Some examples of mathematical modelling in glass industry

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by

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The evolution of viscous gobs is considered as they appear in glass morphology. It is shown how this can be modelled by a Stokes equation. Two applications are considered in more details. One is the densification process through sintering and the other the pressing of a glass in a mould.

1. Introduction

Glass is an interesting material. It is easy to make, as there is an abundance of raw material, like silicium and soda; it is 100% recyclable, a strong advantage over competing materials; most of all it can be used in a large variety of products, like a panes, jars or CRT's. Traditionally, craftmenship and more experience have been the main ingredients of expertise. But even for a seemingly low tech production process like bottle making, sharp competition, enviromental requirements (since high temperatures are essential) and esthetics and strength requirements, necessitate a better and more refined understanding of the process involved. To some extent this can be achieved by carrying out experiments. However, mathematical modelling and simulation are proving their strength here. Once a model has been developed, it may prove quite versatile and cheaper than trial and error methods. In this paper we give two examples of such models. The first one deals with viscous sintering and the second one with pressing glass in a mould.

2. Modelling the problem

In order to describe the morphology of glass we use the (standard) conservation laws of momentum, mass and energy. As for the temperature, although it plays an essential role in the morphology, the process considered here can be viewed isothermal and hence the energy equation can be left out. Given a sufficiently high temperature the viscosity is low enough to model the glass as an incompressible Newtonian fluid. Denoting the time dependent glass domain by Ω_t we have the following Navier-Stokes equations

$$\begin{aligned} \rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) - \operatorname{div}\sigma &= \rho\mathbf{f}, \qquad \text{in } \Omega_t, \\ \operatorname{div}\mathbf{v} &= 0, \qquad \text{in } \Omega_t, \end{aligned} \tag{1}$$

where ρ is the mass density and f are the volume forces. The stress tensor σ is related to velocity gradient $\nabla\mathbf{v}$, pressure p and dynamic viscosity μ as follows:

$$\sigma = -p\mathrm{I} + \mu\left(\nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathrm{T}}\right) \tag{2}$$

Substitution of (2) into (1) gives us the following equations

$$\begin{aligned} \rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) &= \rho\mathbf{f} + \nabla\mathrm{p} - \mu\nabla^2\mathbf{v} \quad \text{in } \Omega_t, \\ \operatorname{div}\mathbf{v} &= 0, \qquad \text{in } \Omega_t. \end{aligned} \tag{3}$$

These equations can be rewritten in dimensionless form. Using reference values indicated with a prime we find

$$\mathbf{v} = U\mathbf{v}', \quad p = \frac{1}{Re}\rho U^2 p', \quad Re = \frac{\rho UL}{\mu},$$

$$\mathbf{x} = L\mathbf{x}', \quad t = \frac{L}{U}t',$$

where Re is the Reynolds number and $\mathbf{v}', p', \mathbf{x}', t'$ are dimensionless variables. Then the previous system of equations

(3) reads as follows:

$$\begin{aligned} Re\left(\frac{\partial \mathbf{v}'}{\partial t'} + \mathbf{v}' \cdot \nabla \mathbf{v}'\right) &= Re\frac{L}{U^2}\mathbf{f} + \nabla \mathbf{p}' - \mu \nabla^2 \mathbf{v}' \quad \text{in } \Omega_t, \\ \operatorname{div} \mathbf{v}' &= 0, \qquad\qquad\qquad\qquad \text{in } \Omega_t. \end{aligned} \tag{4}$$

For smaller domains the Reynolds number is typically quite small and the same holds for the volume forces. Hence we effectively have a Stokes flow

$$\begin{aligned} \mu \nabla^2 \mathbf{v} - \nabla p &= 0 \qquad \text{in } \Omega_t, \\ \operatorname{div} \mathbf{v} &= 0, \qquad \text{in } \Omega_t. \end{aligned} \tag{5}$$

Since (5) is actually a stationary problem the evolution must come from kinematic boundary conditions. We have to complete these equations therefore with boundary conditions for velocity and pressure.

Once we have the velocity field $\mathbf{v}$ we can find the domain from solving

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}(\mathbf{x}). \tag{6}$$

3. Application: 1 Viscous sintering

If Ω_t is a viscous gob we have a free surface. The driving force is then the surface tension which is proportional to the curvature, i.e.

$$\tau \boldsymbol{n} = \kappa \mathbf{n}, \tag{7}$$

where τ is the stress tensor with

$$\tau_{ij} = -p\delta_{ij} + \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \tag{8}$$

$\mathbf{n}$ is the outward normal vector and κ the curvature.

If we let two such gobs coalesce this ariving force will eventualy produce a circular cylinder of ball (in 2-D and 3-D respectively). The coalescence itself is a delicate matter to deal with numerically: initially the curvature at the touching point is infinity, but also further on at the so called *neck* (see fig. 1) we have large curvatures which render numerical computations ineffective. An elegant way out of this is using a sophisticated conformal mapping described by Hopper[3]. He gives a formula for the actual shape which can fruitfully be imployed to be combined with numerical computations.

The nature of the problem, with the need to describe the evolution of the boundary makes it quite naturally to solve the problem by a Boundary Element Method (BEM). Taking into account that we actually have a Neumann problem we should make the solution unique by fixing the translation and rotation of the body. The results of a simple coalescence are displayed in fig. 1.

We model this by considering a so called unit cell, being part of a periodic lattice. In such a unit cell one may design a typical set of larger and smaller particles. In Fig. 2 we have displayed the shrinking of pores in such a process. The actually used numerical method was based on boundary element and sophisticated time stepping [2]: we first solve (5) for the given domain Ω_t and then find the new boundary at $t + \Delta t$ by using a discretised form of (6).

4. Application: 2 Forming of parison

The production of various glass products often contains of a phase where the material is pressed in mould. In Fig 3 we have sketched such a process: a glass blob is falling down and then a stamp is moving up while the mould is closed. Assuming the mould is axisymmetric we can reformulate the equation in cylindrical coordinates and obtain a 2-D problem. Clearly we now have a partially free and a partially fixed boundary for the Stokes problem. The latter poses an additional problem when solving (6). Indeed, small displacements, as e.g. found by an Euler forward step may give points outside the physical domain. Therefore some "clipping" procedure is needed, see fig 4. This, however, implies that material is removed, thus violating the mass conservation law $\operatorname{div} \mathbf{v} = 0$. There are various ways to deal with the latter problem, which will be reported elsewhere. Clearly, if the time step is small enough this problem can be eliminated virtually. In fig 5 we have displayed a numerical simulation using a FEM code to solve

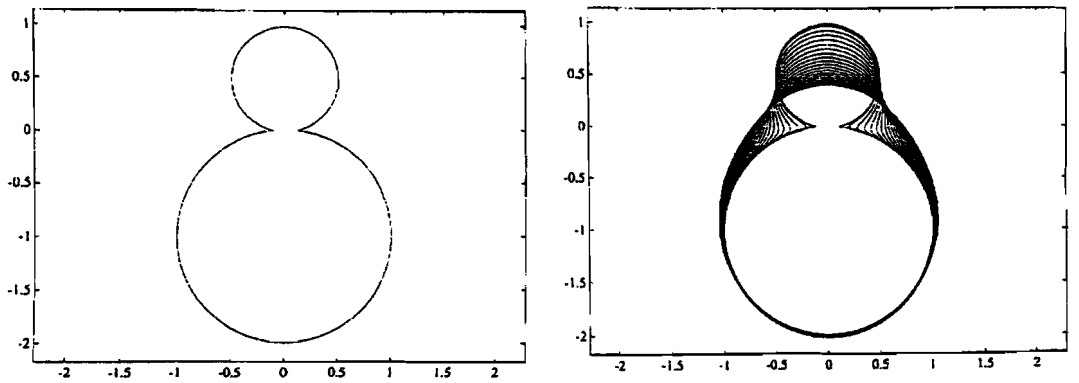

Figure 1: Two cylinders with different diameters and their transformation in time.

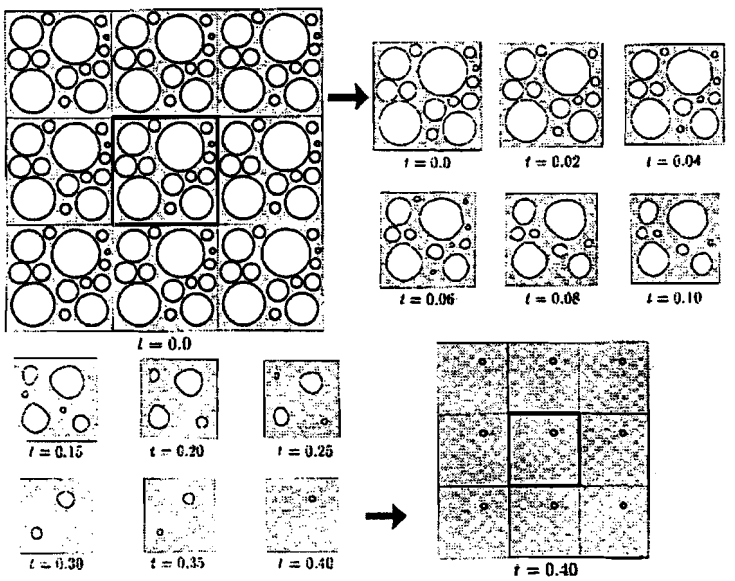

Figure 2: The shape deformation of the unit cell of a period lattice non-uniformly sized cylindrical pores at subsequent times demonstrates that the pores vanish in order of size.

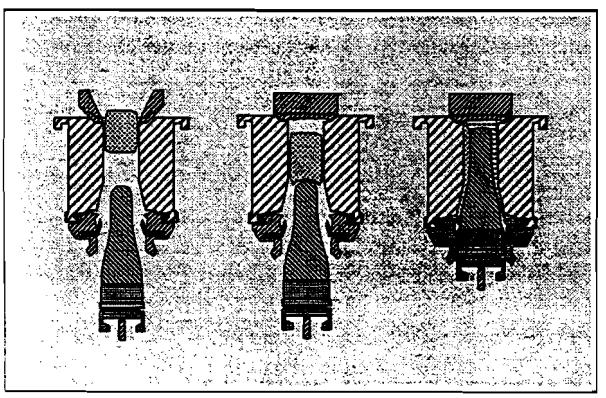

Figure 3: The pressing phase.

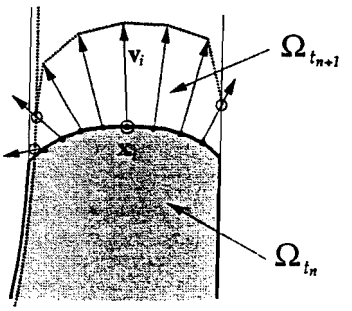

Figure 4: Clip algorithm.

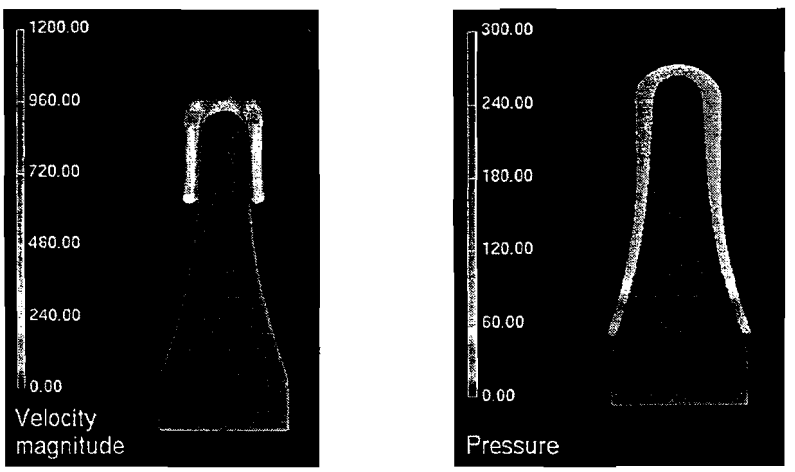

Figure 5: Numerical simulations

the Stokes equations.

Acknowledgements

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