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In this note, we establish sufficient conditions for invertibility of bounded linear operators on Banach spaces. Herewith, we extend and reprove a result of Hilding [Hi] presented in the context of Hilbert spaces.

For a bounded linear operator T on a complex Banach space X by $\sigma(T)$ we denote the spectrum of T and by $\rho(T)$ its resolvent set. So $\rho(T) = \mathbb{C}\backslash\sigma(T)$. We recall that $\sigma(T)$ is non-empty compact subset of $\mathbb{C}$. By $B(X,Y)$, we denote the collection of all bounded linear operators from the Banach space X into the Banach space Y. Also, we write $B(X)$ instead of $B(X,X)$.

Lemma 1. For $T \in B(X)$, suppose there are $\alpha_0 \in \mathbb{R}$ and $\beta_0 > 0$ such that

$$\|Tx - \alpha x\| \geq \beta_0 \|x\|$$

for all $x \in X$ and all $\alpha \leq \alpha_0$. Then the half-infinite interval $(-\infty, \alpha_0]$ is contained in $\rho(T)$.

Proof. We may assume that $\sigma(T) \cap \mathbb{R} \neq \emptyset$. Let

$$\lambda_0 := \min\{\lambda \in \mathbb{R} \mid \lambda \in \sigma(T)\} .$$

Now suppose $\lambda_0 \leq \alpha_0$. Put $\alpha_n = \lambda_0 - \frac{1}{n}$, $n \in \mathbb{N}$. Then for all n, $\alpha_n \in \rho(T)$. Since

$$\|Tx - \lambda_0 x\| \geq \beta_0 \|x\| , \quad x \in X ,$$

$T - \lambda_0$ is injective (and has closed range). To arrive at a contradiction we shall prove that $T - \lambda_0$ is surjective. Let $y \in X$, and define the sequence (x_n) in X by

$$x_n = (T - \alpha_n)^{-1} y , \quad n \in \mathbb{N} .$$

By the second resolvent identity, for all $n, m \in \mathbb{N}$

$$\|x_n - x_m\| = |\frac{1}{n} - \frac{1}{m}| \, \|(T-\alpha_n)^{-1}(T-\alpha_m)^{-1}y\| \leq \frac{1}{\beta_0^2}|\frac{1}{n} - \frac{1}{m}| \, \|y\| \, .$$

So the sequence $(x_n)_{n\in\mathbb{N}}$ converges, to $x \in X$ say. Then for all $n \in \mathbb{N}$

$$\|(T-\lambda_0)x - y\| \leq \|(T-\lambda_0)(x-x_n)\| + \frac{1}{n}\|x_n\| \, ,$$

so that $y = (T-\lambda_0)x$.

We conclude that $\lambda_0 > \alpha_0$. □

Theorem 2. Let $T \in B(X)$. Suppose there are $\alpha_1 \in \mathbb{R}$ and $\beta_1 > 0$ such that

$$\|Tx - \alpha x\| \geq \beta_1 \|x\|$$

for all $x \in X$ and $\alpha \leq \alpha_1$. Then $(-\infty, \alpha_1 + \beta_1) \subset \rho(T)$.

Proof. Let $\alpha_0 = \alpha_1 + \eta$ and $\beta_0 = \beta_1 - \eta$, where $0 < \eta < \beta_1$. Then for all α with $\alpha_1 \leq \alpha \leq \alpha_0$ and all $x \in X$

$$\|Tx - \alpha x\| \geq \|Tx - \alpha_1 x\| - (\alpha - \alpha_1)\|x\| \geq (\beta_1 - \eta)\|x\|$$

We conclude that

$$\|Tx - \alpha x\| \geq \beta_0 \|x\|$$

for all $\alpha \leq \alpha_0$ and $x \in X$. Hence $(-\infty, \alpha_0] \subset \rho(T)$. The result follows, since

$$(-\infty, \alpha_1 + \beta_1) = \bigcup_{\eta < \beta_1} (-\infty, \alpha_1 + \eta] \subset \rho(T) \, .$$

□

Theorem 3. Let $T : X \to X$ be a linear operator. Suppose there are $\lambda_1, \lambda_2 \in \mathbb{R}$ with $0 \leq \lambda_1, \lambda_2 < 1$ such that

$$\|Tx - x\| \leq \lambda_1\|x\| + \lambda_2\|Tx\| \ .$$

Then

a) T is bounded with $\|T\| \leq \dfrac{1+\lambda_1}{1-\lambda_2}$

b) $\Big(-\infty, \dfrac{1-\lambda_1}{1+\lambda_2}\Big) \subset \rho(T)$.

c) T is invertible with $\|T^{-1}\| \leq \dfrac{1+\lambda_2}{1-\lambda_1}$.

Proof.

a) Let $x \in X$. Then

$$\|Tx\| \leq \|x\| + \|Tx - x\| \leq (1+\lambda_1)\|x\| + \lambda_2\|Tx\|$$

so that $\|Tx\| \leq \dfrac{1+\lambda_1}{1-\lambda_2}\|x\|$.

b) Let $\alpha \leq 0$, and $x \in X$. Then

$$\|Tx - \alpha x\| = \|Tx - x + (1-\alpha)x\| \geq (1-\alpha)\|x\| - \lambda_1\|x\| - \lambda_2\|Tx\|$$

$$\geq (1 - \alpha - \lambda_1 + \lambda_2\alpha)\|x\| - \lambda_2\|Tx - \alpha x\| \ .$$

Hence for all $x \in X$ and $\alpha \leq 0$

$$\|Tx - \alpha x\| \geq \frac{1 - \lambda_1 - (1-\lambda_2)\alpha}{1+\lambda_2}\|x\| \geq \frac{1-\lambda_1}{1+\lambda_2}\|x\| \ .$$

By Theorem 2 with $\alpha_1 = 0$ and $\beta_1 = \dfrac{1-\lambda_1}{1+\lambda_2}$, we obtain

$$\Big(-\infty, \frac{1-\lambda_1}{1+\lambda_2}\Big) \subset \rho(T) \ .$$

c) Since $0 \in \rho(T)$ by b), T is invertible. Moreover

$$\|Tx\| \geq \frac{1-\lambda_1}{1+\lambda_2}\|x\| \ , \quad x \in X \ ,$$

so that $\|T^{-1}\| \leq \dfrac{1+\lambda_2}{1-\lambda_1}$. $\square$

We mention some consequences of the above theorem.

Corollary 4. Let $T : X \to X$ be a *bounded* linear opertor. Suppose there is $\lambda \in \mathbb{R}$ with $0 \leq \lambda < 1$ such that

$$\|Tx - x\| \leq \lambda\|x\| + \|Tx\| \ .$$

Then

a) $\left(-\infty, \dfrac{1-\lambda}{2}\right) \subset \rho(T)$.

b) T is invertible with $\|T^{-1}\| \leq \dfrac{2}{1-\lambda}$.

Proof. Let $\varepsilon_0 := \min\left(1, \dfrac{\lambda}{\|T\|}\right)$. Then for $0 < \varepsilon \leq \varepsilon_0$,

$$0 \leq 1 - \varepsilon < 1 \text{ and } 0 \leq \lambda - \varepsilon\|T\| < 1 \ ,$$

and

$$\|Tx - x\| \leq (\lambda - \varepsilon\|T\|)\|x\| + (1-\varepsilon)\|Tx\| \ .$$

Hence by the preceding theorem

$$\left(-\infty, \frac{1-\lambda+\varepsilon\|T\|}{2-\varepsilon}\right) \subset \rho(T)$$

and

$$\|T^{-1}\| \leq \frac{2-\varepsilon}{1-\lambda-\varepsilon\|T\|} \ .$$

By letting $\varepsilon \downarrow 0$ the assertions a) and b) follow. □

Corollary 5. Let X and Y be Banach spaces and let $U : X \to Y$ be a bounded invertible operator. Let $\lambda_1, \lambda_2 \in I\!R$ with $0 \le \lambda_1 < 1$ and $0 \le \lambda_2 \le 1$. Then for all $V \in B(X,Y)$ satisfying that for all $x \in X$

$$(*) \qquad \|Ux - Vx\| \le \lambda_1 \|Ux\| + \lambda_2 \|Vx\| \ .$$

a) $V - \alpha U$ is invertible for all $\alpha \in \left(-\infty, \frac{1-\lambda_1}{1+\lambda_2}\right)$.

b) V is invertible and $\|V^{-1}\| \le \frac{1+\lambda_2}{1-\lambda_1}\|U^{-1}\|$

Proof. Let $V \in B(X,Y)$ satisfy condition $(*)$. Put $S = VU^{-1}$. Then for all $y \in Y$

$$\|y - Sy\| \le \lambda_1 \|y\| + \lambda_2 \|Sy\| \ .$$

We conclude from the above results that $S - \alpha I$ is invertible for all $\alpha \in \left(-\infty, \frac{1-\lambda_1}{1+\lambda_2}\right)$ and that $\|S^{-1}\| \le \frac{1+\lambda_2}{1-\lambda_1}$. Now the assertions a) and b) follow from the observation that $V - \alpha U = (S - \alpha I)U$. □

Definition 6. For X and Y Banach spaces, $U \in B(X,Y)$ is said to be right invertible if there is $U^+ \in B(Y,X)$ such that $UU^+y = y$ for all $y \in Y$. U^+ is called a right inverse of U.

Remark. Recall that for X and Y Hilbert spaces the following are equivalent

1) $U \in B(X,Y)$ is right invertible.

2) $U \in B(X,Y)$ is surjective.

3) $U \in B(X,Y)$ and $UU^* \in B(Y)$ invertible.

Corollary 7. Let X and Y be Banach spaces and $U \in B(X,Y)$ be right invertible with right inverse U^+. Then all $V \in B(X,Y)$ for which there are $\lambda_1, \lambda_2, \lambda_3 \in I\!R$ such that for all $x \in X$,

$$\|Ux - Vx\| \le \lambda_1 \|Ux\| + \lambda_2 \|Vx\| + \lambda_3 \|x\|$$

where $0 \le \lambda_1 + \lambda_3 \|U^+\| < 1$ and $0 \le \lambda_2 \le 1$, are right invertible.

Proof. Let V satisfy the condition, stated in the assertion. Then for all $y \in Y$,

$$\begin{aligned} \|VU^+y - y\| &\le \lambda_1 \|y\| + \lambda_2 \|VU^+y\| + \lambda_3 \|U^+y\| \\ &\le (\lambda_1 + \lambda_3 \|U^+\|)\|y\| + \lambda_2 \|VU^+y\| \ . \end{aligned}$$

By Theorem 3, VU^+ is invertible in $B(Y)$. Hence $V^+ := U^+(VU^+)^{-1}$ is a right inverse of V. Observe that

$$\|V^+\| \le \left(\frac{1 + \lambda_2}{1 - \lambda_1 - \lambda_3 \|U^+\|} \right) \|U^+\| \ .$$

□

Remark. Let X and Y be Banach spaces, and $U \in B(X, Y)$ be right invertible. Let X_0 be a dense subspace of X and let $V_0 : X_0 \to X$ satisfy for all $x \in X_0$

$$\|Ux - V_0 x\| \le \lambda_1 \|Ux\| + \lambda_2 \|Vx\| + \mu \|x\| \ ,$$

where $0 \le \lambda_2 < 1$. Then V_0 extends to a bounded operator $V \in B(X, Y)$ with

$$\|V\| \le \frac{\mu + \lambda_1 \|U\|}{1 - \lambda_2}$$

and for all $x \in X$

$$\|Ux - Vx\| \le \lambda_1 \|Ux\| + \lambda_2 \|Vx\| + \mu \|x\| \ .$$

[Hi] Hilding, S.; Note on completeness theorems of Paley-Wiener type. Ann. of Math. 49, no. 4 (1948), pp. 953-955.

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