongelijkheden uit de doorsnede van deze klassen aan de initiële LP-formulering volstaat om het 1-DSP op te lossen wanneer de kosten van het Wagner-Whitin type zijn. Dit is een veel gebruikte kostenstruktuur in serie-grootte problemen waarvoor het niet optimaal is om een nieuwe produktieserie te starten wanneer er nog uit voorraad geleverd kan worden.

In hoofdstuk 3 worden eerst enkele resultaten met betrekking tot de complexiteit van het DSP gegeven. Vervolgens worden toegelaten ongelijkheden voor het probleem met meerdere goederen afgeleid. Tenslotte wordt de ontwikkeling van een branch-and-cut algoritme besproken en wordt verslag gedaan van uitgebreide rekenexperimenten.

In hoofdstuk 4 wordt het DSP met volgorde-onafhankelijke opstarttijden bestudeerd. Bij dit probleem legt het opstarten van de machine de produktie gedurende een aantal perioden stil. We modelleren het probleem met opstarttijden zodanig dat toegelaten ongelijkheden voor het gewone DSP eenvoudig kunnen worden omgezet in toegelaten ongelijkheden voor het probleem met opstarttijden. Verder wordt voor het 1-DSP met opstarttijden een multicommodity flow formulering besproken. Hiervoor wordt een volledige lineaire beschrijving gegeven.

Het laatste hoofdstuk staat enigszins los van de voorgaande hoofdstukken. Het geeft een eerste aanzet tot een polyhedrale aanpak van het delivery man probleem. Dit is een variant van het handelsreizigersprobleem waarbij de totale wachttijd van de klanten geminimaliseerd moet worden. Er wordt een formulering gegeven waarmee ook tijdvensters eenvoudig gemodelleerd kunnen worden. Zowel voor het gewone delivery man probleem als voor het probleem met tijdvensters worden extra klassen toegelaten ongelijkheden afgeleid en rekenresultaten gepresenteerd.

Acknowledgements

At the end of this thesis, I want to express my gratitude to those who, in one way or another, have contributed to its realization.

First of all, I thank Stan van Hoesel for his involvement in my work over the past few years and for his ever (too) optimistic view on the progress I made. I want to thank Stan, Jan Karel Lenstra, Yves Pochet, Martin Savelsbergh, and Karen Aardal for their efforts made at proofreading and reading the proofs. I am especially grateful to Yves for giving me more insight in my own work.

I owe thanks to Miguel Constantino, not only for his inequalities, but also for inciting me to study the complexity of DLSP. With respect to that subject also Gerhard Woeginger is thanked for his help. Furthermore, I acknowledge Marc Salomon for providing me with the Fleischmann instances and Norbert Ascheuer for solving my TSPTW instances.

Last but not least, my heartfelt thanks to Marc for never failing to remind me that one day I would be able to write down these words.

Cleola

En tot slot

Ik wil niets
meer bewijzen,

vooral
mezelf niet,

ik wil reizen
door de tijd,

een hoed op
vol gaten,

licht
in het hoofd.

Hans Andreus, *Holte van licht*

Stellingen

behorende bij het proefschrift

A Polyhedral Approach to the Discrete Lot-sizing and Scheduling Problem

van

CLEOLA VAN EIJL

I

Beschouw de formulering voor het 1-item DLSP uit hoofdstuk 2 van dit proefschrift. Zij $\alpha x + \beta y \geq \gamma$ een facet-definiërende reguliere-blokongelijkheid met $p_B = 1$ voor ieder blok B en met $t^* = \max\{t : \alpha_t + \beta_t > 0\}$ zodanig dat $d_{t^*+1,T} \geq 1$ (zie paragraaf 2.2.3 voor definities). Definieer $\hat{t} = \max\{t : \alpha_t = 1\}$ en laat s_j de $j$de vraagperiode na t^* zijn. Zij verder $(\bar{x}, \bar{y})$ de met het algoritme FILL_BLOCKS bepaalde oplossing. Als $\bar{x}_{l_B} = 1$ voor ieder blok B, als $\bar{x}_t = 0$ voor ten minste één periode met $\alpha_t = 1$ en als $s_1 \neq \hat{t} + 1$, dan definieert de ongelijkheid

$$\alpha x + \beta y + \sum_{j \in J} (x_{\hat{t}+j} + y_{\hat{t}+j+1,s_j}) \geq \gamma + |J|$$

voor iedere $J \subseteq \{1, \ldots, d_{t^*+1,T}\}$ een facet.

II

In [1] en [2] worden gemengd geheeltallige programmeringsformuleringen voor het handelsreizigersprobleem met tijdvensters voorgesteld waarbinnen verschillende typen doelstellingsfunkties gehanteerd kunnen worden. De effectiviteit van sommige typen sneden blijkt sterk af te hangen van het type doelstellingsfunktie. Bij de ontwikkeling van goede snedemethoden gebaseerd op een dergelijke formulering zal dan ook informatie over de geldende doelstellingsfunktie benut moeten worden.

[1] A. Langevin et al., A two-commodity flow formulation for the traveling salesman and the makespan problems with time windows, *Networks* 23 (1993), 631-640.

[2] Hoofdstuk 5 van dit proefschrift.

III

Zij A de verbindingsmatrix van de Higman-Sims graaf, de unieke sterk reguliere graaf met parameters $(100, 22, 0, 6)$. Zij I de 100×100-eenheidsmatrix en J de 100×100-matrix waarvan alle elementen gelijk aan 1 zijn. Voor iedere $b \in \mathbb{Z}$ is de rang van $A + bJ$ over $\mathbb{F}_2$ gelijk aan 22 en de rang van $A - 2I + bJ$ over $\mathbb{F}_5$ gelijk aan 23.

[3] A.E. Brouwer en C.A. van Eijl, On the p-rank of the adjacency matrices of strongly regular graphs, *Journal of Algebraic Combinatorics* 1 (1992), 329-346.

IV

Beschouw een voetbalpool waarin voor n wedstrijden moet worden voorspeld wat de uitslag wordt: winst voor de thuisspelende ploeg, winst voor de uitspelende ploeg of gelijkspel. Wanneer men niet meer dan één uitslag verkeerd voorspeld heeft, ontvangt men een prijs. Als $M(n)$ het minimum aantal voorspellingen is dat men moet inleveren om zeker te zijn van een prijs, dan geldt: $M(7) \geq 153$, $M(8) \geq 399$, $M(9) \geq 1062$ en $M(11) \geq 7826$.

[4] M. Struik, *Covering radius problems*, Ongepubliceerd manuscript.

V

Om bij de lezer geen ijdele hoop op de bruikbaarheid van een referentie te wekken, dient men de mededeling 'personal communication' niet in de referentielijst maar in de tekst zelf op te nemen.

VI

Volgens [5] komt het woord 'logistiek' weliswaar van het Griekse 'logistikos' (bedreven in het rekenen) maar heeft het in zijn huidige betekenis weinig van doen met rekenkunst. Gezien het feit dat de 'logistika' vooral de praktische rekenkunst betrof ([6]) en gezien het belang van kwantitatieve modellen en methoden in de moderne logistiek, is deze bewering onjuist.

[5] *The new encyclopaedia Brittanica*, Volume 29, 15th edition, 1994.
[6] D.J. Struik, *Geschiedenis van de wiskunde*, Het Spectrum, 1990.

VII

Een optimale permutatie van de getallen 0 tot en met 9 met betrekking tot de kostenfunktie die Paul Clark in zijn theatervoorstelling 'Bohemian from 9 to 5' introduceerde, is

0 2 3 7 1 4 9 5 6 8.

De waarde van deze permutatie bedraagt 16.

VIII

Teveel openheid wekt afgrijzen op. Voorbeelden hiervan kunnen in ruime mate op de televisie gevonden worden en op de website *http://www.cam-orl.co.uk/cgi-bin/ab.*

IX

Telefoontoestellen op perrons zouden zodanig moeten worden afgesteld dat gesprekken niet langer dan 45 seconden kunnen duren.

X

De meeste elektrische apparaten worden aangeschaft om energie te besparen.