

## Multiobjective control : an overview

***Citation for published version (APA):***

Vroemen, B. G., & Jager, de, A. G. (1997). Multiobjective control : an overview. In *Proceedings of the 36th IEEE Conference on Decision and Control (San Diego CA, USA, December 10-12, 1997)* (pp. 440-445). Institute of Electrical and Electronics Engineers. <https://doi.org/10.1109/CDC.1997.650664>

***DOI:***

[10.1109/CDC.1997.650664](https://doi.org/10.1109/CDC.1997.650664)

***Document status and date:***

Published: 01/01/1997

***Document Version:***

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

***Please check the document version of this publication:***

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

[Link to publication](#)

***General rights***

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license above, please follow below link for the End User Agreement:

[www.tue.nl/taverne](http://www.tue.nl/taverne)

***Take down policy***

If you believe that this document breaches copyright please contact us at:

[openaccess@tue.nl](mailto:openaccess@tue.nl)

providing details and we will investigate your claim.

## Multiobjective control: An overview

Bas Vroemen

Bram de Jager<sup>1</sup>

Faculty of Mechanical Engineering, Eindhoven University of Technology  
P.O. Box 513, 5600 MB Eindhoven, The Netherlands

### Abstract

An overview of a number of approaches to the multiobjective control problem is given. In practice, this problem usually boils down to a mixed-norm optimization, where traditionally the norms of interest are  $\mathcal{H}_2$ ,  $\mathcal{H}_\infty$  and  $\ell_1$ . To capture different, often conflicting, design specifications a single-norm form is usually not enough and therefore a mixed-norm formalism combining these norms would be of considerable interest. Although it would be nice to have all three norms present, most approaches focus on the two-norm problem. Frequently encountered is the  $\mathcal{H}_2/\mathcal{H}_\infty$  mixed-norm optimization problem, but combinations of  $\ell_1$  and the other two norms are starting to get more attention. It will be seen that the solution to the mixed-norm optimization problem has not yet reached a final shape, since most methods still exhibit problems, like not being able to find a solution if performance specifications are tight, or generating high-order or too conservative controllers, etc.  
**Keywords:**  $\mathcal{H}_2/\mathcal{H}_\infty/\ell_1$ , multiobjective control, robust control, LMI, linear systems.

### 1. Introduction

During the past decades, attention has been paid to many single objective control problems. Several important controller synthesis problems have been formulated as optimization problems. Notably, LQG or  $\mathcal{H}_2$ ,  $\mathcal{H}_\infty$  and  $\ell_1$ <sup>2</sup> control theory have provided us with basic synthesis tools. The underlying premise behind these theories is that all the design objectives can be translated into minimizing a suitably weighted norm of a closed-loop transfer function matrix.

The LQG approach proved particularly suited to meet performance specifications while guaranteeing closed-loop stability in the presence of disturbances. Despite of this, LQG control was shown to possess no guaranteed robustness margins if applied in conjunction with an observer or Kalman filter. This resulted in the development of  $\mathcal{H}_\infty$  control theory which could deal with the problem of robust stability: obtaining closed-loop stability in the presence of system uncertainty. For systems with structured uncertainty the  $\mathcal{H}_\infty$  framework can be refined to  $\mu$ -synthesis which has been successfully applied to a number of hard practical control problems (see, e.g., [Skogestad et al. 88]). However, despite its significance,  $\mathcal{H}_\infty$  control—being a frequency domain method—cannot directly address time domain specifications. Recently,  $\ell_1$  optimal control problems have been studied, where the signals involved are bounded in magnitude. This presents a method to accommodate the time domain specifications, although of course it cannot directly accommodate common classes of frequency domain specifications (such as  $\mathcal{H}_2$  or  $\mathcal{H}_\infty$  bounds).

Obviously, different, often conflicting design specifications such as simultaneous rejection of disturbances having different characteristics (white noise, bounded energy, persistent); good tracking of classes of inputs or satisfaction of bounds on the peak values of outputs, cannot always

be cast into a single norm form. It is therefore natural to expect a mixed-norm formalism to be of considerable interest. This paper aims at presenting an overview of the current state of affairs in this area.

The paper is organized as follows: In Section 2 some notions to be used are introduced, whereas the possible problem statements are given in Section 3. In Section 4 an overview of the many approaches to solve the problem is given. Finally, Section 5 concludes with some remarks.

Throughout this paper, the time  $t$ , the shift-operator  $z$ , and the Laplace transform variable  $s$  will often be omitted for clarity. The notation used in this paper was chosen in such a way that a uniform description was possible. Some symbols used here will therefore inevitably differ from those used in the literature. However, it is fairly standard otherwise.  $M^T$  denotes the transpose of  $M$  and  $M^*$  the complex conjugate transpose. The trace of  $M$  is denoted by  $\text{tr}(M)$ .  $I_{q_2}$  is a unit matrix of dimensions  $q_2 \times q_2$  and finally two abbreviations, MI and ARE, mean Matrix Inequality and Algebraic Riccati Equation, respectively. Further notational issues are explained within the text.

### 2. Preliminaries

#### 2.1. System norms

Given a stable strictly proper (in order to keep the norms finite, see, e.g., [Doyle et al. 92, p. 16]) transfer function matrix  $G(s)$  with state space realization  $(A, B, C, D)$ , the following performance measures can be defined.

- The  $\mathcal{H}_2$ -norm of a transfer function  $G(s)$  is defined as:

$$\|G(s)\|_2 = \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{tr}[G^T(-j\omega)G(j\omega)] d\omega \right)^{1/2} \quad (1)$$

for the continuous-time case and

$$\|G(z)\|_2 = \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{tr}[G(e^{j\theta})G^T(e^{-j\theta})] d\theta \right)^{1/2} \quad (2)$$

for the discrete-time case.

The 2-norm can be computed with Lyapunov equations:

$$\begin{aligned} \|G\|_2^2 &= \text{tr}[SC^T C] = \text{tr}[PBB^T] \\ &= \text{tr}[CSC^T] = \text{tr}[B^T P B] \end{aligned} \quad (3)$$

where  $S$  is the controllability Gramian and  $P$  is the observability Gramian solving

$$AS + SA^T + BB^T = 0 \quad A^T P + PA + C^T C = 0 \quad (4)$$

- The  $\mathcal{H}_\infty$ -norm of a transfer function  $G(s)$  is defined as, with  $\bar{\sigma}$  the maximum singular value:

$$\|G(s)\|_\infty = \sup_{\omega \in \mathbb{R}} \bar{\sigma}(G(j\omega)) \quad (5)$$

for the continuous-time case and

$$\|G(z)\|_\infty = \sup_{0 \leq \theta \leq \pi} \bar{\sigma}(G(e^{j\theta})) \quad (6)$$

for the discrete-time case.

1. Corresponding author. Email: A.G.de.Jager@wfw.wtb.tue.nl  
2.  $\ell_1$  denotes the discrete-time case, whereas  $L_1$  is used for continuous-time. In the following, whenever  $\ell_p$  is mentioned, the same can be assumed to be true for the continuous-time case, unless stated otherwise (where  $p$  can be 1, 2 or  $\infty$ ). This means that whenever  $L_p$  is mentioned, only the continuous-time case is referred to.

- The  $\ell_1$ -norm of a transfer function matrix  $G(z)$  is not as easy to define as the other two. Recall the 1-norm of a sequence  $x(t)$  being  $\|x\|_1 = \sum_{t=-\infty}^{\infty} |x(t)|$ . Then, given a matrix  $g$  with elements  $g_{ij}$ , representing a linear operator defined by the usual discrete-time convolution  $y = g * u$  (and with a corresponding transfer function matrix  $G$ ), its 1-norm is defined as:

$$\|G\|_1 = \max_{1 \leq i \leq m} \sum_{j=0}^n \|g_{ij}\|_1 \quad (7)$$

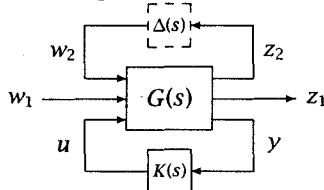
for the discrete-time case. The definition for the continuous-time case requires more notation and can be found in, e.g., [Sznaier and Blanchini 94]. Since the interpretation of the 1-norm (see Subsection 2.2) is of more use to us than the formal definition, this will not be repeated here.

## 2.2. Norm interpretations

- The  $\mathcal{H}_2$ -norm:
  1. For SISO systems, the induced norm from  $\ell_2$  to  $\ell_\infty$ .
  2. The square root of the average power (is RMS-value or "power-norm") of the response to a white input signal of unit spectral density or the spectrum/power gain.
  3. The square root of the energy contained in the impulse response.
- The  $\mathcal{H}_\infty$ -norm:
  1. The induced norm from  $\ell_2$  to  $\ell_2$ .
  2. The power/power gain (RMS gain).
  3. The spectrum/spectrum gain.
  4. An upper bound on the  $\ell_\infty$ /power gain, assuming that the input is a persistent sinusoidal signal.
  5. The peak amplitude of the Bode singular value plot.
- The  $\ell_1$ -norm:
  1. The induced norm from  $\ell_\infty$  to  $\ell_\infty$ .

## 3. Statement of the problem

The general problem can be posed as follows. Suppose the plant is given by its transfer function matrix  $G(s)$  with three sets of inputs and outputs:



with

$$\dot{x} = Ax + B_1 w_1 + B_2 w_2 + B_3 u \quad (8)$$

$$z_1 = C_1 x + D_{11} w_1 + D_{12} w_2 + D_{13} u \quad (9)$$

$$z_2 = C_2 x + D_{21} w_1 + D_{22} w_2 + D_{23} u \quad (10)$$

$$y = C_3 x + D_{31} w_1 + D_{32} w_2 + D_{33} u \quad (11)$$

or equivalently, using packed notation

$$G = \begin{bmatrix} A & B_1 & B_2 & B_3 \\ C_1 & D_{11} & D_{12} & D_{13} \\ C_2 & D_{21} & D_{22} & D_{23} \\ C_3 & D_{31} & D_{32} & D_{33} \end{bmatrix} \quad (12)$$

Furthermore

$$\begin{aligned} n &= \dim(x) & q_1 &= \dim(z_1) & d_1 &= \dim(w_1) & l &= \dim(y) \\ m &= \dim(u) & q_2 &= \dim(z_2) & d_2 &= \dim(w_2) \end{aligned}$$

In the system equations (8)–(11)  $u$  represent the control actions,  $w$  ( $= [w_1, w_2]$ ) the exogenous disturbances,  $y$  the measurements and  $z$  ( $= [z_1, z_2]$ ) the regulated outputs. The

signal sets  $\{w_1, z_1\}$  are related to performance criteria (measured by what we will call the  $p_1$ -norm), whereas  $\{w_2, z_2\}$  are related to  $p_2$ -norm constraints. These norms will usually be either  $\mathcal{H}_2$  and  $\mathcal{H}_\infty$ ,  $\ell_1$  and  $\mathcal{H}_\infty$ , or  $\mathcal{H}_2$  and  $\ell_1$  (seldom used). In control problems involving  $\mathcal{H}_2$  minimization,  $D_{11}$  is always taken to be zero to prevent the  $\mathcal{H}_2$ -norm from becoming infinite.

For the system as defined above the mixed two-norm problem, as encountered in the literature, can be written as (when  $T_{w_1 \rightarrow z_1}$  denotes the transfer function from  $w_1$  to  $z_1$  ( $i = 1, 2$ ) and  $\gamma \in \mathbb{R}$  is the positive  $p_2$ -norm bound):

1. Find an internally stabilizing controller  $K$  which minimizes  $\|T_{w_1 \rightarrow z_1}\|_{p_1}$  while maintaining  $\|T_{w_2 \rightarrow z_2}\|_{p_2} \leq \gamma$ , where  $p_1$  ( $= 2$  or  $1$ ) can denote either the  $\mathcal{H}_2$  or  $\ell_1$ -norm and  $p_2$  ( $= \infty$  or  $2$ ) denotes the  $\mathcal{H}_\infty$  or the  $\mathcal{H}_2$ -norm. Another formulation, used by [Steinbuch and Bosgra 94, Stoorvogel 93], is the following:
2. Minimize the  $p_1$ -norm of the transfer function from  $w_1$  to  $z_1$  using the internally stabilizing controller  $K(s)$ , while maximizing the  $p_1$ -norm of that same transfer function over the allowable,  $p_2$ -norm bounded, uncertainties:

$$\min_{\| \Delta \|_{p_2} \leq 1/\gamma} \sup_{K(s)} \|T_{w_1 \rightarrow z_1}(K, \Delta)\|_{p_1}$$

where, in case of the problem addressed by [Steinbuch and Bosgra 94, Stoorvogel 93]  $p_1 = 2$  and  $p_2 = \infty$ .

Finally, [Elia *et al.* 93, Dahleh and Diaz-Bobillo 95, Voulgaris 94] use the following formulation:

3. Find an internally stabilizing controller  $K(s)$  which minimizes  $\|T_{w \rightarrow z}\|_{p_1}$  and satisfies a set of linear constraints given by  $\mathcal{A}$  and  $b$ :

$$\inf_{K(s) \text{ stabilizing}} \|T_{w \rightarrow z}\|_{p_1} \quad \text{such that} \quad \mathcal{A} T_{w \rightarrow z} \leq b$$

with  $\mathcal{A}$  a linear operator from  $\ell_{p_1}^{q \times d}$  to  $\ell_{p_2}^{n_b \times m_b}$ , and  $b \in \ell_{p_2}^{n_b \times m_b}$  a fixed element (possibly containing the " $\gamma$ -bound"). In this formulation usually  $w_1 = w_2 =: w$  and  $z_1 = z_2 =: z$ , see Section 4.1.

Of course, slight variations with respect to these problem statements occur, though never essentially influencing the rest of the approach. For instance, some approaches instead of using  $\|T_{w_2 \rightarrow z_2}\|_{p_2} \leq \gamma$  use the strict inequality.

As mentioned before, most approaches focus on solving the mixed  $\mathcal{H}_2/\mathcal{H}_\infty$  control problem, while the other two problems ( $\ell_1/\mathcal{H}_\infty$  and  $\mathcal{H}_2/\ell_1$ ) so far have received little attention. The  $\mathcal{H}_2/\ell_1$  problem actually is a special case of the approach followed by [Dahleh and Diaz-Bobillo 95, Voulgaris 94, Elia *et al.* 93, Elia and Dahleh 96] which provides a method (originating from  $\ell_1$  optimal control theory) that either minimizes or constrains the  $\ell_1$ -norm combined with  $\mathcal{H}_2$  and/or  $\mathcal{H}_\infty$ -norm minimization or constraints. Apart from this, mainly Sznaier [Sznaier 94, Sznaier and Blanchini 94, Sznaier 93, Hrissagis *et al.* 96, Sznaier and Bu 96] addresses the mixed  $\ell_1/\mathcal{H}_\infty$  problem, both for the discrete-time and the continuous-time case. The widest variety can be found in the approaches to the  $\mathcal{H}_2/\mathcal{H}_\infty$  problem, eventually to be divided into five categories. One other distinction can be made based on the number of sets of in- and outputs used in the statement of the problem. The distinction discrete-time versus continuous-time, however non-trivial it might be, will not be made explicitly since it doesn't essentially alter the approach used.

Finally, as a counterpart of the  $\mathcal{H}_\infty$ -norm constraint can be mentioned the Extended Strictly Positive Real (ESPR) stability criterion (see, e.g., [Shim 94]). Positive realness is an old, but important concept in system and control theory and is used in various areas, like network analysis, adaptive control, nonlinear control and robust control. It is well-known that positive realness is closely related to absolute stability. This criterion will however not be treated here.

Another, rather different, approach to the mixed-norm problem, is based on the so-called "behavioral setting." This methodology can be characterized by the fact that all variables are considered a priori on an equal footing, without a distinction between inputs and outputs, and the behavior is defined as a subset of the possible time trajectories. Because of the fact that this setting, which is so unlike the others, was hardly ever encountered (but is becoming popular), it will not be treated here, but can be found in, e.g., [Paganini *et al.* 94] and references therein.

#### 4. Overview of approaches

We will now describe a number of approaches to the solution of the mixed-norm optimization problem. This overview can of course not be exhaustive, but an attempt was made to (briefly) describe the approaches most frequently encountered in the literature.

The following classification was used:

Approach		Section
$\ell_1/\mathcal{H}_2$		4.1
$\ell_1/\mathcal{H}_\infty$		4.2
$\mathcal{H}_2/\mathcal{H}_\infty$	MI's	4.3
	ARE's	
	$w_1 = w_2$ $z_1 = z_2$	4.4
	$w_1 \neq w_2$ $z_1 \neq z_2$	4.5
	$w_1 \neq w_2$ $z_1 = z_2$	4.6
	$w_1 \neq w_2$ $z_1 \neq z_2$	4.7

Subsections 4.1 and 4.5 will be treated in somewhat more detail. It must be stressed that this classification is arbitrary and other classifications can be used as well. In fact, there are some approaches that don't actually fit in any one of these classes, for instance [Sznaier *et al.* 95] addresses the mixed  $\mathcal{H}_2/\mathcal{L}_1$  problem (where  $w_1 = w_2$ ,  $z_1 = z_2$ ) in a way not really fitting into Section 4.1. The same can be said for [Wu and Chu 96] which treats the  $\mathcal{H}_2/\ell_1$  problem using Youla-parameterizations. Finally, [Chen and Wen 96] considers the  $\mathcal{L}_1/\mathcal{H}_\infty$  problem and uses LMI's combined with the so called *Laguerre polynomial augmentation* method. Although similarities with the methods of Section 4.1 exist, again this paper does not really fit into this section. However, for the approaches discussed below, this classification suffices.

##### 4.1. $\ell_1/\mathcal{H}_2$ : a linear programming approach

This approach uses the problem statement (3) from Section 3, where most commonly  $w_1 = w_2 =: w$  and  $z_1 = z_2 =: z$ , although different linear constraints can be defined for different closed-loop maps  $T_{w_j-z_i}$ , i.e., on the map between the  $j^{\text{th}}$  input set and the  $i^{\text{th}}$  output set. Using that formulation, either  $p_1$  or  $p_2$  is taken to be 1 and the remaining  $p=1, 2$  or  $\infty$ . Most common is the  $\ell_1$  minimization combined with  $\mathcal{H}_2$  and/or  $\mathcal{H}_\infty$  constraints [Dahleh and Diaz-Bobillo 95, Elia *et al.* 93]. The  $\mathcal{H}_2/\ell_1$  problem is not so often encountered [Voulgaris 94]. All these approaches use the technique of *Linear Programming (LP)* combined with *duality theory*. An LP problem is an optimization problem in  $\mathbb{R}^n$ , where the objective function is linear in the unknowns, and the unknowns have to satisfy a set of linear equality and/or inequality constraints. This can be stated in the following standard form:

$$\begin{aligned} \min_x & c^T x \\ \text{subject to} & Ax = b \\ & x_i \geq 0 \quad i = 1, \dots, n \end{aligned} \quad (13)$$

where  $x, c \in \mathbb{R}^n$ ,  $b \in \mathbb{R}^m$  and  $A \in \mathbb{R}^{m \times n}$ .

Realizing that a large class of specifications can be expressed in terms of linear constraints leads to the following approach. The idea is to simply augment the constraint of the linear program, derived from the  $\ell_1$  optimal control, with the linear specifications constraints and solve the new linear program. These constraints can be  $\mathcal{H}_\infty$ -constraints and/or time-domain (template) constraints and they will be combined with the feasibility- (or interpolation-) constraints (see [Dahleh and Diaz-Bobillo 95, pp. 123-126]). This way, even the three-norm problem can be handled. The infinite-dimensional  $\mathcal{H}_\infty$ -constraint will have to be approximated by a finite number of constraints. Unfortunately this may prevent finding a solution if the performance specifications are tight.

Given the standard form minimization (13), which we will call the primal problem, it is always possible to define an associated linear maximization problem, known as the dual problem. The corresponding primal-dual pair is given by

$$\begin{aligned} \text{(primal)} \quad & \min_x c^T x \\ \text{subject to} \quad & Ax = b \\ & x_i \geq 0, \quad i = 1, \dots, n \end{aligned} \quad \begin{aligned} \text{(dual)} \quad & \max_\eta \eta^T b \\ \text{subject to} \quad & \eta^T A \leq c^T \end{aligned}$$

where  $\eta$  is the vector of dual variables  $\in \mathbb{R}^m$  (i.e., in "dual space"). Duality theory is used for instance in the solution of the multiblock problem (i.e., a problem in which  $d > l$  and/or  $q > m$ , whereas for a one-block problem  $d = l$  and  $q = m$ ). This problem, which has infinitely many variables and constraints, can in fact be shown to be partly finite-dimensional, by taking a close look at the structure of the dual problem. The part which still is infinite-dimensional can be (attempted to be) approximated by an appropriate truncation of the original problem. There are basically three approximation methods:

1. **Finitely Many Variables (FMV):** provides a suboptimal polynomial feasible solution by constraining the number of (primal) variables to be finite.
2. **Finitely Many Equations (FME):** provides a superoptimal infeasible solution by including only a finite number of (primal) equality constraints. It is to be combined with FMV to get an idea of the achieved accuracy. The FME/FMV method does unfortunately result in controllers of high order, related to the order of the approximation.
3. **Delay Augmentation (DA):** provides both a suboptimal and a superoptimal solution by embedding the problem into a one-block problem through augmenting the operators  $U$  and  $V$  with delays (where  $T_{w-z} = H - UQV$  is an equivalent form of the Youla-parameterization as used in [Dahleh and Diaz-Bobillo 95]). This method is used more often since it doesn't necessarily suffer from order-inflation when in- and outputs are (re)ordered properly (see [Dahleh and Diaz-Bobillo 95, p. 283]).

For a more thorough treatment on these methods the reader is referred to [Dahleh and Diaz-Bobillo 95, Chapter 12].

A recent paper in this area worth mentioning is [Elia and Dahleh 96], where it is shown that the approximation methods mentioned above may fail to converge to the optimal  $\ell_1$  cost.

##### 4.2. $\ell_1/\mathcal{H}_\infty$ : using the Youla-parameterization

This approach uses the more general description where  $w_1 \neq w_2$  and  $z_1 \neq z_2$  in both the discrete-time- and the continuous-time case (see [Sznaier 93] (SISO), [Sznaier 94] (MIMO) for discrete-time and [Sznaier and Blanchini 94] (MIMO) for continuous-time). The main result shows that a suboptimal solution to the  $\ell_1/\mathcal{H}_\infty$  problem, with performance arbitrarily close to the optimum, can be obtained by solving a finite-dimensional convex optimization problem and an unconstrained  $\mathcal{H}_\infty$  problem. One important

concept in this approach is “replacing”  $\mathcal{RH}_\infty$  by  $\mathcal{RH}_{\infty,\delta}$  which is defined as follows:  $\mathcal{RH}_{\infty,\delta}$  denotes the subspace of transfer matrices in  $\mathcal{RH}_\infty$  which are analytic outside the disc of radius  $\delta$ ,  $0 < \delta < 1$ , equipped with the norm

$$\|G(z)\|_{\infty,\delta} := \sup_{0 \leq \theta \leq \pi} \overline{\sigma}(G(\delta e^{j\theta}))$$

The discrete-time  $\ell_1/\mathcal{H}_\infty$  problem (using the Youla-parameterization with parameter  $Q$ ), stated as:

• *Mixed  $\ell_1/\mathcal{H}_\infty$  control problem*

1. Find the optimal value of the performance measure:

$$\begin{aligned} \mu^0 &= \inf_{Q \in \mathcal{RH}_\infty} \|T_{w_1-z_1}\|_1 \\ &= \inf_{Q \in \mathcal{RH}_\infty} \|V_{11} + V_{12}QV_{21}\|_1 \end{aligned} \quad (14)$$

subject to

$$\|T_{w_2-z_2}\|_\infty = \|T_{11} + T_{12}QT_{21}\|_\infty \leq \gamma$$

2. Given  $\varepsilon > 0$ , synthesize a controller such that

$$\|T_{w_2-z_2}\|_\infty \leq \gamma \quad \text{and} \quad \|T_{w_1-z_1}\|_1 \leq \mu^0 + \varepsilon$$

can then be solved by considering a sequence of modified problems, involving the  $(\infty, \delta)$ -norm, where  $\delta_i$  is taken to be an increasing sequence approaching one. Next, the continuous-time problem can be solved by solving a sequence of discrete-time  $\ell_1/\mathcal{H}_\infty$  problems. The most severe limitation of the proposed method is that it may result in high order controllers, necessitating some type of model reduction. More recent work includes [Sznajder and Bu 96, Hrisagis *et al.* 96], the latter concentrating on robust predictive control.

**4.3.  $\mathcal{H}_2/\mathcal{H}_\infty$ : convex optimization using matrix inequalities**

The approach described here has received a great deal of attention (see, e.g., [Boyd and Barratt 91, Boyd *et al.* 93, Geromel *et al.* 92, Khargonekar and Rotea 91, Halikias 94, Scherer 95, Kapila and Haddad 95, Fujisaki and Yoshida 96, Chilali *et al.* 96, D’Andrea 96, El Ghaoui and Følcher 96, Scherer *et al.* 97]). However, although all these approaches use matrix inequalities to arrive at a convex optimization problem there still exists a wide variety among these approaches. For instance, some, but not all approaches use the Youla-parameterization; some set  $w_1 = w_2$  where others take all four sets of inputs and outputs to be different; the MI’s involved can be Linear (most common) or Quadratic MI’s and some approaches use the performance measure of Bernstein and Haddad (see Subsection 4.5) where others don’t.

The solution to many convex optimization problems can be computed in a time which is comparable to the time required to evaluate a “closed-form” solution for a similar problem. Nowadays, a control engineering problem that reduces to solving two algebraic Riccati equations is generally regarded as “solved.” When a control engineering problem reduces to solving even a large number of convex algebraic Riccati inequalities the growing belief is this should also be regarded as “solved,” even though there is no “analytic” solution (see [Boyd *et al.* 93]). Hence a large number of approaches focuses on making the optimization a convex one, mostly by using some suitable parameterization. There are effective and powerful algorithms for the solution of these problems, that is, algorithms that compute the global optimum, with non-heuristic stopping criteria. A number of general algorithms exist, for example the ellipsoid algorithm (see, e.g., [Boyd and Barratt 91, Bland *et al.* 81]) and the recently developed efficient interior point methods for solving LMI-based problems, based on

the work of Nesterov and Nemirovsky [Nesterov and Nemirovsky 93].

For instance, [Khargonekar and Rotea 91] uses the performance measure of Bernstein and Haddad and more importantly they use a change of variables  $K = WY^{-1}$  (which essentially is an over-parameterization), where  $Y$  is the symmetric solution to the quadratic matrix inequality that characterizes the infinity norm constraint. Although this necessitates a search over  $(W, Y)$  instead of  $K$  being the only decision variable, it converts the original problem (which is not necessarily convex) into a convex one.

The concept of LMI-based convex optimization is treated extensively in control literature and it has great potential, since there exist effective and powerful algorithms for the solution of these problems, as described earlier.

**4.4.  $\mathcal{H}_2/\mathcal{H}_\infty$ : optimizing an entropy cost functional**

This subsection refers to the work done mainly by Mustafa and Glover in [Mustafa 89, Glover and Mustafa 89, Mustafa and Glover 88, Mustafa *et al.* 91]. They address the problem where  $w_1 = w_2 =: w$  and  $z_1 = z_2 =: z$ . The entropy of  $T_{w-z}$ , where  $\|T_{w-z}\|_\infty < \gamma$ , is defined by

$$\begin{aligned} \mathcal{I}(T_{w-z}, \gamma) &:= \lim_{s_0 \rightarrow \infty} \left\{ \frac{\gamma^2}{2\pi} \int_{-\infty}^{\infty} \ln |\det(I \right. \\ &\quad \left. - \gamma^2 T_{w-z}(j\omega) T_{w-z}^*(j\omega))| \left[ \frac{s_0}{|s_0 - j\omega|} \right]^2 d\omega \right\}, \end{aligned}$$

where  $s_0 \in \mathbb{R}^+$ .

The maximum entropy/ $\mathcal{H}_\infty$  control problem [Glover and Mustafa 89, Mustafa and Glover 88] can then be stated as: Find, for the plant  $G$ , a feedback controller  $K$  such that:

1.  $K$  stabilizes  $G$
2. The closed-loop transfer function  $T_{w-z}$  satisfies the  $\mathcal{H}_\infty$ -norm bound  $\|T_{w-z}\|_\infty < \gamma$ , where  $\gamma \in \mathbb{R}$  is given
3. The closed-loop entropy  $\mathcal{I}(T_{w-z}, \gamma)$  is maximized.

The key result they establish states that minus the entropy equals the auxiliary cost (as will be defined in Subsection 4.5).

While the maximum value of the entropy can be expressed in terms of the solutions to two algebraic Riccati equations, the minimum auxiliary cost in addition to this requires the solution to a third algebraic Riccati equation coupled to the other two. Since the two optimal values were said to be equal, we will be able to discard of the minimum auxiliary cost expression and the corresponding algebraic Riccati equation as redundant.

**4.5.  $\mathcal{H}_2/\mathcal{H}_\infty$ : fixed-order controller design using the auxiliary cost**

This class of approaches refers mainly to the work done by Bernstein and Haddad in [Bernstein and Haddad 89, Haddad and Bernstein 90, Haddad *et al.* 91] and some methods based on it [Ge *et al.* 94]. They address the problem of synthesizing a reduced- (or fixed-) order controller. To describe this problem we have to consider  $n_c$ th-order dynamic compensators

$$\dot{x}_c = A_c x_c + B_c y \quad (15)$$

$$u = C_c x_c + D_c y \quad (16)$$

Now the closed-loop (8)-(11)+(15)-(16) can be written as

$$\dot{\tilde{x}} = \tilde{A} \tilde{x} + \tilde{B} \tilde{w} \quad (17)$$

$$\tilde{z} = \tilde{C} \tilde{x} + \tilde{D} \tilde{w} \quad (18)$$

where

$$\tilde{x} = \begin{bmatrix} x \\ x_c \end{bmatrix}, \quad \tilde{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \quad \tilde{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \quad \tilde{n} = n + n_c$$

In the papers mentioned above,  $w_1 = w_2 =: w$  which simplifies the problem significantly. However, in [Vroemen 96],

the more general case of  $w_1 \neq w_2$  is described, in order to allow for all existing approaches using the performance measure of Bernstein and Haddad to derive the appropriate expressions. The final results, which can then be found in [Haddad and Bernstein 90, Bernstein and Haddad 89], are valid for the simplified case only.

The LQG controller synthesis problem with an  $\mathcal{H}_\infty$  constraint can now be stated as follows:

Find an  $n_c^{\text{th}}$  order dynamic compensator described by (15)–(16) which satisfies the following criteria

1. the closed-loop system (17)–(18) is asymptotically stable, i.e.,  $\tilde{A}$  is asymptotically stable;
2. the closed-loop transfer function  $T_{w_2 \rightarrow z_2}$  satisfies the constraint  $\|T_{w_2 \rightarrow z_2}\|_\infty \leq \gamma$  where  $\gamma > 0$  is a given constant;
3. the performance functional (LQG cost)  $J(A_c, B_c, C_c, D_c)$  is minimized.

Then, for a given compensator the performance is given by

$$J(A_c, B_c, C_c, D_c) = \text{tr}[\tilde{S}\tilde{R}_1] \quad (19)$$

where  $\tilde{R}_1 = \tilde{c}_1^T \tilde{c}_1$  ( $\tilde{C} =: [\tilde{c}_1]$ ) and  $\tilde{S}$  satisfies the Lyapunov equation

$$\tilde{A}\tilde{S} + \tilde{S}\tilde{A}^T + \tilde{V} = 0 \quad (20)$$

with  $\tilde{V} = \tilde{B}\tilde{B}^T$ . Note that (19) and (20) are similar to (3) and (4). The auxiliary cost  $J(A_c, B_c, C_c, D_c, S)$  is then defined as  $J(A_c, B_c, C_c, D_c, S) := \text{tr}[\tilde{S}\tilde{R}_1]$ , with  $S \in \mathbb{R}^{\tilde{n} \times \tilde{n}}$  the positive-semidefinite ( $S \geq 0$ ) solution to

$$\tilde{A}S + S\tilde{A}^T + \gamma^{-2}(\tilde{B}D_{22}^T + S\tilde{c}_2^T)M_{q_2}^{-1}(\tilde{B}D_{22}^T + S\tilde{c}_2^T)^T + \tilde{V} = 0 \quad (21)$$

where  $M_{q_2} := I_{q_2} - \gamma^{-2}D_{22}D_{22}^T$  is positive-definite.

This leads to the *auxiliary minimization problem*: determine  $(A_c, B_c, C_c, D_c, S)$  which minimizes the auxiliary cost  $J(A_c, B_c, C_c, D_c, S)$  subject to (21) with  $S \geq 0 \in \mathbb{R}^{\tilde{n} \times \tilde{n}}$ .

The auxiliary minimization problem can be solved by using Lagrange multipliers as was done in [Haddad and Bernstein 90, Bernstein and Haddad 89]. Like them, we will take  $D_c$  to be zero from now on.

Then, to optimize  $J(A_c, B_c, C_c, S)$  over some set  $\mathcal{X}$  (reflecting technical assumptions) subject to the constraint that positive-definite  $S$  satisfies (21), the following Lagrangian is formed:

$$\mathcal{L}(A_c, B_c, C_c, S, \mathcal{M}) := \text{tr}\{\tilde{S}\tilde{R}_1 + [\tilde{A}S + S\tilde{A}^T + \gamma^{-2}(\tilde{B}D_{22}^T + S\tilde{c}_2^T)M_{q_2}^{-1}(\tilde{B}D_{22}^T + S\tilde{c}_2^T)^T + \tilde{V}]\mathcal{M}\} \quad (22)$$

where  $\mathcal{M} \in \mathbb{R}^{\tilde{n} \times \tilde{n}}$  is a Lagrange multiplier.

After setting the partial derivatives  $\frac{\partial \mathcal{L}}{\partial S}$ ,  $\frac{\partial \mathcal{L}}{\partial A_c}$ ,  $\frac{\partial \mathcal{L}}{\partial B_c}$  and  $\frac{\partial \mathcal{L}}{\partial C_c}$  to zero, the solution can be obtained either analytically or numerically. Bernstein and Haddad take the first approach and derive solutions to the problems of finding fixed- as well as full-order controllers for both the  $\mathcal{H}_2/\mathcal{H}_\infty$  and the pure  $\mathcal{H}_\infty$  problem for the case where  $B_1, D_{21}, D_{31}, D_{12}$  and  $D_c$  (and  $D_{11}$ ) are taken to be zero. They find  $A_c, B_c$  and  $C_c$  (and  $S$ ) in terms of the solutions to four coupled ARE's.

Ge *et al.* choose the numerical approach and use homotopy techniques (see [Ge *et al.* 94] and references in [Bernstein and Haddad 89, Haddad and Bernstein 90]). They basically take the LQG solution as a starting point and iterate “towards” the  $\mathcal{H}_2/\mathcal{H}_\infty$  problem.

#### 4.6. $\mathcal{H}_2/\mathcal{H}_\infty$ : using a bounded power characterization

In this subsection the “power-norm” and the “spectrum-norm” are used. These semi-norms are defined as the square root of the average power of a signal and the square root of the  $\infty$ -norm of the spectral density respectively. The corresponding signal spaces (containing all signals having

a finite power-/spectrum-norm) are denoted by  $\mathcal{P}$  and  $\mathcal{S}$ . The problem addressed [Doyle *et al.* 89, Zhou *et al.* 90] sets  $w_1 \neq w_2$ ,  $z_1 = z_2 =: z$  where  $w_1$  is assumed to be fixed and white, and  $w_2$  is assumed to be bounded in power. The design objective is to minimize the power of the output error signal  $z$ , i.e., compute

$$\sup_{w_1 \in \mathcal{S}, w_2 \in \mathcal{P}} \|z\|_{\mathcal{P}}^2 \quad (23)$$

and minimize this. In their approach they use (cross-) spectral density relations to solve the following cases:

1. The orthogonal case, i.e., the cross-spectral density  $S_{w_1 w_2} = 0$
2. The white and causal case, i.e.,  $w_1$  is assumed to be white with  $S_{w_1 w_1} = I$  and  $S_{w_2 w_1} = S(s)$  with  $S(s)$  strictly causal (i.e., we assume that  $w_2(t)$  can be generated from  $w_1$  through a strictly causal filter)
3. The non-white and non-causal case
4. The white and non-causal case: this problem appears to be equal to the 3<sup>rd</sup> problem, i.e., the worst-case signal  $w_1$  in the 3<sup>rd</sup> problem is shown to be white.

Their approach mainly focuses on the 2<sup>nd</sup> case. Eventually they obtain both necessary and sufficient conditions for the mixed  $\mathcal{H}_2/\mathcal{H}_\infty$  optimal control problem.

#### 4.7. $\mathcal{H}_2/\mathcal{H}_\infty$ : minimizing the worst-case $\mathcal{H}_2$ -norm

In this section the problem with four different sets of inputs and outputs  $w_1, w_2, z_1$  and  $z_2$  is considered, where  $p_1=2, p_2=\infty$ . Based on problem statement (1) from Section 3, [Rotea and Khargonekar 91] give explicit formulae for a state-feedback controller.

Finally, we detail methods from [Steinbuch and Bosgra 94, Stoorvogel 93], who use problem statement (2). This still allows for a considerable variety in the approach followed. In [Steinbuch and Bosgra 94] a “lossless bounded real (LBR) formulation” is used to parameterize the uncertainty  $\Delta(s)$ , thereby reducing the original constrained optimization to an unconstrained one. This results in a “ $\Delta_k$ - $K^*$ ”-iteration, similar to the  $D$ - $K$ -iteration known from  $\mu$ -synthesis: if  $\Delta_k$  qualifies as a worst-case perturbation, for fixed  $\Delta_k$  we can determine a  $K^*$  solving

$$\min_K \|T_{w_1 \rightarrow z_1}(u = K(s)y)\|_2$$

By computing an  $\mathcal{H}_2$ -optimal  $K^*(s)$  for each  $\Delta_k(s)$ , we iterate over  $\Delta_k(s)$  until it satisfies the conditions for a worst-case disturbance. The LBR-parameterization, which characterizes all real rational causal stable transfer functions  $\Delta(s)$  of order  $n_\Delta$  having  $\|\Delta\|_\infty < 1$ , is then used to formulate an unconstrained optimization problem.

Stoorvogel [Stoorvogel 93] uses a Lagrange multiplier  $\varphi$  for the same purpose, resulting in a “ $\varphi$ - $K$ ”-iteration. One disadvantage of the approach followed by Stoorvogel is that it is conservative in the sense that the disturbance system is not assumed to be causal. Furthermore, the uncertainty is assumed to be unstructured.

## 5. Conclusions

We have presented a number of approaches to solve the mixed-norm optimization problem. It was seen that all but one focus on solving the two-norm problem, although this one approach (see Subsection 4.1), which considers a three-norm problem setting does not really exploit this possibility and ends up giving no more than a description of what the methodology would look like.

It was also seen that the  $\mathcal{H}_2/\mathcal{H}_\infty$  problem received the greatest deal of attention. This is due to the fact that the need for a mixed-norm formalism originates from the separate  $\mathcal{H}_2$  and  $\mathcal{H}_\infty$  control theories not being able to accommodate all practical design specifications. To accommodate bounded-magnitude signals, the  $\ell_1$  optimal control theory was developed, but not until a few years ago,

which explains the relatively small number of approaches to this problem.

Most of the approaches tend to have an ad hoc character, but the same is said for  $\mu$ -synthesis [Zhou et al. 90], which has been successfully applied in recent years. It is not clear which one of these approaches qualifies as most promising. The future will point out which methods are best suited for practical application, but all efforts will undoubtedly contribute to what must become a clean closed-form solution to the mixed-norm optimization problem.

It is recommended that a new overview of mixed-norm optimization techniques is carried out in a few years, since the work in this area is growing and for some time to come not finished, and it is of interest for industrial applications, see, e.g., [De Jager 95].

## References

- [Bernstein and Haddad 89] D. S. Bernstein and W. M. Haddad, "LQG control with an  $\mathcal{H}_\infty$  performance bound: A Riccati equation approach," *IEEE Trans. Automat. Contr.*, vol. 34, no. 3, pp. 293-305, 1989
- [Bland et al. 81] R. G. Bland, D. Goldfarb and M. J. Todd, "The ellipsoid method: A survey," *Operations research*, vol. 29, no. 6, pp. 1039-1091, 1981
- [Boyd et al. 93] S. Boyd, V. Balakrishnan, E. Feron and L. El Ghaoui, "Control system analysis and synthesis via linear matrix inequalities," *Proc. Amer. Contr. Conf.*, vol. 2, pp. 2147-2154, 1993
- [Boyd and Barratt 91] S. P. Boyd and C. H. Barratt, "Linear controller design: Limits of performance," *Prentice-Hall*, 1991
- [Chen and Wen 96] X. Chen and J. T. Wen, "Rational  $\mathcal{L}_1$  and mixed  $\mathcal{L}_1/\mathcal{H}_\infty$  compensators for continuous time multiblock problems," *IFAC World Congress*, vol. G, pp. 221-226, 1996
- [Chilali et al. 96] M. Chilali, P. Gahinet and C. Scherer, "Multi-objective output-feedback control via LMI optimization," *IFAC World Congress*, vol. D, pp. 249-254, 1996
- [Dahleh and Diaz-Bobillo 95] M. A. Dahleh and I. J. Diaz-Bobillo, "Control of uncertain systems: A linear programming approach," *Prentice-Hall*, 1995
- [D'Andrea 96] R. D'Andrea, "LMI approach to mixed  $\mathcal{H}_2$  and  $\mathcal{H}_\infty$  performance objective controller design," *IFAC World Congress*, vol. G, pp. 327-332, 1996
- [De Jager 95] B. de Jager, "Multiobjective suspension control problem," *Proc. IEEE Conf. Decision Contr.*, vol. 4, pp. 3652-3657, 1995
- [Doyle et al. 92] J. C. Doyle, B. A. Francis and A. R. Tannenbaum, "Feedback control theory," *Macmillan*, 1992
- [Doyle et al. 89] J. C. Doyle, K. Zhou and B. Bodenheimer, "Optimal control with mixed  $\mathcal{H}_2$  and  $\mathcal{H}_\infty$  performance objectives," *Proc. Amer. Contr. Conf.*, vol. 3, pp. 2065-2070, 1989
- [El Ghaoui and Folcher 96] L. El Ghaoui and J. P. Folcher, "Robust multiobjective LTI control design for systems with structured perturbations," *IFAC World Congress*, vol. H, pp. 179-184, 1996
- [Elia and Dahleh 96] N. Elia and M. A. Dahleh, " $\ell_1$  minimization with magnitude constraints in the frequency domain," *IFAC World Congress*, vol. C, pp. 397-402, 1996
- [Elia et al. 93] N. Elia, M. A. Dahleh and I. J. Diaz-Bobillo, "Controller design via infinite-dimensional linear programming," *Proc. Amer. Contr. Conf.*, vol. 3, pp. 2165-2169, 1993
- [Fujisaki and Yoshida 96] Y. Fujisaki and T. Yoshida, "A linear matrix inequality approach to mixed  $\mathcal{H}_2/\mathcal{H}_\infty$  control," *IFAC World Congress*, vol. C, pp. 361-366, 1996
- [Ge et al. 94] Y. Ge, E. G. Collins, L. T. Watson and D. S. Bernstein, "Probability-one homotopy algorithms for full and reduced order  $\mathcal{H}_2/\mathcal{H}_\infty$  controller synthesis," *Proc. IEEE Conf. Decision Contr.*, vol. 3, pp. 2672-2677, 1994
- [Geromel et al. 92] J. C. Geromel, P. L. D. Peres and S. R. Souza, "Mixed  $\mathcal{H}_2/\mathcal{H}_\infty$  control for continuous-time linear systems," *Proc. IEEE Conf. Decision Contr.*, pp. 3717-3722, 1992
- [Glover and Mustafa 89] K. Glover and D. Mustafa, "Derivation of the maximum entropy  $\mathcal{H}_\infty$ -controller and a state-space formula for its entropy," *Int. J. Contr.*, vol. 50, no. 3, pp. 899-916, 1989
- [Haddad and Bernstein 90] W. M. Haddad and D. S. Bernstein, "Generalized Riccati equations for the full- and reduced-order mixed-norm  $\mathcal{H}_2/\mathcal{H}_\infty$  standard problem," *Syst. Contr. Lett.*, vol. 14, no. 3, pp. 185-197, 1990
- [Haddad et al. 91] W. M. Haddad, D. S. Bernstein and D. Mustafa, "Mixed-norm  $\mathcal{H}_2/\mathcal{H}_\infty$  regulation and estimation: The discrete-time case," *Syst. Contr. Lett.*, vol. 16, no. 4, pp. 235-247, 1991
- [Halikias 94] G. D. Halikias, "The hierarchical  $\mathcal{H}_\infty/\mathcal{H}_2$ -optimal control problem," *Proc. IEEE Conf. Decision Contr.*, vol. 4, pp. 3165-3166, 1994
- [Hrissagis et al. 96] K. Hrissagis, O. D. Crisalle and M. Sznaier, "Robust predictive control with constraints via  $\ell_1/\mathcal{H}_\infty$  design," *IFAC World Congress*, vol. G, pp. 237-242, 1996
- [Kapila and Haddad 95] V. Kapila and W. M. Haddad, " $\mathcal{H}_2$  and mixed- $\mathcal{H}_2/\mathcal{H}_\infty$  stable stabilization," *Proc. IEEE Conf. Decision Contr.*, vol. 2, pp. 1911-1916, 1995
- [Khargonekar and Rotea 91] P. P. Khargonekar and M. A. Rotea, "Mixed  $\mathcal{H}_2/\mathcal{H}_\infty$  control: A convex optimization approach," *IEEE Trans. Automat. Contr.*, vol. 36, no. 7, pp. 824-837, 1991
- [Mustafa 89] D. Mustafa, "Relations between maximum-entropy/ $\mathcal{H}_\infty$  control and combined  $\mathcal{H}_\infty$ /LQG control," *Syst. Contr. Lett.*, vol. 12, no. 3, pp. 193-203, 1989
- [Mustafa and Glover 88] D. Mustafa and K. Glover, "Controllers which satisfy a closed-loop  $\mathcal{H}_\infty$ -norm bound and maximize an entropy integral," *Proc. IEEE Conf. Decision Contr.*, vol. 2, pp. 959-964, 1988
- [Mustafa et al. 91] D. Mustafa, K. Glover and D. J. N. Limebeer, "Solutions to the  $\mathcal{H}_\infty$  general distance problem which minimize an entropy integral," *Automatica*, vol. 27, pp. 193-199, 1991.
- [Nesterov and Nemirovsky 93] Y. Nesterov and A. Nemirovsky, "Interior point polynomial methods in convex programming: Theory and applications," *SIAM*, 1993
- [Paganini et al. 94] F. Paganini, R. D'Andrea and J. C. Doyle, "Behavioral approach to robustness analysis," *Proc. Amer. Contr. Conf.*, vol. 3, pp. 2782-2786, 1994
- [Rotea and Khargonekar 91] M. A. Rotea and P. P. Khargonekar, " $\mathcal{H}_2$ -optimal control with an  $\mathcal{H}_\infty$ -constraint: the state feedback case," *Automatica*, vol. 27, pp. 307-316, 1991
- [Scherer 95] C. W. Scherer, "Mixed  $\mathcal{H}_2/\mathcal{H}_\infty$  control for linear parametrically varying systems," *Proc. IEEE Conf. Decision Contr.*, vol. 3, pp. 3182-3187, 1995
- [Scherer et al. 97] C. Scherer, P. Gahinet, and M. Chilali, "Multi-objective Output-Feedback Control via LMI Optimization," *IEEE Trans. Automat. Contr.*, vol. 42, no. 7, pp. 896-910, 1997
- [Shim 94] D. Shim, "Mixed  $\mathcal{H}_2$ /ESPR (Extended Strictly Positive Real) control for state-feedback case," *Proc. IEEE Conf. Decision Contr.*, vol. 4, pp. 3175-3180, 1994
- [Skogestad et al. 88] S. Skogestad, M. Morari and J. Doyle, "Robust control of ill-conditioned plants: High-purity distillation," *IEEE Trans. Automat. Contr.*, vol. 33, no. 12, pp. 1092-1105, 1988
- [Steinbuch and Bosgra 94] M. Steinbuch and O. H. Bosgra, "Robust performance  $\mathcal{H}_2/\mathcal{H}_\infty$  optimal control," *Proc. IEEE Conf. Decision Contr.*, vol. 4, pp. 3167-3172, 1994
- [Stoorvogel 93] A. A. Stoorvogel, "The robust  $\mathcal{H}_2$  control problem: A worst-case design," *IEEE Trans. Automat. Contr.*, vol. 38, no. 9, pp. 1358-1370, 1993
- [Sznaier 93] M. Sznaier, "Mixed  $\ell_1/\mathcal{H}_\infty$  controllers for SISO discrete time systems," *Proc. IEEE Conf. Decision Contr.*, vol. 1, pp. 34-38, 1993
- [Sznaier 94] M. Sznaier, "Mixed  $\ell_1/\mathcal{H}_\infty$  controllers for MIMO discrete-time systems," *Proc. IEEE Conf. Decision Contr.*, vol. 4, pp. 3187-3192, 1994
- [Sznaier and Blanchini 94] M. Sznaier and F. Blanchini, "Mixed  $\mathcal{L}_1/\mathcal{H}_\infty$  controllers for MIMO continuous-time systems," *Proc. IEEE Conf. Decision Contr.*, vol. 3, pp. 2702-2707, 1994
- [Sznaier and Bu 96] M. Sznaier and J. Bu, "On the properties of the solution to mixed  $\ell_1/\mathcal{H}_\infty$  control problems," *IFAC World Congress*, vol. G, pp. 249-254, 1996
- [Sznaier et al. 95] M. Sznaier, M. Holmes and J. Bu, "Mixed  $\mathcal{H}_2/\mathcal{L}_1$  control with low order controllers: A linear matrix inequality approach," *Proc. IEEE Conf. Decision Contr.*, vol. 2, pp. 1352-1357, 1995
- [Voulgaris 94] P. Voulgaris, "Optimal  $\mathcal{H}_2/\ell_1$  control: The SISO case," *Proc. IEEE Conf. Decision Contr.*, vol. 4, pp. 3181-3186, 1994
- [Vroemen 96] B. G. Vroemen, "Multiobjective control: A survey," Technical Report WFW 96.013, Eindhoven University of Technology, 1996
- [Wu and Chu 96] J. Wu and J. Chu, "Mixed  $\mathcal{H}_2/\ell_1$  control for discrete-time systems," *IFAC World Congress*, vol. G, pp. 453-457, 1996
- [Zhou et al. 90] K. Zhou, J. Doyle, K. Glover and B. Bodenheimer, "Mixed  $\mathcal{H}_2$  and  $\mathcal{H}_\infty$  control," *Proc. Amer. Contr. Conf.*, vol. 3, pp. 2502-2507, 1990