Non-separable Gabor schemes : their design and implementation

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Non-separable Gabor Schemes
Their Design and Implementation

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Adriaan Johan van Leest

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Dit proefschrift is goedgekeurd door de promotoren:

prof.dr.ir. J.W.M. Bergmans
en
prof.dr. H.G. Feichtinger

Copromotor:

dr.ir. M.J. Bastiaans

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promotiecommissie:

dr.ir. M.J. Bastiaans

prof.dr.ir. J.W.M. Bergmans

prof.dr.ir. W.M.G. van Bokhoven

prof.dr.ir. P.P.J. van den Bosch

dr.ir. A.C. den Brinker

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prof.dr. H.G. Feichtinger

prof.dr. LJ. Stanković

dr.ir. F.M.J. Willems

prof.dr.ir. A.P.M. Zwamborn

Summary

Gabor's signal expansion represents a signal as a series of shifted and modulated versions of a window. Gabor chose a Gaussian window, since it is optimal with respect to localization in the time-frequency plane. The Gabor expansion coefficients can be calculated by taking inner products of the signal with a dual window, and can be interpreted as samples from the windowed Fourier transformed signal based on this dual window. As a consequence, if this dual window is well-localized in the time-frequency plane as well, then the coefficients of Gabor's signal expansion exhibit the time-frequency information of the signal.

Gabor considered the separable (rectangular) case, i.e., the Gabor scheme where the set of shifted and modulated versions of a window corresponds to a separable lattice in the time-frequency plane. In this thesis, it is shown that in the sense of stability a non-separable (hexagonal) Gabor scheme with a Gaussian window is better than a separable one. Conceivably, other non-separable Gabor schemes than the hexagonal Gabor scheme in connection with different types of windows yield a higher stability than a separable one. Multi-dimensional non-separable Gabor schemes for continous-time and discrete-time signals is the topic of this thesis. The one-dimensional non-separable Gabor scheme is treated separately for illustrative purposes.

The Zak transformation plays a central role in this thesis. The Zak transformation has proved its value in connection with the separable Gabor scheme. It can be used to calculate the dual window for a given window and to calculate the Gabor expansion coefficients, and to reconstruct the signal. In this thesis it is shown that the Zak transformation can also be used in the case of non-separable Gabor schemes.

Non-separable lattices can be obtained, for instance, via a scaled rotation operation or a shear operation on a rectangular lattice. A scaled rotation in the time-frequency plane can be associated with the fractional Fourier transform, and a shear can be associated with multiplications by quadratic phase terms. In this thesis it is shown how these operations can be used to re-use algorithms designed for separable Gabor schemes in the non-separable case.

The separable Gabor scheme can be implemented in the form of a filter bank. It is shown that the non-separable Gabor scheme can be implemented in the form of a filter bank, as well.

Contents

List of symbols

symbol	description	first used on page
$\mathbb{C}$	the set of complex numbers	
$\mathbb{R}$	the set of real numbers	
$\mathbb{Z}$	the set of integers	
$L_2(\mathbb{R})$	the Hilbert space of square-integrable functions on $\mathbb{R}$	
$\ell_2(\mathbb{Z})$	the Hilbert space of square-summable functions on $\mathbb{Z}$	
$\mathcal{W}_h$	the short time Fourier transformation based on window h	2
Δ_h	the spread of h	3
$\mathcal{F}$	the Fourier transformation	4
$\langle x, y \rangle$	the inner product of x and y in the Hilbert space where they belong to	5
$\delta[k]$	a Kronecker delta, with $\delta[0] = 1$ and $\delta[k] = 0$ for $k \neq 0$	5
$\|x\|$	the norm of x in the normed space where x belongs to	8
$\mathcal{T}_\tau$	the time shift operator	8
$\mathcal{M}_\omega$	the frequency shift operator	8
A, B	the frame bounds	8
$\mathcal{V}$	the frame generator	9
$\mathcal{S}$	the frame operator	9
$\mathcal{I}$	the identity operator	10
$\mathcal{R}_\tau$	the continuous fundamental rectangle	11
$\mathcal{Z}$	the Zak transformation	11
$\langle K \rangle$	a finite interval of K successive integers	15
$\mathcal{R}_{N,M}$	the discrete fundamental rectangle	17
$x * y$	the convolution of x and y	18
$\mathcal{C}_h$	the convolution operator	18
$\mathcal{Z}$	the z-transformation	19
$\mathbb{T}$	the unit circle in the complex plane	19
$(\downarrow N)$	the decimaton operator	20
$(\uparrow N)$	the interpolation operator	20
$\rho^{\pm}_{R;m}$	the mth polyphase component	22
$\boldsymbol{M}$	the matrix $\boldsymbol{M}$	23

$\boldsymbol{M}^T$	the transpose of $\boldsymbol{M}$	23
$\underline{v}$	the vector $\underline{v}$	23
$\boldsymbol{I}_d$	the $d \times d$ identity matrix	23
diag	a concatenation of matrices in a diagonal matrix	23
row	a concatenation of matrices in a row	23
col	a concatenation of matrices in a column	23
$\boldsymbol{M}^*$	the complex conjugate and transpose of $\boldsymbol{M}$	24
$\sigma_{\left[\begin{smallmatrix}\tau\\ \omega\end{smallmatrix}\right]}$	the translation operator	32
$P_{\boldsymbol{\Lambda}}(m,k)$	the Poisson summation formula	38
$\mathcal{F}^{(2)}$	the two-dimensional Fourier expansion	40
$\mathcal{F}_{\theta,\xi}$	the fractional Fourier transformation	44
$\mathcal{Q}_{\omega_a}$	the shear operator	48
$\mathbf{0}_d$	the $d \times d$ zero matrix	58
part	a partition of $\mathbb{Z}^d$	63
cont	a partition of $\mathbb{R}^d$	63
fund	a fundamental region of $\mathbb{Z}^d$	63
$\boldsymbol{M}^{-T}$	the transpose and inverse of $\boldsymbol{M}$	64
$\mathcal{F}^{(2d)}$	the $2d$-dimensional Fourier expansion	67
$\mathcal{F}^{(2)}_{dis}$	the two-dimensional discrete Fourier expansion	80
$\phi(n)$	Euler's totient function	97
$\sigma_k(n)$	the divisor function	98
$\underline{1}_d$	the $d \times 1$ vector containing only ones	106
$\mathcal{F}^{(2d)}_{dis}$	the $2d$-dimensional discrete Fourier expansion	110

Chapter 1

Introduction

In Section 1.1, we introduce Gabor's signal expansion for continuous-time signals and for discrete-time signals. Gabor's signal expansion represents a signal as a series of properly shifted and modulated versions of a window. Gabor chose a Gaussian window, since it is optimal with respect to localization in the time-frequency plane. The members of a complete set of shifted and modulated Gaussians are not orthogonal. As a consequence, the Gabor expansion coefficients cannot be simply obtained by taking the inner product of the window with the signal. The Gabor expansion coefficients can be calculated by taking inner products of the signal with a dual window, and can be interpreted as samples from the windowed Fourier transform based on this dual window. As a consequence, if this dual window is also well-localized in the time-frequency plane, then the expansion coefficients exhibit the time-frequency information of the signal. To have a dual window that is well-localized in the time-frequency plane, we consider the case of oversampling. Due to this oversampling, the set of shifted and modulated windows is 'overcomplete', i.e., if a member is excluded the orthogonal complement remains zero. In addition, as mentioned, the members of a complete set of shifted and modulated Gaussians are not orthogonal. This makes it more difficult to calculate the Gabor expansion coefficients. Mathematically, this problem can be solved by using the concept of frame. Frame theory along with the concept of Gabor frame is introduced in Section 1.2 to deal with this problem.

The Zak transformation, which plays a central role throughout this thesis, has proved its value in connection with Gabor's signal expansion. It provides an efficient method to calculate the dual window of a given window and to calculate the Gabor expansion coefficients, and to reconstruct the signal. The Zak transformation for continuous-time signals is introduced in Section 1.3 and for discrete-time signals in Section 1.5.

The Gabor scheme can also be implemented in the form of a filter bank. In Section 1.6, we introduce filter banks in the context of Gabor's signal expansion. These filter banks are uniform DFT filter banks, i.e., filter banks where the filters are modulated versions of a prototype, which can be implemented efficiently by using the polyphase representation. This polyphase representation results in an implementation where only the impulse responses of the prototypes and FFT's are needed.

Gabor considered the separable (rectangular) Gabor scheme, i.e., the set of shifted and modulated versions of the window corresponds to a separable (rectangular) lattice. The topic of this thesis is the non-separable Gabor scheme, i.e., the Gabor scheme where the set of shifted and modulated versions of a window corresponds to a non-separable lattice. In Section 1.7, we show that a non-separable Gabor scheme with a Gaussian window on a hexagonal (quincunx) lattice is better in the sense of stability than a separable one with the same oversampling rate. This observation serves as a motivation to consider more general non-separable lattices.

We end this chapter with an overview of the thesis.

1.1 Gabor's signal expansion

In 1946, Gabor proposed to expand a signal φ as a series of properly scaled shifted and modulated versions of an elementary signal g (see [40] for Gabor's original paper, and [37] for a modern treatment of Gabor analysis). That is the signal φ is represented as

$$\varphi = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a_{mk} g_{mk} \quad \varphi \in L_2(\mathbb{R}), \tag{1.1}$$

where

$$g_{mk}(t) = e^{jk\Omega t} g(t - mT), \tag{1.2}$$

and with coefficients a_{mk} properly chosen. The time shift parameter T and frequency shift parameter Ω were supposed by Gabor to satisfy the relationship $\Omega T = 2\pi$. Gabor's original choice for g was a Gaussian elementary signal

$$g(t) = 2^{\frac{1}{4}} e^{-\pi (t/T)^2}. \tag{1.3}$$

Gabor suggested this signal expansion to describe the degrees of freedom of a signal, where the choice of elementary signals g_{mk} is more related to the actual building up of a signal than the Fourier expansion. Each expansion coefficient a_{mk} represents one complex degree of freedom, where g_{mk} corresponds to a frequency $k\Omega$ present at a time mT (see [8, 40]). This becomes clear if the windowed Fourier transform of the Gaussian elementary signal g is considered. The windowed Fourier transform or short time Fourier transform $\mathcal{W}_h$ is defined as

$$(\mathcal{W}_h g)(\tau, \omega) = \frac{1}{T} \int_{-\infty}^{\infty} g(t) h^*(t - \tau) e^{-j\omega t} \, dt, \tag{1.4}$$

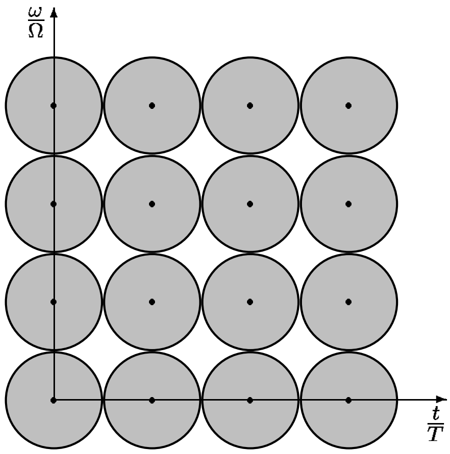

Figure 1.1: *The windowed Fourier transforms $|\mathcal{W}_g g_{mk}|$ of the shifted and modulated versions g_{mk} of the elementary signal g correspond to shifted circular contour lines along a rectangular lattice $\Lambda = \mathbb{Z} \times \mathbb{Z}$.*

where h is a window function. By taking the window function h equal to the Gaussian elementary signal g, the windowed Fourier transformation applied to the signal g takes the form

$$(\mathcal{W}_g g)(\tau, \omega) = e^{-j\frac{1}{2}\omega\tau} e^{-\frac{1}{2}\pi\left[(\tau/T)^2 + (\omega/\Omega)^2\right]}.$$

So $|\mathcal{W}_g g(\tau/T, \omega/\Omega)|$ has circular contour lines in the time-frequency plane. As a consequence, the windowed Fourier transformation with window g applied to g_{mk} has shifted circular contour lines along a rectangular lattice $\Lambda = \mathbb{Z} \times \mathbb{Z}$ in the time-frequency plane

$$(\mathcal{W}_g g_{mk})(\tau, \omega) = e^{-jm\omega T}(\mathcal{W}_g g)(\tau - mT, \omega - k\Omega),$$

which is illustrated in Fig. 1.1. Each g_{mk} occupies a certain region in the time-frequency plane, and represents a quantum of information (Gabor called this a logon). Gabor chose a Gaussian elementary signal g, since this g is optimal with respect to localization in the time-frequency plane. The spread Δ_g, defined as the square root of the normalized second-order moment (see [69])

$$\Delta_g = \frac{1}{\|g\|}\left\{\int_{-\infty}^{\infty} t^2 |g(t)|^2\, dt\right\}^{1/2},$$

of a Gaussian g and the spread $\Delta_{\mathcal{F}g}$ of its Fourier transform $\mathcal{F}g$ (a Gaussian, as well), where the Fourier transformation is defined by

$$(\mathcal{F}\varphi)(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(t) e^{-j\omega t}\, dt,$$

read $\Delta_g = T/2\sqrt{\pi}$ and $\Delta_{\mathcal{F}g} = \Omega/2\sqrt{\pi}$, respectively. If a signal is largely concentrated on the time interval $|t| < \frac{1}{2}a$ while its Fourier transform is largely concentrated on the frequency interval $|\omega| < \frac{1}{2}b$, the number of complex degrees of freedom is less than or equal to the number of expansion coefficients a_{mk} in the time-frequency rectangle with area ab, this number being approximately equal to the time-bandwidth product $(a/T)(b/\Omega) = ab/2\pi$.

For accurate time-frequency localization, one has to choose an elementary signal g such that the spread Δ_g of this signal and the spread $\Delta_{\mathcal{F}g}$ of its Fourier transform are as small as possible. However, the Heisenberg uncertainty inequality states that it is not possible to find arbitrarily well-localized g:

$$\Delta_g \Delta_{\mathcal{F}g} \geq \tfrac{1}{2}.$$

The shifted and modulated Gaussians are the only functions attaining equality in the above inequality.

In general, the members of the set of shifted and modulated versions g_{mk} of the elementary signal g are not orthogonal. Two well-known exceptions are the rectangular-shaped function, $g_r(t) = 1$ for $-\frac{1}{2}T < t \leq \frac{1}{2}T$ and $g_r(t) = 0$ outside that time interval, and the sinc-shaped function, $g_s(t) = \sin(\pi t/T)/(\pi t/T)$. However, although the spread of the rectangular-shaped function is short, the spread of its Fourier transform (a sinc function) is infinite ($\Delta_{g_r}\Delta_{\mathcal{F}g_r} = \infty$). The opposite holds for the sinc-function; the spread of the sinc-function is infinite in time, the spread of its Fourier transform (a rectangular-shaped function) is short ($\Delta_{g_s}\Delta_{\mathcal{F}g_s} = \infty$). Thus, both examples lead to systems with bad localization properties in either time or frequency. Note that it is obvious that g_s also leads to a bad localization, since g_s is proportional to the Fourier transform of g_r.

The Gaussian attains equality in the Heisenberg uncertainty inequality. However, as stated, two elements g_{mk} and $g_{n\ell}$ from the set of shifted and modulated versions of a Gaussian elementary signal g are not orthogonal:

$$\langle g_{mk}, g_{n\ell}\rangle = T(-1)^{(n-m)(\ell-k)} e^{-\frac{1}{2}\pi[(n-m)^2 + (\ell-k)^2]}.$$

As a consequence, the array of Gabor coefficients a_{mk} cannot be calculated in the usual way, i.e., $a_{mk} \neq \langle \varphi, g_{mk}\rangle$ in general. Gabor's paper includes an iterative method to approximate the array of Gabor coefficients a_{mk}. However, quite recently, in [41], it was shown that the convergence of the suggested algorithm depends on the elementary signal g chosen. Although Gabor gave an algorithm to approximate the array of Gabor coefficients a_{mk}, no specific practical use was made of the Gabor expansion due to the difficulty of its calculation. The Gabor expansion lost the interest of the researchers, until in 1980, Bastiaans [5–7] and later on Janssen (see [47]) published an analytical method to compute the Gabor expansion coefficients a_{mk}. A set of shifted and modulated versions γ_{mk} [similar to g_{mk}, as defined in Eq. (1.2)] of a dual elementary signal γ was introduced, nowadays called 'Bastiaans' function', satisfying the bi-orthonormal condition

$$\langle \gamma_{n\ell}, g_{mk} \rangle = \delta[n-m]\delta[\ell-k],$$

where $\delta[k]$ is a Kronecker delta, with $\delta[0] = 1$ and $\delta[k] = 0$ for $k \neq 0$. With this dual elementary signal γ, the Gabor coefficients can be calculated by taking inner products

$$a_{mk} = \langle \varphi, \gamma_{mk} \rangle = \int_{-\infty}^{\infty} \varphi(t)\gamma^*(t-mT)e^{-jk\Omega t}\,dt.$$

This transform is known as the Gabor transform. From this expression, it follows that the Gabor coefficients can be interpreted as samples from the windowed Fourier transform with window γ [cf. Eq. (1.4)]

$$a_{mk} = T(\mathcal{W}_\gamma \varphi)(mT, k\Omega).$$

As a consequence, if the dual elementary signal γ is well-localized both in time and frequency, the array of Gabor coefficients a_{mk} contains time-frequency information of the signal φ. In Fig. 1.2, the dual elementary signal γ is depicted for the case of a Gaussian elementary signal g. This dual elementary signal γ is bounded, but is not in $L_2(\mathbb{R})$ as was pointed out by Janssen (see [49]). Furthermore, it has a poor time-frequency localization, and even worse, there is no numerically stable algorithm to reconstruct the signal φ. In fact, in general it is not possible to have a dual elementary signal γ with a good time-frequency localization. This statement is made precise by the Balian-Low theorem (see [4, 63]). This theorem was originally stated by Balian in 1981 and, independently, by Low in 1985. The Balian-Low theorem holds for a Gabor scheme $\{g_{mk} | m, k \in \mathbb{Z}\}$ that forms an orthonormal basis for $L_2(\mathbb{R})$. The extension of the Balian-Low theorem to bounded unconditional bases was accomplished by Daubechies and Janssen in 1993 (see [27]).

In the above discussion we considered the case of critical sampling. We have seen that in this case, there are no elementary signals g which are well-localized in both time and frequency as well as the corresponding dual elementary signal γ. Oversampling, i.e., $\Omega T < 2\pi$, is one possible way to remedy this situation. The set of shifted and modulated windows g_{mk} now corresponds to a denser rectangular

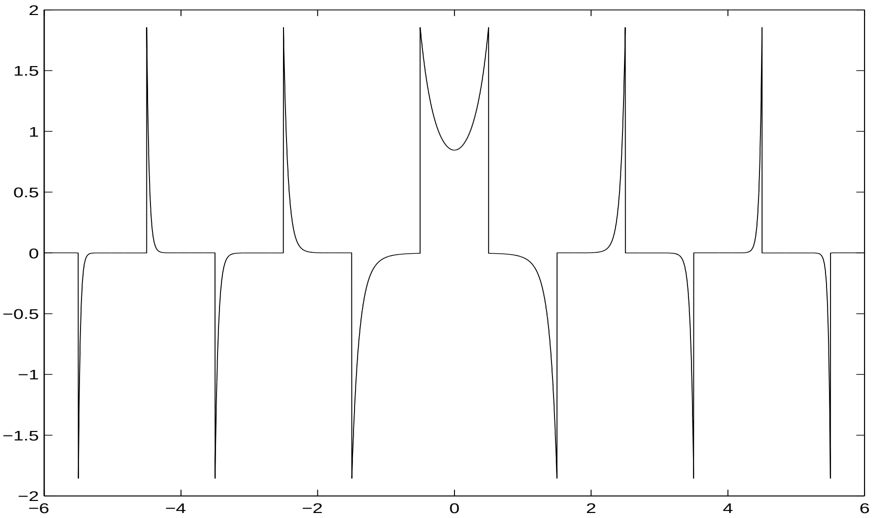

Figure 1.2: *The corresponding dual elementary signal γ of a Gaussian elementary signal g in the case of critical sampling.*

lattice in the time-frequency space:

$$g_{mk}(t) = e^{jk\Omega t} g(t - mT),$$

with $\Omega T < 2\pi$ [cf. Eq. (1.2)]. We shall assume throughout the thesis that due to the oversampling the set is overcomplete, which implies that Gabor's expansion coefficients a_{mk} become non-unique and can no longer be identified as degrees of freedom. We are aware of the fact that this assumption is not trivial. Even for simple windows g such as Gaussians, one-sided and two-sided exponentials, hyperbolic secants, characteristic functions of an interval, this problem is quite hard to solve (see [23, 24, 52, 54, 56]). However, it can be proved (see [23, 52, 78]) that for undersampling, i.e., $\Omega T > 2\pi$, there are no elementary signals g such that the set $\{g_{mk} | m, k \in \mathbb{Z}\}$ is complete in $L_2(\mathbb{R})$. In Fig. 1.3, some dual elementary signals γ of a Gaussian elementary signal g for different oversampling rates $2\pi/\Omega T$ are depicted to illustrate the effect of oversampling. In practice, the oversampling rate has to be kept as low as possible to avoid many calculations and memory storage. An oversampling rate between one and two is used often (see [37]).

In the discussion above we used a Gaussian elementary signal g. However, other choices for g in the case of oversampling are possible, yielding overcomplete sets of shifted and modulated windows.

In this thesis we use the following notation for the shifted and modulated elemen-

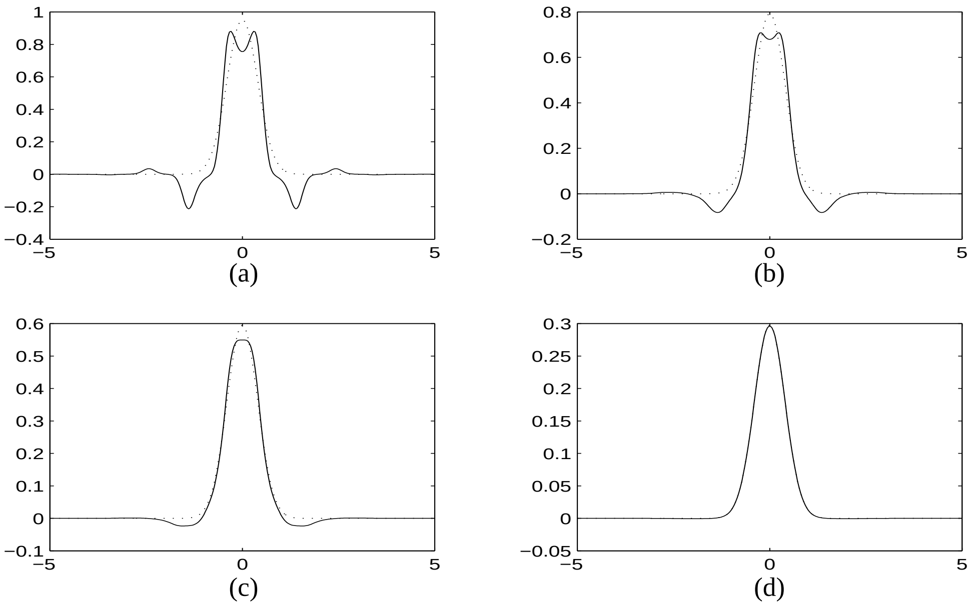

Figure 1.3: *Some dual elementary signals γ corresponding to a Gaussian elementary signal g for different oversampling rates $2\pi/\Omega T$. The solid lines correspond to the dual elementary signal γ and the dotted line correspond to the scaled Gaussian elementary signal g. (a) $2\pi/\Omega T = 5/4$ where $T = \sqrt{4/5}$ (b) $2\pi/\Omega T = 3/2$ where $T = \sqrt{2/3}$ (c) $2\pi/\Omega T = 2$ where $T = \sqrt{1/2}$ (d) $2\pi/\Omega T = 4$ where $T = \sqrt{1/4}$.*

tary signals g_{mk}

$$g_{mk} = e^{jk\beta\Omega t} g(t - m\alpha T), \tag{1.5}$$

where $\alpha\beta \leq 1$ and $\Omega T = 2\pi$. The oversampling rate is given by $2\pi/\alpha\beta\Omega T = 1/\alpha\beta$. The use of parameters α and β while keeping $\Omega T = 2\pi$ stems from the following observation. The Fourier transform $\mathcal{F}g$ of the Gaussian (1.3) reads

$$(\mathcal{F}g)(\omega) = 2^{1/4} \frac{\sqrt{2\pi}}{\Omega} e^{-\pi(\omega/\Omega)^2},$$

i.e., Ω is related to the spread of $\mathcal{F}g$ similarly as T for g. Put differently, Ω and T can be seen as the units of the Gaussian elementary signal that is used in the Gabor scheme. The variable α is the time-shift parameter, and β is the frequency-shift parameter. Note that the use of T and Ω makes only sense in connection with a Gaussian elementary signal. Because of the importance of the Gaussian for Gabor schemes, we will use this notation (1.5) for the shifted and modulated elementary signals g_{mk}.

In modern literature, the elementary signal g is called the window g and the dual elementary signal γ is called the dual window γ. In the sequel, we will adopt this convention too.

1.2 Gabor frames

As mentioned in the previous section, for a general window g the members of a complete set $\{g_{mk}|m,k \in \mathbb{Z}\}$ in $L_2(\mathbb{R})$ of shifted and modulated versions of the window g, i.e., a set $\{g_{mk}|m,k \in \mathbb{Z}\}$ whose span is dense in $L_2(\mathbb{R})$, are in general not orthogonal. If the set $\{g_{mk}|m,k \in \mathbb{Z}\}$ is an orthonormal basis in $L_2(\mathbb{R})$, then we have the representation

$$\varphi = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \langle \varphi, g_{mk} \rangle g_{mk},$$

In general this is not the case. Due to the oversampling, the set is 'overcomplete', i.e., if a member is excluded the orthogonal complement remains zero. So now the following questions arise: a) if the set is not orthogonal how can we compute an admissible family of coefficients a_{mk} for a Gabor expansion (1.1) of φ, and b) how can this be done in a numerically stable way. Mathematically, these questions can be tackled by using the concept of frame. The concept of frame was originally developed in 1952 by Duffin and Schaeffer (see [29]), but became popular in signal analysis after publications of [25] in 1986 and [23] in 1990 by Daubechies et al. In this section, we introduce frame theory along with the concept of Gabor frames. For more mathematical details about frames we refer to [14, 24, 37, 88].

For convenience, we introduce the time translation operator $\mathcal{T}_\tau$ and the modulation operator $\mathcal{M}_\omega$ defined as

$$(\mathcal{T}_\tau \varphi)(t) = \varphi(t-\tau) \quad \text{and} \quad (\mathcal{M}_\omega \varphi)(t) = e^{j\omega t}\varphi(t),$$

respectively. We will use the same notation for the modulation operator $\mathcal{M}_\omega$ and the time translation operator $\mathcal{T}_\tau$ for the discrete-time setting.

The collection $\{g_{mk}|m,k \in \mathbb{Z}\}$ in $L_2(\mathbb{R})$ is called a Gabor frame (or Weyl-Heisenberg frame), if there are $B \geq A > 0$ such that for all $\varphi \in L_2(\mathbb{R})$ the sequence $(\langle \varphi, g_{mk} \rangle)_{m,k\in\mathbb{Z}}$ belongs to $\ell_2(\mathbb{Z}^2)$ with

$$A\|\varphi\|^2 \leq \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |\langle \varphi, g_{mk} \rangle|^2 \leq B\|\varphi\|^2.$$

The largest A and the smallest B are called the frame bounds. We shall assume g to be normalized such that $A + B = 1$, and so, $B = 1 - A$ and $0 < A \leq \frac{1}{2}$.

Let $\{e_{mk}|m, k \in \mathbb{Z}\}$ denote the standard orthonormal basis in $\ell_2(\mathbb{Z}^2)$. Then for all $h \in \ell_2(\mathbb{Z}^2)$

$$h = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \langle h, e_{mk} \rangle e_{mk}.$$

Since the sequence $(\langle \varphi, g_{mk} \rangle)_{m,k \in \mathbb{Z}}$ belongs to $\ell_2(\mathbb{Z}^2)$, we can introduce the frame generator (analysis mapping) $\mathcal{V} : L_2(\mathbb{R}) \to \ell_2(\mathbb{Z}^2)$

$$\mathcal{V}\varphi = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \langle \varphi, g_{mk} \rangle e_{mk}, \quad \varphi \in L_2(\mathbb{R}).$$

Its Hilbert adjoint $\mathcal{V}^*$ (synthesis mapping) is given by

$$\mathcal{V}^* h = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \langle h, e_{mk} \rangle g_{mk}, \quad h \in \ell_2(\mathbb{Z}^2).$$

We see that

$$A\|\varphi\|^2 \leq \|\mathcal{V}\varphi\|^2 = \langle \mathcal{S}\varphi, \varphi \rangle \leq B\|\varphi\|^2,$$

where $\mathcal{S} = \mathcal{V}^*\mathcal{V} : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ is the frame operator:

$$\mathcal{S}\varphi = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \langle \varphi, g_{mk} \rangle g_{mk}.$$

So $\mathcal{S}$ is invertible and self-adjoint. Boundedness below of $\mathcal{V}$ implies that for $\varphi \in L_2(\mathbb{R})$ significantly not equal to 0, i.e., $\|\varphi\| \geq \epsilon$, the sequence $\mathcal{V}\varphi$ is significantly not equal to the null sequence in $\ell_2(\mathbb{Z}^2)$. Boundedness of $\mathcal{V}$ implies that a small perturbation $\Delta\varphi$ of a signal φ results in a small perturbation $\mathcal{V}(\Delta\varphi)$. A frame is said to be a tight frame if the lower bound A is equal to the upper bound B, i.e., $A = B$. Frames that are almost tight, i.e., B/A close to one, are called snug frames.

It is not difficult to show that $\mathcal{S}$ commutes with $\mathcal{T}_{\alpha T}$ and $\mathcal{M}_{\beta\Omega}$:

$$\mathcal{T}_{\alpha T}\mathcal{S} = \mathcal{S}\mathcal{T}_{\alpha T} \quad \text{and} \quad \mathcal{M}_{\beta\Omega}\mathcal{S} = \mathcal{S}\mathcal{M}_{\beta\Omega}.$$

From this it follows that $\mathcal{S}^{-1}$ also commutes with the time shift operator $\mathcal{T}_{\alpha T}$ and frequency shift operator $\mathcal{M}_{\beta\Omega}$. Define $\gamma = \mathcal{S}^{-1}g$. Then

$$\mathcal{S}^{-1}g_{mk} = \mathcal{S}^{-1}\mathcal{M}_{\beta\Omega}^{k}\mathcal{T}_{\alpha T}^{m}g = \mathcal{M}_{\beta\Omega}^{k}\mathcal{T}_{\alpha T}^{m}\mathcal{S}^{-1}g = \mathcal{M}_{\beta\Omega}^{k}\mathcal{T}_{\alpha T}^{m}\gamma = \gamma_{mk}.$$

This means, the elements of the dual frame $\{\gamma_{mk}|m,k \in \mathbb{Z}\}$ are also shifted and modulated versions of a window, in this case the dual window γ. The frame bounds for the dual frame are B^{-1} and A^{-1}. Now we have for all $\varphi \in L_2(\mathbb{R})$

$$\varphi = \mathcal{S}^{-1}\mathcal{S}\varphi = [\mathcal{S}^{-1}\mathcal{V}^*]\mathcal{V}\varphi = \sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} \langle \varphi, g_{mk}\rangle \gamma_{mk}.$$

Since the coefficients $\langle \varphi, g_{mk}\rangle$ are square summable the frame conditions imply that the double sum is unconditionally convergent, i.e., independently of the order of summation. Moreover, since g_{mk} and γ_{mk} are dual we also have

$$\varphi = \sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} \langle \varphi, \gamma_{mk}\rangle g_{mk},$$

i.e., the roles of g_{mk} and γ_{mk} are interchangeable.

With the frame operator $\mathcal{S}$, we now know how to obtain the dual window γ; the dual window γ is obtained by calculating $\mathcal{S}^{-1}g$. The problem is that the inverse frame operator $\mathcal{S}$ often cannot be determined easily. However, if the frame bounds are known (or are approximated well) the inverse frame operator can be approximated by a Neumann series (see [23])

$$\mathcal{S}^{-1} = 2\sum_{n=0}^{\infty} (\mathcal{I} - 2\mathcal{S})^n ,$$

where $\mathcal{I}$ is the identity operator. And thus

$$\gamma = \mathcal{S}^{-1}g = 2\sum_{n=0}^{\infty} (\mathcal{I} - 2\mathcal{S})^n g.$$

The closer B/A to 1, the faster the convergence. This follows from the fact that

$$A\mathcal{I} \leq \mathcal{S} \leq B\mathcal{I},$$

where the inequality $\mathcal{O}_1 \leq \mathcal{O}_2$, with $\mathcal{O}_1$ and $\mathcal{O}_2$ self-adjoint operators in $L_2(\mathbb{R})$, denotes $\langle \varphi, \mathcal{O}_1\varphi\rangle \leq \langle \varphi, \mathcal{O}_2\varphi\rangle$ for all $\varphi \in L_2(\mathbb{R})$. And so

$$-(B-A)\mathcal{I} \leq (\mathcal{I} - 2\mathcal{S}) \leq (B-A)\mathcal{I}.$$

Since $B = 1 - A$ and $0 < A \leq \frac{1}{2}$, we find that

$$\|\mathcal{I} - 2\mathcal{S}\| = \sup_{\substack{\varphi\in L_2(\mathbb{R})\\ \|\varphi\|=1}} \|(\mathcal{I} - 2\mathcal{S})\varphi\| < 1.$$

Consequently, the Neumann series always converges. In general, the frame bounds cannot be calculated exactly except for some windows like the Gaussian and the one- and double-sided exponential in the case of integer oversampling (see [23, 54]). A bad approximation of the frame bounds or a large ratio B/A leads to a poor rate of convergence of the Neumann series. A method to calculate the dual window γ without knowing the frame bounds exactly (of course the set $\{g_{mk}|m,k \in \mathbb{Z}\}$ should be a frame) is the method that uses the Zak transformation. The Zak transformation is the subject of the next section.

1.3 Zak transform

In 1967, Zak introduced (see [89]) a transformation in the setting of solid-state physics. This transformation, called kq-representation by Zak, has been popularized in the field of signal analysis by Janssen (see [50]) and became known as the Zak transformation. In literature it is also known as Weil-Brezin transform or as Gel'fand mapping. The Zak transform plays an important role in the theory of Gabor expansions. It has been shown (see [10, 53, 91]) that the Zak transform leads to efficient algorithms to calculate the dual window and the Gabor expansion coefficients, and to reconstruct the signal.

The Zak transform $(\mathcal{Z}\varphi)(t,\omega;\tau)$ of a signal φ is defined by

$$(\mathcal{Z}\varphi)(t,\omega;\tau) = \sum_{m=-\infty}^{\infty} \varphi(t+m\tau)e^{-jm\omega\tau}. \tag{1.6}$$

We see that the Zak transform $(\mathcal{Z}\varphi)(t,\omega;\tau)$ is periodic in the frequency variable ω with period $2\pi/\tau$ and quasi-periodic in the time variable t with quasi-period τ:

$$(\mathcal{Z}\varphi)(t,\omega+2\pi/\tau;\tau) = (\mathcal{Z}\varphi)(t,\omega;\tau)$$

and

$$(\mathcal{Z}\varphi)(t+\tau,\omega;\tau) = (\mathcal{Z}\varphi)(t,\omega;\tau)e^{j\omega\tau}.$$

So the Zak transform of φ is determined by its values in the fundamental rectangle $\mathcal{R}_\tau$

$$\mathcal{R}_\tau : (-\tfrac{1}{2}\tau < t \leq \tfrac{1}{2}\tau, -\pi/\tau < \omega \leq \pi/\tau).$$

A φ can be recovered from its Zak transform according to

$$\varphi(t) = \varphi(t'+m\tau) = \frac{\tau}{2\pi}\int_{2\pi/\tau} (\mathcal{Z}\varphi)(t',\omega;\tau)e^{jm\omega\tau}\,d\omega, \tag{1.7}$$

where $-\frac{1}{2}\tau < t' < \frac{1}{2}\tau$ and m is an integer. Note that $\sqrt{\tau/2\pi}\mathcal{Z}$ is unitary. Thus we have

$$\langle \varphi_1, \varphi_2 \rangle = \frac{\tau}{2\pi} \langle \mathcal{Z}\varphi_1, \mathcal{Z}\varphi_2 \rangle,$$

or in an explicit form

$$\int_{-\infty}^{\infty} \varphi_1(t)\varphi_2^*(t)\, dt = \frac{\tau}{2\pi} \iint\limits_{\mathcal{R}_\tau} (\mathcal{Z}\varphi_1)(t,\omega;\tau)\, (\mathcal{Z}\varphi_2)^*(t,\omega;\tau)\, dt\, d\omega.$$

In particular we have Parseval's energy theorem

$$\|\varphi\|^2 = \frac{\tau}{2\pi}\|\mathcal{Z}\varphi\|^2.$$

The Zak transformation applied to g_{mk} in the case of critical sampling [see Eq. (1.2)] with $\tau = T$ yields

$$(\mathcal{Z}g_{mk})(t,\omega;T) = e^{-j2\pi(m\omega T - kt\Omega)}\, (\mathcal{Z}g)(t,\omega;T). \tag{1.8}$$

Using this expression it is not difficult to show that in the case of critical sampling we have

$$\mathcal{Z}\mathcal{S}\mathcal{Z}^* h = T|\mathcal{Z}g|^2 h \qquad \text{with} \quad h \in L_2(\mathcal{R}_\tau).$$

Thus the Zak-transformed frame operator is in essence a multiplication by $|\mathcal{Z}g|^2$. Now we find with $h = \sqrt{\tau/2\pi}\mathcal{Z}\varphi$

$$\langle \mathcal{S}\varphi, \varphi \rangle = \langle \mathcal{Z}\mathcal{S}\mathcal{Z}^* h, h \rangle = \frac{T}{\Omega}\langle |\mathcal{Z}g|^2 \mathcal{Z}\varphi, \mathcal{Z}\varphi \rangle.$$

From this it follows that

$$A = \text{ess inf } T|\mathcal{Z}g|^2 \quad \text{and} \quad B = \text{ess sup } T|\mathcal{Z}g|^2.$$

Thus, if the Zak transformed window g has a zero in the fundamental rectangle $\mathcal{R}_T$, then $A = 0$ and the set $\{g_{mk} | m, k \in \mathbb{Z}\}$ is not a frame. By taking $h = \mathcal{Z}\mathcal{S}^{-1}g = \mathcal{Z}\gamma$, we find the relationship between the Zak transformed windows g and γ

$$\mathcal{Z}g = T|\mathcal{Z}g|^2 \mathcal{Z}\gamma,$$

or

$$\mathcal{Z}\gamma = \frac{1}{T\,(\mathcal{Z}g)^*}.$$

Problems arise when the Zak transform $(\mathcal{Z}g)$ has a zero in the fundamental rectangle $\mathcal{R}_T$. In particular, the Zak transform of a Gaussian (see Eq. 1.3) has a zero in the fundamental rectangle $\mathcal{R}_T$. As a consequence, Gabor's signal expansion (1.1) with a Gaussian window g does not converge in a weak $L_2(\mathbb{R})$ sense (see [27, 49]); the set of shifted and modulated Gaussians $\{g_{mk}|m,k\in\mathbb{Z}\}$ does not constitute a frame.

By using the Zak transformation (1.6) and Eq. (1.8), the Zak transformed Gabor expansion (1.1) takes the form

$$(\mathcal{Z}\varphi)\,(tT,\omega\Omega;T) = (\mathcal{Z}g)\,(tT,\omega\Omega;T)\sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} a_{mk}e^{-j2\pi(m\omega-kt)},$$

where the variables t and ω range through an interval of length 1. Hence, the array $\{a_{mk}\}$ of Gabor coefficients can be calculated, at least formally, by using the Fourier transform

$$a_{mk} = \int_0^1\int_0^1 \frac{(\mathcal{Z}\varphi)\,(tT,\omega\Omega;T)}{(\mathcal{Z}g)\,(tT,\omega\Omega;T)}e^{j2\pi(m\omega-kt)}\,dt\,d\omega,$$

i.e., a_{mk} are the Fourier coefficients of the periodic function $\mathcal{Z}\varphi/\mathcal{Z}g$. Again, problems arise when the Zak transform $1/\mathcal{Z}g$ has a non-removable singularity in the fundamental rectangle $\mathcal{R}_T$. In general, it will be the case that

$$\sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}|a_{mk}|^2 = \infty,$$

even when φ and g are well-behaved, rapidly decaying, and finite-energy signals (see [50]).

In order to have frames, we have to oversample the Gabor scheme, i.e., we consider sets of shifted and modulated windows g_{mk} corresponding to denser rectangular lattices in the time-frequency space

$$g_{mk}(t) = e^{jk\beta\Omega t}g(t-m\alpha T),$$

where $\alpha\beta\le 1$ and $\Omega T = 2\pi$. In 1988, Daubechies and Grossmann conjectured in [26] that the set $\{g_{mk}|m,k\in\mathbb{Z}\}$ is a frame when g is the Gaussian (1.3) and $\alpha\beta<1$. In 1992, this was proved by Lyubarskii [64] and, independently, by Seip and Wallstén (see [79]) using advanced methods from entire function theory. In 1994, Janssen [52] presented a new and short proof based on the Wexler-Raz condition [85], and moreover, he gave a short proof for the fact that a set $\{g_{mk}|m,k\in\mathbb{Z}\}$ cannot be a frame when $\alpha\beta>1$ for any $g\in L_2(\mathbb{R})$. In addition to this, when $\alpha\beta<1$, the set of shifted and modulated Gaussians multiplied by a quadratic phase

$$g(t) = 2^{\frac{1}{4}}e^{jct^2}e^{-\pi(t/T)^2},$$

where c is a real number, constitutes a frame as well.[1] The latter result is important

[1] Communicated by A.J.E.M. Janssen.

if we consider Gabor schemes on non-separable lattices (see Section 2.4).

In the case of oversampling, the Zak transformation can also be used to calculate the dual window and the Gabor coefficients, and to reconstruct the signal. However, the method with the Zak transformation is restricted to the case of rational oversampling, i.e., the case that $\alpha\beta = q/p < 1$ with p and q positive integers and relatively prime. For more details about the relationship between the Gabor scheme and the Zak transformation in the case of rational oversampling we refer to [10, 53, 91] and Chapter 1 in [37].

1.4 Discrete Gabor scheme

Until now, we have only considered continuous-time signals. Ultimately, the aim of this thesis is to implement Gabor schemes for discrete-time signals as a software package. This means that we need a Gabor scheme for discrete-time signals. In 1985, Bastiaans introduced (see [9]) the Gabor scheme for discrete-time signals in the case of critical sampling. The Gabor transform was a result of sampling the discrete windowed Fourier transform (see [75]). The reconstruction of the signal, the Gabor expansion, followed directly from the continuous-time case. The case of oversampling was introduced in 1990 by Wexler and Raz (see [85]), which was based on sampling continuous-time periodic signals. This yielded an oversampled Gabor scheme for periodic signals. Orr (see [67]) derived the finite discrete Gabor scheme by periodization and sampling of time-continuous signals in the case of critical sampling. In 1994, Janssen showed (see [51, 55]) under which conditions a Weyl-Heisenberg (Gabor) frame for finite energy discrete-time signals or for discrete-time periodic signals is generated by sampling a Weyl-Heisenberg frame for $L_2(\mathbb{R})$. In this section we introduce the Gabor scheme for finite energy discrete-time signals and for periodic discrete-time signals.

The Gabor expansion represents a signal as a series of properly shifted and modulated versions of a window. In the discrete-time setting these windows are given as

$$g_{mk}[n] = e^{j2\pi kn/K} g[n - mN], \tag{1.9}$$

where N and K are positive integers and where we used the square brackets [] to denote discrete-time signals [cf. Eq. (1.5) on page 7 with αT and $\beta\Omega$ corresponding to N and $2\pi/K$, respectively]. Note that g_{mk} is periodic in the variable k with period K. Similar to the continuous-time case, the elements of the dual frame $\{\gamma_{mk}\}$ are the shifted and modulated versions of the dual window γ [see Eq. (1.9)]. The Gabor expansion for a discrete-time signal φ now has the form

$$\varphi = \sum_{m=-\infty}^{\infty} \sum_{k=\langle K \rangle} a_{mk} g_{mk} \quad \varphi \in \ell_2(\mathbb{Z}), \tag{1.10}$$

where the expression $k = \langle K \rangle$ denotes a finite interval of K successive integers k, and where the array $\{a_{mk}\}$ of Gabor coefficients is obtained by the Gabor transform

$$a_{mk} = \langle \varphi, \gamma_{mk} \rangle. \tag{1.11}$$

Note that due to the periodicity of γ_{mk} in the k-direction, the array $\{a_{mk}\}$ is periodic in the k-direction with period K as well. The rate of oversampling is equal to K/N. We will use the Gabor expansion (1.10) and the Gabor transform (1.11) for signals with infinite support in connection with filter banks (see Sections 1.6 and 4.3).

In the discussion above, the signals can have an infinite support. In practice we can only deal with finite length signals, i.e., if the signal has an infinite support or a very long support, we have to split the signal in parts of finite length. Furthermore, if we want to use fast algorithms like the FFT to calculate the array $\{a_{mk}\}$ and to reconstruct the signal φ, we have to periodize the Gabor transform and the Gabor expansion due to the periodicity property of the discrete Fourier transform. In this thesis we will mainly focus our attention on periodic Gabor schemes for discrete-time signals. Now consider the Gabor transform (1.11) and assume that the signal φ and the dual window γ have a finite length N_φ and N_γ, respectively. Under these conditions of finite support, the array $\{a_{mk}\}$ has a finite support of length M in the m-variable, where the support of length M satisfies the condition

$$MN \geq N_\varphi + N_\gamma + 1. \tag{1.12}$$

Now we periodize the signal φ and the dual window γ with period MN. We shall write a capital letter to indicate that we deal with the periodized version. We denote the space of MN-periodic signals by $\mathcal{P}_{MN}$ with inner product

$$\langle \Phi_1, \Phi_2 \rangle = \sum_{\ell=\langle MN \rangle} \Phi_1[\ell] \Phi_2^*[\ell], \qquad \Phi_1, \Phi_2 \in \mathcal{P}_{MN}.$$

The periodized version Φ of the signal φ and the periodized version Γ of the dual window γ are defined by

$$\Phi[n] = \sum_{i=-\infty}^{\infty} \varphi[n + iMN] \quad \text{and} \quad \Gamma[n] = \sum_{i=-\infty}^{\infty} \gamma[n + iMN],$$

respectively. The shifted and modulated periodic windows Γ_{mk} defined by

$$\Gamma_{mk}[n] = e^{j2\pi kn/K} \Gamma[n - mN]$$

are, due to the exponential, only periodic with period MN, i.e.,

$$\Gamma_{mk}[n + MN] = \Gamma_{mk}[n],$$

if and only if K is a divisor of MN. We shall assume that K is a divisor of MN. For convenience, we write $N = qJ$ and $K = pJ$ with $J = \gcd(K, N)$. Now it follows that $M = pL$, where L is a positive integer such that condition (1.12) is fulfilled. Substituting the periodized versions Φ and Γ_{mk} into the Gabor transform (1.11) yields a periodic array $\{A_{mk}\}$ which is periodic in the m-variable with period M and periodic in the k-variable with period K, i.e., the periodic Gabor transform

$$A_{mk} = \langle \Phi, \Gamma_{mk} \rangle.$$

The array $\{a_{mk}\}$ of Gabor coefficients of the finite signal φ then follows as one period of the periodic array A_{mk}. The periodic signal Φ can be reconstructed using the periodized Gabor expansion as

$$\Phi[n] = \sum_{k=\langle K \rangle} \sum_{m=\langle M \rangle} A_{mk} G_{mk}[n].$$

On the other hand, Φ is the periodized version of the signal φ

$$\Phi[n] = \sum_{i=-\infty}^{\infty} \varphi[n + iMN] = \sum_{i=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{k=\langle K \rangle} a_{mk} g_{mk}[n + iMN].$$

Now we have a periodized Gabor transform and a periodized Gabor expansion, which can be implemented very efficiently by using the discrete Fourier transform and the discrete Zak transformation for periodic signals. The discrete Zak transformation will be the subject of the next section. For other methods to implement the finite and discrete Gabor scheme we refer to [72–74, 87] and to Chapter 8 in [37].

As mentioned before, if the signal φ has an infinite support or a very long support, we can apply an overlap-add technique by dissecting the signal φ and treating the parts separately. The procedure is as follows. The signal φ is represented as a sequence of partial signals φ_s of length N_φ

$$\varphi = \sum_{s=-\infty}^{\infty} \varphi_s, \quad \text{with} \quad \varphi_s = \sum_{i=0}^{N_\varphi - 1} \varphi[sN_\varphi + i] e_{sN_\varphi + i}, \tag{1.13}$$

where $\{e_i | i \in \mathbb{Z}\}$ denotes the standard basis of $\ell_2(\mathbb{Z})$. Upon substituting the expansion (1.13) into the Gabor transform (1.11) we get

$$a_{mk} = \langle \varphi, \gamma_{mk} \rangle = \langle \sum_{s=-\infty}^{\infty} \varphi_s, \gamma_{mk} \rangle = \sum_{s=-\infty}^{\infty} \langle \varphi_s, \gamma_{mk} \rangle = \sum_{s=-\infty}^{\infty} a_{s;mk}, \tag{1.14}$$

where each partial Gabor transform

$$a_{s;mk} = \langle \varphi_s, \gamma_{mk} \rangle$$

can be calculated by periodizing the signals and using algorithms for periodic signals. Note that the signals are periodized with period $MN > N_\varphi$, i.e., there is an overlap between adjacent partial arrays $a_{s;mk}$ in the m-variable. This overlap has to be taken into account to calculate the array a_{mk} in Eq. (1.14) correctly.

1.5 Discrete Zak transform

In 1991, the discrete Zak transformation (DZT) for periodic signals was introduced by Auslander et al. (see [3]). The DZT allows efficient, FFT-based algorithms to calculate the (periodic) dual window and the array of Gabor coefficients, and to reconstruct the (periodic) signal (see [3, 10, 13, 51, 68, 90, 91]). In contrast to the continuous-time case, the discrete-time case is simpler. Although the discrete Zak transformation can also be defined for $\ell_2(\mathbb{Z})$ (see [44]), we will restrict our discussion to the Zak transformation for periodic signals with a fixed period. A systematic study of the discrete Zak transform in the context of signal analysis as an analysis or description tool can be found in [15].

The DZT for periodic signals with period MN (associated to this factorization) is defined as

$$\left(\mathcal{Z}\Phi\right)[n, \ell; N, M] = \sum_{m=\langle M \rangle} \Phi[n + mN] e^{-j2\pi m\ell/M}, \quad \Phi \in \mathcal{P}_{MN}. \tag{1.15}$$

The DZT maps a periodic signal Φ with period MN to a function $\left(\mathcal{Z}\Phi\right)[n, \ell; N, M]$ of two discrete variables n and ℓ defined on the fundamental rectangle $\mathcal{R}_{N,M}$

$$\mathcal{R}_{N,M} = \{n, \ell | 0 \leq n \leq N-1, 0 \leq m \leq M-1, n, \ell \in \mathbb{Z}\}.$$

We see that that the Zak transform $\left(\mathcal{Z}\Phi\right)[n, \ell; N, M]$ is periodic in the frequency variable ℓ with period M and quasi-periodic in the time variable n with quasi-period N:

$$\left(\mathcal{Z}\Phi\right)[n, \ell + M; N, M] = \left(\mathcal{Z}\Phi\right)[n, \ell; N, M]$$

and

$$\left(\mathcal{Z}\Phi\right)[n + N, \ell; N, M] = \left(\mathcal{Z}\Phi\right)[n, \ell; N, M] e^{j2\pi\ell/M}.$$

A Φ can be recovered from its Zak transform according to

$$\Phi[n] = \Phi[n' + mN] = \frac{1}{M} \sum_{\ell=\langle M \rangle} \left(\mathcal{Z}\Phi\right)[n', \ell; N, M] e^{j2\pi m\ell/M},$$

where $n' = 0 \ldots N-1$ and m integer. The discrete Zak transform can be calculated by using N FFT's of length M.

Since the relationship between the Zak transformation and the Gabor scheme in the discrete-time case looks very similar to the continuous-time case (see Section 1.3) we only give the results. The relationship between the Zak transformed windows $\mathcal{Z}G$ and $\mathcal{Z}\Gamma$ is given by

$$N\left(\mathcal{Z}G\right)[n,\ell;N,M]\left(\mathcal{Z}\Gamma\right)^{*}[n,\ell;N,M] = 1.$$

The Zak transformed periodized Gabor expansion is given by

$$\left(\mathcal{Z}\Phi\right)[n,\ell;N,M] = \left(\mathcal{Z}G\right)[n,\ell;N,M] \sum_{m=\langle M\rangle} \sum_{k=\langle K\rangle} A_{mk} e^{-j2\pi(m\ell/M - kn/N)}.$$

The periodized array $\{A_{mk}\}$ can be calculated, at least formally, by using the Fourier transform

$$A_{mk} = \frac{1}{MN} \sum_{n=\langle N\rangle} \sum_{\ell=\langle M\rangle} \frac{\left(\mathcal{Z}\Phi\right)[n,\ell;N,M]}{\left(\mathcal{Z}G\right)[n,\ell;N,M]} e^{j2\pi(m\ell/M - kn/N)}.$$

Again problems arise when the Zak transformed window G has zeros in the fundamental rectangle $\mathcal{R}_{N,M}$.

For more details about the relationship between the Zak transformation and the Gabor scheme in the case of oversampling we refer to [3, 10, 13, 51, 68, 90, 91].

1.6 Filter banks

The Gabor scheme for signals with infinite support [see Eqs. (1.10) and (1.11)] can be implemented by using filter banks. In this section, we introduce the concept of filter bank specialized to Gabor schemes. For generalities about filter banks, we refer to [20, 83], while [16, 21, 22] starts from a more frame theoretical approach.

For convenience we recall our notation for the modulation operator $\mathcal{M}_\omega$

$$(\mathcal{M}_\omega f)[n] = e^{j2\pi n\omega} f[n], \quad \omega \in \mathbb{R},$$

the time translation operator $\mathcal{T}_\tau$

$$(\mathcal{T}_\tau f)[n] = f[n-\tau], \quad \tau \in \mathbb{Z},$$

and the convolution operator $\mathcal{C}_h$ for discrete-time signals

$$\mathcal{C}_h x = \sum_{k=-\infty}^{\infty} \left(\sum_{\ell=-\infty}^{\infty} h[k-\ell]x[\ell] \right) e_k = h * x \quad h \in \ell_1(\mathbb{Z}),$$

where $\{e_k | k \in \mathbb{Z}\}$ is the standard orthonormal basis in $\ell_2(\mathbb{Z})$. The convolution $\mathcal{C}_h x$ can be interpreted as a filtered version of the signal x, using a filter with impulse response h. As mentioned in Section 1.2, we use the same notation for the discrete modulation operator $\mathcal{M}_\omega$ and the discrete time translation operator $\mathcal{T}_\tau$, similar to the continuous-time case. Furthermore, for a better understanding we use the z-transformation. The z-transformation is defined by

$$(\mathcal{Z}x)(z) = \sum_{k=-\infty}^{\infty} x[k] z^{-k} \qquad x \in \ell_2(\mathbb{Z}), z \in \mathbb{T},$$

where $\mathbb{T}$ denotes the unit circle in the complex plane

$$\mathbb{T} = \left\{ z \in \mathbb{C} \middle| |z| = 1 \right\}.$$

An important property of the z-transformation is that it transforms a convolution into a product form

$$\mathcal{Z}\mathcal{C}_h x = (\mathcal{Z}h)(\mathcal{Z}x). \tag{1.16}$$

Furthermore, the z-transform of a modulated and a time translated sequence x are given by

$$(\mathcal{Z}\mathcal{M}_\omega x)(z) = (\mathcal{Z}x)\left(e^{-j2\pi\omega} z\right) \quad \text{and} \quad (\mathcal{Z}\mathcal{T}_\tau x)(z) = z^{-\tau}(\mathcal{Z}x)(z), \tag{1.17}$$

respectively.

To make the connection between filter banks and the Gabor scheme, we first consider the windowed Fourier transform $\mathcal{W}_\gamma$ for discrete-time signals φ, which is defined by

$$(\mathcal{W}_\gamma \varphi)[m, \omega] = \sum_{\ell=-\infty}^{\infty} \varphi[\ell] \gamma^*[\ell - m] e^{-j2\pi\omega\ell}, \tag{1.18}$$

where γ is a window function. Note that $(\mathcal{W}_\gamma \varphi)[m, \omega]$ is periodic in the continuous variable ω with period 1. The windowed Fourier transform can be interpreted as the Fourier transform of the signal φ viewed through the time-shifted window γ^*. Rearrangement of the expression (1.18) yields

$$\begin{aligned}(\mathcal{W}_\gamma \varphi)[m, \omega] &= e^{-j2\pi\omega m} \sum_{\ell=-\infty}^{\infty} \varphi[\ell] \gamma^*[-(m - \ell)] e^{j2\pi\omega(m - \ell)} \\ &= e^{-j2\pi\omega m} (\mathcal{C}_{(\mathcal{M}_\omega h_{(a)})} \varphi)[m],\end{aligned}$$

where $h_{(a)}[n] = \gamma^*[-n]$. From this expression we see that the windowed Fourier transform $(\mathcal{W}_\gamma \varphi)[m, \omega]$ of a signal φ as a function of m with ω fixed can be obtained by filtering the signal φ with a filter with impulse response $\mathcal{M}_\omega h_{(a)}$. Since the array $\{a_{mk}\}$ of Gabor coefficients consists of samples of the windowed Fourier transform [see Eq. (1.11)]

$$a_{mk} = (\mathcal{W}_\gamma \varphi)[mN, k/K],$$

we see that the array $\{a_{mk}\}$ of Gabor coefficients can be obtained, apart from a phase factor $\exp(-j2\pi mkN/K)$, by filtering the signal φ with K filters with impulse responses $h_k^{(a)} = \mathcal{M}_{1/K}^k h_{(a)}$, with $h_{(a)}[n] = \gamma^*[-n]$ and $k = 0 \ldots K-1$, and taking only those samples of the K filtered signals which occur at time points equal to multiples of N. The latter operation is a basic operation in filter bank analysis; this operation is the decimation operation (also called downsampling). The decimation operator $(\downarrow N) : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z})$ is defined by

$$(\downarrow N)x = \sum_{k=-\infty}^{\infty} x[kN] e_k.$$

The z-transform of a decimated signal x is given by

$$(\mathcal{Z}(\downarrow N)x)(z) = \frac{1}{N} \sum_{k=0}^{N-1} (\mathcal{Z}x) \left(F_N^k z^{1/N} \right),$$

with $F_N = \exp(-j2\pi/N)$. The Hilbert adjoint of the decimation operator, the interpolation operator $(\uparrow N)$ (also called upsampling), is defined by

$$(\downarrow N)^* x = (\uparrow N)x = \sum_{k=-\infty}^{\infty} x[k] e_{kN}.$$

The interpolation operator inserts $N-1$ zeros after each entry of its input sequence. The z-transform of an interpolated signal x is given by

$$(\mathcal{Z}(\uparrow N)x)(z) = (\mathcal{Z}x) \left(z^N \right).$$

Note that the phase factor $\exp(-j2\pi mkN/K)$ is due to the non-commutativity of the time translation operator and the modulation operator:

$$\mathcal{M}_{1/K}^k \mathcal{T}_N^m = e^{-j2\pi mkN/K} \mathcal{T}_N^m \mathcal{M}_{1/K}^k.$$

The phase factor could be avoided by using modulated and shifted (dual) windows in the Gabor scheme instead of shifted and modulated windows. However, from a historical point of view, this is not desirable. In Fig. 1.4a it is shown schematically how Gabor's signal expansion coefficients $c_k[m] = \exp(j2\pi mkN/K) a_{mk}$ can be

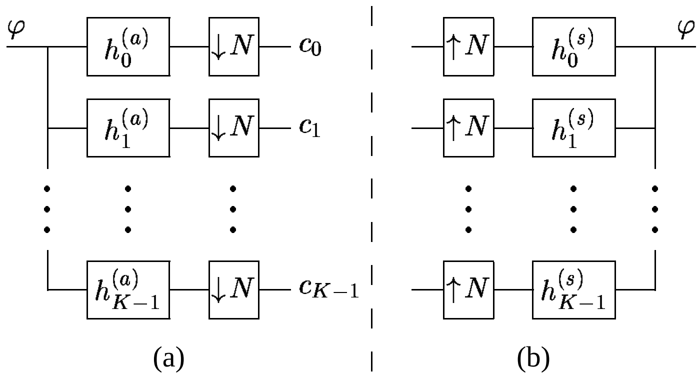

Figure 1.4: *The filter bank consists of two sections: (a) the analysis bank and (b) the synthesis bank.*

calculated with a collection of K parallel filters $h_k^{(a)}$. This collection of K parallel filters is called the analysis bank of the filter bank.

The Gabor expansion can also obtained by a collection of K filters, but now preceded by an interpolation of the K input signals. Consider Gabor's signal expansion (1.10) without the summation over k

$$\sum_{m=-\infty}^{\infty} a_{mk} g_{mk}[n] = \sum_{m=-\infty}^{\infty} c_k[m] g[n-mN] e^{j2\pi k(n-mN)/K}$$
$$= \sum_{m=-\infty}^{\infty} ((\uparrow N)c_k)[m] g[n-m] e^{j2\pi k(n-m)/K} = \left(\mathcal{C}_{h_k^{(s)}} (\uparrow N) c_k\right)[n], \quad (1.19)$$

where $h_k^{(s)} = \mathcal{M}_{1/K}^k h_{(s)}$, with $h_{(s)} = g$. Summation over k yields the signal φ. The reconstruction, the synthesis bank in filter bank analysis, is depicted schematically in Fig. 1.4b.

The analysis and the synthesis filters of the filter bank are modulated versions of $h_{(a)}$ and $h_{(s)}$:

$$h_k^{(a)} = \mathcal{M}_{1/K}^k h_{(a)} \quad \text{and} \quad h_k^{(s)} = \mathcal{M}_{1/K}^k h_{(s)}.$$

Such a filter bank is called a uniform DFT filter bank. The filters with the impulse responses $h_{(a)}$ and $h_{(s)}$ are called the prototypes. Uniform DFT filter banks can be implemented very efficiently. For this efficient implementation we need the following identities of the decimator and the interpolator operator

$$(\uparrow N)(x * y) = ((\uparrow N)x * (\uparrow N)y) \quad (1.20)$$

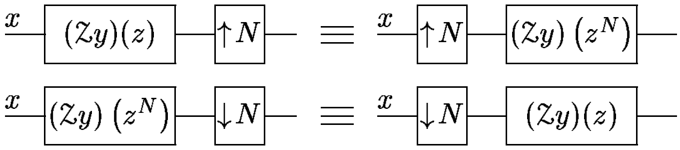

Figure 1.5: *The noble identities.*

and

$$(\downarrow N)\left(x * (\uparrow N)y\right) = \left((\downarrow N)x * y\right), \tag{1.21}$$

which can easily be verified with the help of the z-transformation. These identities are known as the noble identities. In Fig. 1.5 these identities are depicted in a schematical way. In addition to the noble identities we also need the polyphase representation of a sequence. The polyphase representation represents a sequence x in the following way

$$x = \sum_{m=0}^{R-1} \mathcal{T}_m(\uparrow R)(\downarrow R)\mathcal{T}_{-m}x = \sum_{m=0}^{R-1} \mathcal{T}_m(\uparrow R)\rho^-_{R;m}x, \tag{1.22}$$

where $\rho^-_{R;m}x = (\downarrow R)\mathcal{T}_{-m}x$ are the R polyphase components of x, or alternatively

$$x = \sum_{m=0}^{R-1} \mathcal{T}_{-m}(\uparrow R)(\downarrow R)\mathcal{T}_m x = \sum_{m=0}^{R-1} \mathcal{T}_{-m}(\uparrow R)\rho^+_{R;m}x, \tag{1.23}$$

where $\rho^+_{R;m}x = (\downarrow R)\mathcal{T}_m x$ are the R polyphase components of x. The z-transforms are given by

$$(\mathcal{Z}x)(z) = \sum_{m=0}^{R-1} z^{-m}(\mathcal{Z}\rho^-_{R;m}x)\left(z^R\right), \quad \text{and} \quad (\mathcal{Z}x)(z) = \sum_{m=0}^{R-1} z^{m}(\mathcal{Z}\rho^+_{R;m}x)\left(z^R\right),$$

respectively.

Now we have introduced the necessary tools to implement the uniform DFT filter bank efficiently and we will continue by showing how these tools can be used. The z-transformed polyphase representations of the modulated analysis filters with impulse responses $h_k^{(a)} = \mathcal{M}^k_{1/K}h_{(a)}$ are given by

$$\begin{aligned}\left(\mathcal{Z}h_k^{(a)}\right)(z) &= \left(\mathcal{Z}h_{(a)}\right)\left(e^{-j2\pi k/K}z\right)\\ &= \sum_{m=0}^{R-1} e^{-j2\pi mk/K}z^m\left(\mathcal{Z}\rho^+_{R;m}h_{(a)}\right)\left(e^{-j2\pi kR/K}z^R\right).\end{aligned}$$

From this expression, we see that if we take R a multiple of K, we only need the R polyphase components and a $K \times K$ DFT matrix to implement the K analysis filters. In fact, this is the reason why the polyphase representation is used. We shall assume that R is a multiple of K. Then we have

$$\left(\mathcal{Z}h_k^{(a)}\right)(z) = \sum_{m=0}^{K-1} e^{-j2\pi mk/K} \sum_{\ell=0}^{R/K-1} z^{m+\ell K} \left(\mathcal{Z}\rho^{+}_{R;m+\ell K} h_{(a)}\right)(z^R).$$

Or in matrix notation

$$\underline{h}_{(a)}(z) = \boldsymbol{F}_K \boldsymbol{E}_{(a)}(z^R)\underline{d}_R(z), \tag{1.24}$$

where $\underline{h}_{(a)}(z) \in \mathbb{C}^{K\times 1}$ is a vector containing the z-transformed impulse responses $h_k^{(a)}$

$$\underline{h}_{(a)}(z) = \left[(\mathcal{Z}h_{(a)})(z), (\mathcal{Z}h_1^{(a)})(z), \dots, (\mathcal{Z}h_{K-1}^{(a)})(z)\right]^T,$$

and $\boldsymbol{F}_K \in \mathbb{C}^{K\times K}$ is the DFT matrix with elements $[\boldsymbol{F}_K]_{ik} = F_K^{ik}$. The matrix $\boldsymbol{E}_{(a)}(z) \in \mathbb{C}^{K\times R}$ is the polyphase matrix

$$\begin{aligned}\boldsymbol{E}_{(a)}(z) = &\operatorname{row}(\boldsymbol{I}_K, \dots, \boldsymbol{I}_K) \\ &\times \operatorname{diag}\left(\left(\mathcal{Z}\rho^{+}_{R;0}h_{(a)}\right)(z), \left(\mathcal{Z}\rho^{+}_{R;1}h_{(a)}\right)(z), \dots, \left(\mathcal{Z}\rho^{+}_{R;R-1}h_{(a)}\right)(z)\right),\end{aligned} \tag{1.25}$$

where we used $\operatorname{row}(\boldsymbol{I}_K, \dots, \boldsymbol{I}_K)$ to denote a row concatenation of the identity matrices $\boldsymbol{I}_K$. Or generally, we use $\operatorname{row}(\boldsymbol{A}_0, \dots, \boldsymbol{A}_{K-1})$ to denote a row concatenation of the matrices $\boldsymbol{A}_0, \dots, \boldsymbol{A}_{K-1}$:

$$\operatorname{row}(\boldsymbol{A}_0, \boldsymbol{A}_1, \dots, \boldsymbol{A}_{K-1}) = [\boldsymbol{A}_0|\boldsymbol{A}_1|\cdots|\boldsymbol{A}_{K-1}],$$

where all matrices $\boldsymbol{A}_i$ have the same number of rows. In this thesis, we also use the expression $\operatorname{col}(\boldsymbol{A}_0, \dots, \boldsymbol{A}_{K-1})$ to denote a column concatenation of the matrices $\boldsymbol{A}_0, \dots, \boldsymbol{A}_{K-1}$:

$$\operatorname{col}(\boldsymbol{A}_0, \boldsymbol{A}_1, \dots, \boldsymbol{A}_{K-1}) = \begin{bmatrix} \boldsymbol{A}_0 \\ \hline \boldsymbol{A}_1 \\ \hline \vdots \\ \hline \boldsymbol{A}_{K-1} \end{bmatrix},$$

where all matrices $\boldsymbol{A}_i$ have the same number of columns. The vector $\underline{d}_R(z) \in \mathbb{C}^{R\times 1}$ in Eq. (1.24) is the 'delay' vector

$$\underline{d}_R(z) = [1\, z, \dots, z^{R-1}]^T.$$

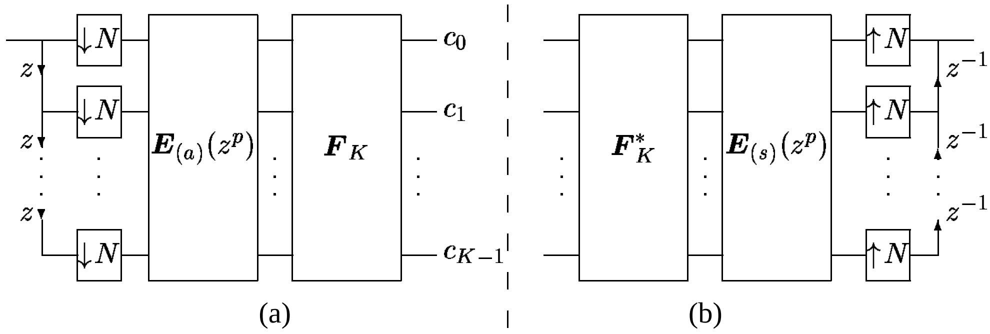

Figure 1.6: *Efficient implementation of the uniform DFT filter bank. (a) the analysis bank. (b) the synthesis bank.*

Furthermore, if we also take R a multiple of N we can use noble identity (1.21) and we can shift the downsamplers through the polyphase components to the front end of the analysis bank. Thus we choose $R = \mathrm{lcm}(K, N)l = pqJl$, where l is an arbitrary positive integer. In Fig. 1.6a we depicted the analysis bank of the uniform DFT filter bank schematically. Comparing this implementation with the implementation we started with (see Fig. 1.4a), we see that the implementation in Fig. 1.6a is indeed more efficient; besides the locations of the downsamplers, we only need the impulse response $h_{(a)}$ of the prototype and a DFT instead of all K impulse responses $h_k^{(a)}$. However, we observe that the implementation in Fig. 1.4a is easier to understand than the implementation shown in Fig. 1.6a.

The synthesis bank can be implemented similarly; z-transforming the synthesis operation (1.19), using property (1.16) and using the polyphase representation yields

$$(\mathcal{Z}\varphi)(z) = \sum_{m=0}^{K-1} \sum_{\ell=0}^{ql-1} z^{-(m+\ell K)} \left(\mathcal{Z}\rho^{-}_{R;m+\ell K} h_{(s)}\right)(z^R) \sum_{k=0}^{K-1} (\mathcal{Z}c_k)(z^N) e^{j2\pi mk/K},$$

or in matrix notation

$$(\mathcal{Z}\varphi)(z) = \underline{d}_R^T(z^{-1}) \boldsymbol{E}_{(s)}(z^R) \boldsymbol{F}_K^* \underline{c}(z^N), \tag{1.26}$$

where $\underline{c}(z) \in \mathbb{C}^{K\times 1}$ is a vector with the z-transformed subband signals c_k

$$\underline{c}(z) = [(\mathcal{Z}c_0)(z), (\mathcal{Z}c_1)(z), \ldots, (\mathcal{Z}c_{K-1})(z)]^T,$$

and where $\boldsymbol{E}_{(s)}(z) \in \mathbb{C}^{R\times K}$ is the polyphase matrix

$$\boldsymbol{E}_{(s)}(z) = \mathrm{diag}\left(\left(\mathcal{Z}\rho^{+}_{R;0} h_{(s)}\right)(z), \left(\mathcal{Z}\rho^{+}_{R;1} h_{(s)}\right)(z), \ldots, \left(\mathcal{Z}\rho^{+}_{R;R-1} h_{(s)}\right)(z)\right) \times \mathrm{col}(\boldsymbol{I}_K \cdots \boldsymbol{I}_K]).$$

Noble identity (1.20) can be used to shift the upsamplers through the polyphase components to the back end of the synthesis bank. In Fig. 1.6b, the synthesis bank is depicted schematically.

In the discussion above, we showed how the uniform DFT filter bank can be implemented efficiently by using the polyphase representation. The polyphase representation and the z-transformation can also be used to transform the frame operator into a matrix-vector product. This matrix-vector product immediately yields a relationship between the window g and its dual window γ. Furthermore, it provides a method to calculate the frame bounds of the frame operator $\mathcal{S}$. We now continue by showing how this matrix-vector product can be obtained. The frame operator is given by

$$\mathcal{S}\varphi = \sum_{k=\langle K\rangle} \sum_{m=-\infty}^{\infty} \langle \varphi, g_{mk}\rangle g_{mk}.$$

As we have seen, the inner product can be calculated with the help of an analysis bank, but now by choosing for the prototype $h_{(a)} = g^*[-n]$ with polyphase matrix $E_{(a)}(z)$ equal to $E^*_{(s)}(z)$, where $E_{(s)}(z)$ is the polyphase matrix of the prototype $h_{(s)} = g$. Using the matrix notation (1.24) with $\boldsymbol{E}_{(a)}(z)$ replaced by $\boldsymbol{E}^*_{(s)}(z)$, and using Eq. (1.26) with

$$\underline{c}(z^N) = \boldsymbol{F}_K \boldsymbol{E}^*_{(s)}(z^R) \frac{1}{N} \sum_{i=0}^{N-1} (\mathcal{Z}\varphi)(F_N^i z) \underline{d}_R(F_N^i z),$$

yields

$$\begin{aligned}(\mathcal{Z}\mathcal{S}\varphi)(z) =& \frac{K}{N} \underline{d}_N^T(z^{-1}) \operatorname{row}(\boldsymbol{I}_N, z^{-N}\boldsymbol{I}_N, \ldots, z^{-R+N}\boldsymbol{I}_N) \boldsymbol{E}_{(s)}(z^R) \boldsymbol{E}^*_{(s)}(z^R) \\ &\times \operatorname{col}(\boldsymbol{I}_N, z^N\boldsymbol{I}_N, \ldots, z^{R-N}\boldsymbol{I}_N]) \sum_{i=0}^{N-1} \underline{d}_N(F_N^i z)(\mathcal{Z}\varphi)(F_N^i z), \quad (1.27)\end{aligned}$$

where we used

$$\underline{d}_R(z) = \operatorname{col}(\boldsymbol{I}_N, z^N \boldsymbol{I}_N, \ldots, z^{R-N}\boldsymbol{I}_N) \underline{d}_N(z).$$

Here we recognize the upsampled z-transformed polyphase components $\rho^-_{N;m}\varphi$ of the signal φ

$$(\mathcal{Z}\rho^-_{N;m}\varphi)(z^N) = \frac{1}{N} \sum_{i=0}^{N-1} F_N^{im} z^m (\mathcal{Z}\varphi)(F_N^i z).$$

Rewriting Eq. (1.27) yields the polyphase representation of the z-transformed frame operator $\mathcal{S}$

$$(\mathcal{Z}\mathcal{S}\varphi)(z) = \underline{d}_N^T(z^{-1})\boldsymbol{S}(z^N)\underline{\rho}^-_{N;\varphi}(z^N), \tag{1.28}$$

where the matrix $\boldsymbol{S}(z) \in \mathbb{C}^{N\times N}$ is defined by

$$\begin{aligned}\boldsymbol{S}(z) =& K\,\mathrm{row}(\boldsymbol{I}_N, z^{-1}\boldsymbol{I}_N, \ldots, z^{-lp+1}\boldsymbol{I}_N)\boldsymbol{E}_{(s)}(z^{lp})\\ &\times \boldsymbol{E}^*_{(s)}(z^{lp})\,\mathrm{col}(\boldsymbol{I}_N, z\boldsymbol{I}_N, \ldots, z^{lp-1}\boldsymbol{I}_N),\end{aligned}$$

and where the vector $\underline{\rho}^-_{N;\varphi}(z) \in \mathbb{C}^{N\times 1}$ contains the z-transformed polyphase components

$$\underline{\rho}^-_{N;\varphi}(z) = [(\mathcal{Z}\rho^-_{N;0}\varphi)(z), (\mathcal{Z}\rho^-_{N;1}\varphi)(z), \ldots, (\mathcal{Z}\rho^-_{N;N-1}\varphi)(z)]^T.$$

So we find that

$$\underline{\rho}^-_{N;\mathcal{S}\varphi}(z) = \boldsymbol{S}(z)\underline{\rho}^-_{N;\varphi}(z).$$

In particular, we have the following relationship between the window g and its dual window γ

$$\underline{\rho}^-_{N;g}(z) = \boldsymbol{S}(z)\underline{\rho}^-_{N;\gamma}(z),$$

or

$$\underline{\rho}^-_{N;\gamma}(z) = \boldsymbol{S}(z)^{-1}\underline{\rho}^-_{N;g}(z).$$

With the help of the matrix $\boldsymbol{S}(z)$, we can also find the frame bounds A and B of the frame operator $\mathcal{S}$. By using Eq. (1.28) and taking $z = \exp(j2\pi\theta)$ we find that

$$\langle \mathcal{S}\varphi, \varphi\rangle = \langle \underline{d}_N^T\boldsymbol{S}\underline{\rho}^-_{N;\varphi}, \underline{d}_N^T\underline{\rho}^-_{N;\varphi}\rangle = \langle \boldsymbol{S}\underline{\rho}^-_{N;\varphi}, \underline{\rho}^-_{N;\varphi}\rangle.$$

From this identity we see that the frame bounds follow from:

$$A = \min_{|z|=1}\min_{\substack{\underline{u}\in\mathbb{C}^N\\ ||\underline{u}||=1}} \langle \boldsymbol{S}(z)\underline{u}, \underline{u}\rangle \qquad \text{and} \qquad B = \max_{|z|=1}\max_{\substack{\underline{u}\in\mathbb{C}^N\\ ||\underline{u}||=1}} \langle \boldsymbol{S}(z)\underline{u}, \underline{u}\rangle.$$

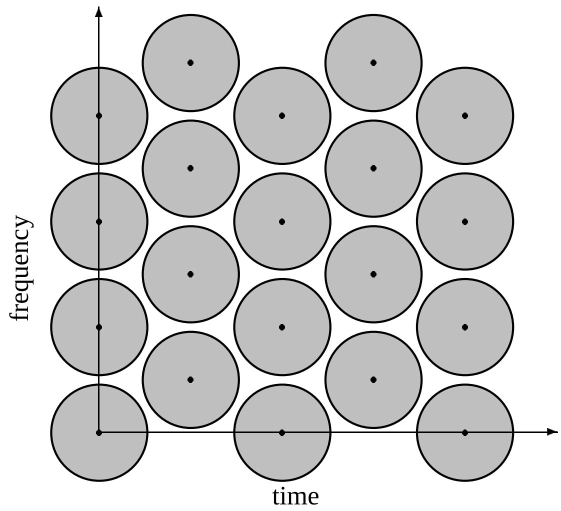

Figure 1.7: *Hexagonal tiling of the time-frequency plane with circles.*

1.7 Motivation

Frequently the question arises why one would consider the Gabor scheme on a lattice different from the rectangular one. To give an answer to this question, we consider Fig. 1.1 on page 3 once again. As shown in Section 1.1, the Gabor expansion (1.1) with a Gaussian window g can be seen as a rectangular tiling of the time-frequency plane with circles, i.e., a signal φ with its corresponding time-frequency behaviour is written as a sum of these circles with appropriate weights:

$$(\mathcal{W}_g\varphi)(\tau,\omega) = \sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} a_{mk}\,(\mathcal{W}_g g_{mk})\,(\tau,\omega),$$

where $|\mathcal{W}_g g_{mk}|$ has circular contour lines in the time-frequency plane. Now suppose that the energy of a windowed Fourier transformed signal φ is mainly concentrated in the area between the circles. Since the height of the circular contour lines decreases rapidly, one can expect that the Gabor coefficients become more sensitive in these areas where the contributions of the signal φ is high. Of course, choosing a higher oversampling rate, i.e., shifting the circles towards each other, is an option. An other option would be a different tiling of the time-frequency plane with the circles. We obtain a higher packing density if we place the circles in a hexagonal way. This hexagonal or quincunx lattice is depicted in Fig. 1.7. We immediately see, that the white areas between the circles are spread out in the most optimal way. Therefore we expect that the Gabor expansion with a Gaussian window g on a quincunx lattice is less sensitive than the Gabor expansion with a Gaussian window g on a rectangular lattice with the same rate of oversampling. Put differently, the Gabor scheme

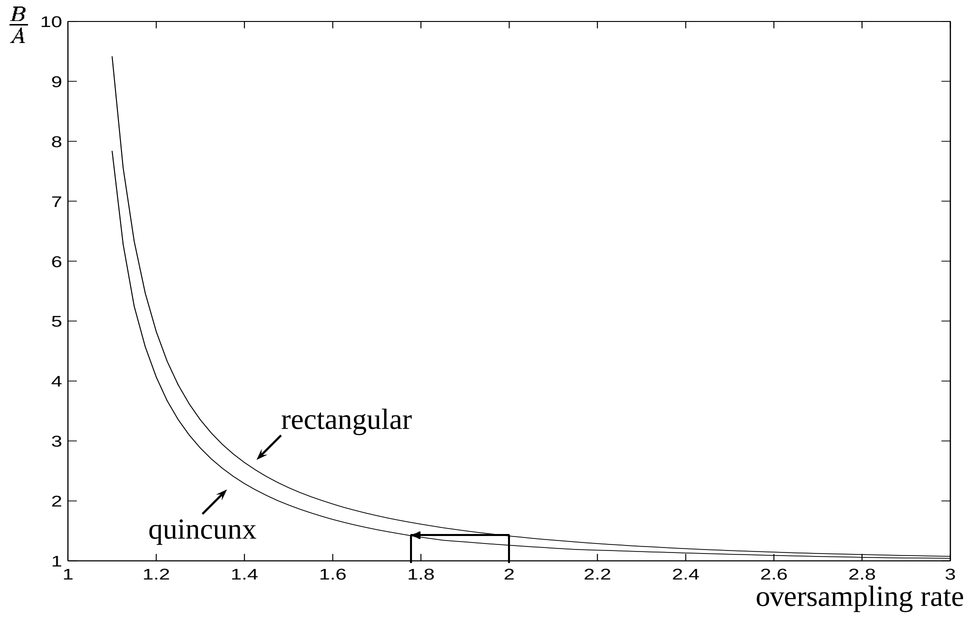

Figure 1.8: *The bound ratios B/A for the rectangular and the quincunx Gabor frame for different values of oversampling.*

with a Gaussian window g on a quincunx lattice would lead to a tighter frame. To illustrate this, we estimated the bounds of Gabor frames with a Gaussian window $g(t) = \pi^{-1/4} \exp(-\frac{1}{2}t^2)$ for a rectangular and a quincunx lattice for different values of oversampling. For the rectangular case we choose the optimal setting $\alpha = \beta$ (see [11,12]). We also choose an optimal setting for the quincunx case ($\beta = \sqrt{\frac{2}{3}\sqrt{3}/o}$ and $\alpha = \frac{1}{2}\sqrt{3}\beta$ in Eq. (2.8) with o the rate of oversampling), which is based on the method presented in [11,12]. The bounds are calculated with the help of Theorem 2.1 in [19] and the shear operator as described in Section 2.4 to approximate the bounds in the quincunx case. In Fig. 1.8, the ratios of the higher bound B and the lower bound A of the frames for values of oversampling between 1.1 and 3 are depicted. In this figure we can see that the line corresponding to the quincunx Gabor frame lies under the line of the rectangular Gabor frame, i.e., the quincunx Gabor frame is indeed tighter. For example, in the case of a rectangular Gabor scheme and an oversampling rate 2 the ratio B/A is approximately 1.41, whereas in the quincunx case, the ratio B/A is approximately 1.26. If we allow a ratio of 1.41, we can decrease the oversampling rate to approximately 1.77.

In the discussion above we considered the Gaussian window which has circular contour lines in the time-frequency plane. Other types of windows, like the one- and two-sided exponentials, yield different shapes in the time-frequency plane. Conceivably, in these situations other lattices are more suitable than the rectangular lattice, i.e., other lattices yield tighter frames. This motivates us to consider Gabor schemes on more general lattices.

1.8 Outline

The subject matter of this thesis can be divided into two parts; 1) the non-separable Gabor schemes for continuous-time signals and 2) the non-separable Gabor schemes for discrete-time signals. In both cases, the one-dimensional as well as the multi-dimensional case are discussed. Chapters 2 and 3 cover the continuous-time case. Chapter 2 contains the non-separable Gabor scheme for univariate signals. Chapter 3 contains the multi-dimensional case. Of course, the one-dimensional Gabor scheme is a special case of the multi-dimensional one. The distinction between of one- and multi-dimensional Gabor schemes is made for illustrative purposes. The concept of the multi-dimensional Gabor scheme then follows in a more or less straightforward fashion from the one-dimensional Gabor scheme. Moreover, the one-dimensional Gabor scheme allows for techniques to transform a one-dimensional non-separable Gabor scheme into a separable one; these techniques are based on the fractional Fourier transformation and shearing (see Sections 2.3 and 2.4, respectively). Transforming a non-separable multi-dimensional Gabor scheme into a separable one is much more complicated and could be a subject for further research (see [57]).

Chapter 4 covers the one-dimensional non-separable Gabor scheme for discrete-time signals, and Chapter 5 contains the multi-dimensional case. In order to use fast algorithms such as the FFT, we have to dissect and thereafter periodize the signals. This is a difference between the continuous-time and the discrete-time case setting. Due to the periodization of the signals, extra conditions have to be fulfilled which make things less convenient. Again the one-dimensional case is discussed separately for the same reason as outlined above.

We finish the thesis with the conclusions (see Chapter 6).

Chapter 2

Non-separable 1-D Gabor scheme for continuous-time signals

In Section 2.1, the Gabor scheme for the rectangular (separable) lattice for time-continuous signals is extended to the general, non-separable lattice in a structured way; this is achieved by describing the non-separable lattice by means of a lattice generator matrix. The lattice generator matrix is written in the Hermite normal form to obtain a shear representation on the shifted and modulated windows and to obtain an alternative expression of the shifted and modulated windows. This alternative expression follows from the separable lattice that refines the non-separable lattice. The set of shifted and modulated versions of the window, which corresponds to the non-separable lattice, is obtained by masking the separable lattice; a shifted and modulated version of the window in this separable lattice is multiplied by one if it belongs to the non-separable lattice and is multiplied by zero otherwise.

In Section 2.2, we show that the Zak transformation can also be used in the case of a non-separable Gabor scheme. Again, similar to the separable case, the Zak transformation is useful to calculate the dual window and the Gabor expansion coefficients, and to reconstruct the signal. Since the alternative expression for the shifted and modulated versions of the window is obtained by masking a separable lattice, we can exploit the known expressions for the separable case within the context of the Zak transformation.

The non-separable lattices can be obtained, for instance, via a scaled rotation operation or a shear operation on the rectangular lattice. In Section 2.3, we show that the fractional Fourier transform, which corresponds to a rotation in the time-frequency plane, can be used to transform a non-separable lattice into a separable one. In Section 2.4, the shear operator is employed to reshear a non-separable lattice into a separable one. As a result, methods for the separable case to calculate the dual window and the Gabor expansion coefficients, and to reconstruct the signal can be re-used in the non-separable case. As an example, we calculate the dual window of a Gaussian window for a hexagonal (quincunx) Gabor scheme in the case of critical sampling by using the shear operator.

2.1 Gabor's signal expansion on a non-separable lattice

The rectangular (or separable) lattice considered in the previous chapter can be obtained by integer combinations of two orthogonal vectors

$$\underline{v}_0 = [\alpha T,\ 0]^T \quad \text{and} \quad \underline{v}_1 = [0,\ \beta\Omega]^T, \tag{2.1}$$

where $\alpha, \beta \in \mathbb{R}^+$, and $\Omega T = 2\pi$. Thus the lattice Λ is expressed in the form

$$\Lambda = \left\{ \mathbf{\Lambda}\underline{n} | \underline{n} \in \mathbb{Z}^2 \right\}, \tag{2.2}$$

where $\mathbf{\Lambda}$ is the lattice generator matrix with column vectors $\underline{v}_0$ and $\underline{v}_1$. The shifted and modulated windows $g_{\Lambda;mk}$ on this rectangular lattice are given by

$$g_{\Lambda;mk} = \sigma_{\mathbf{\Lambda}\left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right]} g_\Lambda = \mathcal{M}_{\mathbf{\Lambda}_{2*}\left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right]} \mathcal{T}_{\mathbf{\Lambda}_{1*}\left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right]} g_\Lambda = \mathcal{M}^k_{\beta\Omega} \mathcal{T}^m_{\alpha T} g_\Lambda,$$

where $\mathbf{\Lambda}_{n*}$ denotes the nth row of the matrix $\mathbf{\Lambda}$, and where we used the time-frequency shift operator $\sigma_{\left[\begin{smallmatrix} \tau \\ \omega \end{smallmatrix}\right]}$ defined as

$$\sigma_{\left[\begin{smallmatrix} \tau \\ \omega \end{smallmatrix}\right]} = \mathcal{M}_\omega \mathcal{T}_\tau,$$

with the modulation operator

$$\left(\mathcal{M}_\omega f\right)(t) = e^{j\omega t} f(t),$$

and time translation operator

$$\left(\mathcal{T}_\tau f\right)(t) = f(t-\tau).$$

Note that this modulation operator $\mathcal{M}_\omega$ and this time translation operator $\mathcal{T}_\tau$ are unitary on $L_2(\mathbb{R})$, with corresponding Hilbert adjoints $\mathcal{M}^*_\omega = \mathcal{M}_{-\omega}$ and $\mathcal{T}^*_\tau = \mathcal{T}_{-\tau}$. For convenience, we will use the same notation for the time-frequency shift operator $\sigma_{\left[\begin{smallmatrix} \tau \\ \omega \end{smallmatrix}\right]}$, the modulation operator $\mathcal{M}_\omega$ and the time translation operator $\mathcal{T}_\tau$ for the discrete-time setting (see Chapters 4 and 5).

In this chapter, we consider Gabor's signal expansion on a time-frequency lattice that is no longer separable. We call a time-frequency lattice non-separable, if the time-shifts and the modulations in the shifted and modulated windows $g_{\Lambda;mk}$ are not independent operations anymore. Such a lattice is obtained by integer combinations of two linearly independent, but no longer orthogonal vectors, which are expressed in the forms

$$\underline{v}_0 = [a\alpha T,\ c\beta\Omega/D]^T \quad \text{and} \quad \underline{v}_1 = [b\alpha T,\ d\beta\Omega/D]^T, \tag{2.3}$$

with a, b, c and d integers, $\alpha, \beta \in \mathbb{R}^+$, $D = ad - bc$, and $\Omega T = 2\pi$ again. The first component in the vectors $\underline{v}_0$ or $\underline{v}_1$ corresponds to a time-shift $a\alpha T$ or $b\alpha T$, respectively, while the second component corresponds to a modulation by a frequency $c\beta\Omega/D$ or $d\beta\Omega/D$, respectively.

Each point $\underline{\lambda} \in \Lambda$ can be obtained by a matrix-vector product [see Eqs. (2.2) and (2.3)]

$$\forall_{\underline{\lambda}\in\Lambda}\exists_{\underline{n}\in\mathbb{Z}^2}\;\; \underline{\lambda} = \boldsymbol{\Lambda}\underline{n} = \boldsymbol{U}\boldsymbol{L}\underline{n}, \tag{2.4}$$

with

$$\boldsymbol{U} = \frac{1}{D}\begin{bmatrix}\alpha T D & 0\\ 0 & \beta\Omega\end{bmatrix} \quad \text{and} \quad \boldsymbol{L} = \begin{bmatrix}a & b\\ c & d\end{bmatrix}.$$

The column vectors $\underline{v}_0$ and $\underline{v}_1$ are the columns of the lattice generator matrix $\boldsymbol{UL}$. Note, moreover, that D is equal to the determinant of the matrix $\boldsymbol{L}$. We shall assume that the possible common divisor of the integers a and b and the possible common divisor of the integers c and d are handled by the matrix $\boldsymbol{U}$, i.e.,

$$\gcd(a, b) = 1 \quad \text{and} \quad \gcd(c, d) = 1.$$

The area of a cell (a parallelogram spanned by $\underline{v}_0$ and $\underline{v}_1$) in the time-frequency plane is equal to the determinant of the lattice generator matrix $\boldsymbol{UL}$, which equals $\alpha\beta\Omega T = 2\pi\alpha\beta$. It is well known that the set of shifted and modulated versions of the window g_Λ is not complete in $L_2(\mathbb{R})$ in the case that $\alpha\beta > 1$. Equality, $\alpha\beta = 1$, corresponds to critical sampling, and $\alpha\beta < 1$ corresponds to oversampling (see [33]). Note that we only consider those lattice generator matrices $\boldsymbol{\Lambda}$ that can be decomposed as $\boldsymbol{\Lambda} = \boldsymbol{UL}$. Only then, the corresponding lattices have samples on the time- and frequency-axes and are therefore suitable for a discrete-time approach as well (see Chapter 4).

2.1.1 Hermite normal form

For a given matrix $\boldsymbol{U}$, there are many matrices $\boldsymbol{L}$ that generate the same lattice Λ. One form, the Hermite normal form [45], is very interesting, since the Hermite normal form is unique, and we need only two parameters to describe a non-separable lattice Λ instead of the four parameters a,b,c and d. Given the $n \times n$ nonsingular integer-valued matrix $\boldsymbol{L}$, there exists an $n \times n$ unimodular matrix $\boldsymbol{K}$ such that $\boldsymbol{LK} = \boldsymbol{L}'$, the Hermite normal form of $\boldsymbol{L}$, whose entries satisfy

$$\begin{array}{lll} L'_{ij} = 0 & & \forall_{j>i},\\ L'_{ii} > 0 & & \forall_i,\\ L'_{ij} \leq 0 & \text{and } |L'_{ij}| < L'_{ii} & \forall_{j<i}. \end{array} \tag{2.5}$$

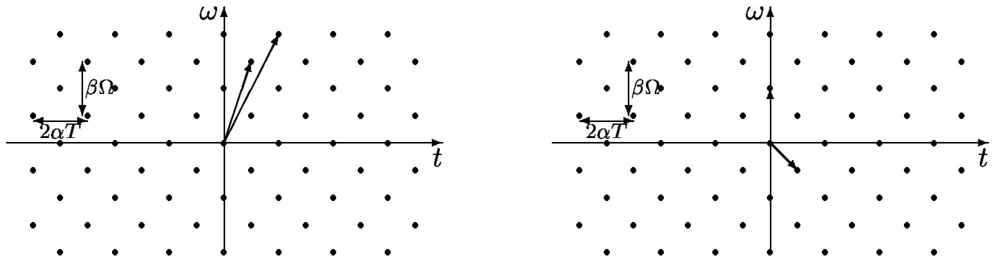

Figure 2.1: *The quincunx lattice and two possible combinations of generator vectors in the time-frequency plane.*

Note that the Hermite normal form of a matrix is lower triangular. In fact, the Hermite normal form $\boldsymbol{L}'$ of the matrix $\boldsymbol{L}$ is given by

$$\boldsymbol{L}' = \boldsymbol{L}\boldsymbol{K} = \boldsymbol{L}\begin{bmatrix}(h_0 - wb) & -b\\ (h_1 + wa) & a\end{bmatrix} = \begin{bmatrix}1 & 0\\ -r & D\end{bmatrix}, \tag{2.6}$$

where the integer $-r$ equals $h_0c + h_1d + wD$, with the integers h_0 and h_1 such that $\det(\boldsymbol{K}) = h_0a + h_1b = 1$ and where w is an integer chosen such that $0 \le r < D$. Note that these integers h_0 and h_1 exist, since $\gcd(a, b) = 1$, and that they can be obtained by the Euclidean division algorithm (see, for instance, [30]). Note, moreover, that an arbitrary integer w yields an equivalent lattice generator matrix in the sense that both matrices generate the same lattice, or more generally, the matrices $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ generate the same lattice if and only if there exists a unimodular matrix $\boldsymbol{K}_0$ such that $\boldsymbol{L}_1 = \boldsymbol{L}_2\boldsymbol{K}_0$. The matrix $\boldsymbol{U}$ remains the same. As an example, two possible combinations of lattice generators in the case of a quincunx lattice ($r = 1$ and $D = 2$) are plotted in Figs. 2.1 (a) and 2.1 (b). Fig. 2.1 (b) corresponds to the Hermite normal form. Since the matrices $\boldsymbol{L}$ and $\boldsymbol{L}'$ are equivalent in the sense that both matrices correspond to the same lattice, the integers r and D are relatively prime, as well:

$$\gcd(r, D) = 1. \tag{2.7}$$

A determinant $D = 1$ corresponds to a rectangular lattice; $\boldsymbol{U} = \left[\begin{smallmatrix}\alpha T & 0\\ 0 & \beta\Omega\end{smallmatrix}\right]$ and $\boldsymbol{L} = \boldsymbol{I}_2$ the 2×2 identity matrix [cf. Eq. (2.1)]. The columns of the matrix $\boldsymbol{U}\boldsymbol{L}'$ are equal to $[\alpha T,\ -r\beta\Omega/D]^T$ and $[0,\ \beta\Omega]^T$, respectively. Consequently, the shifted and modulated versions $g_{\Lambda;mk}$ of the window g_Λ on the lattice Λ take the form

$$g_{\Lambda;mk} = \sigma_{\boldsymbol{\Lambda}'\left[\begin{smallmatrix}m\\ k\end{smallmatrix}\right]} g_\Lambda = \mathcal{M}^m_{-r\beta\Omega/D}\left[\mathcal{M}^k_{\beta\Omega}\mathcal{T}^m_{\alpha T} g_\Lambda\right], \tag{2.8}$$

where $\mathbf{\Lambda}' = \boldsymbol{UL}'$. These shifted and modulated windows $g_{\Lambda;mk}$ are based on a separable lattice via a shear of the frequency variable due to the modulation $\mathcal{M}^m_{-r\beta\Omega/D}$. A different expression for the shifted and modulated windows $g_{\Lambda;mk}$ can also be obtained by a shear of the time variable. To obtain this expression, we need a different lattice generator matrix $\mathbf{\Lambda}''$ of the form

$$\mathbf{\Lambda}'' = \boldsymbol{UL}'' = \frac{1}{D}\begin{bmatrix}\alpha TD & 0\\ 0 & \beta\Omega\end{bmatrix}\begin{bmatrix}D & -r'\\ 0 & 1\end{bmatrix}, \tag{2.9}$$

where $\boldsymbol{L}''$ is a modified Hermite normal form of the matrix $\boldsymbol{L}$ [cf. Eqs. (2.4) and (2.6)]

$$\boldsymbol{L}'' = \boldsymbol{L}\begin{bmatrix}d & (h_2 + w'd)\\ -c & (h_3 - w'c)\end{bmatrix} = \begin{bmatrix}D & -r'\\ 0 & 1\end{bmatrix}, \tag{2.10}$$

where $-r' = ah_2 + bh_3 + w'D$, with the integers h_2 and h_3 such that $\det(\boldsymbol{K}) = h_2c + h_3d = 1$ (recall that $\gcd(c,d) = 1$, and so, these integers exist), and w' an integer chosen such that $0 \leq r' < D$. Note that the integers r and r' do not have the same value, in general. The shifted and modulated windows $g_{\Lambda;mk}$ of the window g_Λ are now given by [cf. Eq. (2.8)]

$$g_{\Lambda;mk} = \sigma_{\mathbf{\Lambda}''\left[\begin{smallmatrix}m\\k\end{smallmatrix}\right]} g_\Lambda = \mathcal{M}^k_{\beta\Omega/D}\mathcal{T}^k_{-r'\alpha T}\mathcal{T}^m_{D\alpha T} g_\Lambda. \tag{2.11}$$

In the sequel, we assume that the matrix $\boldsymbol{L}$ is written in the Hermite normal form (2.6) with corresponding shifted and modulated windows $g_{\Lambda;mk}$ of the form (2.8) and we will drop the primes.

The Gabor frame operator $\mathcal{S}_\Lambda$ is defined as

$$\mathcal{S}_\Lambda\varphi = \sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\langle\varphi, g_{\Lambda;mk}\rangle g_{\Lambda;mk}, \quad \varphi \in L_2(\mathbb{R}).$$

For convenience, we will use the same notation for the Gabor frame operator $\mathcal{S}_\Lambda$ for the discrete-time setting (see Chapters 4 and 5). Now the question arises whether the Gabor frame operator commutes with the time-frequency shift operator $\sigma_{\mathbf{\Lambda}\left[\begin{smallmatrix}m\\k\end{smallmatrix}\right]}$, i.e.,

$$\mathcal{S}_\Lambda\sigma_{\mathbf{\Lambda}\left[\begin{smallmatrix}m\\k\end{smallmatrix}\right]} = \sigma_{\mathbf{\Lambda}\left[\begin{smallmatrix}m\\k\end{smallmatrix}\right]}\mathcal{S}_\Lambda.$$

Then the elements of the dual Gabor frame $\{\mathcal{S}_\Lambda^{-1}g_{\Lambda;mk} = \gamma_{\Lambda;mk}\}$ are generated by a single function γ_Λ, i.e., $\gamma_{\Lambda;mk} = \sigma_{\mathbf{\Lambda}\left[\begin{smallmatrix}m\\k\end{smallmatrix}\right]}\gamma_\Lambda$, similar to the separable case.

In order to prove this, we need the properties of the modulation operator $\mathcal{M}_\omega$ and the time translation operator $\mathcal{T}_\tau$, which are tabulated in Table 2.1. We have

$$\sigma_{\mathbf{\Lambda}\left[\begin{smallmatrix}n\\\ell\end{smallmatrix}\right]}\mathcal{S}_\Lambda\varphi = \sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\langle\sigma_{\mathbf{\Lambda}\left[\begin{smallmatrix}n\\\ell\end{smallmatrix}\right]}\varphi, \sigma_{\mathbf{\Lambda}\left[\begin{smallmatrix}n\\\ell\end{smallmatrix}\right]}g_{\Lambda;mk}\rangle\sigma_{\mathbf{\Lambda}\left[\begin{smallmatrix}n\\\ell\end{smallmatrix}\right]}g_{\Lambda;mk}, \tag{2.12}$$

Table 2.1: *Some properties of the modulation and time translation operator.*

$\mathcal{T}_{\tau_0}\mathcal{T}_{\tau_1} = \mathcal{T}_{\tau_1}\mathcal{T}_{\tau_0}$
$\mathcal{M}_{\omega_0}\mathcal{M}_{\omega_1} = \mathcal{M}_{\omega_1}\mathcal{M}_{\omega_0}$
$\mathcal{M}_{\omega}\mathcal{T}_{\tau} = e^{j\omega\tau}\mathcal{T}_{\tau}\mathcal{M}_{\omega}$

where n and ℓ are integers. Since

$$\begin{aligned}\sigma_{\mathbf{\Lambda}\left[\begin{smallmatrix}n\\ \ell\end{smallmatrix}\right]} g_{\Lambda;mk} &= \mathcal{M}^{\ell}_{\beta\Omega}\mathcal{M}^{n}_{-r\beta\Omega/D}\mathcal{T}^{n}_{\alpha T}\mathcal{M}^{k}_{\beta\Omega}\mathcal{M}^{m}_{-r\beta\Omega/D}\mathcal{T}^{m}_{\alpha T} g_{\Lambda} \\ &= e^{-jn\alpha T(k\beta\Omega - mr\beta\Omega/D)} g_{\Lambda;m+n,k+\ell}\end{aligned}$$

Eq. (2.12) reduces to

$$\begin{aligned}\sigma_{\mathbf{\Lambda}\left[\begin{smallmatrix}n\\ \ell\end{smallmatrix}\right]}\mathcal{S}_{\Lambda}\varphi &= \sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\langle\sigma_{\mathbf{\Lambda}\left[\begin{smallmatrix}n\\ \ell\end{smallmatrix}\right]}\varphi, g_{\Lambda;m+n,k+\ell}\rangle g_{\Lambda;m+n,k+\ell} \\ &= \sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\langle\sigma_{\mathbf{\Lambda}\left[\begin{smallmatrix}n\\ \ell\end{smallmatrix}\right]}\varphi, g_{\Lambda;mk}\rangle g_{\Lambda;mk} \\ &= \mathcal{S}_{\Lambda}\sigma_{\mathbf{\Lambda}\left[\begin{smallmatrix}n\\ \ell\end{smallmatrix}\right]}\varphi.\end{aligned}$$

From this it follows that the elements of the dual Gabor frame $\{\mathcal{S}_{\Lambda}^{-1}g_{\Lambda;mk}\}$ are indeed the shifted and modulated versions $\gamma_{\Lambda;mk}$ of the dual window γ_{Λ}. As a consequence, the Gabor expansion on a non-separable lattice Λ is given by

$$\varphi = \sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} a_{mk} g_{\Lambda;mk}, \tag{2.13}$$

where the array of Gabor coefficients $\{a_{mk}\}$ is obtained by the Gabor transform

$$a_{mk} = \langle\varphi, \gamma_{\Lambda;mk}\rangle = \int_{-\infty}^{\infty}\varphi(t)\gamma^{*}_{\Lambda;mk}(t)\,dt. \tag{2.14}$$

Note that the array $\{a_{mk}\}$ is sheared in the frequency variable k.

The Zak transformation can be very useful to calculate the dual window γ_{Λ} and the array of Gabor coefficients $\{a_{mk}\}$, and to reconstruct the signal φ in the case of a separable lattice Λ. The shifted and modulated windows $g_{\Lambda;mk}$ in the expression (2.8) are based on the separable (rectangular) lattice via a shear of the frequency variable due to the modulation $\mathcal{M}^{m}_{-r\beta\Omega/D}$. In order to exploit the known expressions for the separable case within the scope of the Zak transformation, we need a different expression for the shifted and modulated windows presented in Eq. (2.8). We consider

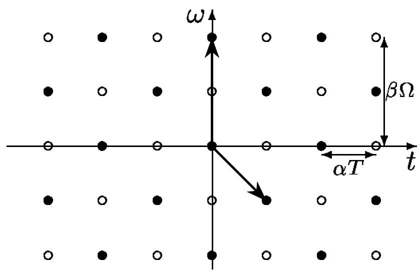

Figure 2.2: *The separable (rectangular) lattice that refines the quincunx lattice.*

the separable lattice Λ_s

$$\Lambda_\text{s} = \{\boldsymbol{\Lambda}_\text{s}\underline{n} | \underline{n} \in \mathbb{Z}^2\}, \quad \text{with} \quad \boldsymbol{\Lambda}_\text{s} = \boldsymbol{U} = \frac{1}{D}\begin{bmatrix}\alpha T D & 0 \\ 0 & \beta\Omega\end{bmatrix}.$$

This lattice Λ_s refines the non-separable lattice Λ, i.e., Λ is a sub-lattice of Λ_s,

$$\{\sigma_{\boldsymbol{\Lambda}\left[\begin{smallmatrix}n\\ \ell\end{smallmatrix}\right]} g_\Lambda | n, \ell \in \mathbb{Z}\} \subset \{\sigma_{\boldsymbol{\Lambda}_\text{s}\left[\begin{smallmatrix}m\\ k\end{smallmatrix}\right]} g_\Lambda | m, k \in \mathbb{Z}\},$$

and

$$\forall_{n,\ell\in\mathbb{Z}} \exists!_{m,k\in\mathbb{Z}} \;\; \boldsymbol{\Lambda}\left[\begin{smallmatrix}n\\ \ell\end{smallmatrix}\right] = \boldsymbol{\Lambda}_\text{s}\boldsymbol{L}\left[\begin{smallmatrix}n\\ \ell\end{smallmatrix}\right] = \boldsymbol{\Lambda}_\text{s}\left[\begin{smallmatrix}m\\ k\end{smallmatrix}\right], \quad \text{with} \quad \left[\begin{smallmatrix}m\\ k\end{smallmatrix}\right] = \boldsymbol{L}\left[\begin{smallmatrix}n\\ \ell\end{smallmatrix}\right].$$

The non-separable lattice Λ is obtained by assigning the value zero to the shifted and modulated window $g_{\Lambda_\text{s};mk}$ on the separable lattice, which do not belong to the non-separable lattice Λ, i.e., $\left[\begin{smallmatrix}m\\ k\end{smallmatrix}\right] \neq \boldsymbol{L}\left[\begin{smallmatrix}n\\ \ell\end{smallmatrix}\right]$. As an example, in Fig. 2.2, the quincunx lattice and the rectangular lattice that refines the quincunx lattice are depicted. The filled circles belong to the quincunx lattice. We assign the value zero to the open circles. So we have to find all (m, k) for which (n, ℓ) exist with

$$\boldsymbol{L}\left[\begin{smallmatrix}n\\ \ell\end{smallmatrix}\right] = \left[\begin{smallmatrix}m\\ k\end{smallmatrix}\right].$$

We have the following equivalent expressions

$$\boldsymbol{L}^{-1}\left[\begin{smallmatrix}m\\ k\end{smallmatrix}\right] = \frac{1}{D}\begin{bmatrix}D & 0 \\ r & 1\end{bmatrix}\begin{bmatrix}m \\ k\end{bmatrix} \in \mathbb{Z}^2$$

$\equiv$

$$mr + k \quad \text{is divisible by } D$$

$\equiv$

$$P_{\boldsymbol{\Lambda}}(m, k) = 1,$$

where $P_{\mathbf{\Lambda}}(m,k)$ is given by the Poisson summation formula

$$P_{\mathbf{\Lambda}}(m,k) = \frac{1}{D} \sum_{i=\langle D \rangle} e^{j2\pi i(mr+k)/D}. \qquad (2.15)$$

Here the expression $i = \langle D \rangle$ denotes a finite interval of D successive integers i. We find that

$$\mathbf{\Lambda}_{\mathrm{s}} \left[\begin{smallmatrix} m \\ k \end{smallmatrix} \right] \in \Lambda \quad \equiv \quad P_{\mathbf{\Lambda}}(m,k) = 1,$$

or

$$\mathbf{\Lambda}_{\mathrm{s}} \left[\begin{smallmatrix} m \\ k \end{smallmatrix} \right] \notin \Lambda \quad \equiv \quad P_{\mathbf{\Lambda}}(m,k) = 0.$$

The shifted and modulated versions of the window g_{Λ} on the lattice Λ now take the form [cf. Eq. (2.8)]

$$\tilde{g}_{\Lambda_{\mathrm{S}};mk} = P_{\mathbf{\Lambda}}(m,k) \left(\sigma_{\mathbf{\Lambda}_{\mathrm{S}} \left[\begin{smallmatrix} m \\ k \end{smallmatrix} \right]} g_{\Lambda} \right). \qquad (2.16)$$

We put a tilde on top of $g_{\Lambda_{\mathrm{S}};mk}$ to indicate that the multiplication operator $P_{\mathbf{\Lambda}}(m,k)$ is involved. With this modified definition (2.16) of the shifted and modulated windows, Gabor's signal expansion and the Gabor transform take the form [cf. Eqs. (2.13) and (2.14)]

$$\varphi = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \tilde{a}_{mk} \tilde{g}_{\Lambda_{\mathrm{S}};mk}, \quad \varphi \in L_2(\mathbb{R}) \qquad (2.17)$$

and

$$\tilde{a}_{mk} = \langle \varphi, \tilde{\gamma}_{\Lambda_{\mathrm{S}};mk} \rangle, \qquad (2.18)$$

respectively. Note that, due to the multiplication operator $P_{\mathbf{\Lambda}}(m,k)$, the array $\{\tilde{a}_{mk}\}$ of Gabor expansion coefficients obtained with the shifted and modulated dual windows $\tilde{\gamma}_{\Lambda_{\mathrm{S}};mk}$ contains many zeros.

Assuming that the set of shifted and modulated versions $g_{\Lambda;mk}$ of the window g_{Λ} constitutes a frame, then the relationship between the window g_{Λ} and the dual window γ_{Λ} follows from substituting the Gabor transform (2.14) into Gabor's signal expansion (2.13) or from substituting the Gabor transform (2.18) into Gabor's signal expansion (2.17). In Appendix A.1, it is shown how after some manipulation the condition

$$\sum_{m=-\infty}^{\infty} e^{-j2\pi mkr/D} \gamma_{\Lambda}^{*} \left(t - k\frac{T}{\beta} - m\alpha T \right) g_{\Lambda}\left(t - m\alpha T\right) = \frac{\beta}{T} \delta[k], \qquad (2.19)$$

is obtained. This condition should hold for $k \in \mathbb{Z}$ and $t \in \mathbb{R}$. Thus for a given window g_{Λ}, we have to find a function γ_{Λ} such that this condition (2.19) is fulfilled.

2.2 Zak transform

It has been shown (see [6, 10, 53, 91]) that, to calculate the dual window γ_Λ corresponding to a given window g_Λ and the Gabor expansion coefficients, and to reconstruct the signal in the case of a rectangular lattice, the Zak transformation can be very useful. The method based on the Zak transformation is restricted to the case that the parameters α and β satisfy the relation $\alpha\beta = q/p \leq 1$, where p and q are positive integers, $p \geq q \geq 1$, and are relatively prime: the case of rational oversampling. This section extends this idea to the non-separable case, again assuming rational oversampling.

Condition (2.19) can be used to find the dual window γ_Λ. However, this is not very practical. By using the Zak transform

$$(\mathcal{Z}\varphi)(t,\omega;\tau) = \sum_{m=-\infty}^{\infty} \varphi(t+m\tau)e^{-jm\omega\tau}, \tag{2.20}$$

the condition (2.19) can be transformed into the following sum-of-products form (see Appendix A.2)

$$\sum_{k=\langle fp \rangle} g_{ik}(x,y)\gamma^*_{nk}(x,y) = \frac{fq}{\alpha T}\delta[i-n], \tag{2.21a}$$

where

$$g_{ik}(x,y) = (\mathcal{Z}g_\Lambda)\left((Dx+i)\frac{T}{\beta}, \left[y - \frac{k}{fp} - \frac{ir}{D}\right]\frac{\Omega}{\alpha}; \alpha T\right), \tag{2.21b}$$

and

$$\gamma_{ik}(x,y) = (\mathcal{Z}\gamma_\Lambda)\left((Dx+k)\frac{T}{\beta}, \left[y - \frac{k}{fp} - \frac{ir}{D}\right]\frac{\Omega}{\alpha}; \alpha T\right), \tag{2.21c}$$

with $f = D/\gcd(D,q)$, $i = 0\ldots fq-1$, and x and y extending over an interval of length $1/D$ and $1/fp$, respectively. Now we combine these functions g_{ik} and γ_{ik}, into the $(fq \times fp)$ matrices of functions

$$\boldsymbol{G}(x,y) = \begin{bmatrix} g_{00}(x,y) & g_{01}(x,y) & \cdots & g_{0,fp-1}(x,y) \\ g_{10}(x,y) & g_{11}(x,y) & \cdots & g_{1,fp-1}(x,y) \\ . & . & \cdots & . \\ . & . & \cdots & . \\ g_{fq-1,0}(x,y) & g_{fq-1,1}(x,y) & \cdots & g_{fq-1,fp-1}(x,y) \end{bmatrix} \tag{2.22}$$

and

$$\boldsymbol{\Gamma}(x,y)=\begin{bmatrix}\gamma_{00}(x,y) & \gamma_{01}(x,y) & \cdots & \gamma_{0,fp-1}(x,y)\\ \gamma_{10}(x,y) & \gamma_{11}(x,y) & \cdots & \gamma_{1,fp-1}(x,y)\\ \cdot & \cdot & \cdots & \cdot\\ \cdot & \cdot & \cdots & \cdot\\ \gamma_{fq-1,0}(x,y) & \gamma_{fq-1,1}(x,y) & \cdots & \gamma_{fq-1,fp-1}(x,y)\end{bmatrix},$$

respectively. With the help of these matrices $\boldsymbol{G}$ and $\boldsymbol{\Gamma}$, Eqs. (2.21a-c) can now be expressed in the matrix-product

$$\boldsymbol{G}\boldsymbol{\Gamma}^{*}=\frac{fq}{\alpha T}\boldsymbol{I}_{fq}, \tag{2.23}$$

where $\boldsymbol{I}_{fq}$ is the $(fq\times fq)$ identity matrix. Note that matrix $\boldsymbol{G}$ in Eq. (2.23) is not a square matrix in the case of oversampling ($p>q$) and does not have an inverse, but in general has a right inverse. It is well known that the optimum solution in the sense of minimum L_2-norm can be found with the help of the generalized (Moore-Penrose) inverse (see [10, 53] for the separable case) $\boldsymbol{G}^{\dagger}$, defined by (see [18])

$$\boldsymbol{G}^{\dagger}=\boldsymbol{G}^{*}(\boldsymbol{G}\boldsymbol{G}^{*})^{-1}. \tag{2.24}$$

The optimum solution $\boldsymbol{\Gamma}_{opt}$ then reads

$$\boldsymbol{\Gamma}_{opt}=\frac{fq}{\alpha T}(\boldsymbol{G}^{\dagger})^{*},$$

which corresponds to the minimum L_2-norm dual window γ_{Λ}.

Suppose the signal φ has the Gabor expansion (2.17). Then by using the Fourier expansion of the array of Gabor expansion coefficients $\tilde{a}_{mk}$, defined by

$$(\mathcal{F}^{(2)}\tilde{a})(x,y)=\sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\tilde{a}_{mk}e^{-j2\pi(my-kx)},$$

with inverse

$$\tilde{a}_{mk}=\int_0^1\int_0^1(\mathcal{F}^{(2)}\tilde{a})(x,y)e^{j2\pi(my-kx)}\,dx\,dy,$$

the Zak transform [see Eq. (2.20)] and the shifted and modulated windows $\tilde{\gamma}_{\Lambda_S;mk}$ [see Eq. (2.16)], it can be shown (see Appendix A.3) that the Gabor transform (2.18) can also be transformed into a sum-of-products form

$$(\mathcal{F}^{(2)}\tilde{a})\left(x,y-\frac{k}{fp}\right)=\alpha T\frac{p}{q}\sum_{i=\langle fq\rangle}\gamma_{ik}^{*}(x,y)\varphi_i(x,y), \tag{2.25a}$$

where

$$\varphi_i(x,y) = (\mathcal{Z}\varphi)\left((Dx+i)\frac{T}{\beta}, \left[y - \frac{ir}{D}\right]\frac{\Omega}{\alpha}; fp\alpha T\right), \tag{2.25b}$$

$k = 0\ldots fp-1$, and where the variables x and y extend over an interval of length $1/D$ and over an interval of length $1/fp$, respectively. The Fourier transform $(\mathcal{F}^{(2)}\tilde{a})$ is completely determined by the fp functions

$$a_k(x,y) = (\mathcal{F}^{(2)}\tilde{a})\left(x, y - \frac{k}{fp}\right).$$

The functions a_k with $k = 0\ldots fp-1$, are now combined into the fp-dimensional column vector of functions

$$\underline{a}(x,y) = [a_0(x,y), a_1(x,y), \ldots, a_{fp-1}(x,y)]^T$$

and, likewise, the functions φ_i with $i = 0\ldots fq-1$, into the fq-dimensional column vector of functions

$$\underline{\phi}(x,y) = [\varphi_0(x,y), \varphi_1(x,y), \ldots, \varphi_{fq-1}(x,y)]^T.$$

With the help of the vectors $\underline{a}$ and $\underline{\phi}$, Eq. (2.25a-b) can now be expressed as the matrix-vector product

$$\underline{a} = \alpha T\frac{p}{q}\mathbf{\Gamma}^*\underline{\phi}. \tag{2.26}$$

The relation (2.23) applied to an arbitrary vector $\underline{\phi}$ leads to the condition

$$\boldsymbol{G}\mathbf{\Gamma}^*\underline{\phi} = \frac{fq}{\alpha T}\underline{\phi}.$$

Substitution of (2.26) into the previous expression yields

$$\underline{\phi} = \frac{1}{fp}\boldsymbol{G}\underline{a}. \tag{2.27}$$

Note that this vector $\underline{\phi}$ is unique, since $\mathbf{\Gamma}^*$ is injective, and $\boldsymbol{G}$ is proportional to the left inverse of $\mathbf{\Gamma}^*$. The expression (2.26) provides now a method to calculate the array $\{a_{mk}\}$ of Gabor expansion coefficients:

- From the signal φ and the dual window γ_Λ we derive their Zak transforms $(\mathcal{Z}\varphi)(t,\omega; fp\alpha T)$ and $(\mathcal{Z}\gamma_\Lambda)(t,\omega;\alpha T)$, respectively, according to the definition (2.20).

- We construct the vector $\underline{\phi}$, and the matrix $\mathbf{\Gamma}$ and obtain the vector $\underline{a}$ by the matrix-vector product (2.26).
- The array of Gabor expansion coefficients $\{\tilde{a}_{mk}\}$ follows from the function $(\mathcal{F}^{(2)}\tilde{a})$ with the help of the inverse Fourier transform.

A similar procedure can be applied to reconstruct the signal φ with the help of the matrix-vector product (2.27).

The sum-of-products forms in the non-separable case are very similar to the sum-of-products forms in the case of a rectangular lattice. However, the number of elements in the sum-of-product forms, now not only depends on the oversampling p/q, but on the determinant D, as well. Unlike in the separable case (rectangular lattice), the sum-of-products forms do not reduce to product forms in the case of critical sampling and a non-separable lattice, due to the integer $f = D/\gcd(D, q) = D$. As a consequence, the dual window in the case of a non-separable lattice and critical sampling is more difficult to calculate.

The result achieved in this section is obtained by using the shifted and modulated dual windows $\tilde{\gamma}_{\Lambda_s;mk}$ [see Eq. (2.16)] instead of $\gamma_{\Lambda;mk}$ [see Eq. (2.8)]. It is not possible to use these windows $\gamma_{\Lambda;mk}$ to obtain sum-of-products forms that are similar to the separable case, since using this form leads to the following expression for the Fourier transformed array $\{a_{mk}\}$ of Gabor coefficients:

$$
\begin{aligned}
&(\mathcal{F}^{(2)}a)(x, y - k/fp) = \\
&\alpha T \frac{p}{q} \sum_{i=\langle fq \rangle} \sum_{\ell=-\infty}^{\infty} \varphi\left((x+i)\frac{T}{\beta} + \ell\alpha p T\right) e^{-j2\pi\ell f p(y - xr/D - ir/D)} \\
&\times \sum_{m=-\infty}^{\infty} \gamma^*\left((x+i)\frac{T}{\beta} + m\alpha T\right) e^{j2\pi m(y - k/fp - xr/D - ir/D)},
\end{aligned}
$$

in which the Zak transform does not appear. Unlike the Zak transforms in expression (2.25a-b), the exponent in the previous expression depends on the variable x. Apparently, the Zak transform can only be used in the case of a rectangular lattice. In fact, the shifted and modulated windows $\tilde{\gamma}_{\Lambda_s;mk}$ correspond to a rectangular lattice; the non-separable lattice is obtained by leaving out (assigning the value zero to be precise) the shifted and modulated windows that are not a part of the non-separable lattice.

2.3 Fractional Fourier transform

In the previous section, we derived the connection between the Gabor scheme on a non-separable lattice and the Zak transformation. In the non-separable case, the number of elements in the sum-of-products forms not only depends on the oversampling p/q, but on the determinant D as well. In particular, unlike in the separable case, the sum-of-products forms do not reduce to a product form in the case of critical sampling. Since the non-separable lattice can be obtained by, for example, a rotation or a shear, it is most likely that these operations can be useful to transform a non-separable Gabor scheme into a separable one. In this section, we show that the fractional Fourier transform, which can be associated with a rotation of the lattice in the time-frequency plane, can be useful to transform the Gabor scheme on a non-separable lattice into a Gabor scheme on a rectangular lattice. As a result, the fractional Fourier transformation makes it possible to exploit the known properties of the Gabor scheme on a rectangular lattice. In particular, more can be said about the frame bounds in the case of a Gabor scheme on a non-separable lattice. Moreover, the fractional Fourier transformation makes it possible to re-use 'old' methods to calculate the dual window γ_Λ and the array of Gabor coefficients, and to reconstruct the signal φ in the case of a non-separable lattice.

The fractional Fourier transformation (see [2, 17, 58, 65, 66] [1]) is a generalization of the classical Fourier transformation. Kober observed (see [58]) that the eigenvalues of the Fourier transform are $\{\exp -j\frac{1}{2}\pi n\}_{n=0}^{\infty}$ with corresponding eigenfunctions $\exp(-\frac{1}{2}t^2)H_n(t)$, where H_n is the Hermite polynomial of degree n, i.e.,

$$H_n(t) = (-1)^n e^{t^2} \frac{d^n}{dt^n} e^{-t^2}.$$

The orthonormal set of Hermite functions

$$\psi_n(t) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} e^{-t^2/2} H_n(t), \quad n = 0, 1, \ldots$$

is total in $L_2(\mathbb{R})$, and therefore, every signal $h \in L_2(\mathbb{R})$ has the representation

$$h(t) = \sum_{n=0}^{\infty} \langle h, \psi_n \rangle \psi_n(t).$$

From Kober's observation, it follows that the Fourier transform $\mathcal{F}h$,

$$(\mathcal{F}h)(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t) e^{-j\omega t}\, dt,$$

can be written as

$$(\mathcal{F}h)(t) = \sum_{n=0}^{\infty} e^{-j\frac{1}{2}\pi n} \langle h, \psi_n \rangle \psi_n(t).$$

[1] The author would like to thank A.J.E.M. Janssen for drawing his attention to [58] and [17].

Table 2.2: *Two important properties of the fractional Fourier transform.*

$\mathcal{F}_{\theta,\xi}\mathcal{T}_\tau h = e^{j\frac{1}{2}\xi\tau^2\sin(\theta)\cos(\theta)}\mathcal{M}_{-\xi\tau\sin(\theta)}\mathcal{T}_{\tau\cos(\theta)}\mathcal{F}_{\theta,\xi}h$
$\mathcal{F}_{\theta,\xi}\mathcal{M}_\omega h = e^{-j\frac{1}{2}\omega^2\sin(\theta)\cos(\theta)/\xi}\mathcal{M}_{\omega\cos(\theta)}\mathcal{T}_{\omega\sin(\theta)/\xi}\mathcal{F}_{\theta,\xi}h$

Kober then defined the operator $\mathcal{F}_{\theta,1}$ by the eigenvalue equation

$$(\mathcal{F}_{\theta,1}h)(t) = \sum_{n=0}^{\infty} e^{-j\theta n}\langle h, \psi_n\rangle\psi_n(t)$$

with the integral representation

$$(\mathcal{F}_{\theta,\xi}h)(v) = \sqrt{\frac{\xi}{2\pi\sin(\theta)}}e^{j\frac{1}{2}(\theta - \frac{1}{2}\pi)}e^{j\frac{1}{2}\xi v^2\cot(\theta)} \times \int_{-\infty}^{\infty} e^{-j\frac{\xi}{\sin(\theta)}uv} e^{j\frac{1}{2}\xi u^2\cot(\theta)}h(u)\,du, \tag{2.28}$$

where a normalization scale ξ is taken into account with dimension rad/s^2. Namias (see [66]) called this integral representation the fractional Fourier transform. The fractional Fourier transform can be seen as a rotation by an angle θ in the time-frequency plane and corresponds to expressing a signal h in terms of a generalized basis formed by chirps, i.e., complex exponentials with linearly varying instantaneous frequencies. The fractional Fourier transformation is a unitary transformation, i.e., the fractional Fourier transform $\mathcal{F}_{\theta,\xi}$ is bijective and isometric. A more precise formulation can be found in terms of metaplectic representations (see [39]).

Two other important properties of the fractional Fourier transform concerning translation and modulation are tabulated in Table 2.2. By using these properties, the relation between rotation of the sampling lattice and the fractional Fourier transform becomes apparent when the fractional Fourier transform of a shifted and modulated version $\mathcal{M}_\omega\mathcal{T}_\tau h$ is considered:

$$\begin{aligned}\mathcal{F}_{\theta,\xi}\mathcal{M}_\omega\mathcal{T}_\tau h &= e^{j\phi_0(\tau,\omega)}\mathcal{M}_{\omega\cos(\theta)}\mathcal{T}_{\omega\sin(\theta)/\xi}\mathcal{M}_{-\xi\tau\sin(\theta)}\mathcal{T}_{\tau\cos(\theta)}\mathcal{F}_{\theta,\xi}h \\ &= e^{j\phi(\tau,\omega)}\mathcal{M}_{-\xi\tau\sin(\theta)+\omega\cos(\theta)}\mathcal{T}_{\tau\cos(\theta)+\omega\sin(\theta)/\xi}\mathcal{F}_{\theta,\xi}h \\ &= e^{j\phi(\tau,\omega)}\mathcal{M}_{\overline{\omega}}\mathcal{T}_{\overline{\tau}}\mathcal{F}_{\theta,\xi}h \end{aligned} \tag{2.29}$$

where the additional phase terms $\phi_0(\tau,\omega)$ and $\phi(\tau,\omega)$ read

$$\phi_0(\tau,\omega) = \tfrac{1}{2}\frac{\sin(\theta)}{\xi}(\xi^2\tau^2\cos(\theta) - \omega^2\cos(\theta))$$

and

$$\phi(\tau,\omega)=\tfrac{1}{2}\frac{\sin(\theta)}{\xi}(\xi^2\tau^2\cos(\theta)+2\xi\tau\omega\sin(\theta)-\omega^2\cos(\theta)), \tag{2.30}$$

respectively, and where the translation $\overline{\tau}$ and the modulation $\overline{\omega}$ are related to the shift τ and the modulation ω through the matrix relationship

$$\begin{bmatrix}\overline{\tau}\\ \overline{\omega}\end{bmatrix}=\boldsymbol{M}_{\theta,\xi}\begin{bmatrix}\tau\\ \omega\end{bmatrix} \tag{2.31}$$

with a matrix

$$\boldsymbol{M}_{\theta,\xi}=\begin{bmatrix}\frac{1}{\sqrt{\xi}} & 0\\ 0 & \sqrt{\xi}\end{bmatrix}\begin{bmatrix}\cos(\theta) & \sin(\theta)\\ -\sin(\theta) & \cos(\theta)\end{bmatrix}\begin{bmatrix}\sqrt{\xi} & 0\\ 0 & \frac{1}{\sqrt{\xi}}\end{bmatrix}=\boldsymbol{N}_{\xi}^{-1}\boldsymbol{M}_{\theta}\boldsymbol{N}_{\xi}$$

that corresponds to a normalization and a rotation.

In the case of a non-separable lattice, the time shift τ and the modulation ω are equal to $m\alpha T$ and $(k-m\frac{r}{D})\beta\Omega$, respectively [see Eq. (2.8)]. Substitution of this into Eq. (2.31) and replacing ξ by $\frac{\beta\Omega}{\alpha T}\xi'$ yields

$$\begin{bmatrix}\overline{\tau}\\ \overline{\omega}\end{bmatrix}=\begin{bmatrix}\alpha\overline{T} & 0\\ 0 & \beta\overline{\Omega}\end{bmatrix}\begin{bmatrix}\frac{[D\xi'\cos(\theta)-r\sin(\theta)]\cos(\theta)}{D\xi'} & \frac{\sin(\theta)\cos(\theta)}{\xi'}\\ -\frac{D\xi'\sin(\theta)+r\cos(\theta)}{D\cos(\theta)} & 1\end{bmatrix}\begin{bmatrix}m\\ k\end{bmatrix},$$

with $\overline{T}=T/\cos(\theta)$, $\overline{\Omega}=\Omega\cos(\theta)$ and $0<\theta<\pi/2$. This expression can be compared with Eq. (2.4):

$$\begin{bmatrix}\overline{\tau}\\ \overline{\omega}\end{bmatrix}=\overline{\boldsymbol{U}}\,\overline{\boldsymbol{L}}\begin{bmatrix}m\\ k\end{bmatrix},\quad\text{where}\quad\overline{\boldsymbol{U}}=\begin{bmatrix}\alpha\overline{T} & 0\\ 0 & \beta\overline{\Omega}\end{bmatrix}$$

and

$$\overline{\boldsymbol{L}}=\begin{bmatrix}\frac{[D\xi'\cos(\theta)-r\sin(\theta)]\cos(\theta)}{D\xi'} & \frac{\sin(\theta)\cos(\theta)}{\xi'}\\ -\frac{D\xi'\sin(\theta)+r\cos(\theta)}{D\cos(\theta)} & 1\end{bmatrix}.$$

Although this matrix $\overline{\boldsymbol{L}}$ has determinant equal to one, it does not necessarily correspond to a rectangular lattice. This is only the case if the elements in this matrix are integers. Then the elements in each row in $\overline{\boldsymbol{L}}$ do not have a common divisor, since the determinant of $\overline{\boldsymbol{L}}$ is equal to one. Writing $\overline{\boldsymbol{L}}$ in its Hermite normal form $\overline{\boldsymbol{L}}'$ (assuming that $\overline{\boldsymbol{L}}$ contains only integers) yields

$$\overline{\boldsymbol{L}}'=\begin{bmatrix}\frac{[D\xi'\cos(\theta)-r\sin(\theta)]\cos(\theta)}{D\xi'} & 0\\ -\frac{D\xi'\sin(\theta)+r\cos(\theta)}{D\cos(\theta)} & \frac{D\xi'}{[D\xi'\cos(\theta)-r\sin(\theta)]\cos(\theta)}\end{bmatrix}.$$

This matrix $\overline{\boldsymbol{L}}'$ corresponds to a rectangular lattice if the angle θ and the parameter ξ' are chosen such that the conditions

1. $D\xi'\cos^2(\theta) - r\sin(\theta)\cos(\theta) = D\xi'$;
2. $D\xi'\sin(\theta) + r\cos(\theta) = 0$;
3. $\sin(\theta)\cos(\theta)/\xi'$ is an integer;

are fulfilled, i.e., the matrix $\overline{\boldsymbol{L}}'$ reduces to the identity matrix and $\overline{\boldsymbol{L}}$ is a matrix containing only integers. The first two conditions are met if

$$\xi'_\theta = -\frac{r}{D}\cot(\theta). \tag{2.32}$$

From this it follows that $D\sin^2(\theta)/r$ has to be an integer (see condition 3). Since $0 \le r < D$, the possible solutions are

$$\theta_\ell = \pm\arcsin\left(\sqrt{\frac{r}{D}\ell}\right) \quad \text{with} \quad \ell = 1\ldots\left\lceil\frac{D}{r}\right\rceil - 1,$$

where $\lceil . \rceil$ means the smallest integer greater or equal to x. Using one of the angles θ_ℓ and the corresponding parameter $\xi_\ell := \xi'_{\theta_\ell}$ [see Eq. (2.32)], the fractional Fourier transformed set of shifted and modulated windows $\mathcal{F}_{\theta_\ell,\xi_\ell} g_{\Lambda;mk}$ now corresponds to a rectangular lattice [see Eqs. (2.29) and (2.30)]:

$$\begin{aligned}\mathcal{F}_{\theta_\ell,\xi_\ell} g_{\Lambda;mk} = \mathcal{F}_{\theta_\ell,\xi_\ell}\sigma_{\boldsymbol{\Lambda}\left[\begin{smallmatrix}m\\k\end{smallmatrix}\right]} g_\Lambda &= e^{j\phi\left(m\alpha T,\,\left[k-\frac{r}{D}m\right]\beta\Omega\right)}\sigma_{\boldsymbol{M}_{\theta_\ell,\xi_\ell}\boldsymbol{\Lambda}\left[\begin{smallmatrix}m\\k\end{smallmatrix}\right]}\mathcal{F}_{\theta_\ell,\xi_\ell} g_\Lambda \\ &= e^{j\phi\left(m\alpha T,\,\left[k-\frac{r}{D}m\right]\beta\Omega\right)}\sigma_{\overline{\boldsymbol{U}}\,\overline{\boldsymbol{L}}\left[\begin{smallmatrix}m\\k\end{smallmatrix}\right]}\mathcal{F}_{\theta_\ell,\xi_\ell} g_\Lambda.\end{aligned} \tag{2.33}$$

By applying the unitary fractional Fourier transformation with angle θ_ℓ to the Gabor expansion (2.13), and using the expression (2.33) of the fractional Fourier transformed shifted and modulated windows $\mathcal{F}_{\theta_\ell,\xi_\ell} g_{\Lambda;mk}$, it becomes clear how the fractional Fourier transformation transforms a non-separable Gabor expansion into a separable one:

$$\begin{aligned}\mathcal{F}_{\theta_n,\xi}\varphi &= \sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\langle\mathcal{F}_{\theta_\ell,\xi_\ell}\varphi, \mathcal{F}_{\theta_\ell,\xi_\ell}\gamma_{\Lambda;mk}\rangle\mathcal{F}_{\theta_\ell,\xi_\ell} g_{\Lambda;mk} \\ &= \sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\langle\mathcal{F}_{\theta_\ell,\xi_\ell}\varphi, \sigma_{\overline{\boldsymbol{U}}\,\overline{\boldsymbol{L}}\left[\begin{smallmatrix}m\\k\end{smallmatrix}\right]}\mathcal{F}_{\theta_\ell,\xi_\ell}\gamma\rangle\sigma_{\overline{\boldsymbol{U}}\,\overline{\boldsymbol{L}}\left[\begin{smallmatrix}m\\k\end{smallmatrix}\right]}\mathcal{F}_{\theta_\ell,\xi_\ell} g_\Lambda \\ &= \sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\hat{a}_{mk}\sigma_{\overline{\boldsymbol{U}}\,\overline{\boldsymbol{L}}\left[\begin{smallmatrix}m\\k\end{smallmatrix}\right]}\mathcal{F}_{\theta_\ell,\xi_\ell} g_\Lambda,\end{aligned} \tag{2.34}$$

where

$$\hat{a}_{mk} = a_{mk} e^{j\phi\left(m\alpha T,\, \left[k - \frac{r}{D}m\right]\beta\Omega\right)} = \langle \mathcal{F}_{\theta_\ell,\xi_\ell}\varphi, \sigma_{\overline{\boldsymbol{U}}\,\overline{\boldsymbol{L}}\left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right]} \mathcal{F}_{\theta_\ell,\xi_\ell}\gamma_\Lambda \rangle$$

is the array of the (modified) Gabor coefficients. Note that the additional phase terms are canceled in Eq. (2.34). Since Eq. (2.34) corresponds to a (modified) Gabor expansion on a rectangular lattice, it is possible to exploit methods of the separable case to calculate the dual window γ_Λ and the Gabor expansion coefficients, and to reconstruct the signal φ.

For example, the Zak transformation can be used to calculate the dual window γ_Λ. Eq. (2.21a-c) now takes the form

$$\begin{aligned} &\sum_{k=\langle p \rangle} (\mathcal{Z}\mathcal{F}_{\theta_\ell,\xi_\ell} g_\Lambda) \left((x+i)\frac{\overline{T}}{\beta}, \left[y - \frac{k}{p}\right]\frac{\overline{\Omega}}{\alpha}; \alpha\overline{T}\right) \\ &\quad \times (\mathcal{Z}\mathcal{F}_{\theta_\ell,\xi_\ell}\gamma_\Lambda)^* \left((x+n)\frac{\overline{T}}{\beta}, \left[y - \frac{k}{p}\right]\frac{\overline{\Omega}}{\alpha}; \alpha\overline{T}\right) = \frac{q}{\alpha\overline{T}}\delta[i-n], \end{aligned}$$

with $i, n = 0 \ldots q-1$, and x and y extending over an interval of length 1 and $1/p$, respectively. Note that this sum-of-products form indeed reduces to a product form in the case of critical sampling ($p = q = 1$). The procedure to obtain the dual window γ_Λ is clear:

- We calculate the fractional Fourier transform $\mathcal{F}_{\theta_\ell,\xi_\ell} g_\Lambda$ of the window g_Λ.
- We calculate its Zak transform and construct the matrix $\boldsymbol{G}$ [see Eq. (2.22)].
- The matrix $\boldsymbol{\Gamma}$ is obtained by the matrix-product (2.23).
- The fractional Fourier transform $\mathcal{F}_{\theta_\ell,\xi_\ell}\gamma_\Lambda$ of the dual window γ_Λ is now obtained by applying the inverse Zak transform.
- The dual window γ_Λ follows now from the inverse fractional Fourier transform.

Assume that the set $\{g_{\Lambda;mk} | m, k \in \mathbb{Z}\}$, corresponding to a set of shifted and modulated windows on a non-separable lattice, constitutes a frame with frame bounds A and B. Then we have for all $h \in L_2(\mathbb{R})$

$$\begin{aligned} & A\|h\|^2 \le \sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} |\langle h, g_{\Lambda;mk}\rangle|^2 \le B\|h\|^2, \\ \equiv\quad & A\|h\|^2 \le \sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} |\langle h, \mathcal{F}_{\theta_\ell,\xi_\ell} g_{\Lambda;mk}\rangle|^2 \le B\|h\|^2 \\ \overset{\text{Eq. (2.33)}}{\equiv}\quad & A\|h\|^2 \le \sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} |\langle h, \sigma_{\overline{\boldsymbol{U}}\left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right]} \mathcal{F}_{\theta_\ell,\xi_\ell} g_\Lambda\rangle|^2 \le B\|h\|^2, \end{aligned}$$

Table 2.3: *Two important properties of the shear operator.*

$\mathcal{Q}_\omega \mathcal{T}_\tau = e^{-j\omega\tau^2} \mathcal{M}_{2\omega\tau} \mathcal{T}_\tau \mathcal{Q}_\omega$
$\mathcal{Q}_{\omega_0} \mathcal{M}_{\omega_1} = \mathcal{M}_{\omega_1} \mathcal{Q}_{\omega_0}$

i.e., the frame bounds A and B of the frame constituted by the set $\{g_{\Lambda;mk} | m, k \in \mathbb{Z}\}$, corresponding to a non-separable lattice Λ, are equal to the frame bounds of the frame constituted by the set $\{\sigma_{\overline{U}\left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right]} \mathcal{F}_{\theta_\ell, \xi_\ell} g_\Lambda | m, k \in \mathbb{Z}\}$ which corresponds to a rectangular lattice. In the case of a Gaussian window g_Λ, which is an eigenfunction of the fractional Fourier transformation, these frame bounds can be calculated exactly in the case of integer oversampling (see [23, 54]).

2.4 Shearing

In the previous section, the fractional Fourier transformation, which can be associated with a rotation in the time-frequency plane, is used to transform a non-separable lattice into a separable lattice. A shear of the time variable or the frequency variable, is able to perform such a transformation, as well. First we consider a shear of the frequency variable. We define the shear operator $\mathcal{Q}_{\omega_a}$ as

$$(\mathcal{Q}_{\omega_a} \varphi)(t) = e^{j\omega_a t^2} \varphi(t), \tag{2.35}$$

which is unitary on $L_2(\mathbb{R})$ with corresponding Hilbert adjoint $\mathcal{Q}^*_{\omega_a} = \mathcal{Q}_{-\omega_a}$. For convenience, we will use the same notation for the shear operator $\mathcal{Q}_{\omega_a}$ for the discrete-time setting. The increase in modulation per second caused by the shear operator is equal to $\frac{d}{dt}\omega_a t^2 = 2\omega_a t$. A non-separable lattice Λ is sheared in the frequency direction with $-r\beta\Omega/\alpha T D$ [see Eq. (2.8)]. Thus, to reshear the non-separable lattice into a separable lattice we have to shear with

$$\omega_{a,0} = r\beta\Omega/2\alpha T D.$$

Note that $\omega_{a,0} t^2$ is dimensionless. In Table 2.3, two properties of this shear operator $\mathcal{Q}_{\omega_a}$ are tabulated concerning translation and modulation. By applying the shear operator $\mathcal{Q}_{\omega_a}$ and its properties to the shifted and modulated windows $g_{\Lambda;mk}$, we obtain

$$\mathcal{Q}_{\omega_{a,0}} g_{\Lambda;mk} = \mathcal{M}^k_{\beta\Omega} \mathcal{M}^m_{\frac{-r\beta\Omega}{D}} \mathcal{Q}_{\omega_{a,0}} \mathcal{T}^m_{\alpha T} g_\Lambda = e^{-j\pi\alpha\beta r m^2/D} \mathcal{M}^k_{\beta\Omega} \mathcal{T}^m_{\alpha T} \mathcal{Q}_{\omega_{a,0}} g_\Lambda. \tag{2.36}$$

So if the collection $\{g_{\Lambda;mk} | m, k \in \mathbb{Z}\}$, which corresponds to a non-separable lattice, establishes a frame, then the collection $\{\mathcal{Q}_{\omega_{a,0}} g_{\Lambda;mk} | m, k \in \mathbb{Z}\}$ establishes a frame

Table 2.4: *Some properties of the Fourier transformation.*

$\mathcal{F}\mathcal{M}_{\omega_0} = \mathcal{T}_{\omega_0}\mathcal{F}$
$\mathcal{F}\mathcal{T}_{\tau} = \mathcal{M}_{-\tau}\mathcal{F}$

on a rectangular lattice. Applying the unitary shear operator $\mathcal{Q}_{\omega_{a,0}}$, the sheared Gabor expansion (2.13) is now on a rectangular lattice, as well:

$$\mathcal{Q}_{\omega_{a,0}}\varphi = \sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} \langle \mathcal{Q}_{\omega_{a,0}}\varphi, \mathcal{M}^k_{\beta\Omega}\mathcal{T}^m_{\alpha T}\mathcal{Q}_{\omega_{a,0}}\gamma_\Lambda\rangle \mathcal{M}^k_{\beta\Omega}\mathcal{T}^m_{\alpha T}\mathcal{Q}_{\omega_{a,0}}g_\Lambda$$
$$= \sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} \check{a}_{mk}\mathcal{M}^k_{\beta\Omega}\mathcal{T}^m_{\alpha T}\mathcal{Q}_{\omega_{a,0}}g_\Lambda, \tag{2.37}$$

where

$$\check{a}_{mk} = a_{mk}e^{-j\pi\alpha\beta r m^2/D} = \langle \mathcal{Q}_{\omega_{a,0}}\varphi, \mathcal{M}^k_{\beta\Omega}\mathcal{T}^m_{\alpha T}\mathcal{Q}_{\omega_{a,0}}\gamma_\Lambda\rangle$$

is the array of (modified) Gabor expansion coefficients. Note that the additional phase terms are canceled in Eq. (2.37).

As mentioned before, a shear of the time variable is able to translate a non-separable lattice into a separable lattice, as well. Now we need the shifted and modulated windows of the form (2.11), with corresponding Fourier transform

$$\mathcal{F}g_{\Lambda;mk} = \mathcal{F}\mathcal{M}^k_{\beta\Omega/D}\mathcal{T}^k_{-r'\alpha T}\mathcal{T}^m_{D\alpha T}g_\Lambda = \mathcal{T}^k_{\beta\Omega/D}\mathcal{M}^k_{r'\alpha T}\mathcal{M}^m_{-D\alpha T}\mathcal{F}g_\Lambda$$
$$= e^{j2\pi\alpha\beta(mk - k^2r'/D)}\mathcal{M}^k_{r'\alpha T}\mathcal{M}^m_{-D\alpha T}\mathcal{T}^k_{\beta\Omega/D}\mathcal{F}g_\Lambda, \tag{2.38}$$

where we used the properties tabulated in Table 2.4. In order to reshear the corresponding lattice into a separable lattice, we need the shear operator $\mathcal{Q}_{t_a}$ defined as

$$(\mathcal{Q}_{t_a}\varphi)(\omega) = e^{jt_a\omega^2}\varphi(\omega). \tag{2.39}$$

From expression (2.38), it follows that we have to shear with $-r'\alpha TD/\beta\Omega$, and thus we have to choose

$$t_{a,0} = -r'\alpha TD/2\beta\Omega$$

in order to reshear the non-separable into a separable lattice. The sheared shifted and modulated Fourier transformed windows $\mathcal{Q}_{t_{a,0}}\mathcal{F}g_{\Lambda;mk}$ are indeed on a rectangular lattice:

$$\mathcal{Q}_{t_{a,0}}\mathcal{F}g_{\Lambda;mk} = e^{j2\pi\alpha\beta(mk - k^2r'/D)}\mathcal{Q}_{t_{a,0}}\mathcal{M}^m_{-D\alpha T}\mathcal{M}^k_{r'\alpha T}\mathcal{T}^k_{\beta\Omega/D}\mathcal{F}g_\Lambda$$
$$= e^{j2\pi\alpha\beta(mk - k^2r'/2D)}\mathcal{M}^m_{-D\alpha T}\mathcal{T}^k_{\beta\Omega/D}\mathcal{Q}_{t_{a,0}}\mathcal{F}g_\Lambda.$$

Note that the phase $\exp(j2\pi\alpha\beta mk)$ is due to the exchange of the modulation operator $\mathcal{M}^m_{-D\alpha T}$ and the time shift operator $\mathcal{T}^k_{\beta\Omega/D}$. As a consequence, applying the unitary Fourier transformation $\mathcal{F}$ and the unitary shear operator $\mathcal{Q}_{t_a,0}$, the sheared and Fourier transformed Gabor expansion (2.13) is on a rectangular lattice, as well:

$$\mathcal{Q}_{t_a,0}\mathcal{F}\varphi = \sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} \breve{a}_{mk}\mathcal{M}^m_{-D\alpha T}\mathcal{T}^k_{\beta\Omega/D}\mathcal{Q}_{\omega_{a_1}}\mathcal{F}g_\Lambda, \tag{2.40}$$

where

$$\begin{aligned}\breve{a}_{mk} &= a_{mk}e^{j2\pi\alpha\beta(mk - k^2r'/2D)} \\ &= \langle \mathcal{Q}_{t_a,0}\mathcal{F}\varphi, \mathcal{M}^m_{-D\alpha T}\mathcal{T}^k_{\beta\Omega/D}\mathcal{Q}_{t_a,0}\mathcal{F}\gamma_\Lambda\rangle\end{aligned}$$

is the array of (modified) Gabor coefficients. Since Eqs. (2.37) and (2.40) correspond to a (sheared) Gabor expansion on a rectangular lattice, it is possible to exploit methods of the separable case to calculate the dual window γ_Λ and the Gabor expansion coefficients, and to reconstruct the signal φ.

Assume that the set $\{g_{\Lambda;mk}|m,k \in \mathbb{Z}\}$, corresponding to a set of shifted and modulated windows on a non-separable lattice, constitutes a frame with frame bounds A and B. Then we have for all $h \in L_2(\mathbb{R})$

$$\begin{aligned} & A\|h\|^2 \le \sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} |\langle h, g_{\Lambda;mk}\rangle|^2 \le B\|h\|^2, \\ \equiv\quad & A\|h\|^2 \le \sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} |\langle h, \mathcal{Q}_{\omega_a,0}g_{\Lambda;mk}\rangle|^2 \le B\|h\|^2 \\ \overset{\text{Eq. (2.36)}}{\equiv}\quad & A\|h\|^2 \le \sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} |\langle h, \sigma_{\left[\begin{smallmatrix} m\alpha T \\ k\beta\Omega\end{smallmatrix}\right]}\mathcal{Q}_{\omega_a,0}g_\Lambda\rangle|^2 \le B\|h\|^2, \end{aligned}$$

i.e., the frame bounds A and B of the frame constituted by the set $\{g_{\Lambda;mk}|m,k \in \mathbb{Z}\}$, corresponding to a non-separable lattice Λ, are equal to the frame bounds of the frame constituted by the set $\{\sigma_{\left[\begin{smallmatrix} m\alpha T \\ k\beta\Omega\end{smallmatrix}\right]}\mathcal{Q}_{\omega_a,0}g_\Lambda|m,k \in \mathbb{Z}\}$ which corresponds to a rectangular lattice. On the other hand, if the set $\{\sigma_{\left[\begin{smallmatrix} m\alpha T \\ k\beta\Omega\end{smallmatrix}\right]}\mathcal{Q}_{\omega_a,0}g_\Lambda|m,k \in \mathbb{Z}\}$ constitutes a frame, then the set $\{g_{\Lambda;mk}|m,k \in \mathbb{Z}\}$ constitutes a frame as well. Put differently, we can use the window g_Λ in the non-separable case, if the set $\{\sigma_{\left[\begin{smallmatrix} m\alpha T \\ k\beta\Omega\end{smallmatrix}\right]}\mathcal{Q}_{\omega_a,0}g_\Lambda|m,k \in \mathbb{Z}\}$ constitutes a frame. A similar result can be obtained with the shear operator $\mathcal{Q}_{t_a,0}$.

The Zak transformation can be used to calculate the dual window γ_Λ. Eq. (2.21) now takes the form (corresponding to a frequency shear)

$$\sum_{k=\langle p\rangle} \left(\mathcal{Z}\mathcal{Q}_{\omega_{a,0}} g_\Lambda\right)\left((x+i)\frac{T}{\beta},\left[y-\frac{k}{p}\right]\frac{\Omega}{\alpha};\alpha T\right) \times \left(\mathcal{Z}\mathcal{Q}_{\omega_{a,0}}\gamma_\Lambda\right)^*\left((x+n)\frac{T}{\beta},\left[y-\frac{k}{p}\right]\frac{\Omega}{\alpha};\alpha T\right)=\frac{q}{\alpha T}\delta[i-n],$$

with $i, n = 0\ldots q-1$, and x and y extending over an interval of length 1 and $1/p$, respectively. Note that this sum-of-products form indeed reduces to a product form in the case of critical sampling ($p = q = 1$). The procedure to obtain the dual window γ_Λ is clear:

- We multiply the window g_Λ by a quadratic phase term to obtain the sheared window $\mathcal{Q}_{\omega_{a,0}} g_\Lambda$.
- We calculate its Zak transform and construct the matrix $\boldsymbol{G}$ [see Eq. (2.22)].
- The matrix $\boldsymbol{\Gamma}$ is obtained by the matrix-product (2.23).
- The sheared dual window $\mathcal{Q}_{\omega_{a,0}}\gamma_\Lambda$ is now obtained by applying the inverse Zak transform.
- And finally, we get the dual window γ_Λ by multiplying the sheared dual window $\mathcal{Q}_{\omega_{a,0}}\gamma_\Lambda$ by a quadratic phase term.

2.4.1 Calculation generalized window in the case of critical sampling

In the early eighties, Bastiaans (see [5–7]) and Janssen (see [48]) calculated analytically the generalized function γ_Λ of a Gaussian window

$$g_\Lambda(t) = 2^{\frac{1}{4}} e^{-\pi(t/T)^2},$$

such that

$$\varphi = \sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\langle\varphi,\gamma_{\Lambda;mk}\rangle g_{\Lambda;mk} \tag{2.41}$$

in the case of a rectangular lattice Λ and critical sampling. Although the set of shifted and modulated versions $\{g_{\Lambda;mk}|m,k \in \mathbb{Z}\}$ is complete in $L_2(\mathbb{R})$, this sum does not converge in a weak $L_2(\mathbb{R})$-sense (see [27, 49]); the set $\{g_{\Lambda;mk}|m,k \in \mathbb{Z}\}$ does not constitute a frame. Janssen showed that convergence holds only in the sense of distributions (see [47]). In Fig. 2.3 (a), the well-known function γ_Λ,

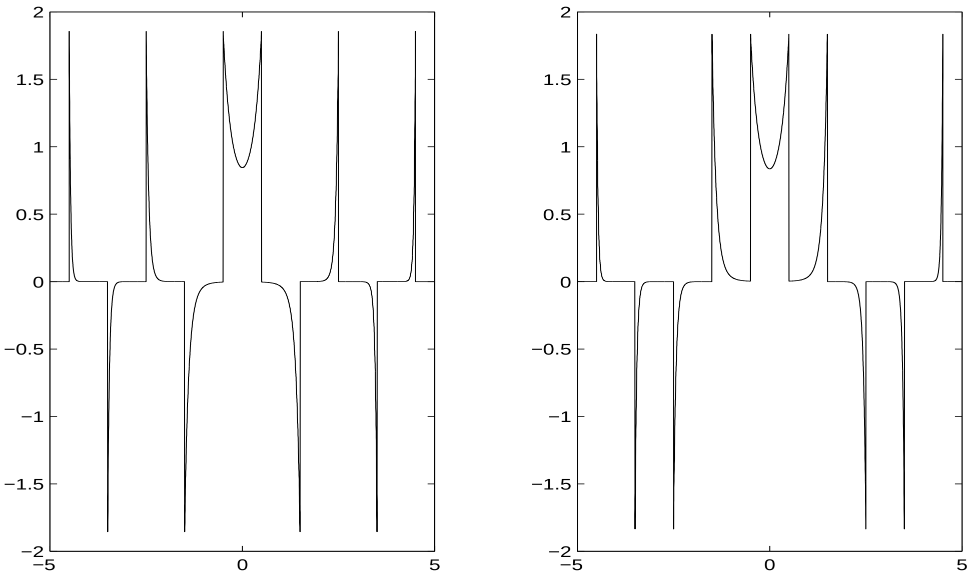

Figure 2.3: *The generalized functions γ_Λ in (a) the case of a rectangular lattice and in (b) the case of a quincunx lattice.*

$$\gamma_\Lambda(t) = \frac{2^{-\frac{1}{4}}}{\alpha T}\left(\frac{K_0}{\pi}\right)^{-3/2}(-1)^{m_t} e^{\pi[(t/\alpha T)^2 - (m_t + \frac{1}{2})^2]} \times \sum_{u=m_t}^{\infty}(-1)^{u-m_t} e^{-\pi[(u+\frac{1}{2})^2 - (m_t+\frac{1}{2})^2]},$$

where $K_0 = 1.85407468$ and m_t is an integer such that $(m_t - \frac{1}{2})\alpha T < |t| < (m_t + \frac{1}{2})\alpha T$, with its typical peaks is depicted for $\alpha = \beta = T = 1$. In order to calculate this function γ_Λ, Bastiaans used the Zak transform and the sum-of-products form (2.21a) in the case of critical sampling, which reduces then to the product form

$$(\mathcal{Z}g_\Lambda)\left(t\frac{T}{\beta}, \omega\frac{\Omega}{\alpha}; \alpha T\right)(\mathcal{Z}\gamma_\Lambda)^*\left(t\frac{T}{\beta}, \omega\frac{\Omega}{\alpha}; \alpha T\right) = \frac{1}{\alpha T},$$

where the variables t and ω extend over an interval of length 1, or equivalently

$$\alpha T\,(\mathcal{Z}g_\Lambda)(t,\omega;\alpha T)\,(\mathcal{Z}\gamma_\Lambda)^*(t,\omega;\alpha T) = 1, \tag{2.42}$$

where the variable t extends over an interval αT and the variable ω over an interval $2\pi/\alpha T$. The function γ_Λ can be formally found in the following way:

- from the window g_Λ we derive its Zak transform $\mathcal{Z}g_\Lambda$;

- under assumption that division by $\mathcal{Z}g_\Lambda$ is allowed, the function $\mathcal{Z}\gamma_\Lambda$ can be found with the help of relation (2.42);
- finally, the function γ_Λ follows from its Zak transform $\mathcal{Z}\gamma_\Lambda$ by means of the inverse Zak transformation.

Although we are aware of the fact that the expansion (2.41) in the case of critical sampling converges only in the sense of distributions, to illustrate usefulness of the shear operator, we calculate in this section the generalized function γ_Λ in the case of a non-separable lattice and critical sampling. In the non-separable case, the sum-of-products form (2.21a) does not reduce to a simple product form, due to the integer $f = D > 1$. As a consequence, the generalized function γ_Λ is more difficult to calculate. In order to transform this sum-of-products into a product form, we can use the fractional Fourier transform $\mathcal{F}_{\theta,\xi}$ or the shear operators $\mathcal{Q}_{\omega_a}$ and $\mathcal{Q}_{t_a}$, as shown in the Sections 2.3 and 2.4, respectively. In this section we use the shear operator $\mathcal{Q}_{\omega_a}$, as an example.

Following the procedure outlined in the previous section and substituting the shear operator $\mathcal{Q}_{\omega_{a,0}}$ in Eq. (2.42) yields

$$\alpha T \left(\mathcal{Z}\mathcal{Q}_{\omega_{a,0}} g_\Lambda\right)(t,\omega;\alpha T)\left(\mathcal{Z}\mathcal{Q}_{\omega_{a,0}}\gamma_\Lambda\right)^*(t,\omega;\alpha T) = 1. \tag{2.43}$$

The Zak transform of the Gaussian g_Λ multiplied by the quadratic phase term with $\alpha = 1/\beta$ and $\Omega T = 2\pi$

$$(\mathcal{Q}_{\omega_{a,0}} g_\Lambda)(t) = g_\Lambda(t) e^{j\pi(r/D)(t/\alpha T)^2} = 2^{\frac{1}{4}} e^{-\pi[\alpha^2 - j(r/D)](t/\alpha T)^2}$$

reads

$$\begin{aligned}\left(\mathcal{Z}\mathcal{Q}_{\omega_{a,0}} g_\Lambda\right)(t,\omega;\alpha T) &= 2^{\frac{1}{4}} (v^*)^{(t/\alpha T)^2} \sum_{m=-\infty}^{\infty} (v^*)^{m^2} e^{-j2mz^*(t,\omega)} \\ &= 2^{\frac{1}{4}} (v^*)^{(t/\alpha T)^2} \theta_3(z^*(t,\omega), v^*),\end{aligned}$$

where

$$\theta_3(z(t,\omega), v) = \sum_{m=-\infty}^{\infty} v^{m^2} e^{-j2mz(t,\omega)}$$

is a theta function (see, for instance, [86]), in this case with nome $v = \exp\{-\pi[\alpha^2 + j(r/D)]\}$, and with $z(t,\omega) = \pi\{\alpha\omega/\Omega + j[\alpha^2 + j(r/D)](t/\alpha T)\}$. The Zak transform of the generalized function $\mathcal{Q}_{\omega_{a,0}}\gamma_\Lambda$, follows now from Eq. (2.43)

$$\begin{aligned}&\alpha T 2^{\frac{1}{4}} v^{(t/\alpha T)^2} \left(\mathcal{Z}\mathcal{Q}_{\omega_{a,0}}\gamma_\Lambda\right)(t,\omega;\alpha T) = \frac{2^{\frac{1}{4}} v^{(t/\alpha T)^2}}{\left[\left(\mathcal{Z}\mathcal{Q}_{\omega_{a,0}} g\right)(t,\omega;\alpha T)\right]^*} \\ &= \frac{1}{\theta_3(z(t,\omega), v)} = \frac{1}{\theta_4(z(t,\omega) + \frac{1}{2}\pi, v)}.\end{aligned} \tag{2.44}$$

In the fundamental rectangle $(-\frac{1}{2}\alpha T < t < \frac{1}{2}\alpha T,\ -\pi/\alpha T < \omega < \pi/\alpha T)$ the inverse of the theta function $\theta_4(z(t,\omega)+\frac{1}{2}\pi, v)$ can be expressed as (see, for instance, [86], p. 489)

$$\frac{1}{\theta_4(z(t,\omega)+\frac{1}{2}\pi, v)} = \frac{2}{\theta_2\theta_3\theta_4}\left[c_0 + 2\sum_{m=1}^{\infty}(-1)^m c_m \cos(2mz(t,\omega))\right]$$
$$= \frac{2}{\theta_2\theta_3\theta_4}\sum_{m=-\infty}^{\infty}(-1)^m c_{|m|} e^{-j2mz(t,\omega)},$$

where the constants θ_2, θ_3 and θ_4 are equal to the theta functions $\theta_2(z(t,\omega),v)$, $\theta_3(z(t,\omega),v)$ and $\theta_4(z(t,\omega),v)$ evaluated in $z(t,\omega) = 0$, respectively, and where the coefficients c_m are defined by

$$c_m = \sum_{n=0}^{\infty}(-1)^n v^{(n+\frac{1}{2})(2m+n+\frac{1}{2})}. \tag{2.45}$$

In Appendix A.4 it is shown that the generalized function γ_Λ has the form

$$\gamma_\Lambda(t) = \frac{2^{\frac{3}{4}}}{\alpha T\theta_2\theta_3\theta_4} e^{-jr\beta\Omega t^2/2\alpha TD} v^{-(t/\alpha T)^2} \sum_{u=m_t}^{\infty}(-1)^u\left[v^{(u+\frac{1}{2})^2}\right], \tag{2.46}$$

where m_t is an integer such that $(m_t-\frac{1}{2})\alpha T < |t| < (m_t+\frac{1}{2})\alpha T$. From a computational point of view, this expression is not practical; a more practical way to represent this window is in the form (cf. [60] in the case of a rectangular lattice)

$$\gamma_\Lambda(t) = \frac{2^{\frac{3}{4}}}{\alpha T\theta_2\theta_3\theta_4} e^{-jr\beta\Omega t^2/2\alpha TD}(-1)^{m_t}\left[v^{-[(t/\alpha T)^2-(m_t+\frac{1}{2})^2]}\right]$$
$$\times\sum_{u=m_t}^{\infty}(-1)^{u-m_t}\left[v^{(u+\frac{1}{2})^2-(m_t+\frac{1}{2})^2}\right]. \tag{2.47}$$

As an example, the generalized function of a Gaussian window in the case of a quincunx lattice is calculated, i.e., $r = 1$ and $D = 2$ [the nome v takes the value $\exp(-\pi(\alpha^2+\frac{1}{2}j)]$. The generalized function γ_Λ then takes the form [see Eq. (2.47)]

$$\gamma_\Lambda(t) = \frac{2^{\frac{3}{4}}}{\alpha T\theta_2\theta_3\theta_4} e^{-j\frac{1}{2}\pi(t/\alpha T)^2}(-1)^{m_t} e^{\pi(\alpha^2+\frac{1}{2}j)\left[(t/\alpha T)^2-(m_t+\frac{1}{2})^2\right]}$$
$$\times\sum_{u=m_t}^{\infty}(-1)^{u-m_t} e^{-\pi(\alpha^2+\frac{1}{2}j)\left[(u+\frac{1}{2})^2-(m_t+\frac{1}{2})^2\right]}.$$

Rewriting this expression

$$\gamma_\Lambda(t) = \frac{2^{\frac{3}{4}}}{\alpha T \theta_2\theta_3\theta_4}(-1)^{m_t} e^{-j\frac{1}{2}\pi(m_t+\frac{1}{2})^2} e^{\pi\alpha^2[(t/\alpha T)^2 - (m_t+\frac{1}{2})^2]}$$
$$\times \sum_{u=m_t}^{\infty} (-1)^{u-m_t} e^{-j\frac{1}{2}\pi\,[(u+\frac{1}{2})^2 - (m_t+\frac{1}{2})^2]} e^{-\pi\alpha^2\,[(u+\frac{1}{2})^2 - (m_t+\frac{1}{2})^2]},$$

and expanding $\exp(-j\frac{1}{2}\pi(m_t+\frac{1}{2})^2) = \exp(-j\frac{1}{8}\pi)\exp(-j\frac{1}{2}\pi m_t(m_t+1))$, where the expression $\exp(-j\frac{1}{2}\pi m_t(m_t+1))$ is real-valued for all integers m_t, yields

$$\gamma_\Lambda(t) = \frac{2^{\frac{3}{4}}}{\alpha T \theta_2\theta_3\theta_4}(-1)^{m_t} e^{-j\frac{1}{2}\pi m_t(m_t+1)} e^{-j\frac{1}{8}\pi} e^{\pi\alpha^2[(t/\alpha T)^2 - (m_t+\frac{1}{2})^2]}$$
$$\times \sum_{u=m_t}^{\infty} (-1)^{u-m_t} e^{-j\frac{1}{2}\pi[u(u+1) - m_t(m_t+1)]} e^{-\pi\alpha^2\,[(u+\frac{1}{2})^2 - (m_t+\frac{1}{2})^2]}.$$

The constant $\theta_2\theta_3\theta_4$ is equal to $\frac{d}{dz}\theta_1(z,v)\big|_{z=0}$; the type 1 theta function $\theta_1(z,v)$ differentiated with regard to the variable z evaluated at $z = 0$ (see [86]). This constant $\theta_2\theta_3\theta_4$ can be written in the following product form $2v^{\frac{1}{4}}G^3$ with $G = \prod_{n=1}^{\infty}(1 - v^{2n})$. Note that G is real-valued and $v^{\frac{1}{4}} = \exp(-j\frac{1}{8}\pi)\exp(-\frac{1}{4}\pi\alpha^2)$. As a consequence, the generalized function γ_Λ is real-valued in the case of a quincunx lattice:

$$\gamma_\Lambda(t) = \frac{2^{-\frac{1}{4}}}{\alpha T G^3}(-1)^{m_t} e^{-j\frac{1}{2}\pi m_t(m_t+1)} e^{\frac{1}{4}\pi\alpha^2} e^{\pi\alpha^2[(t/\alpha T)^2 - (m_t+\frac{1}{2})^2]}$$
$$\times \sum_{u=m_t}^{\infty} (-1)^{u-m_t} e^{-j\frac{1}{2}\pi[u(u+1) - m_t(m_t+1)]} e^{-\pi\alpha^2\,[(u+\frac{1}{2})^2 - (m_t+\frac{1}{2})^2]}.$$

Choosing $\alpha = 1$ and observing that the summation in this expression yields a result which is close to unity for any value of m_t (1.001867 for $m_t = 0$, 0.999997 for $m_t = 1, \ldots, 1$ for $m_t = \infty$), leads to the approximation

$$\gamma_\Lambda(t) = \frac{2^{-\frac{1}{4}}}{\alpha T G^3}(-1)^{m_t} e^{-j\frac{1}{2}\pi m_t(m_t+1)} e^{\frac{1}{4}\pi} e^{\pi[(t/T)^2 - (m_t+\frac{1}{2})^2]},$$

with m_t defined by $(m_t - \frac{1}{2})T < |t| < (m_t + \frac{1}{2})T$. In Fig. 2.3 (b), this generalized function is depicted with $T = 1$. The generalized function looks very similar to the dual window in the rectangular case, except for the sign of the side peaks, due to the term $\exp(j\frac{1}{2}\pi m_t(m_t+1))$.

2.5 Concluding remarks

In this chapter, the Gabor scheme for the separable lattice for continuous-time signals is extended to the general, non-separable lattice in a structured way; this is achieved by describing the non-separable lattice by means of a lattice generator matrix. The lattice generator matrix is written in the Hermite normal form to obtain a shear representation on the shifted and modulated windows, which shear representation then leads to a modification of the rectangular Gabor scheme and results in the Gabor scheme on a non-separable lattice. The lattice generator matrix written in its Hermite normal form also leads to an alternative expression of the shifted and modulated windows. This expression is based on a rectangular lattice, as well; the non-separable lattice is obtained by deleting the shifted and modulated windows of a refined rectangular lattice, which do not belong to the non-separable lattice.

Since the Zak transform can be very helpful in determining Gabor's expansion coefficients, in finding the dual window and in reconstructing the signal from the given Gabor expansion coefficients in the case of a rectangular time-frequency lattice, this idea is extended to the non-separable case. The results in the non-separable case are very similar to the results in the rectangular case; by using the Fourier transform and the Zak transform, the Gabor transform and Gabor's signal expansion can be written as a sum-of-products form, again. However, the number of elements in the sum now not only depends on the oversampling, but on the determinant of the lattice generator matrix, as well. As a consequence, the sum-of-products form does not reduce to a product form in the case of critical sampling and a non-separable lattice.

The non-separable lattices can be obtained, for instance, via a scaled rotation operation or a shear operation on the rectangular lattice. It is shown, that the fractional Fourier transform, which can be seen as a rotation in the time-frequency plane, and multiplying by quadratic phase terms, associated with the shear operation, translate the non-separable case to the rectangular case. As a consequence, techniques used in the rectangular case can be re-used for the non-separable case. Due to the translation of the non-separable to the rectangular case, the sum-of-products form reduces to a product form in the case of critical sampling. As an example, the generalized window of a Gaussian window is calculated in the case of critical sampling and a non-separable lattice.

Chapter 3

Multi-dimensional non-separable Gabor scheme for continuous-time signals

In this chapter, the one-dimensional non-separable Gabor scheme is extended to the d-dimensional Gabor scheme for possibly non-separable lattices and possibly non-separated windows and the relationship with the Zak transformation and the d-dimensional non-separable Gabor scheme is elucidated. Again, similar to the one-dimensional case, the non-separable lattice is described by means of a lattice generator matrix written in Hermite normal form. In order to show the connection between the d-dimensional non-separable Gabor scheme and the Zak transformation, an alternative expression for the Gabor scheme of shifted and modulated versions of the window is used. This alternative expression is obtained in a similar way as in the one-dimensional case; the Gabor scheme of shifted and modulated versions of the window based on the separable lattice that refines the non-separable lattice is obtained by multiplying by one if it belongs to the non-separable lattice and multiplying by zero otherwise.

In [59] instead, the Kohn-Nirenberg correspondence is used to study the Gabor schemes for the situations of non-separable windows and/or non-separable lattices.

3.1 Gabor's signal expansion on a non-separable lattice

In this section, as an introduction to the remaining part of the section, we construct a d-dimensional non-separable Gabor scheme starting from the one-dimensional non-separable Gabor schemes. Each ith two-dimensional lattice is described by $\boldsymbol{U}_i$ and $\boldsymbol{L}_i$ corresponding to the ith non-separable lattice Λ_i, [cf. Eqs. (2.4) and (2.6)]

$$\boldsymbol{U}_i = \frac{1}{D_i}\begin{bmatrix}\alpha_i T_i D_i & 0\\ 0 & \beta_i \Omega_i\end{bmatrix} \quad \text{and} \quad \boldsymbol{L}_i = \begin{bmatrix}1 & 0\\ -r_i & D_i\end{bmatrix},$$

where $i = 0 \ldots d-1$. Then the $2d$-dimensional lattice Λ is described by the following matrices $\boldsymbol{U} \in \mathbb{R}^{2d\times 2d}$ and $\boldsymbol{L} \in \mathbb{Z}^{2d\times 2d}$ in its Hermite normal form:

$$\boldsymbol{U} = \begin{bmatrix} \boldsymbol{AT} & \boldsymbol{0}_d \\ \boldsymbol{0}_d & \boldsymbol{B\Omega D}^{-1} \end{bmatrix}, \quad \text{and} \quad \boldsymbol{L} = \begin{bmatrix} \boldsymbol{I}_d & \boldsymbol{0}_d \\ -\boldsymbol{R} & \boldsymbol{D} \end{bmatrix}, \tag{3.1}$$

where the diagonal matrices $\boldsymbol{A}$, $\boldsymbol{T}$, $\boldsymbol{B}$, $\boldsymbol{\Omega} \in \mathbb{R}^{d\times d}$, and the diagonal matrices $\boldsymbol{D}$ and $\boldsymbol{R} \in \mathbb{Z}^{d\times d}$ are given by

$$\begin{aligned} &\boldsymbol{A} = \operatorname{diag}(\alpha_0, \ldots, \alpha_{d-1}), \quad \boldsymbol{T} = \operatorname{diag}(T_0, \ldots, T_{d-1}), \\ &\boldsymbol{B} = \operatorname{diag}(\beta_0, \ldots, \beta_{d-1}), \quad \boldsymbol{\Omega} = \operatorname{diag}(\Omega_0, \ldots, \Omega_{d-1}), \\ &\boldsymbol{D} = \operatorname{diag}(D_0, \ldots, D_{d-1}), \quad \text{and} \quad \boldsymbol{R} = \operatorname{diag}(r_0, \ldots, r_{d-1}), \end{aligned}$$

respectively. Moreover, $\boldsymbol{I}_d \in \mathbb{Z}^{d\times d}$ and $\boldsymbol{0}_d \in \mathbb{Z}^{d\times d}$ denote the identity matrix and the zero matrix, respectively. Note that $\boldsymbol{\Omega T} = 2\pi \boldsymbol{I}_d$. For example, a two-dimensional Gabor scheme starting from the two one-dimensional Gabor schemes with matrices

$$\boldsymbol{U}_0 = \frac{1}{2}\begin{bmatrix} \sqrt{2} & 0 \\ 0 & \pi\sqrt{2} \end{bmatrix}, \quad \boldsymbol{U}_1 = \begin{bmatrix} \frac{1}{2}\sqrt{2\pi} & 0 \\ 0 & \frac{1}{2}\sqrt{2\pi} \end{bmatrix},$$

$$\boldsymbol{L}_0 = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}, \quad \text{and} \quad \boldsymbol{L}_1 = \boldsymbol{I}_2,$$

i.e., a Gabor scheme on a quincunx lattice and a Gabor scheme on a rectangular lattice, thus results in the matrices

$$\boldsymbol{U} = \left[\begin{array}{cc|cc} \frac{1}{2}\sqrt{2} & 0 & & \\ 0 & \frac{1}{2}\sqrt{2\pi} & & \\ \hline & & \frac{1}{2}\pi\sqrt{2} & 0 \\ & & 0 & \frac{1}{2}\sqrt{2\pi} \end{array}\right] \quad \text{and} \quad \boldsymbol{L} = \left[\begin{array}{cc|cc} 1 & 0 & & \\ 0 & 1 & & \\ \hline -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

Similar to the one-dimensional case, we assume that the integers in one row appearing in $\boldsymbol{L}$ do not have a common divisor. We could choose a different row order in $\boldsymbol{U}$, however, the similarities between the one-dimensional case and the d-dimensional case are less apparent then. Although the lattice Λ generated by $\boldsymbol{U}$ and $\boldsymbol{L}$ is non-separable, we call this lattice 'not completely non-separable'. It can be separated in d one-dimensional (non-separable) lattices. We shall call these lattices decomposable. Moreover, we call a decomposable lattice Λ separable if the corresponding d one-dimensional lattices are separable, i.e., $\boldsymbol{L} = \boldsymbol{I}_{2d}$.

Each point $\underline{\lambda} \in \Lambda$ can be obtained by a matrix-vector product

$$\forall_{\underline{\lambda}\in\Lambda} \exists_{\underline{n}\in\mathbb{Z}^{2d}} \quad \underline{\lambda} = \boldsymbol{\Lambda}\underline{n} = \boldsymbol{UL}\underline{n}, \quad \text{with } \boldsymbol{U} \text{ and } \boldsymbol{L} \text{ as defined in Eq. (3.1).}$$

Let g_Λ be the d-variate window. If g_Λ can be separated into a product of d univariate windows, we say that g_Λ is a separated window. Otherwise, g_Λ is called non-separated. A Gabor scheme based on a decomposable lattice with a non-separated window is called non-separable. By using the sub-matrices $\mathbf{\Lambda}_{ik} \in \mathbb{R}^{d\times d}$ of $\mathbf{\Lambda}$,

$$\mathbf{\Lambda} = \boldsymbol{U}\boldsymbol{L} = \begin{bmatrix} \mathbf{\Lambda}_{11} & \mathbf{\Lambda}_{12} \\ \mathbf{\Lambda}_{21} & \mathbf{\Lambda}_{22} \end{bmatrix},$$

it follows that the shifted and modulated versions $g_{\Lambda;\underline{m}\,\underline{k}}$ of the d-dimensional window g_Λ take the form [cf. Eq. (2.8)]

$$\begin{aligned} g_{\Lambda;\underline{m}\,\underline{k}} &= \sigma_{\mathbf{\Lambda}\left[\begin{smallmatrix}\underline{m}\\ \underline{k}\end{smallmatrix}\right]} g_\Lambda = \mathcal{M}_{\mathbf{\Lambda}_{21}\underline{m}} \mathcal{M}_{\mathbf{\Lambda}_{22}\underline{k}} \mathcal{T}_{\mathbf{\Lambda}_{11}\underline{m}} \mathcal{T}_{\mathbf{\Lambda}_{12}\underline{k}} g_\Lambda \\ &= \mathcal{M}_{-\boldsymbol{B\Omega D}^{-1}\boldsymbol{R}\underline{m}} \mathcal{M}_{\boldsymbol{B\Omega}\underline{k}} \mathcal{T}_{\boldsymbol{AT}\underline{m}} g_\Lambda, \end{aligned} \tag{3.2}$$

where $\left[\begin{smallmatrix}\underline{m}\\ \underline{k}\end{smallmatrix}\right] = \mathrm{col}(\underline{m}, \underline{k})$, and where we used the translation operator $\sigma_{\left[\begin{smallmatrix}\underline{\tau}\\ \underline{\omega}\end{smallmatrix}\right]}$ defined by

$$\sigma_{\left[\begin{smallmatrix}\underline{\tau}\\ \underline{\omega}\end{smallmatrix}\right]} = \mathcal{M}_{\underline{\omega}} \mathcal{T}_{\underline{\tau}},$$

with the modulation operator

$$(\mathcal{M}_{\underline{\omega}} f)(\underline{t}) = e^{j\langle \underline{\omega}, \underline{t}\rangle} f(\underline{t}),$$

and translation operator

$$(\mathcal{T}_{\underline{\tau}} f)(\underline{t}) = f(\underline{t} - \underline{\tau}).$$

The operators $\mathcal{M}_{\underline{\omega}}$ and $\mathcal{T}_{\underline{\tau}}$ are unitary on $L_2(\mathbb{R}^d)$ with Hilbert adjoints $\mathcal{M}^*_{\underline{\omega}} = \mathcal{M}_{-\underline{\omega}}$ and $\mathcal{T}^*_{\underline{\tau}} = \mathcal{T}_{-\underline{\tau}}$.

The way of presenting the matrices $\boldsymbol{U}$ and $\boldsymbol{L}$ in expression (3.1) suggests that in the d-dimensional non-decomposable case, this expression remains valid, but now with a non-diagonal $\boldsymbol{R}$ and a lower triangular $\boldsymbol{D}$. Furthermore, the identity matrix $\boldsymbol{I}_d$ appearing in the matrix $\boldsymbol{L}$ in Eq. (3.1) will be replaced by a lower triangular matrix $\boldsymbol{S}$, which can be unified with the matrix $\boldsymbol{U}$, but the matrices $\boldsymbol{A}$, $\boldsymbol{T}$, $\boldsymbol{B}$ and $\boldsymbol{\Omega}$ will be kept diagonal:

$$\boldsymbol{U} = \begin{bmatrix} \boldsymbol{ATS} & \mathbf{0}_d \\ \mathbf{0}_d & \boldsymbol{B\Omega D}^{-1} \end{bmatrix} \quad \text{and} \quad \boldsymbol{L} = \begin{bmatrix} \boldsymbol{I}_d & \mathbf{0}_d \\ -\boldsymbol{R} & \boldsymbol{D} \end{bmatrix}.$$

Then $\boldsymbol{UL}$ is a non-fully filled lower triangular:

$$\boldsymbol{UL} = \begin{bmatrix} \boldsymbol{ATS} & \mathbf{0}_d \\ -\boldsymbol{B\Omega D}^{-1}\boldsymbol{R} & \boldsymbol{B\Omega} \end{bmatrix},$$

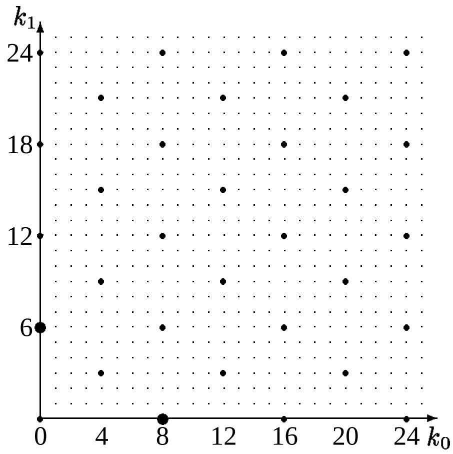

Figure 3.1: *The lattice generated by $\boldsymbol{D} = \left[\begin{smallmatrix} 4 & 0 \\ 3 & 6 \end{smallmatrix}\right]$. The lattice points on the axes closest to the origin are indicated by heavier dots.*

which means that we cannot have a non-separable lattice in the frequency direction. This limitation can be overcome by replacing the sub-matrix $\boldsymbol{D}$ of $\boldsymbol{U}$ by the diagonal matrix $\boldsymbol{C} \in \mathbb{Z}^{d \times d}$, where the $\imath$th column vector corresponds to the point on the lattice generated by $\boldsymbol{D}$ on the $\imath$th axis and closest to the origin. Since these points on the axes belong to the lattice, it follows that $\boldsymbol{D}^{-1}\boldsymbol{C}$ is a matrix containing only integers. As an example, in Fig. 3.1, the lattice generated by the matrix $\boldsymbol{D} = \left[\begin{smallmatrix} 4 & 0 \\ 3 & 6 \end{smallmatrix}\right]$ is depicted. From this figure, we see that $\boldsymbol{C} = \mathrm{diag}(8, 6)$.

This small change in the matrix $\boldsymbol{U}$ yields the $2d$-dimensional non-separable lattice Λ with lattice generator matrix $\boldsymbol{\Lambda} = \boldsymbol{U}\boldsymbol{L}$, where

$$\boldsymbol{U} = \begin{bmatrix} \boldsymbol{ATS} & \boldsymbol{0}_d \\ \boldsymbol{0}_d & \boldsymbol{B\Omega C}^{-1} \end{bmatrix} \quad \text{and} \quad \boldsymbol{L} = \begin{bmatrix} \boldsymbol{I}_d & \boldsymbol{0}_d \\ -\boldsymbol{R} & \boldsymbol{D} \end{bmatrix}. \tag{3.3}$$

Note that $\boldsymbol{U}\boldsymbol{L}$ can be a fully filled lower triangular. For example, the matrices

$$\boldsymbol{A} = \begin{bmatrix} \frac{1}{2}\sqrt{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} \frac{1}{2}\sqrt{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad \boldsymbol{D} = \begin{bmatrix} 4 & 0 \\ -3 & 6 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} 8 & 0 \\ 0 & 6 \end{bmatrix},$$

$$\boldsymbol{R} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad \boldsymbol{S} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}, \quad \boldsymbol{\Omega} = \begin{bmatrix} 2\pi & 0 \\ 0 & \sqrt{2\pi} \end{bmatrix}, \quad \text{and} \quad \boldsymbol{T} = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{2\pi} \end{bmatrix}$$

correspond to a lattice generator matrix $\boldsymbol{\Lambda} = \boldsymbol{U}\boldsymbol{L}$ of a non-separable lattice with

$$\boldsymbol{U} = \left[\begin{array}{cc|cc} \frac{1}{2}\sqrt{2} & 0 & & \\ -\frac{1}{2}\sqrt{2\pi} & \sqrt{2\pi} & & \\ \hline & & \frac{1}{8}\pi\sqrt{2} & 0 \\ & & 0 & \frac{1}{12}\sqrt{2\pi} \end{array}\right] \quad \text{and} \quad \boldsymbol{L} = \left[\begin{array}{cc|cc} 1 & 0 & & \\ 0 & 1 & & \\ \hline -1 & -2 & 4 & 0 \\ -2 & -1 & -3 & 6 \end{array}\right].$$

The shifted and modulated versions $g_{\Lambda;\underline{m}\,\underline{k}}$ of the window g_Λ are given by

$$g_{\Lambda;\underline{m}\,\underline{k}} = \mathcal{M}_{\mathbf{\Lambda}_{21}\underline{m}}\mathcal{M}_{\mathbf{\Lambda}_{22}\underline{k}}\mathcal{T}_{\mathbf{\Lambda}_{11}\underline{m}}\mathcal{T}_{\mathbf{\Lambda}_{12}\underline{k}}g_\Lambda, \tag{3.4}$$

with $\mathbf{\Lambda}_{21} = -\boldsymbol{B\Omega C}^{-1}\boldsymbol{R}$, $\mathbf{\Lambda}_{22} = \boldsymbol{B\Omega C}^{-1}\boldsymbol{D}$, $\mathbf{\Lambda}_{11} = \boldsymbol{ATS}$ and $\mathbf{\Lambda}_{12} = \mathbf{0}_d$. Note that in the decomposable case ($\boldsymbol{R}$ and $\boldsymbol{D}$ are diagonal) the matrix $\boldsymbol{C}$ is equal to $\boldsymbol{D}$. Actually, the matrix $\boldsymbol{C}$ is not really necessary, since it can be unified with $\boldsymbol{B}$. However, if we unify $\boldsymbol{C}$ with $\boldsymbol{B}$, then the determinant of $\mathbf{\Lambda} = \boldsymbol{UL}$ in the decomposable case equals $(2\pi)^d \det(\boldsymbol{ABD})$. We would like to have, corresponding to the one-dimensional case, that in the decomposable case the expression for the oversampling equals $1/\det(\boldsymbol{AB})$, which is not the case if we unify $\boldsymbol{C}$ with $\boldsymbol{B}$. However, it can be achieved by using the matrix $\boldsymbol{C}$, because then the oversampling indeed reduces to the expression $1/\det(\boldsymbol{AB})$ instead of $1/\det(\boldsymbol{ABD})$. Moreover, although $\boldsymbol{C}$ depends on $\boldsymbol{D}$, the fact that $\boldsymbol{C}$ can be compensated by $\boldsymbol{B}$ and that $\boldsymbol{L}$ is in its Hermite normal form leads to the observation that we now cover all possible lattices with lattice points on all d axes, i.e., the lattice generator matrices that can be decomposed into the two matrices $\boldsymbol{U}$ and $\boldsymbol{L}$. Since the lattice Λ has lattice points on all the axes, it is suitable for a discrete-time approach as well (see Chapter 5).

The volume of a cell (a parallelepiped spanned by the column vectors of the matrix $\mathbf{\Lambda}$) in the position-frequency space is equal to the determinant of the lattice generator matrix $\mathbf{\Lambda}$. This determinant is equal to $\det(\boldsymbol{ATSB\Omega C}^{-1}\boldsymbol{D}) = (2\pi)^d \det(\boldsymbol{ABC}^{-1}\boldsymbol{DS})$. The equality $\det(\mathbf{\Lambda}) = (2\pi)^d$, corresponds to critical sampling, whereas $\det(\mathbf{\Lambda}) > (2\pi)^d$ corresponds to undersampling, and $\det(\mathbf{\Lambda}) < (2\pi)^d$ corresponds to oversampling (see [59]). However, here the term oversampling is misleading compared to the one-dimensional case. In the one-dimensional case, frames with excellent time-frequency localization properties exist. In particular, Gabor frames with a Gaussian window g_Λ always constitute a frame in the case of oversampling (see [64, 79]). This is not necessarily true in the multi-dimensional case. For example, in the case of a two-dimensional separable lattice and a two-dimensional window that is a product of two Gaussians with $\boldsymbol{A} = \mathrm{diag}(\frac{1}{2}, \frac{4}{3})$, $\boldsymbol{B} = \boldsymbol{C} = \boldsymbol{D} = \boldsymbol{S} = \mathrm{diag}(1,1)$, the oversampling is equal to $3/2$, however it consists of a one-dimensional Gabor scheme with oversampling 2 and a one-dimensional Gabor scheme with undersampling $4/3$. As a consequence, although the oversampling is $3/2$, the set of shifted and modulated windows is not complete in $L_2(\mathbb{R}^2)$.

We shall assume that the non-separable lattice Λ is generated with the help of the matrices $\boldsymbol{U}$ and $\boldsymbol{L}$ as defined in Eq. (3.3). The Gabor frame operator $\mathcal{S}_\Lambda$ is defined as

$$\mathcal{S}_\Lambda \varphi = \sum_{\underline{m}\in\mathbb{Z}^d}\sum_{\underline{k}\in\mathbb{Z}^d} \langle \varphi, g_{\Lambda;\underline{m}\,\underline{k}}\rangle g_{\Lambda;\underline{m}\,\underline{k}}, \quad \varphi \in L_2(\mathbb{R}^d).$$

Similar to the proof in Section 2.1, it can be shown, by using the properties that are tabulated in Table 3.1, that the Gabor frame operator $\mathcal{S}_\Lambda$ commutes with the transla-

Table 3.1: *Some properties of the modulation and translation operators.*

$\mathcal{T}_{\underline{\tau}_0}\mathcal{T}_{\underline{\tau}_1} = \mathcal{T}_{\underline{\tau}_1}\mathcal{T}_{\underline{\tau}_0}$
$\mathcal{M}_{\underline{\omega}_0}\mathcal{M}_{\underline{\omega}_1} = \mathcal{M}_{\underline{\omega}_1}\mathcal{M}_{\underline{\omega}_0}$
$\mathcal{M}_{\underline{\omega}}\mathcal{T}_{\underline{\tau}} = e^{j\langle\underline{\omega},\underline{\tau}\rangle}\mathcal{T}_{\underline{\tau}}\mathcal{M}_{\underline{\omega}}$

tion operator $\sigma_{\mathbf{\Lambda}\left[\frac{\underline{m}}{\underline{k}}\right]}$ in the non-separable case, i.e.,

$$\mathcal{S}_\Lambda \sigma_{\mathbf{\Lambda}\left[\frac{\underline{m}}{\underline{k}}\right]} = \sigma_{\mathbf{\Lambda}\left[\frac{\underline{m}}{\underline{k}}\right]}\mathcal{S}_\Lambda.$$

From this it follows that $\mathcal{S}_\Lambda^{-1}$ also commutes with the time-frequency operator $\sigma_{\mathbf{\Lambda}\left[\frac{\underline{m}}{\underline{k}}\right]}$. As a result, the elements of the dual Gabor frame $\{\mathcal{S}_\Lambda^{-1} g_{\Lambda;\underline{m}\,\underline{k}} = \gamma_{\Lambda;\underline{m}\,\underline{k}}\}$ are generated by a single function γ_Λ. Thus the d-dimensional non-separable Gabor expansion on a non-separable lattice Λ is given by

$$\varphi = \sum_{\underline{m}\in\mathbb{Z}^d}\sum_{\underline{k}\in\mathbb{Z}^d} a_{\underline{m}\,\underline{k}}\, g_{\Lambda;\underline{m}\,\underline{k}}, \quad \varphi \in L_2(\mathbb{R}^d) \tag{3.5}$$

where the array of Gabor coefficients $\{a_{\underline{m}\,\underline{k}}\}$ is obtained by the Gabor transform

$$a_{\underline{m}\,\underline{k}} = \langle \varphi, \gamma_{\Lambda;\underline{m}\,\underline{k}}\rangle_d.$$

In Section 2.2, we showed that the Zak transformation can be used in the case of a non-separable Gabor scheme for one-dimensional signals to calculate the dual window of a given window and to calculate the array of Gabor expansion coefficients, and to reconstruct the signal. We will show in the next section that the Zak transformation can be used also in the case of a non-separable Gabor scheme for multi-dimensional signals. Similar to the one-dimensional case (see Section 2.1), we consider the separable lattice Λ_s

$$\Lambda_\mathrm{s} = \left\{\mathbf{\Lambda}_\mathrm{s}\left[\frac{\underline{m}}{\underline{k}}\right] | \underline{m}, \underline{k} \in \mathbb{Z}^d\right\}, \quad \text{with} \quad \mathbf{\Lambda}_\mathrm{s} = \boldsymbol{U} = \begin{bmatrix} \boldsymbol{ATS} & \mathbf{0}_d \\ \mathbf{0}_d & \boldsymbol{B\Omega C}^{-1} \end{bmatrix}.$$

This lattice Λ_s refines the non-separable lattice Λ, i.e., Λ is a sub-lattice of Λ_s,

$$\{\mathbf{\Lambda}\left[\frac{\underline{n}}{\underline{\ell}}\right] | \underline{n}, \underline{\ell} \in \mathbb{Z}^d\} \subset \{\mathbf{\Lambda}_\mathrm{s}\left[\frac{\underline{m}}{\underline{k}}\right] | \underline{m}, \underline{k} \in \mathbb{Z}^d\}.$$

Moreover,

$$\forall_{\underline{n},\underline{\ell}\in\mathbb{Z}^d}\exists!_{\underline{m},\underline{k}\in\mathbb{Z}^d} \;\; \mathbf{\Lambda}\left[\frac{\underline{n}}{\underline{\ell}}\right] = \mathbf{\Lambda}_\mathrm{s}\boldsymbol{L}\left[\frac{\underline{n}}{\underline{\ell}}\right] = \mathbf{\Lambda}_\mathrm{s}\left[\frac{\underline{m}}{\underline{k}}\right], \quad \text{with} \quad \left[\frac{\underline{m}}{\underline{k}}\right] = \boldsymbol{L}\left[\frac{\underline{n}}{\underline{\ell}}\right].$$

The non-separable Gabor scheme of shifted and modulated windows $g_{\Lambda;\underline{n}\,\underline{\ell}}$ is obtained by assigning the value zero to the shifted and modulated windows $g_{\Lambda_\mathrm{s};\underline{m}\,\underline{k}}$ on

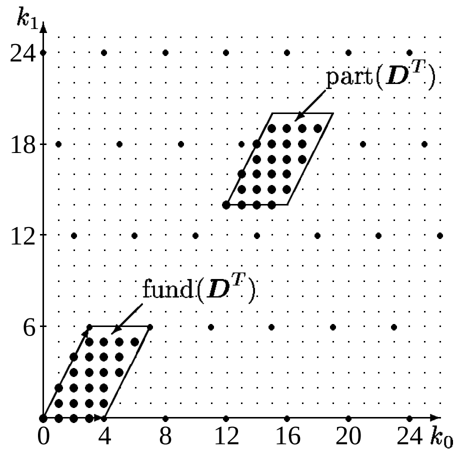

Figure 3.2: *The lattice generated by* $\boldsymbol{D}^T = \left[\begin{smallmatrix}4 & 3\\ 0 & 6\end{smallmatrix}\right]$*, a possible* $\mathrm{part}(\boldsymbol{D}^T)$*, the fundamental region* $\mathrm{fund}(\boldsymbol{D}^T)$*, and the corresponding continuous regions* $\mathrm{cont}(\boldsymbol{D}^T)$ *which are indicated by the lines.*

the separable lattice that are not part of the non-separable lattice Λ, i.e., $\left[\begin{smallmatrix}\underline{m}\\ \underline{k}\end{smallmatrix}\right] \neq \boldsymbol{L}\left[\begin{smallmatrix}\underline{n}\\ \underline{\ell}\end{smallmatrix}\right]$. We have the following equivalent expressions

$$\boldsymbol{L}^{-1}\left[\begin{smallmatrix}\underline{m}\\ \underline{k}\end{smallmatrix}\right] \in \mathbb{Z}^{2d}$$

$\Longleftrightarrow$

$$P_{\boldsymbol{\Lambda}}(\underline{m},\underline{k}) = 1,$$

where $P_{\boldsymbol{\Lambda}}(\underline{m},\underline{k})$ is given by the expression

$$P_{\boldsymbol{\Lambda}}(\underline{m},\underline{k}) = \frac{1}{\det(\boldsymbol{D})} \sum_{\underline{\ell} \in \mathrm{part}(\boldsymbol{D}^T)} \exp\left(j2\pi\underline{\ell}^T\boldsymbol{D}^{-1}\,\mathrm{row}(\boldsymbol{R},\boldsymbol{I}_d)\left[\begin{smallmatrix}\underline{m}\\ \underline{k}\end{smallmatrix}\right]\right). \tag{3.6}$$

Here $\mathrm{part}(\boldsymbol{D}^T)$ denotes a set of $\det(\boldsymbol{D})$ points in $\mathbb{Z}^d$ for which $\cup_{\underline{n}\in\mathbb{Z}^d}\big(\boldsymbol{D}^T\underline{n} + \mathrm{part}(\boldsymbol{D}^T)\big)$ is a partition of $\mathbb{Z}^d$. The corresponding continuous region will be indicated by $\mathrm{cont}(\boldsymbol{D}^T) = \{\boldsymbol{D}^T\underline{x}|\underline{x} \in [0,1)^d\} + \underline{y}$, where $\underline{y} \in \mathbb{R}^d$, with volume $\det(\boldsymbol{D})$. We shall also use $\mathrm{cont}(\boldsymbol{D}^T)$ for non-integer matrices $\boldsymbol{D}$. The fundamental region, i.e., the set $\{\boldsymbol{D}^T\underline{x}|\underline{x} \in [0,1)^d\} \cap \mathbb{Z}^d$ will be denoted by $\mathrm{fund}(\boldsymbol{D}^T)$. Note that $\mathrm{fund}(\boldsymbol{D}^T)$ is a possible set for $\mathrm{part}(\boldsymbol{D}^T)$. As an example, in Fig. 3.2, we show the lattice generated by $\boldsymbol{D}^T = \left[\begin{smallmatrix}4 & 3\\ 0 & 6\end{smallmatrix}\right]$, a possible $\mathrm{part}(\boldsymbol{D}^T)$, the continuous region $\mathrm{cont}(\boldsymbol{D}^T)$, and the fundamental region $\mathrm{fund}(\boldsymbol{D}^T)$. Having established this notation we continue our discussion. We find that

$$\boldsymbol{\Lambda}_s\left[\begin{smallmatrix}\underline{m}\\ \underline{k}\end{smallmatrix}\right] \in \Lambda \quad \equiv \quad P_{\boldsymbol{\Lambda}}(\underline{m},\underline{k}) = 1,$$

or

$$\boldsymbol{\Lambda}_{\mathrm{S}}\left[\begin{smallmatrix}\underline{m}\\ \underline{k}\end{smallmatrix}\right] \notin \Lambda \quad \equiv \quad P_{\boldsymbol{\Lambda}}(\underline{m},\underline{k}) = 0.$$

The expression for the shifted and modulated versions of the window g_Λ on the lattice Λ is now given by [cf. Eq. (3.4)]

$$\tilde{g}_{\Lambda_{\mathrm{S}};\underline{m}\underline{k}} = P_{\boldsymbol{\Lambda}}(\underline{m},\underline{k})\left(\sigma_{\boldsymbol{\Lambda}_{\mathrm{S}}\left[\begin{smallmatrix}\underline{m}\\ \underline{k}\end{smallmatrix}\right]} g_\Lambda\right). \tag{3.7}$$

Here, we put a tilde on top of $g_{\Lambda_{\mathrm{S}};\underline{m}\underline{k}}$ to indicate that the multiplication with $P_{\boldsymbol{\Lambda}}(\underline{m},\underline{k})$ is involved. The shifted and modulated versions $\tilde{\gamma}_{\Lambda_{\mathrm{S}};\underline{m}\underline{k}}$ of the dual window γ_Λ are defined similarly. Here $\{\tilde{\gamma}_{\Lambda_{\mathrm{S}};\underline{m}\underline{k}}|\underline{m},\underline{k} \in \mathbb{Z}^d\}$ is the dual frame of $\{\tilde{g}_{\Lambda_{\mathrm{S}};\underline{m}\underline{k}}|\underline{m},\underline{k} \in \mathbb{Z}^d\}$. With this definition (3.7) for the shifted and modulated versions $\tilde{g}_{\Lambda_{\mathrm{S}};\underline{m}\underline{k}}$ of the window g_Λ, the d-dimensional Gabor signal expansion on a non-separable lattice Λ takes the form

$$\varphi = \sum_{\underline{k}\in\mathbb{Z}^d}\sum_{\underline{m}\in\mathbb{Z}^d} \tilde{a}_{\underline{m}\underline{k}}\tilde{g}_{\Lambda_{\mathrm{S}};\underline{m}\underline{k}}, \quad \varphi \in L_2(\mathbb{R}^d) \tag{3.8}$$

where the array $\{\tilde{a}_{\underline{m}\underline{k}}\}$ of Gabor coefficients is obtained by the Gabor transform

$$\tilde{a}_{\underline{m}\underline{k}} = \langle \varphi, \tilde{\gamma}_{\Lambda_{\mathrm{S}};\underline{m}\underline{k}}\rangle_d. \tag{3.9}$$

Note that due to the multiplication operator $P_{\boldsymbol{\Lambda}}(\underline{m},\underline{k})$, the array $\{\tilde{a}_{\underline{m}\underline{k}}\}$ of Gabor coefficients contains many zeros. Furthermore, the array $a_{\underline{m}\underline{k}}$ is sheared in the frequency variable $\underline{k}$, where the array $\tilde{a}_{\underline{m}\underline{k}}$ is not.

Assuming that the set of shifted and modulated versions $g_{\Lambda;\underline{m}\underline{k}}$ constitutes a frame, then the relationship between the window g_Λ and the dual window γ_Λ follows from substituting the Gabor transform (3.9) into Gabor's signal expansion (3.8). In Appendix B.1, it is shown by which manipulations the biorthogonality condition [cf. Eq (2.19)]

$$\begin{aligned}&\sum_{\underline{m}\in\mathbb{Z}^d} \exp\left(-j2\pi\underline{k}^T\boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}\right)g_\Lambda(\underline{t}-\boldsymbol{ATS}\underline{m})\gamma_\Lambda^*\left(\underline{t}-\boldsymbol{B}^{-1}\boldsymbol{TCD}^{-T}\underline{k}-\boldsymbol{ATS}\underline{m}\right)\\ &= \frac{\det(\boldsymbol{BD})}{\det(\boldsymbol{CT})}\delta[\underline{k}].\end{aligned} \tag{3.10}$$

is obtained. This condition should hold for $\underline{k} \in \mathbb{Z}^d$ and $\underline{t} \in \mathbb{R}^d$.

3.2 Zak transform

In this section, the connection between the multi-dimensional Gabor scheme introduced in the previous section and the Zak transformation for $\mathbb{R}^d$ is shown. Similar to the one-dimensional case, we shall use the Zak transformation to calculate the dual window γ_Λ and the array of Gabor coefficients $\{\tilde{a}_{mk}\}$, and to reconstruct the signal φ. The use of the Zak transformation in this context is limited to the case that the diagonal matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ satisfy the condition $\boldsymbol{AB} = \boldsymbol{QP}^{-1}$, where $\boldsymbol{Q}$ and $\boldsymbol{P}$ are diagonal matrices containing only integers, i.e, the oversampling $1/\det(\boldsymbol{ABC}^{-1}\boldsymbol{DS})$ is rational.

The Zak transform $(\mathcal{Z}\varphi)\,(\underline{t},\underline{\omega};\boldsymbol{M})$ of a d-dimensional signal φ is defined as

$$(\mathcal{Z}\varphi)\,(\underline{t},\underline{\omega};\boldsymbol{M}) = \sum_{\underline{m}\in\mathbb{Z}^d} \varphi(\underline{t}+\boldsymbol{M}\underline{m})e^{-j\langle\underline{\omega},\,\boldsymbol{M}\underline{m}\rangle} \quad \varphi \in L_2(\mathbb{R}^d), \tag{3.11}$$

where $\boldsymbol{M} \in \mathbb{R}^{d\times d}$ is nonsingular. The inverse formula of the Zak transformation is defined by

$$\varphi(\underline{t}) = \varphi(\underline{t}'+\boldsymbol{M}\underline{m}) = \frac{\det(\boldsymbol{M})}{(2\pi)^d} \int\limits_{\mathrm{cont}(2\pi\boldsymbol{M}^{-T})} (\mathcal{Z}\varphi)\,(\underline{t}',\underline{\omega};\boldsymbol{M})e^{j\langle\underline{\omega},\,\boldsymbol{M}\underline{m}\rangle}\,d\underline{\omega},$$

where the variable $\underline{t}'$ extends over a region $\mathrm{cont}(\boldsymbol{M})$ and $\underline{m} \in \mathbb{Z}^d$. The Zak transform $(\mathcal{Z}\varphi)\,(\underline{t},\underline{\omega};\boldsymbol{M})$ is periodic in the frequency variable ω with respect to the regular partition of $\mathbb{R}^d$ generated by $\mathrm{cont}(2\pi\boldsymbol{M}^{-T})$ and quasi-periodic in the variable $\underline{t}$ with $\mathrm{cont}(\boldsymbol{M})$, namely,

$$(\mathcal{Z}\varphi)\,(\underline{t},\underline{\omega}+2\pi\boldsymbol{M}^{-T}\underline{\ell};\boldsymbol{M}) = (\mathcal{Z}\varphi)\,(\underline{t},\underline{\omega};\boldsymbol{M}),$$

and

$$(\mathcal{Z}\varphi)\,(\underline{t}+\boldsymbol{M}\underline{k},\underline{\omega};\boldsymbol{M}) = (\mathcal{Z}\varphi)\,(\underline{t},\underline{\omega};\boldsymbol{M})e^{j\langle\underline{\omega},\,\boldsymbol{M}\underline{k}\rangle}.$$

By taking $\boldsymbol{M} = \boldsymbol{ATS}$ in the Zak transformation (3.11), it can be shown (see Appendix B.2) that the bi-orthogonality condition (3.10) can be transformed into the following sum-of-products form

$$\sum_{\underline{v}\in\mathrm{part}(\boldsymbol{V}^T)} g_{\underline{i}\,\underline{v}}(\underline{x},\underline{y})\gamma^*_{\underline{n}\,\underline{v}}(\underline{x},\underline{y}) = \frac{\det(\boldsymbol{BDV})}{\det(\boldsymbol{CT})}\delta[\underline{i}-\underline{n}], \tag{3.12a}$$

with

$$g_{\underline{i}\,\underline{v}}(\underline{x},\underline{y}) = (\mathcal{Z}g_\Lambda)\big(\boldsymbol{B}^{-1}\boldsymbol{TC}(\underline{x}+\boldsymbol{D}^{-T}\underline{i}), 2\pi(\boldsymbol{ATS})^{-T}(\underline{y}-\boldsymbol{V}^{-T}\underline{v}-(\boldsymbol{D}^{-1}\boldsymbol{R})^T\underline{i}); \boldsymbol{ATS}\big), \tag{3.12b}$$

and

$$\gamma_{\underline{i}\,\underline{v}}(\underline{x},\underline{y}) = \\ (\mathcal{Z}\gamma_\Lambda)\big(\boldsymbol{B}^{-1}\boldsymbol{T}\boldsymbol{C}(\underline{x}+\boldsymbol{D}^{-T}\underline{i}), 2\pi(\boldsymbol{A}\boldsymbol{T}\boldsymbol{S})^{-T}(\underline{y}-\boldsymbol{V}^{-T}\underline{v}-(\boldsymbol{D}^{-1}\boldsymbol{R})^T\underline{i}); \boldsymbol{A}\boldsymbol{T}\boldsymbol{S}\big). \tag{3.12c}$$

Here the vector $\underline{i}$ extends over the region $\mathrm{fund}(\boldsymbol{\Psi}\boldsymbol{V})$ where $\boldsymbol{\Psi} = \boldsymbol{D}^T\boldsymbol{C}^{-1}\boldsymbol{A}\boldsymbol{B}\boldsymbol{S}$, and $\underline{x}$ and $\underline{y}$ extend over regions $\mathrm{cont}(\boldsymbol{D}^{-T})$ and $\mathrm{cont}(\boldsymbol{V}^{-T})$, respectively. The non-singular integer matrix $\boldsymbol{V}$, introduced in Appendix B.2, is taken such that $\boldsymbol{\Psi}\boldsymbol{V}$ and $(\boldsymbol{D}^{-1}\boldsymbol{R})^T\boldsymbol{\Psi}\boldsymbol{V}$ are matrices containing only integers. Ideally, this matrix $\boldsymbol{V}$ has the smallest possible determinant; the determinant of $\boldsymbol{V}$ directly influences the number of elements in the sum-of-products form (3.12a). Let the vectors $\underline{i}_0 \cdots \underline{i}_{\det(\boldsymbol{\Psi}\boldsymbol{V})-1}$ and $\underline{v}_0 \cdots \underline{v}_{\det(\boldsymbol{V})-1}$ be the vectors corresponding to the points in the regions $\mathrm{fund}(\boldsymbol{\Psi}\boldsymbol{V})$ and $\mathrm{part}(\boldsymbol{V}^T)$, respectively. Then we combine the functions $g_{\underline{i}_k\,\underline{v}_\ell}$ and $\gamma_{\underline{i}_k\,\underline{v}_\ell}$ in the matrices $\boldsymbol{G}$ and $\boldsymbol{\Gamma}$ of functions

$$\boldsymbol{G}(\underline{x},\underline{y}) = \begin{bmatrix} g_{\underline{i}_0\underline{v}_0}(\underline{x},\underline{y}) & \cdots & g_{\underline{i}_0,\underline{v}_{\det(\boldsymbol{V})-1}}(\underline{x},\underline{y}) \\ g_{\underline{i}_1\underline{v}_0}(\underline{x},\underline{y}) & \cdots & g_{\underline{i}_1,\underline{v}_{\det(\boldsymbol{V})-1}}(\underline{x},\underline{y}) \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ g_{\underline{i}_{\det(\boldsymbol{\Psi}\boldsymbol{V})-1},\underline{v}_0}(\underline{x},\underline{y}) & \cdots & g_{\underline{i}_{\det(\boldsymbol{\Psi}\boldsymbol{V})-1},\underline{v}_{\det(\boldsymbol{V})-1}}(\underline{x},\underline{y}) \end{bmatrix},$$

and

$$\boldsymbol{\Gamma}(\underline{x},\underline{y}) = \begin{bmatrix} \gamma_{\underline{i}_0\underline{v}_0}(\underline{x},\underline{y}) & \cdots & \gamma_{\underline{i}_0,\underline{v}_{\det(\boldsymbol{V})-1}}(\underline{x},\underline{y}) \\ \gamma_{\underline{i}_1\underline{v}_0}(\underline{x},\underline{y}) & \cdots & \gamma_{\underline{i}_1,\underline{v}_{\det(\boldsymbol{V})-1}}(\underline{x},\underline{y}) \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \gamma_{\underline{i}_{\det(\boldsymbol{\Psi}\boldsymbol{V})-1},\underline{v}_0}(\underline{x},\underline{y}) & \cdots & \gamma_{\underline{i}_{\det(\boldsymbol{\Psi}\boldsymbol{V})-1},\underline{v}_{\det(\boldsymbol{V})-1}}(\underline{x},\underline{y}) \end{bmatrix},$$

respectively. With the help of these matrices $\boldsymbol{G}$ and $\boldsymbol{\Gamma}$, Eq. (3.12a) can now be expressed as

$$\boldsymbol{G}\boldsymbol{\Gamma}^* = \frac{\det(\boldsymbol{B}\boldsymbol{D}\boldsymbol{V})}{\det(\boldsymbol{C}\boldsymbol{T})}\boldsymbol{I}_{\det(\boldsymbol{\Psi}\boldsymbol{V})}. \tag{3.13}$$

Note that, similar to the one-dimensional case, the matrix $\boldsymbol{G}$ is not a square matrix in the case of oversampling and does not have an inverse, but in general has a right inverse. The optimum solution in the sense of minimum L_2-norm can be found with the help of the generalized (Moore-Penrose) inverse $\boldsymbol{G}^\dagger$. The optimum solution $\boldsymbol{\Gamma}_{opt}$ then reads

$$\boldsymbol{\Gamma}_{opt} = \frac{fq}{\alpha T}(\boldsymbol{G}^\dagger)^*,$$

which corresponds to the minimum L_2-norm dual window γ_Λ. So, given g_Λ, we determine the matrix $\boldsymbol{G}$ by means of the Zak transformation. From that we can determine the matrix $\boldsymbol{\Gamma}_{opt}$. Then with the aid of $\boldsymbol{\Gamma}_{opt}$ we can determine the Zak transform $(\mathcal{Z}\gamma_\Lambda)\,(\underline{x},\underline{y};\boldsymbol{ATS})$ and finally from that, the dual window γ_Λ.

Suppose that the signal φ has the Gabor expansion (3.8). Then, by using the multi-dimensional Fourier expansion $(\mathcal{F}^{(2d)}\tilde{a})$ of the array of Gabor coefficients, defined as

$$(\mathcal{F}^{(2d)}\tilde{a})(\underline{x},\underline{y}) = \sum_{\underline{m}\in\mathbb{Z}^d}\sum_{\underline{k}\in\mathbb{Z}^d}\tilde{a}_{mk}e^{-j2\pi(\underline{y}^T\underline{m}\,-\,\underline{x}^T\underline{k})}, \tag{3.14}$$

with inverse

$$\tilde{a}_{\underline{m}\,\underline{k}} = \int_{\mathrm{cont}(\boldsymbol{I}_d)}\int_{\mathrm{cont}(\boldsymbol{I}_d)}(\mathcal{F}^{(2d)}\tilde{a})(\underline{x},\underline{y})e^{j2\pi(\underline{y}^T\underline{m}\,-\,\underline{x}^T\underline{k})}\,d\underline{x}\,d\underline{y},$$

it can be shown (see Appendix B.3) that the Gabor transform (3.9) can also be transformed into a sum-of-products form

$$(\mathcal{F}^{(2d)}\tilde{a})(\underline{x},\underline{y}-\boldsymbol{V}^{-T}\underline{v}) = \frac{\det(\boldsymbol{CT})}{\det(\boldsymbol{BD})}\sum_{\underline{i}\in\mathrm{fund}(\boldsymbol{\Psi V})}\varphi_{\underline{i}}(\underline{x},\underline{y})\gamma^*_{\underline{i}\,\underline{v}}(\underline{x},\underline{y}), \tag{3.15a}$$

where

$$\begin{aligned}&\varphi_{\underline{i}}(\underline{x},\underline{y})\\ &= (\mathcal{Z}\varphi)\left(\boldsymbol{B}^{-1}\boldsymbol{TC}(\underline{x}+\boldsymbol{D}^{-T}\underline{i}),2\pi(\boldsymbol{ATS})^{-T}(\underline{y}-(\boldsymbol{D}^{-1}\boldsymbol{R})^T\underline{i});\boldsymbol{ATSV}\right),\end{aligned} \tag{3.15b}$$

$\gamma_{\underline{i}\,\underline{v}}(\underline{x},\underline{y})$ as defined in Eq. (3.12c), and where the variables $\underline{v}$, $\underline{x}$ and $\underline{y}$ extend over regions $\mathrm{part}(\boldsymbol{V}^T)$, $\mathrm{cont}(\boldsymbol{D}^{-T})$ and $\mathrm{cont}(\boldsymbol{V}^{-T})$, respectively. Note that the Fourier expansion $(\mathcal{F}^{(2d)}\tilde{a})(\underline{x},\underline{y})$ is quasi-periodic in $\underline{x}$ with respect to the regular partition of $\mathbb{R}^d$ generated by $\mathrm{cont}(\boldsymbol{D}^{-T})$. The Fourier expansion $(\mathcal{F}^{(2d)}\tilde{a})(\underline{x},\underline{y})$ is completely determined by the functions

$$a_{\underline{v}}(\underline{x},\underline{y}) = (\mathcal{F}^{(2d)}\tilde{a})(\underline{x},\underline{y}-\boldsymbol{V}^{-T}\underline{v}).$$

The functions $a_{\underline{v}}(\underline{x},\underline{y})$ can be combined into a column vector $\underline{a}$ of functions

$$\underline{a}(\underline{x},\underline{y}) = [a_{\underline{v}_0}(\underline{x},\underline{y}),a_{\underline{v}_1}(\underline{x},\underline{y}),\ldots,a_{\underline{v}_{\det(\boldsymbol{V})-1}}(\underline{x},\underline{y})]^T, \tag{3.16}$$

and likewise, the functions $\varphi_{\underline{i}}(\underline{x},\underline{y})$ can be combined into a column vector $\underline{\phi}$ of functions

$$\underline{\phi}(\underline{x},\underline{y}) = [\varphi_{\underline{i}_0}(\underline{x},\underline{y}),\varphi_{\underline{i}_1}(\underline{x},\underline{y}),\ldots,\varphi_{\underline{i}_{\det(\boldsymbol{\Psi V})-1}}(\underline{x},\underline{y})]^T. \tag{3.17}$$

With the help of the vectors $\underline{a}$ and $\underline{\phi}$, Eq. (3.15a) can now be expressed in the matrix-vector product

$$\underline{a} = \frac{\det(\boldsymbol{CT})}{\det(\boldsymbol{BD})}\, \boldsymbol{\Gamma}^* \underline{\phi}. \tag{3.18}$$

The relation (3.13) applied to an arbitrary vector $\underline{\phi}$ leads to

$$\boldsymbol{G}\boldsymbol{\Gamma}^* \underline{\phi} = \frac{\det(\boldsymbol{BDV})}{\det(\boldsymbol{CT})}\, \underline{\phi}.$$

Substitution of Eq. (3.18) into the previous expression yields

$$\underline{\phi} = \frac{1}{\det(\boldsymbol{V})}\, \boldsymbol{G}\underline{a}.$$

The result in the multi-dimensional case, looks very similar to the result obtained in the one-dimensional case (see Section 2.2). Again, the sum-of-products forms can be written in matrix-matrix and matrix-vector products. And as a consequence, the same procedures as in the one-dimensional case (see section 2.2) can be used to calculate the dual window and the Gabor coefficients $\tilde{a}_{\underline{m}\,\underline{k}}$, and to reconstruct the signal φ.

3.3 Concluding remarks

In this chapter, the one-dimensional non-separable Gabor scheme has been extended to the multi-dimensional Gabor scheme for possibly non-separable lattices and possibly non-separated windows. Similar to the one-dimensional case, as presented in Section 2.1, the lattice generator matrix $\boldsymbol{\Lambda}$ is factorized in the matrices $\boldsymbol{U}$ and $\boldsymbol{L}$ written in the Hermite normal form. By using the Hermite normal form, we come to a shear representation of the shifted and modulated windows. Although this shear representation is not used explicitly, most likely, it can be used to transform the non-separable Gabor scheme into a Gabor scheme where the matrix $\boldsymbol{R}$ is a zero matrix, as will be elaborated in more detail in Chapter 6.

The connection between the multi-dimensional non-separable Gabor scheme and the Zak transformation is shown. In order to show this connection, an alternative expression for the shifted and modulated windows is used; a separable lattice that refines the non-separable lattice, the inverse of the lattice generator matrix $\boldsymbol{L}$ and the Poisson summation formula lead to the alternative expression. In the multi-dimensional case, using the Fourier and the Zak transformation leads to sum-of-products forms. In matrix notation, these sum-of-products forms become matrix-matrix products and matrix-vector products, similar to the one-dimensional case.

Chapter 4

Non-separable 1-D Gabor scheme for discrete-time signals

In this chapter, we adapt the continuous-time non-separable Gabor scheme as outlined in Chapter 2 to the discrete-time setting. The concept is very similar to the one in the continuous-time setting. Again, we describe the non-separable lattice with the help of a lattice generator matrix. The expressions for the shifted and modulated versions of the (dual) window on a non-separable lattice follow from this lattice generator matrix. We consider two possible expressions. The first expression forms the non-separable lattice by shearing a rectangular lattice, while the second expression forms the non-separable lattice by leaving out lattice points in a refined lattice. In order to use fast algorithms such as the FFT, we have to periodize the signals. In fact, this is a difference between the continuous-time and the discrete-time setting. Due to the periodization of the signals, extra conditions have to be fulfilled which makes things less convenient. Recent related work within the scope of group theory can be found in [31, 32, 35–38, 57, 71].

In Section 4.2, we show the connection between the Zak transformation and the non-separable periodic Gabor scheme. It is shown, that the Zak transformation is very useful to calculate the window of a given dual window and to calculate Gabor expansion coefficients, and to reconstruct the signal.

As shown in Section 1.6, the separable Gabor scheme can be implemented with the help of a uniform DFT filter bank. In Section 4.3, we show that a non-separable lattice Λ generated by the lattice generator matrix $\mathbf{\Lambda} = \boldsymbol{U}\boldsymbol{L}$ consists of $D = \det(\boldsymbol{L})$ separable lattices. As a consequence, a non-separable Gabor scheme can be implemented with the help of D uniform DFT filter banks in parallel.

The non-separable lattice can be obtained by applying a shear operator on a rectangular lattice. In section 4.4, we show that a non-separable lattice can be resheared into a rectangular lattice by multiplying by quadratic phase terms, which is associated with the shear operation. This technique allows a re-use of algorithms which are designed for the separable case explicitly to calculate the window given the dual window and the array of Gabor expansion coefficients, and to reconstruct the signal in the non-separable case. Reshearing a non-separable Gabor scheme into a separable one is possible both for periodic and non-periodic Gabor schemes. However, in the periodic case, additional conditions have to be fulfilled in order to reshear a non-separable Gabor scheme into a separable one.

In the case of a non-separable periodic Gabor scheme, the possible lattices for a given period length are limited. In Section 4.5, we show how many and which lattices are possible for a given period length.

4.1 Gabor's signal expansion on a non-separable lattice

As mentioned above, the concept to obtain the expressions of the shifted and modulated windows in the case of a non-separable lattice for discrete-time signals is similar to the continuous-time case (see Chapter 2). Again, we express the non-separable lattice Λ in the form

$$\Lambda = \{\mathbf{\Lambda}\underline{n} | \underline{n} \in \mathbb{Z}^2\},$$

but now the column vectors of the lattice generator matrix $\mathbf{\Lambda}$, the vectors $\underline{v}_0$ and $\underline{v}_1$, are given by

$$\underline{v}_0 = [aN,\ c/DK]^T \quad \text{and} \quad \underline{v}_1 = [bN,\ d/DK]^T, \tag{4.1}$$

whereas in the continuous-time case [see Eq. (2.3)]

$$\underline{v}_0 = [a\alpha T,\ c\beta\Omega/D]^T \quad \text{and} \quad \underline{v}_1 = [b\alpha T,\ d\beta\Omega/D]^T,$$

i.e., $\alpha T = N$ and $\beta\Omega = 2\pi/K$ (the factor 2π is handled by the modulation operator). The constants a, b, c and d are integers, as well as N and K are, and $D = ad - bc$. The first component in the vectors $\underline{v}_0$ and $\underline{v}_1$ in Eq. (4.1) corresponds to a time-shift aN and bN, respectively, while the second component corresponds to a modulation by a frequency c/DK and d/DK, respectively.

Each point $\underline{\lambda} \in \Lambda$ can be obtained by a matrix-vector product

$$\forall_{\underline{\lambda} \in \Lambda} \exists_{\underline{n} \in \mathbb{Z}^2} \ \ \underline{\lambda} = \mathbf{\Lambda}\underline{n} = \boldsymbol{U}\boldsymbol{L}\underline{n},$$

with

$$\boldsymbol{U} = \frac{1}{DK}\begin{bmatrix} NDK & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{L} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{4.2}$$

The column vectors $\underline{v}_0$ and $\underline{v}_1$ are the columns of the lattice generator matrix $\boldsymbol{U}\boldsymbol{L}$. Note that D is equal to the determinant of the matrix $\boldsymbol{L}$. Again, we shall assume that the possible common divisor of the integers a and b and the possible common divisor of the integers c and d are handled by the matrix $\boldsymbol{U}$, i.e., $\gcd(a,b) = 1$ and $\gcd(c,d) = 1$. The area of a cell (a parallelogram spanned by $\underline{v}_0$ and $\underline{v}_1$) in the time-frequency plane is equal to the determinant $\det(\boldsymbol{\Lambda}) = N/K$. Equality $N/K = 1$ corresponds to critical sampling, $N/K < 1$ corresponds to oversampling, and $N/K > 1$ corresponds to undersampling (see [35]). For convenience, we introduce the three integers p, q and J for which the relationships $J = \gcd(K,N)$, $K = pJ$ and $N = qJ$ hold. Note that the integers p and q are relatively prime, and that the oversampling equals $K/N = p/q$.

Again, we use the Hermite normal form of the matrix $\boldsymbol{L}$ [see Eqs. (2.5) and (2.6)]

$$\boldsymbol{L}' = \begin{bmatrix} 1 & 0 \\ -r & D \end{bmatrix}, \tag{4.3}$$

where $0 \le r < D$. Using this Hermite normal form we define the shifted and modulated windows on the non-separable lattice Λ by

$$g_{\Lambda;mk} = \sigma_{\boldsymbol{\Lambda}'\left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right]} g_\Lambda = \mathcal{M}^m_{-r/DK} \mathcal{M}^k_{1/K} \mathcal{T}^m_N g_\Lambda, \tag{4.4}$$

where $\boldsymbol{\Lambda}' = \boldsymbol{U}\boldsymbol{L}'$, and where we used the discrete time-frequency shift operator $\sigma_{\left[\begin{smallmatrix} \tau \\ \omega \end{smallmatrix}\right]}$ defined by

$$\sigma_{\left[\begin{smallmatrix} \tau \\ \omega \end{smallmatrix}\right]} = \mathcal{M}_\omega \mathcal{T}_\tau,$$

with the modulation operator

$$(\mathcal{M}_\omega f)[n] = e^{j2\pi n\omega} f[n], \quad \omega \in \mathbb{R},$$

and time translation operator

$$(\mathcal{T}_\tau f)[n] = f[n-\tau], \quad \tau \in \mathbb{Z}.$$

For convenience, as mentioned in Chapter 2, we use the same notation for the discrete time-frequency shift operator $\sigma_{\left[\begin{smallmatrix} \tau \\ \omega \end{smallmatrix}\right]}$, the modulation operator $\mathcal{M}_\omega$ and the time translation operator $\mathcal{T}_\tau$ as in the continuous-time setting. Note that the modulation operator $\mathcal{M}_\omega$ and the time translation operator $\mathcal{T}_\tau$ are unitary on $\ell_2(\mathbb{Z})$, with corresponding Hilbert adjoints $\mathcal{M}^*_\omega = \mathcal{M}_{-\omega}$ and $\mathcal{T}^*_\tau = \mathcal{T}_{-\tau}$. Note, moreover, that the shifted and modulated versions $g_{\Lambda;mk}$ in Eq. (4.4) are periodic in the frequency variable k with period K. In the sequel, we assume that the matrix $\boldsymbol{L}$ is written in the Hermite normal form (4.3) and we will drop the primes.

The (discrete) Gabor frame operator $\mathcal{S}_\Lambda$ is defined as

$$\mathcal{S}_\Lambda \varphi = \sum_{k=\langle K\rangle} \sum_{m=-\infty}^{\infty} \langle \varphi, g_{\Lambda;mk}\rangle g_{\Lambda;mk}, \quad \varphi \in \ell_2(\mathbb{Z}).$$

Similar to the continuous-time case (see Section 2.1), it can be shown that the Gabor frame operator $\mathcal{S}_\Lambda$ commutes with the time-frequency shift operator $\sigma_{\mathbf{\Lambda}\left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right]}$, i.e.,

$$\mathcal{S}_\Lambda \sigma_{\mathbf{\Lambda}\left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right]} = \sigma_{\mathbf{\Lambda}\left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right]} \mathcal{S}_\Lambda.$$

From this it follows that $\mathcal{S}_\Lambda^{-1}$ also commutes with the time-frequency operator $\sigma_{\mathbf{\Lambda}\left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right]}$. Define $\gamma_\Lambda = \mathcal{S}^{-1} g_\Lambda$. Then

$$\mathcal{S}_\Lambda^{-1} g_{\Lambda;mk} = \mathcal{S}_\Lambda^{-1} \sigma_{\mathbf{\Lambda}\left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right]} g_\Lambda = \sigma_{\mathbf{\Lambda}\left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right]} \mathcal{S}_\Lambda^{-1} g_\Lambda = \sigma_{\mathbf{\Lambda}\left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right]} \gamma_\Lambda = \gamma_{\Lambda;mk}.$$

This means that the elements of the dual Gabor frame $\{\gamma_{\Lambda;mk} | m \in \mathbb{Z}, k = \langle K \rangle\}$ with lattice generator matrix $\mathbf{\Lambda}$ are the shifted and modulated versions $\gamma_{\Lambda;mk}$ of the dual window γ_Λ. Gabor's signal expansion on a non-separable lattice takes the form

$$\varphi = \sum_{k=\langle K \rangle} \sum_{m=-\infty}^{\infty} a_{mk} g_{\Lambda;mk}, \tag{4.5}$$

where

$$a_{mk} = \langle \varphi, \gamma_{\Lambda;mk} \rangle \tag{4.6}$$

is the array $\{a_{mk}\}$ of Gabor coefficients, which is periodic in the variable k with period K. We use the dual frame $\{\gamma_{\Lambda;mk} | m \in \mathbb{Z}, k = \langle K \rangle\}$ for the analysis part and the frame $\{g_{\Lambda;mk} | m \in \mathbb{Z}, k = \langle K \rangle\}$ for the synthesis part. Note that the array $\{a_{mk}\}$ is periodic in the variable k with period K. Note, moreover, that the array $\{a_{mk}\}$ is sheared in the frequency variable k. In the continuous-time case, this shearing is not a problem. However, it becomes a problem in the discrete-time case if the Gabor transform and Gabor's signal expansion are periodized. The periodization will be the subject of the next subsection.

As in the continuous-time case, the Zak transformation can be used to calculate the dual window and the Gabor expansion coefficients, and to reconstruct the signal in the discrete-time case. In order to show the connection between the Gabor scheme and the Zak transformation, we follow the same procedure as outlined in Section 2.1; the alternative expression for the shifted and modulated versions with the added zeros is obtained by considering the separable lattice Λ_s

$$\Lambda_\text{s} = \{\mathbf{\Lambda}_\text{s} \underline{n} | \underline{n} \in \mathbb{Z}^2\}, \quad \text{with} \quad \mathbf{\Lambda}_\text{s} = \boldsymbol{U} = \frac{1}{DK} \begin{bmatrix} NDK & 0 \\ 0 & 1 \end{bmatrix}.$$

By using the multiplication operator $P_{\mathbf{\Lambda}}(m,k)$ [see Eq. (2.15)], the shifted and modulated versions of the window g_Λ on the lattice Λ now take the form

$$\tilde{g}_{\Lambda_\text{s};mk} = P_{\mathbf{\Lambda}}(m,k) \left(\sigma_{\mathbf{\Lambda}_\text{s}\left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right]} g_\Lambda \right) = \tilde{\sigma}_{\mathbf{\Lambda}_\text{s}\left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right]} g_\Lambda, \tag{4.7}$$

where

$$\tilde{\sigma}_{\mathbf{\Lambda}_\mathrm{S}\left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right]} = P_{\mathbf{\Lambda}}(m,k)\left(\sigma_{\mathbf{\Lambda}_\mathrm{S}\left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right]}\right),$$

and where we put a tilde on top of $g_{\Lambda_\mathrm{S};mk}$ and $\sigma_{\mathbf{\Lambda}_\mathrm{S}\left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right]}$ to indicate that the multiplication operator $P_{\mathbf{\Lambda}}(m,k)$ is involved. Note that the shifted and modulated versions $\tilde{g}_{\Lambda_\mathrm{S};mk}$ are periodic in the frequency variable k with period DK. With the modified expression (4.7) for the shifted and modulated versions $\tilde{g}_{\Lambda_\mathrm{S};mk}$ of the window g_Λ, Gabor's signal expansion takes the form

$$\varphi = \sum_{k=\langle DK\rangle} \sum_{m=-\infty}^{\infty} \tilde{a}_{mk} \tilde{g}_{\Lambda_\mathrm{S};mk}, \tag{4.8}$$

where

$$\tilde{a}_{mk} = \langle \varphi, \tilde{\gamma}_{\Lambda_\mathrm{S};mk} \rangle \tag{4.9}$$

is the array of Gabor expansion coefficients with

$$\tilde{\gamma}_{\Lambda_\mathrm{S};mk} = \tilde{\sigma}_{\mathbf{\Lambda}_\mathrm{S}\left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right]} \gamma_\Lambda.$$

Here $\{\tilde{\gamma}_{\Lambda_\mathrm{S};mk} | m \in \mathbb{Z}, k = \langle DK\rangle\}$ is the dual frame of $\{\tilde{g}_{\Lambda_\mathrm{S};mk} | m \in \mathbb{Z}, k = \langle DK\rangle\}$. Note that the array $\tilde{a}_{mk}$ is periodic in the variable k with period DK. Due to the operator $P_{\mathbf{\Lambda}}(m,k)$, the array $\{\tilde{a}_{mk}\}$ of Gabor expansion coefficients contains many zeros.

4.1.1 Periodization

In order to use the Fourier transformation and the Zak transformation for periodic signals, Gabor's signal expansions (4.5) and (4.8), and the Gabor transforms (4.6) and (4.9) have to be periodized. Therefore, we restrict the class of signals φ and dual windows γ_Λ to signals that have a finite support of length not more than N_φ and N_γ, respectively. Note that we could also restrict the window g_Λ to a class of signals that have a finite support. However, we prefer to restrict the dual window γ_Λ to have a finite support, since in practice the dual window is chosen first in most cases. Note, moreover, that we could also start directly with signals of the cyclic group $\mathbb{Z}_{MN}$ of order MN. This is usually done in the context of groups (see [38]). However, we prefer to start with signals that have a finite support, because we want to use overlap-add techniques. These overlap-add techniques make it possible to handle signals that have an infinite or a very long support (see Section 1.4). The condition of finite support implies that the arrays $\{a_{mk}\}$ and $\{\tilde{a}_{mk}\}$ have a finite support in the m-variable for all signals φ in the class. First we consider Gabor's signal expansion (4.8) and the Gabor transform (4.9). We shall denote the support of length of the array $\{\tilde{a}_{mk}\}$ in the m-variable by M, where the support of length M satisfies the condition

$$MN \geq N_\varphi + N_\gamma - 1. \tag{4.10}$$

Note that this condition is necessary if we want to use overlap-add techniques. From a group theoretical point of view, this condition is not necessary. Then the condition

$$MN \geq \max(N_\varphi, N_\gamma)$$

is sufficient, but for this choice of M, due to time-aliasing, it is not possible to use overlap-add techniques. As mentioned above, we want to use overlap-add techniques and therefore assume that M satisfies condition (4.10). The period of the signals will be MN. We shall write capital letters Φ and Γ_Λ to indicate that we deal with the periodized version of φ and γ_Λ. So, the periodized version Φ of the signal φ is defined by

$$\Phi[n] = \sum_{i=-\infty}^{\infty} \varphi[n + MNi].$$

We define the shifted and modulated versions $\tilde{\Gamma}_{\Lambda_S;mk}$ by

$$\tilde{\Gamma}_{\Lambda_S;mk} = P_{\mathbf{\Lambda}}(m,k)\left(\sigma_{\mathbf{\Lambda}_S\left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right]}\Gamma_\Lambda\right). \tag{4.11}$$

In order to periodize Gabor's signal expansion (4.8) and the Gabor transform (4.9), the shifted and modulated versions $\tilde{\Gamma}_{\Lambda_S;mk}$ as defined above have to be periodic with MN, as well. For this, the variable M has to be chosen carefully such that the conditions

$$\tilde{\Gamma}_{\Lambda_S;mk}[n] = \tilde{\Gamma}_{\Lambda_S;mk}[n + MN] \tag{4.12}$$

and (4.10) are fulfilled. From condition (4.12), we find that M has to be chosen such that $DK|MN$. This, we shall assume. Now we find with $K = pJ$ and $N = qJ$ that M is a multiple of fp where $f = D/\gcd(D,q)$, i.e., $M = fpL_0$, where L_0 is a positive integer such that condition (4.10) is fulfilled. The array $\{\tilde{a}_{mk}\}$ of Gabor coefficients can be calculated with the periodized Gabor transform

$$\tilde{a}_{mk} = \langle \varphi, \tilde{\gamma}_{\Lambda_S;mk}\rangle = \langle \Phi, \tilde{\Gamma}_{\Lambda_S;mk}\rangle,$$

where $k = \langle DK\rangle$ and $m = 0 \ldots M-1$, and the signal φ could be recovered from one period of the periodized signal Φ. Here Φ should satisfy the expansion

$$\Phi = \sum_{k=\langle DK\rangle} \sum_{m=0}^{M-1} \langle \Phi, \tilde{\Gamma}_{\Lambda_S;mk}\rangle \tilde{G}_{\Lambda_S;mk},$$

where G_Λ is dual to Γ_Λ if this expansion is satisfied. However, with $M = fpL_0$, the periodized frame operator, which we will also indicate by $\mathcal{S}_\Lambda$, defined by

$$\mathcal{S}_\Lambda \Phi = \sum_{k=\langle DK \rangle} \sum_{m=0}^{M-1} \langle \Phi, \tilde{\Gamma}_{\Lambda_S;mk} \rangle \tilde{\Gamma}_{\Lambda_S;mk},$$

does not always commute with the time-frequency shift operator $\tilde{\sigma}_{\mathbf{\Lambda}_S \left[\begin{smallmatrix} m \\ k \end{smallmatrix} \right]}$ for all $m = 0 \dots M-1$ and $k = \langle DK \rangle$, i.e., the elements of the dual frame are not the shifted and modulated versions of a dual window. As we will see, we need an additional condition to make the time-frequency shift operator commute with the frame operator. Let $P_{\mathbf{\Lambda}}(m_0, k_0) = 1$, with $(m_0, k_0) \in \mathbb{Z}^2$. Then $P_{\mathbf{\Lambda}}(m + m_0, k + k_0) = P_{\mathbf{\Lambda}}(m, k)$ for all $(m, k) \in \mathbb{Z}^2$, and so

$$\begin{aligned}
\tilde{\sigma}_{\mathbf{\Lambda}_S \left[\begin{smallmatrix} m_0 \\ k_0 \end{smallmatrix} \right]} \mathcal{S}_\Lambda \Phi &= \sum_{k=\langle DK \rangle} \sum_{m=0}^{M-1} \langle \tilde{\sigma}_{\mathbf{\Lambda}_S \left[\begin{smallmatrix} m_0 \\ k_0 \end{smallmatrix} \right]} \Phi, \tilde{\sigma}_{\mathbf{\Lambda}_S \left[\begin{smallmatrix} m_0 \\ k_0 \end{smallmatrix} \right]} \tilde{G}_{\Lambda_S;mk} \rangle \tilde{\sigma}_{\mathbf{\Lambda}_S \left[\begin{smallmatrix} m_0 \\ k_0 \end{smallmatrix} \right]} \tilde{G}_{\Lambda_S;mk} \\
&= \sum_{k=\langle DK \rangle} \sum_{m=0}^{M-1} \langle \tilde{\sigma}_{\mathbf{\Lambda}_S \left[\begin{smallmatrix} m_0 \\ k_0 \end{smallmatrix} \right]} \Phi, \tilde{G}_{\Lambda_S;m+m_0,k+k_0} \rangle \tilde{G}_{\Lambda_S;m+m_0,k+k_0} \\
&= \sum_{k=\langle DK \rangle} \sum_{m=m_0}^{M+m_0-1} \langle \tilde{\sigma}_{\mathbf{\Lambda}_S \left[\begin{smallmatrix} m_0 \\ k_0 \end{smallmatrix} \right]} \Phi, \tilde{G}_{\Lambda_S;mk} \rangle \tilde{G}_{\Lambda_S;mk}.
\end{aligned}$$

From this expression, we see that the frame operator only commutes with the time-frequency shift operator, if the shifted and modulated windows $\tilde{\Gamma}_{\Lambda_S;mk}$ are periodic in the m-variable with period M, i.e.,

$$\tilde{\Gamma}_{\Lambda_S;mk} = \tilde{\Gamma}_{\Lambda_S;m+M,k}. \tag{4.13}$$

For this condition to be satisfied, we find the additional condition $D|M$. However, with $M = fpL_0$, the condition $D|M$ is not necessarily fulfilled. Therefore, we redefine M by $M = pLD$, where L is a positive integer such that condition (4.10) is fulfilled. Substituting the periodized versions Φ and $\tilde{\Gamma}_{\Lambda_S;mk}$ into the Gabor transform (4.9) yields the bi-periodic array $\{\tilde{A}_{mk}\}$ which is periodic in the m-variable with period M and periodic in the k-variable with period DK

$$\tilde{A}_{mk} = \langle \Phi, \tilde{\Gamma}_{\Lambda_S;mk} \rangle. \tag{4.14}$$

The signal Φ can be reconstructed with the periodized Gabor expansion (4.8)

$$\Phi = \sum_{m=\langle M \rangle} \sum_{k=\langle DK \rangle} \tilde{A}_{mk} \tilde{G}_{\Lambda_S;mk}. \tag{4.15}$$

On the other hand, Φ is the periodized version of the signal φ

$$\Phi[n] = \sum_{i=-\infty}^{\infty} \varphi[n + MNi] = \sum_{i=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{k=\langle DK \rangle} \tilde{a}_{mk} \tilde{g}_{\Lambda_S;mk}[n + MNi].$$

We will use the periodized Gabor transform (4.14) and Gabor's expansion (4.15) to show the connection between the periodized Gabor scheme and the Zak transformation (see Section 4.2).

In the discussion outlined in the previous paragraph, we considered the case that the dual window γ_Λ and the signal φ have a finite support. Because of the finite support we were able to periodize the non-separable Gabor scheme. In this Gabor scheme we used the shifted and modulated windows of the form (4.11). In this paragraph, we again assume that the dual window γ_Λ and the signal φ have a finite support and we periodize the Gabor scheme, but now with the original expression (4.4) for the shifted and modulated windows. We will use this Gabor scheme to reshear a non-separable Gabor scheme into a separable lattice (see Section 4.4). This shearing makes it possible to re-use algorithms designed for the separable case explicitly to calculate the window G_Λ for a given dual window Γ_Λ and the Gabor expansion coefficients, and to reconstruct the signal Φ in the non-separable case. Let us return to the Gabor scheme with the shifted and modulated windows of the form (4.4) and assume that the dual window γ_Λ and the signal φ are restricted to the class of signals that have a finite support of length not more than N_φ and N_γ, respectively. This condition of finite support implies that the array $\{a_{mk}\}$ has a finite support in the m-variable. We shall denote the length of this support by M', where the support of length M' satisfies the condition

$$M'N \geq N_\varphi + N_\gamma - 1. \tag{4.16}$$

In the previous paragraph, we found that the period of the signals is MN, where $M = pLD$ and L is a positive integer such that condition (4.10) is fulfilled. However, due to the frequency shear $\mathcal{M}^m_{-r/DK}$ in the expression for the shifted and modulated windows $g_{\Lambda;mk}$, we now have to periodize the signal φ and the dual window γ_Λ with a different period $M'N$. The variable M' can be found in a similar way as the variable M in the previous paragraph; in order to periodize Gabor's signal expansion (4.5) and the Gabor transform (4.6), the shifted and modulated versions $\Gamma_{\Lambda;mk}$, defined by

$$\Gamma_{\Lambda;mk} = \sigma_{\mathbf{\Lambda}\left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right]} \Gamma_\Lambda,$$

have to be periodic with $M'N$, as well. For this, the variable M' has to be chosen such that the conditions

$$\Gamma_{\Lambda;mk}[n] = \Gamma_{\Lambda;mk}[n + M'N] \tag{4.17}$$

and (4.16) are fulfilled. From condition (4.17), we find that M' has to be chosen such that $K|M'N$ and $DK|M'Nr$. This, we shall assume. Now we find with $K = pJ$, $N = qJ$ and $\gcd(D,r) = 1$ [see Eq. (2.7)] that M' is a multiple of fp where $f = D/\gcd(D,q)$, i.e., $M' = fpL_0'$, where L_0' is a positive integer such that condition (4.16) is fulfilled. Note that M' can have the same values as M at this stage. However, with $M' = fpL_0'$, the periodized frame operator, defined by

$$\mathcal{S}_\Lambda \Phi = \sum_{k=\langle K\rangle} \sum_{m=0}^{M'-1} \langle \Phi, \Gamma_{\Lambda;mk}\rangle \Gamma_{\Lambda;mk},$$

does not always commute with the time-frequency shift operator $\sigma_{\mathbf{\Lambda}\left[\begin{smallmatrix} m \\ k\end{smallmatrix}\right]}$ for all $m = 0 \ldots M'-1$. In order to make the time-frequency shift operator commute with the frame operator, we find the additional condition [cf. Eq. (4.13)]

$$\Gamma_{\Lambda;mk} = \Gamma_{\Lambda;m+M',k},$$

i.e., $DK|rM'$. This conditions is fulfilled, if we choose $M' = JpL'D/\gcd(r,J)$ with L' such that condition (4.16) is fulfilled as well. Thus M' could be a factor $J/\gcd(r,J)$ larger than $M = pLD$. In practice, the period of the signals should be chosen as small as possible, since this gives the designer the most freedom. In order to be able to periodize the signals with MN instead of $M'N$, we make the shifted and modulated windows $G_{\Lambda;mk}$ and $\Gamma_{\Lambda;mk}$ periodic in the m-variable with period M by replacing m by $\lfloor m \rfloor_M$, where $\lfloor m \rfloor_M$ denotes $m \mod M$. Note that by replacing m by $\lfloor m \rfloor_M$ in $\Gamma_{\Lambda;mk}$ and $G_{\Lambda;mk}$, we only enumerate the shifted and modulated windows in a different way, and therefore, this can be done without loss of information. Note, moreover, that with $M = pLD$, the conditions $K|MN$, $DK|rMN$ are fulfilled, while $DK|rM$ becomes obsolete. Now we can periodize the signal φ and the dual window γ_Λ also with period length MN. Then the periodized frame operator, defined by

$$\mathcal{S}_\Lambda \Phi = \sum_{k=\langle K\rangle} \sum_{m=\langle M\rangle} \langle \Phi, \Gamma_{\Lambda;\lfloor m \rfloor_M k}\rangle \Gamma_{\Lambda;\lfloor m \rfloor_M k},$$

where the signals are periodized with period MN, commutes with the time-frequency shift operator $\sigma_{\mathbf{\Lambda}\left[\begin{smallmatrix} m \\ k\end{smallmatrix}\right]}$. Substituting the periodized versions Φ and $\Gamma_{\Lambda;\lfloor m \rfloor_M k}$ into the Gabor transform (4.6) yields the bi-periodic array $\{A_{mk}\}$ which is periodic in the m-variable with period M and periodic in the k-variable with period K

$$A_{mk} = \langle \Phi, \Gamma_{\Lambda;\lfloor m \rfloor_M k}\rangle. \tag{4.18}$$

The signal Φ can be reconstructed with the periodized Gabor expansion

$$\Phi = \sum_{m=\langle M\rangle} \sum_{k=\langle K\rangle} A_{mk} G_{\Lambda;\lfloor m \rfloor_M k}, \tag{4.19}$$

where

$$G_{\Lambda;\lfloor m\rfloor_M k} = \mathcal{M}_{-r/DK}^{\lfloor m\rfloor_M} \mathcal{M}_{1/K}^{k} \mathcal{T}_N^m G_\Lambda \tag{4.20}$$

and

$$G_\Lambda = \mathcal{S}_\Lambda^{-1} \Gamma_\Lambda.$$

We will use this Gabor scheme in order to reshear a non-separable Gabor scheme into a separable one (see Section 4.4).

Assuming that the set $\{\Gamma_{\Lambda;\lfloor m\rfloor_M k} | m = \langle M\rangle, k = \langle K\rangle\}$ constitutes a frame, the relationship between the periodized window G_Λ and the periodized dual window Γ_Λ follows from substituting the Gabor transform (4.18) into Gabor's signal expansion (4.19). In Appendix C.1, it is shown by which manipulations the condition

$$K \sum_{m=\langle M\rangle} e^{-j2\pi mkr/D} \Gamma_\Lambda^*[n - kK - mN] G_\Lambda[n - mN] = \sum_{\ell=-\infty}^{\infty} \delta[k - \ell qLD] \tag{4.21}$$

can be obtained. This condition should hold for $k =< qLD >$ and $n =< MN >$. This condition is necessary but not sufficient to find the window G_Λ that is dual to Γ_Λ. In the case of oversampling (i.e., $K/N = p/q > 1$), there are more windows G_Λ, besides the dual $\mathcal{S}_\Lambda^{-1}\Gamma_\Lambda$, satisfying this condition. Since $\mathcal{S}^{-1}\Gamma_\Lambda$ is the window with the minimal ℓ_2-norm, we need this minimal ℓ_2-norm as an additional condition. Thus for a given periodic window Γ_Λ, we have to find a periodic function G_Λ with minimal ℓ_2-norm such that this condition (4.21) is fulfilled. The same relationship can also be found by substituting the Gabor transform (4.14) into Gabor's signal expansion (4.15).

In this subsection, we periodized the Gabor scheme, which uses the expression (4.11) or (4.20) for the shifted and modulated windows. Other lattice generator matrices $\mathbf{\Lambda}$ would lead to different modifications in order to be able to periodize the signals with period MN. In this thesis, we only use the two periodized Gabor schemes as introduced in this subsection. The first Gabor scheme, the one with the shifted and modulated windows (4.11), will be used in Section 4.2 to show the connection between this Gabor scheme and the Zak transformation. The added zeroes in this Gabor scheme make it possible to exploit the known expressions for the separable case within the context of the Zak transformation. The second Gabor scheme, the one with the shifted and modulated windows (4.20), will be used in Section 4.4 to reshear a non-separable lattice into a separable one. This makes it possible to re-use algorithms designed for separable lattices in the non-separable case.

4.2 Discrete Zak transform

Similar to the continuous-time case, the discrete Zak transformation can be very useful and efficient to calculate the window G_Λ given the dual window Γ_Λ and the array of Gabor coefficients, and to reconstruct the signal. For the use of the discrete Zak transformation in the case of rectangular lattices in this context we refer to [3, 13, 68]. In this section we will show how the Zak transformation can be used for the non-separable case, also.

Condition (4.21) can be used directly to find the window G_Λ. However this is not very practical. The Zak transformation leads to a more efficient method to find the window G_Λ for a given dual window Γ_Λ. By using the discrete Zak transform

$$(\mathcal{Z}\Phi)\,[n,\ell;N,M] = \sum_{m=\langle M\rangle} \Phi[n+mN]e^{-j2\pi m\ell/M}, \tag{4.22}$$

the condition (4.21) can be transformed into the following sum-of-products form (see Appendix C.2)

$$\sum_{k=\langle fp\rangle} g_{ik}[n,\ell]\gamma^*_{sk}[n,\ell] = \frac{fp}{K}\delta[i-s]. \tag{4.23a}$$

Here

$$g_{ik}[n,\ell] = (\mathcal{Z}G_\Lambda)\left[n+iK,\ell-\frac{M}{fp}k-\frac{rM}{D}i;N,M\right] \tag{4.23b}$$

and

$$\gamma_{ik}[n,\ell] = (\mathcal{Z}\Gamma_\Lambda)\left[n+iK,\ell-\frac{M}{fp}k-\frac{rM}{D}i;N,M\right], \tag{4.23c}$$

with $f = D/\gcd(D,q)$, $i = 0\ldots fq-1$, $k = 0\ldots fp-1$ and n and ℓ extending over an interval of length DK and M/fp, respectively. Note that M/fp and M/D are integers (recall $M = pLD$). Now we combine these functions g_{ik} and γ_{ik} into the $(fq\times fp)$ matrices of functions

$$\boldsymbol{G}[n,\ell] = \begin{bmatrix} g_{00}[n,\ell] & g_{01}[n,\ell] & \cdots & g_{0,fp-1}[n,\ell] \\ g_{10}[n,\ell] & g_{11}[n,\ell] & \cdots & g_{1,fp-1}[n,\ell] \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ g_{fq-1,0}[n,\ell] & g_{fq-1,1}[n,\ell] & \cdots & g_{fq-1,fp-1}[n,\ell] \end{bmatrix}$$

and

$$\boldsymbol{\Gamma}[n,\ell] = \begin{bmatrix} \gamma_{00}[n,\ell] & \gamma_{01}[n,\ell] & \dots & \gamma_{0,fp-1}[n,\ell] \\ \gamma_{10}[n,\ell] & \gamma_{11}[n,\ell] & \dots & \gamma_{1,fp-1}[n,\ell] \\ . & . & \dots & . \\ . & . & \dots & . \\ \gamma_{fq-1,0}[n,\ell] & \gamma_{fq-1,1}[n,\ell] & \dots & \gamma_{fq-1,fp-1}[n,\ell] \end{bmatrix},$$

respectively. With the help of these matrices $\boldsymbol{G}$ and $\boldsymbol{\Gamma}$, Eq. (4.23a-c) can now be expressed as

$$\boldsymbol{G}\boldsymbol{\Gamma}^* = \frac{fp}{K}\boldsymbol{I}_{fq}, \tag{4.24}$$

where $\boldsymbol{I}_{fq}$ is the $(fq \times fq)$ identity matrix. Note that matrix $\boldsymbol{\Gamma}^*$ in Eq. (4.24) is not a square matrix in the case of oversampling ($p > q$) and does not have an inverse, but in general will have a (non-unique) left inverse. Again, the optimum solution $\boldsymbol{G}_{opt}$ in the sense of minimum ℓ_2-norm can be found with the help of the generalized (Moore-Penrose) inverse $\boldsymbol{\Gamma}^\dagger$ [see Eq. (2.24)]

$$\boldsymbol{G}_{opt} = \frac{fp}{K}(\boldsymbol{\Gamma}^\dagger)^*,$$

which corresponds to the minimum ℓ_2-norm window $G_{\Lambda;opt}$.

Using the (two-dimensional) discrete Fourier transformation $\mathcal{F}_{dis}^{(2)}$, defined by

$$(\mathcal{F}_{dis}^{(2)}\tilde{A})[n,\ell;DK,M] = \sum_{m=\langle M\rangle}\sum_{k=\langle DK\rangle}\tilde{A}_{mk}e^{-j2\pi(m\ell/M\,-\,kn/DK)}, \tag{4.25}$$

the discrete Zak transformation [see Eq. (4.22)] and the shifted and modulated windows $\tilde{\Gamma}_{\Lambda_S;mk}$ [see Eq. (4.7)], it can be shown (see Appendix C.3) that the periodized Gabor transform (4.14) can also be transformed into a sum-of-products form

$$(\mathcal{F}_{dis}^{(2)}\tilde{A})\left[n,\ell-\frac{M}{fp}k;DK,M\right] = K\sum_{i=\langle fq\rangle}\gamma_{ik}^*[n,\ell]\varphi_i[n,\ell], \tag{4.26}$$

where

$$\varphi_i[n,\ell] = (\mathcal{Z}\Phi)\left[n+iK,\ell-\frac{rM}{D}i;fpN,\frac{M}{fp}\right],$$

with $i = 0\dots fq-1$, $k = 0\dots fp-1$, and where the variables n and ℓ extend over an interval of length DK and M/fp, respectively. The Fourier transform $(\mathcal{F}_{dis}^{(2)}\tilde{A})[n,\ell;DK,M]$ is completely determined by the functions

$$a_k[n,\ell] = (\mathcal{F}_{dis}^{(2)}\tilde{A})\left[n,\ell-\frac{M}{fp}k;DK,M\right], \quad k = 0\dots fp-1.$$

The functions a_k are now combined into the fp-dimensional column vector of functions

$$\underline{a}[n,\ell] = (a_0[n,\ell], a_1[n,\ell], \dots, a_{fp-1}[n,\ell])^T,$$

and likewise, the functions φ_i with $i = 0 \dots fq-1$ into the column vector of functions

$$\underline{\phi}[n,\ell] = (\varphi_0[n,\ell], \varphi_1[n,\ell], \dots, \varphi_{fq-1}[n,\ell])^T.$$

With the help of the vector functions $\underline{a}$ and $\underline{\phi}$, Eq. (4.26) can now be expressed in the matrix-vector product

$$\underline{a} = K\boldsymbol{\Gamma}^*\underline{\phi}. \tag{4.27}$$

The relation (4.24) applied to an arbitrary vector $\underline{\phi}$ leads to the condition

$$\boldsymbol{G}\boldsymbol{\Gamma}^*\underline{\phi} = \frac{fp}{K}\underline{\phi}.$$

Substitution of Eq. (4.27) into the previous expression yields

$$\underline{\phi} = \frac{1}{fp}\boldsymbol{G}\underline{a}. \tag{4.28}$$

Note that this vector $\underline{\phi}$ is unique, since $\boldsymbol{\Gamma}^*$ is injective, and $\boldsymbol{G}$ is a (proportional) left inverse of $\boldsymbol{\Gamma}^*$. These matrix-vector products (4.27) and (4.28) provide now a method to calculate the array $\{\tilde{A}_{mk}\}$ of Gabor expansion coefficients and to reconstruct the signal Φ (and therefore φ) from a given array $\{\tilde{A}_{mk}\}$. The procedures are very similar to the continuous-time case as described in Section 2.2.

Note that in the separable case ($D = 1$), the sum-of-products forms (4.23a) and (4.26) reduce to product forms in the case of critical sampling. In the non-separable case, $D > 1$, the number of elements in the sum-of-product forms is equal to the determinant D, and so, the sum-of-products do not reduce to product forms.

As an example, we calculated the corresponding optimal windows $G_{\Lambda;opt}$ for the given truncated periodized Gaussian dual window Γ_Λ by using the Zak transformation as outlined above. We start with the truncated Gaussian dual window γ_Λ

$$\gamma_\Lambda[n] = \begin{cases} e^{-\pi(n/10)^2} & \text{if } n \in \{-32 \dots 32\}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus we have a support of length $N_\gamma = 65$. We periodize this dual window γ_Λ with a period size of 288, i.e., $MN = 288$ and we can handle signals φ with a support of length not more than 224. The corresponding optimal windows $G_{\Lambda;opt}$ are calculated

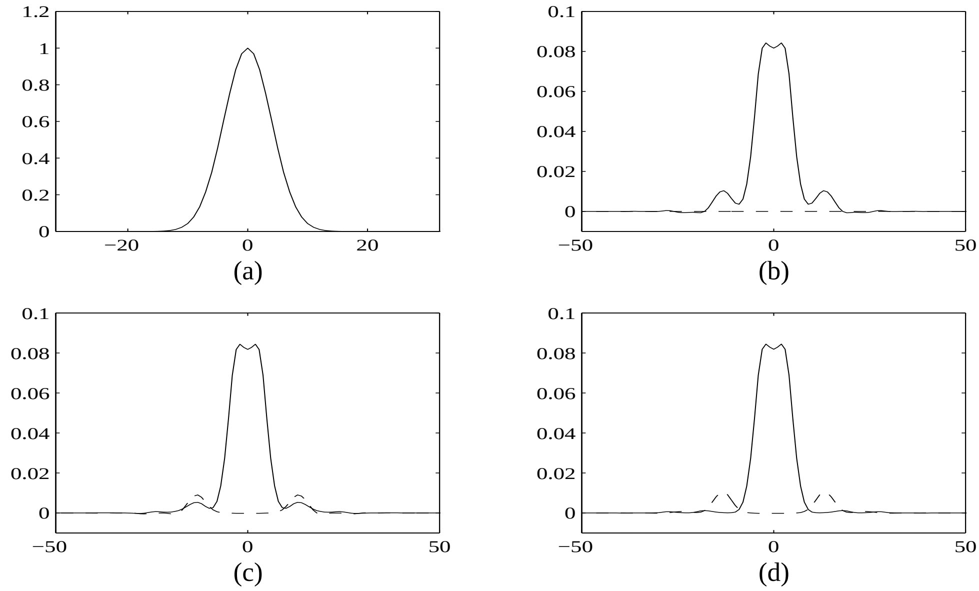

Figure 4.1: *The optimal windows $G_{\Lambda;opt}$ for the given truncated Gaussian dual window Γ_Λ, which is depicted in (a), for different lattices with common parameters $p = 3$, $q = 2$, $J = 4$, $r = 1$, but with different parameters D and L: (b) $D = 2$, $L = 6$, (c) $D = 3$, $L = 4$, (d) $D = 4$, $L = 3$. The solid line corresponds to the real part and the dashed line to the imaginary part.*

for the three lattices with the parameters (r, D) in the matrix $\boldsymbol{L}$ in its Hermite normal form

$$\boldsymbol{L} = \begin{bmatrix} 1 & 0 \\ -r & D \end{bmatrix},$$

equal to $(1, 2)$, $(1, 3)$ and $(1, 4)$. We take a fixed oversampling $p/q = 3/2$ and a fixed time-shift $N = 8$, and so $J = 4$. With the fixed N and p, we only have one free parameter L left in $M = 36 = pLD$. The windows $G_{\Lambda;opt}$ are depicted in Fig. 4.1. Note that the window $G_{\Lambda;opt}$ in the case of the quincunx lattice ($D = 2$), depicted in Fig. 4.1b, is real-valued in contrast with the other windows for lattices with parameter $D = 3$ and $D = 4$ (see Fig. 4.1c and Fig. 4.1d, respectively). In [38], it is shown that if the lattice Λ is invariant under involution $(m, k) \mapsto (m, -k)$, i.e., if the lattice is symmetric with respect to the time-axis, and the dual window Γ_Λ is real-valued, then the window G_Λ with the minimum ℓ_2-norm ($G_{\Lambda;opt}$) is real-valued, as well. The rectangular lattice ($D = 1$) and the quincunx lattice ($D = 2$) are the only lattices satisfying this condition.

4.3 Filter banks

In Section 1.6 we showed how the separable Gabor scheme can be implemented with the help of a filter bank. In this section, we will show how the non-separable Gabor scheme can be implemented by using filter banks. We start this section by considering a non-separable lattice Λ generated by the lattice generator matrix $\boldsymbol{\Lambda} = \boldsymbol{UL}$ where $\det(\boldsymbol{L}) = D$ in more detail. We want to show that the non-separable lattice can be seen as the union of D separable lattices, which are shifted in time and frequency. Consequently, the non-separable Gabor scheme can be implemented by combining D uniform DFT filter banks, where each uniform DFT filter bank corresponds to the implementation of a separable Gabor scheme. For each filter bank, there will be a different prototype. Put differently, a non-separable Gabor scheme can be seen as a multi-window Gabor scheme on time and frequency shifted separable lattices (for more details about the multi-window Gabor scheme, we refer to [92]). The proof that a non-separable lattice Λ can be seen as the union of D time and frequency shifted separable lattices is based on the description of the lattice Λ which can be generated by the lattice generator matrix $\boldsymbol{UL}$ with the Hermite normal form $\boldsymbol{L}$ [Eq. (4.3)]

$$\boldsymbol{L} = \begin{bmatrix} 1 & 0 \\ -r & D \end{bmatrix},$$

or by $\boldsymbol{UL''}$ with the modified Hermite normal form $\boldsymbol{L''}$ [see Eq. (2.10)]

$$\boldsymbol{L''} = \begin{bmatrix} D & -r' \\ 0 & 1 \end{bmatrix}.$$

Starting from the origin in the time-frequency plane, there are always $\det(\boldsymbol{L}) = D$ time-shifts needed to find a sample on the time-axis. This is because $\gcd(r, D) = 1$. The same holds for the frequency direction, since there are always $\det(\boldsymbol{L''}) = D$ frequency-modulations needed to find a sample on the frequency-axis. This means that the non-separable lattice is built up by a uniform structure of tiles, where the tiles are determined by D pairs of vectors. Because of this uniform structure, the non-separable lattice is formed by D separable lattices. As an example, we plotted in Fig. 4.2 a lattice which is generated by the lattice generator matrix $\boldsymbol{\Lambda}$

$$\boldsymbol{\Lambda} = \begin{bmatrix} 1 & 0 \\ 0 & 1/12 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 3 \end{bmatrix}, \tag{4.29}$$

i.e., $r = 1$, $D = 3$, $N = 1$ and $K = 4$. In this figure, we see that this non-separable lattice indeed consists of $D = 3$ separable lattices.

Now we know that the non-separable Gabor scheme can be implemented by combining D uniform DFT filter banks with D different prototypes corresponding to the

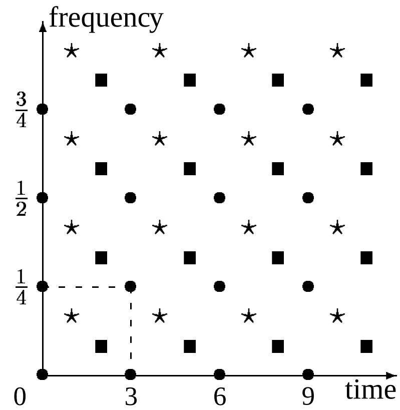

Figure 4.2: *The lattice that is generated by the lattice generator matrix $\mathbf{\Lambda}$ [see Eq. (4.29)], which consists of $D = 3$ separable lattices indicated by the symbols $\bullet$, $\star$, and $\blacksquare$. A tile of the lattice is indicated by dash lines.*

implementation of D separable Gabor schemes. Since we already know how to implement a separable Gabor scheme with the help of a filter bank (see Section 1.6), the implementation of the non-separable Gabor scheme is straightforward if we know the D prototypes. So we continue our discussion by looking for these prototypes. The tiles of the lattice as described above indicate how to find these prototypes; we split the array $\{\tilde{a}_{mk}\}$ of Gabor coefficients [see Eq. (4.9)], with $m \in \mathbb{Z}$ and $k = 0 \dots DK - 1$, in blocks of length D in the m-variable, i.e., in blocks of D time shifts:

$$\tilde{a}_{mD+i,k} = P_{\mathbf{\Lambda}}(i,k) \sum_{\ell=-\infty}^{\infty} \varphi[\ell]\gamma^*[\ell - mDN - iN]e^{-j2\pi k\ell/DK}, \tag{4.30a}$$

with $m \in \mathbb{Z}$, $i = 0 \dots D-1$ and $k = 0 \dots DK - 1$. Recall that

$$P_{\mathbf{\Lambda}}(i,k) = \frac{1}{D} \sum_{v=0}^{D-1} e^{j2\pi v(ir+k)/D}. \tag{4.30b}$$

The integer i corresponds to the ith separable lattice. Due to the operator $P_{\mathbf{\Lambda}}(i,k)$, the array $\{\tilde{a}_{mD+i,k}\}$ contains zeros in the k-direction for given integers i and m. The non-zero elements in the array $\{\tilde{a}_{mD+i,k}\}$ for a given i and m are the K coefficients [see Eqs. (4.30a-b)]

$$\begin{aligned}\tilde{a}_{mD+i,-ir+Du} = {} & e^{-j2\pi(Du-ir)mN/K} \sum_{\ell=-\infty}^{\infty} \varphi[\ell]\gamma^*_{\Lambda}[-(mDN - \ell + iN)] \\ & \times e^{-j2\pi(mDN-\ell)ir/DK} e^{j2\pi(mDN-\ell)u/K},\end{aligned}$$

where $u = 0 \ldots K-1$. From this equation we see that the subband signals $c_{iu}[m]$, the Gabor coefficients apart from a phase, can be obtained by D analysis banks with D prototypes:

$$c_{iu}[m] = \tilde{a}_{mD+i,-ir+Du} \exp(j2\pi(Du-ir)mN/K) = (\downarrow DN)(\mathcal{C}_{h_{iu}^{(a)}}\varphi)[m],$$

where

$$h_{iu}^{(a)} = \mathcal{M}_{1/K}^{u}\mathcal{M}_{r/DK}^{-i}\mathcal{T}_N^{-i}h_{(a)} = \mathcal{M}_{1/K}^{u}h_{(a);i} \tag{4.31}$$

are the DK impulse responses of the filters spread uniformly over D analysis banks with the D prototypes $h_{(a);i} = \mathcal{M}_{r/DK}^{-i}\mathcal{T}_N^{-i}h_{(a)}$ with $h_{(a)}[n] = \gamma_\Lambda^*[-n]$. The D analysis banks are depicted schematically in parallel in Fig. 4.3a. The prototypes of the synthesis banks follow from the Gabor expansion (4.8):

$$\begin{aligned}
\varphi[n] &= \sum_{k=\langle DK\rangle}\sum_{m=-\infty}^{\infty}\tilde{a}_{mk}\tilde{g}_{\Lambda_S;mk}[n] \\
&= \sum_{i=0}^{D-1}\sum_{u=0}^{K-1}\sum_{m=-\infty}^{\infty}\tilde{a}_{mD+i,-ir+Du}\tilde{g}_{\Lambda_S;mD+i,-ir+Du}[n] \\
&= \sum_{i=0}^{D-1}\sum_{u=0}^{K-1}\sum_{m=-\infty}^{\infty}((\uparrow DN)c_{iu})[m]g_\Lambda[n-m-iN]e^{-j2\pi(n-m)ir/DK} \\
&\quad\times e^{j2\pi(n-m)u/K} = \sum_{i=0}^{D-1}\sum_{u=0}^{K-1}\left(\mathcal{C}_{h_{iu}^{(s)}}(\uparrow DN)c_{iu}\right)[n],
\end{aligned} \tag{4.32}$$

where

$$h_{iu}^{(s)} = \mathcal{M}_{1/K}^{u}\mathcal{M}_{r/DK}^{-i}\mathcal{T}_N^{i}h_{(s)} = \mathcal{M}_{1/K}^{u}h_{(s);i}$$

are the DK impulse responses of the filters spread uniformly over D synthesis banks with the D prototypes $h_{(s);i} = \mathcal{M}_{r/DK}^{-i}\mathcal{T}_N^{i}h_{(s)}$ with $h_{(s)} = g_\Lambda$. The D synthesis banks are depicted schematically in parallel in Fig. 4.3b.

In Section 1.6, we showed that the uniform DFT filter bank in the separable case can be implemented efficiently by using the polyphase representations (1.22) and (1.23), and the noble identities (1.20) and (1.21). Since the filter bank in the non-separable case is merely a collection of D filter banks corresponding to D separable Gabor schemes in parallel, the same technique can be used to implement a filter bank in the non-separable case. In the following two subsections, we will show how the filter bank in the non-separable case can be implemented efficiently.

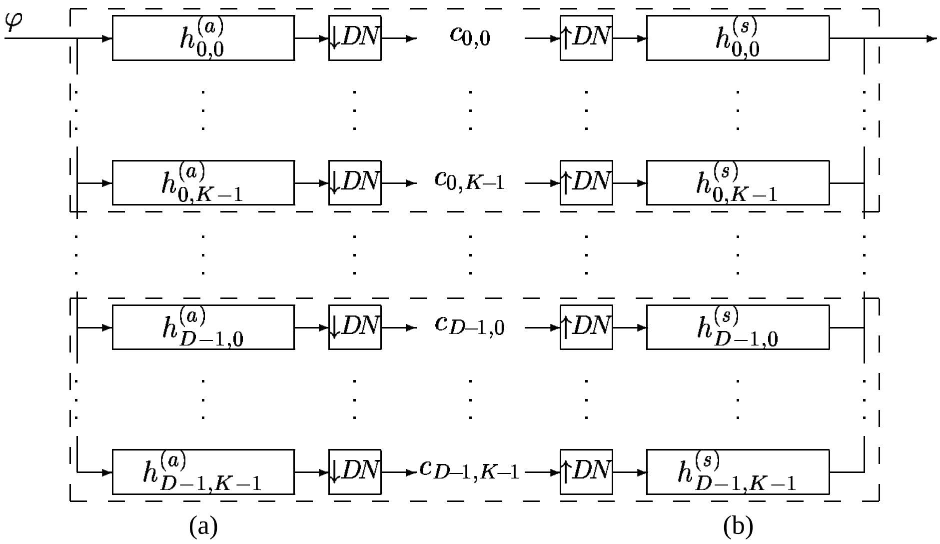

Figure 4.3: *The D uniform DFT filter banks, which are indicated by the dashed lines, consist of D analysis and synthesis banks. (a) The analysis banks. (b) The synthesis banks.*

4.3.1 Analysis bank

In this subsection, we show how the analysis bank corresponding to a non-separable Gabor transform, as depicted in Fig. 4.3a, can be implemented efficiently.

The z-transformed polyphase representations of the D prototypes $h_{(a);i}$ of the analysis bank are given by

$$(\mathcal{Z}h_{(a);i})(z) = (\mathcal{Z}\mathcal{M}^{-i}_{r/DK}\mathcal{T}^{-i}_{N}h_{(a)})(z) = \sum_{m=0}^{R-1} z^m e^{j2\pi mir/DK} \times \left(\mathcal{Z}\rho^{+}_{R;m-iN}h_{(a)}\right)(e^{j2\pi irR/DK}z^R).$$

Note that the time shifts $\mathcal{T}^{-i}_{N}$ are unified with the polyphase components. Note, moreover, that the choice of the number of polyphase components R is still free at this stage. The z-transformed polyphase representations of the modulated analysis filters with impulse responses $h^{(a)}_{ik}$ are given by [see Eq. (4.31)]

$$\left(\mathcal{Z}h^{(a)}_{ik}\right)(z) = \sum_{m=0}^{R-1} z^m e^{-j2\pi m(Dk-ir)/DK} \times \left(\mathcal{Z}\rho^{+}_{R;m-iN}h_{(a)}\right)(e^{-j2\pi(Dk-ir)R/DK}z^R),$$

where we used property (1.17) concerning modulation and the z-transformation. From this expression, we see that if we take R a multiple of DK (recall that D

and r are relatively prime), we only need the R polyphase components and D DFT matrices of size $K \times K$ to implement the DK analysis filters with impulse responses $h_{ik}^{(a)}$:

$$\left(\mathcal{Z}h_{ik}^{(a)}\right)(z) = \sum_{m=0}^{K-1} e^{-j2\pi mk/K} \sum_{n=0}^{D-1} e^{j2\pi ir(m+nK)/DK} \times \sum_{\ell=0}^{R/DK-1} z^{m+nK+\ell DK} \left(\mathcal{Z}\rho_{R;m-iN+nK+\ell DK}^{+} h_{(a)}\right)(z^R),$$

or in matrix notation

$$\underline{H}_{(a);i}(z) = \boldsymbol{F}_K \boldsymbol{E}_{(a);i}(z^R)\underline{d}_R(z), \tag{4.33}$$

where $\underline{H}_{(a);i} \in \mathbb{C}^{K\times 1}$ is a vector containing the z-transformed impulse responses $h_{ik}^{(a)}$

$$\underline{H}_{(a);i}(z) = \left[\left(\mathcal{Z}h_{i,0}^{(a)}\right)(z), \left(\mathcal{Z}h_{i,1}^{(a)}\right)(z), \dots, \left(\mathcal{Z}h_{i,K-1}^{(a)}\right)(z)\right]^T,$$

the matrix $\boldsymbol{F}_K \in \mathbb{C}^{K\times K}$ is the DFT matrix with elements $[\boldsymbol{F}_K]_{ik} = F_K^{ik}$, where $F_K = \exp(-j2\pi/K)$, and $\boldsymbol{E}_{(a);i} \in \mathbb{C}^{K\times R}$ the polyphase matrix

$$\boldsymbol{E}_{(a);i}(z) = \underbrace{\mathrm{row}(\boldsymbol{I}_K, \dots, \boldsymbol{I}_K)}_{K\times DK} \underbrace{\boldsymbol{M}_i}_{DK\times DK} \underbrace{\mathrm{row}(\boldsymbol{I}_{DK}, \dots, \boldsymbol{I}_{DK})}_{DK\times R} \times \underbrace{\mathrm{diag}\left((\mathcal{Z}\rho_{R;-iN}^{+} h_0^{(a)})(z), (\mathcal{Z}\rho_{R;1-iN}^{+} h_0^{(a)})(z), \dots, (\mathcal{Z}\rho_{R;R-1-iN}^{+} h_0^{(a)})(z)\right)}_{R\times R},$$

with the $DK \times DK$ diagonal matrix

$$\boldsymbol{M}_i = \mathrm{diag}\left(1, F_{DK}^{-ir}, \dots, F_{DK}^{-(DK-1)ir}\right).$$

The vector $\underline{d}_R \in \mathbb{C}^{R\times 1}$ is the 'delay' vector

$$\underline{d}_R(z) = [1, z, \dots, z^{R-1}]^T.$$

Concatenating the D vectors $\underline{H}_{(a);i}$, which contain all DK z-transformed impulse responses $h_{ik}^{(a)}$, to a column vector $\underline{H}_{(a)} \in \mathbb{C}^{DK\times 1}$, i.e., we put the impulse responses $h_{ik}^{(a)}$ of the analysis bank in Fig. 4.3a in one column vector, yields

$$\begin{aligned}\underline{H}_{(a)}(z) &= \mathrm{col}\left(\underline{h}_{(a);0}(z), \underline{h}_{(a);1}(z), \dots, \underline{h}_{(a);D-1}(z)\right) \\ &= \underbrace{\mathrm{diag}(\boldsymbol{F}_K, \dots, \boldsymbol{F}_K)}_{DK\times DK} \boldsymbol{E}_{(a)}(z^R)\underline{d}_R(z),\end{aligned} \tag{4.34}$$

where the matrix $\boldsymbol{E}_{(a)} \in \mathbb{C}^{DK\times R}$ is given by

$$\boldsymbol{E}_{(a)}(z) = \mathrm{col}\left(\boldsymbol{E}_{(a);0}(z), \boldsymbol{E}_{(a);1}(z), \ldots, \boldsymbol{E}_{(a);D-1}(z)\right).$$

The vector $\underline{c} \in \mathbb{C}^{DK\times 1}$ containing the z-transformed subband signals c_{ik}, i.e., the subband signals c_{ik} in Fig. 4.3 put in one column vector,

$$\underline{c}(z) = \mathrm{col}(\underline{c}_0(z), \underline{c}_1(z), \ldots, \underline{c}_{D-1}(z)),$$

with

$$\underline{c}_i(z) = [(\mathcal{Z}c_{i0})(z), (\mathcal{Z}c_{i1})(z), \ldots, (\mathcal{Z}c_{i,K-1})(z)]^T,$$

can now be expressed as

$$\underline{c}(z) = \frac{1}{DN}\sum_{v=0}^{DN-1}(\mathcal{Z}\varphi)\left(F_{DN}^{v}z^{1/DN}\right)\underline{H}_{(a)}\left(F_{DN}^{v}z^{1/DN}\right). \tag{4.35}$$

If we also take R a multiple of DN, we can use noble identity (1.21) and we can shift the downsamplers through the polyphase components to the front end of the analysis bank. Thus we choose $R = \mathrm{lcm}(DK, DN)l = DpqJl$, where l is an arbitrary positive integer. Note that, if the impulse response of the dual window γ_Λ is finite, i.e., if we are dealing with FIR filters, then the integer l directly influences the length of the polyphase components. From Eq. (4.34), we see that the analysis bank can be implemented with D DFT matrices $\boldsymbol{F}_K$ and with the R-polyphase components of $h_{(a)}$. In Fig. 4.4a we have depicted the analysis bank schematically for the case that $l = 1$, i.e., $R = DpqJ$. Note that, although we need D matrices $\boldsymbol{F}_K$ and the R-polyphase components of $h_{(a)}$, we downsample by a factor DN, i.e., a factor D larger than in the case of a separable lattice. Therefore, the implementation complexity is comparable to the separable case apart from the $D-1$ matrices $\boldsymbol{M}_i$. Note, moreover, that in the case of a separable Gabor transform, i.e., $D = 1$, the expression (4.33) indeed reduces to expression (1.24).

4.3.2 Synthesis bank

In this subsection, we show how the synthesis bank can be implemented efficiently. We z-transform the synthesis operation (4.32), use property (1.16) of the convolution and use the polyphase representation, which yields

$$\begin{aligned}(\mathcal{Z}\varphi)(z) &= \sum_{u=0}^{K-1}\sum_{i=0}^{D-1}(\mathcal{Z}c_{iu})(z^{DN})(\mathcal{Z}h_{iu}^{(s)})(z) = \sum_{i=0}^{D-1}\sum_{m=0}^{K-1}\sum_{n=0}^{D-1}\sum_{\ell=0}^{ql-1} z^{-(m+nK+\ell DK)}\\ &\times\left(\mathcal{Z}\rho^{-}_{R;m+iN+nK+\ell DK}h_0^{(s)}\right)(z^R)e^{-j2\pi ir(m+nK)/DK}\\ &\times\sum_{u=0}^{K-1}(\mathcal{Z}c_{iu})(z^{DN})e^{j2\pi mu/K},\end{aligned}$$

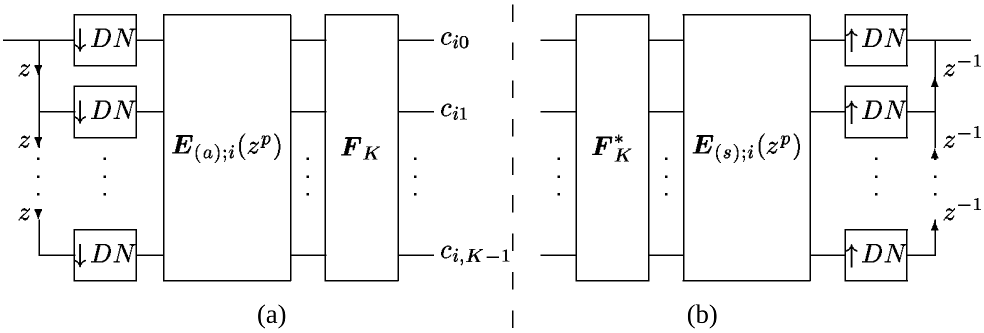

Figure 4.4: *Part i of the polyphase implementation of the DK channel filter bank with D prototypes. (a) The analysis bank. (b) The synthesis bank.*

or in matrix notation

$$(\mathcal{Z}\varphi)(z) = \underline{d}_R^T(z^{-1})\boldsymbol{E}_{(s)}(z^R)\underbrace{\operatorname{diag}(\boldsymbol{F}_K^*,\dots,\boldsymbol{F}_K^*)}_{DK\times DK}\underline{c}(z^{DN}), \tag{4.36}$$

where the matrix $\boldsymbol{E}_{(s)} \in \mathbb{C}^{R\times DK}$ is given by

$$\boldsymbol{E}_{(s)}(z) = \operatorname{row}\left(\boldsymbol{E}_{(s);0}(z), \boldsymbol{E}_{(s);1}(z), \dots, \boldsymbol{E}_{(s);D-1}(z)\right),$$

with

$$\boldsymbol{E}_{(s);i}(z) = \underbrace{\operatorname{diag}\Big((\mathcal{Z}\rho^+_{R;-iN}h_{(s)})(z), (\mathcal{Z}\rho^+_{R;1-iN}h_{(s)})(z), \dots, (\mathcal{Z}\rho^+_{R;R-1-iN}h_{(s)})(z)\Big)}_{R\times R}$$
$$\times \underbrace{\operatorname{col}(\boldsymbol{I}_{DK},\dots,\boldsymbol{I}_{DK})}_{R\times DK}\ \underbrace{\boldsymbol{M}_i^*}_{DK\times DK}\ \underbrace{\operatorname{col}(\boldsymbol{I}_K,\dots,\boldsymbol{I}_K)}_{DK\times K}.$$

Note that expression (4.36) is actually the conjugated and transposed of expression (4.34). From Eq. (4.36), we see that the synthesis bank can be implemented in a similar way as the analysis bank. By using the noble identity (1.20), we can shift the upsamplers through the polyphase components to the back end of the synthesis bank. In Fig. 4.4b we have depicted the synthesis bank schematically for the case that $l = 1$.

4.3.3 Matrix representation of the frame operator

Similar as in the case of a Gabor scheme on a separable lattice (see Section 1.6), by using the z-transformation and the polyphase representation we can find a matrix representation of the frame operator for the non-separable case. This matrix representation can be used to calculate the dual window. It also provides a method to calculate the frame bounds. We continue our discussion and show how this matrix representation of the frame operator can be obtained in the non-separable case. The frame operator $\mathcal{S}_\Lambda$ is defined by

$$\mathcal{S}_\Lambda \varphi = \sum_{k=\langle DK\rangle} \sum_{m=-\infty}^{\infty} \langle \varphi, \tilde{\gamma}_{\Lambda_S;mk}\rangle \tilde{\gamma}_{\Lambda_S;mk}.$$

As we have seen, the non-zero elements of the inner products $\langle \varphi, \tilde{\gamma}_{\Lambda_S;mk}\rangle$ can be calculated with the help of an analysis bank with D prototypes. Using the expressions (4.35) and (4.36) with $\boldsymbol{E}_{(s)}(z)$ replaced by $\boldsymbol{E}^*_{(a)}(z)$ yields

$$\begin{aligned}(\mathcal{Z}\mathcal{S}_\Lambda\varphi)(z) = {}& \frac{K}{DN}\underline{d}^T_{DN}(z^{-1})\,\mathrm{row}(\boldsymbol{I}_{DN}, z^{-DN}\boldsymbol{I}_{DN}, ..., z^{-R+DN}\boldsymbol{I}_{DN})\boldsymbol{E}^*_{(a)}(z^R) \\ & \times \boldsymbol{E}_{(a)}(z^R)\,\mathrm{col}(\boldsymbol{I}_{DN}, z^{DN}\boldsymbol{I}_{DN}, ..., z^{R-DN}\boldsymbol{I}_{DN})\sum_{v=0}^{DN-1}(\mathcal{Z}\varphi)(F^v_{DN}z)\,\underline{d}_{DN}(F^v_{DN}z),\end{aligned} \tag{4.37}$$

where we used

$$\underline{d}_R(z) = \mathrm{col}(\boldsymbol{I}_{DN}, z^{DN}\boldsymbol{I}_{DN}, \dots, z^{R-DN}\boldsymbol{I}_{DN})\underline{d}_{DN}(z).$$

In Eq. (4.37), we recognize the DN z-transformed upsampled polyphase components $\rho^-_{DN;m}\varphi$ of the signal φ

$$(\mathcal{Z}\rho^-_{DN;m}\varphi)(z^{DN}) = \frac{1}{DN}\sum_{v=0}^{DN-1} F^{vm}_{DN} z^m (\mathcal{Z}\varphi)(F^v_{DN}z).$$

Using this expression, Eq. (4.37) now takes the form

$$(\mathcal{Z}\mathcal{S}_\Lambda\varphi)(z) = \underline{d}^T_{DN}(z^{-1})\boldsymbol{S}_\Lambda(z^{DN})\underline{\rho}^-_{DN;\varphi}(z^{DN}), \tag{4.38}$$

where the matrix $\boldsymbol{S}_\Lambda \in \mathbb{C}^{DN\times DN}$ is defined by

$$\begin{aligned}\boldsymbol{S}_\Lambda(z) = {}& K\,\mathrm{row}(\boldsymbol{I}_{DN}\; z^{-1}\boldsymbol{I}_{DN} \dots z^{-p+1}\boldsymbol{I}_{DN})\boldsymbol{E}_{(s)}(z^{lp}) \\ & \times \boldsymbol{E}^*_{(s)}(z^{lp})\,\mathrm{col}(\boldsymbol{I}_{DN}\; z\boldsymbol{I}_{DN} \dots z^{lp-1}\boldsymbol{I}_{DN}),\end{aligned}$$

and where the vector $\underline{\rho}^-_{DN;\varphi} \in \mathbb{C}^{DN\times 1}$ contains the z-transformed polyphase components of the signal φ

$$\underline{\rho}^-_{DN;\varphi}(z) = [(\mathcal{Z}\rho^-_{DN;0}\varphi)(z), (\mathcal{Z}\rho^-_{DN;1}\varphi)(z), \dots, (\mathcal{Z}\rho^-_{DN;DN-1}\varphi)(z)]^T.$$

So we find that

$$\underline{\rho}^{-}_{DN;\mathcal{S}_\Lambda\varphi}(z) = \boldsymbol{S}_\Lambda(z)\underline{\rho}^{-}_{DN;\varphi}(z).$$

In particular, we have the following relationship between the window γ_Λ and its dual window g_Λ

$$\underline{\rho}^{-}_{DN;\gamma_\Lambda}(z) = \boldsymbol{S}_\Lambda(z)\underline{\rho}^{-}_{DN;g_\Lambda}(z),$$

or

$$\underline{\rho}^{-}_{DN;g_\Lambda}(z) = \boldsymbol{S}_\Lambda(z)^{-1}\underline{\rho}^{-}_{DN;\gamma_\Lambda}(z).$$

With the help of the matrix $\boldsymbol{S}_\Lambda$, we can also find the frame bounds A and B of the frame operator $\mathcal{S}_\Lambda$. By using Eq. (4.38) and taking $z = \exp(j2\pi\theta)$ we find that

$$\langle \mathcal{S}_\Lambda\varphi, \varphi\rangle = \langle \underline{d}^T_{DN}\boldsymbol{S}_\Lambda\underline{\rho}^{-}_{DN;\varphi}, \underline{d}^T_{DN}\underline{\rho}^{-}_{DN;\varphi}\rangle = \langle \boldsymbol{S}_\Lambda\underline{\rho}^{-}_{DN;\varphi}, \underline{\rho}^{-}_{DN;\varphi}\rangle.$$

From this expression we see that the frame bounds follow from:

$$A = \min_{|z|=1} \min_{\substack{\underline{u}\in\mathbb{C}^{DN}\\ ||\underline{u}||=1}} \langle \boldsymbol{S}_\Lambda(z)\underline{u}, \underline{u}\rangle \qquad \text{and} \qquad B = \max_{|z|=1} \max_{\substack{\underline{u}\in\mathbb{C}^{DN}\\ ||\underline{u}||=1}} \langle \boldsymbol{S}_\Lambda(z)\underline{u}, \underline{u}\rangle.$$

4.4 Shearing

As we have seen in Section 2.4, the shear operator can be used to reshear a non-separable Gabor scheme for continuous-time signals into a separable one. In the discrete-time setting, it is even more interesting to reshear a non-separable Gabor scheme into a separable one, since many algorithms for the separable case have been designed in the last decade (see, for instance, [10, 31, 32, 35, 37, 73, 74, 76, 84]). In this section we shall consider the discrete-time version of the shear operator for non-periodic and periodic signals.

First we consider the shifted and modulated windows that are obtained by shearing the frequency variable in the non-periodic case [see Eq. (4.4)]. The discrete shear operator $\mathcal{Q}_{\omega_a}$ is defined as

$$(\mathcal{Q}_{\omega_a}\varphi)[n] = e^{j2\pi\omega_a n^2}\varphi[n],$$

which is unitary on $\ell_2(\mathbb{Z})$ with corresponding adjoint operator $\mathcal{Q}^*_{\omega_a} = \mathcal{Q}_{-\omega_a}$. For convenience, as mentioned in Section 2.4, we use the same notation for the shear operator $\mathcal{Q}_{\omega_a}$ for the discrete-time setting. The modulation slope of the continuous

Table 4.1: *Two important properties of the shear operator.*

$\mathcal{Q}_\omega \mathcal{T}_\tau = e^{-j2\pi\omega\tau^2} \mathcal{M}_{2\omega\tau} \mathcal{T}_\tau \mathcal{Q}_\omega$
$\mathcal{Q}_{\omega_0} \mathcal{M}_{\omega_1} = \mathcal{M}_{\omega_1} \mathcal{Q}_{\omega_0}$

line through the origin induced by the shear operator $\mathcal{Q}_{\omega_a}$ is $2\omega_a$ (cf. the continuous-time case). To reshear the non-separable lattice Λ into a separable one, we have to choose

$$\omega_{a,0} = r/2NDK.$$

In Table 4.1, two properties of the shear operator $\mathcal{Q}_{\omega_a}$ are tabulated concerning translation and modulation. By applying the shear operator $\mathcal{Q}_{\omega_a}$ with $\omega_a = \omega_{a,0}$ and its properties to the shifted and modulated windows $g_{\Lambda;mk}$, we obtain

$$\mathcal{Q}_{\omega_{a,0}} g_{\Lambda;mk} = \mathcal{M}^k_{1/K} \mathcal{M}^m_{-r/DK} \mathcal{Q}_{\omega_{a,0}} \mathcal{T}^m_N g_\Lambda = e^{-j\pi m^2 rN/DK} \mathcal{M}^k_{1/K} \mathcal{T}^m_N \mathcal{Q}_{\omega_{a,0}} g_\Lambda.$$

So if the set $\{g_{\Lambda;mk} | m \in \mathbb{Z}, k = \langle K \rangle\}$ is a frame then the set $\{\mathcal{Q}_{\omega_{a,0}} g_{\Lambda;mk} | m \in \mathbb{Z}, k = \langle K \rangle\}$ establishes a frame on a rectangular lattice. Applying the unitary shear operator $\mathcal{Q}_{\omega_{a,0}}$ to the Gabor expansion (4.5) yields a sheared Gabor expansion on a rectangular lattice as well:

$$\begin{aligned} \mathcal{Q}_{\omega_{a,0}} \varphi &= \sum_{m=-\infty}^{\infty} \sum_{k=\langle K \rangle} \langle \mathcal{Q}_{\omega_{a,0}} \varphi, \mathcal{M}^k_{1/K} \mathcal{T}^m_N \mathcal{Q}_{\omega_{a,0}} \gamma_\Lambda \rangle \mathcal{M}^k_{1/K} \mathcal{T}^m_N \mathcal{Q}_{\omega_{a,0}} g_\Lambda \\ &= \sum_{m=-\infty}^{\infty} \sum_{k=\langle K \rangle} \breve{a}_{mk} \mathcal{M}^k_{1/K} \mathcal{T}^m_N \mathcal{Q}_{\omega_{a,0}} g_\Lambda, \end{aligned}$$

where

$$\breve{a}_{mk} = a_{mk} e^{-j\pi m^2 rN/DK} = \langle \mathcal{Q}_{\omega_{a,0}} \varphi, \mathcal{M}^k_{1/K} \mathcal{T}^m_N \mathcal{Q}_{\omega_{a,0}} \gamma_\Lambda \rangle$$

is the array of (modified) Gabor expansion coefficients.

In Section 2.4, we also sheared a non-separable lattice, which is obtained by reshearing the time-variable, into a separable one by using the shear operator $\mathcal{Q}_{t_a}$ [see Eq. (2.39)] and the Fourier transformation. A similar procedure can be followed in the discrete-time case, and therefore we skip further details.

Similar to the continuous-time case, it can be shown that given the collection $\{g_{\Lambda;mk} | m \in \mathbb{Z}, k = \langle K \rangle\}$, corresponding to a set of shifted and modulated windows on a non-separable lattice Λ, is a frame with frame bounds A and B is equivalent to

Table 4.2: *Summary of the constants describing the periodic Gabor scheme.*

$J = \gcd(K, N)$	$N = qJ$	$K = pJ$	$M = pLD$	$\gcd(D, r) = 1$

$\{\mathcal{M}_{1/K}^{k}\mathcal{T}_{N}^{m}\mathcal{Q}_{\omega_{a,0}}g_\Lambda | m \in \mathbb{Z}, k = \langle K \rangle\}$, corresponding to a collection of shifted and modulated windows on a separable lattice, is a frame with the same frame bounds A and B.

In the discussion above, we considered non-periodic signals. In Section 4.1.1, we also considered the case of a signal φ and a dual window γ_Λ that belong to the class of signals with finite support, which resulted in the periodic Gabor scheme. In Table 4.2, we have summarized the constants that describe a periodic Gabor scheme. Applying the shear operator $\mathcal{Q}_{\omega_a}$ to the periodic Gabor scheme is more interesting, since most algorithms are designed for periodic signals. In this paragraph, we consider the lattices that are obtained by shearing the frequency variable. In order to use the shear operator $\mathcal{Q}_{\omega_a}$ for periodic signals, the sheared periodic signals with period MN have to be periodic again with the same period MN, i.e.,

$$(\mathcal{Q}_{\omega_a}\Phi)[n + MN] = (\mathcal{Q}_{\omega_a}\Phi)[n], \tag{4.39}$$

where Φ has period MN. Again we have to choose $\omega_{a,0} = r/2NDK$ to transform a non-separable lattice Λ back into a separable one [see Eq. (4.20)]. In Section 4.1.1, we found an extra condition, which ensures that the frame operator commutes with the time-frequency operator. This condition now takes the form

$$(\mathcal{Q}_{\omega_{a,0}}\Gamma)_{mk} = (\mathcal{Q}_{\omega_{a,0}}\Gamma)_{m+M,k}.$$

This condition is satisfied, since $(\mathcal{Q}_{\omega_{a,0}}\Gamma)_{mk}$ corresponds to a rectangular lattice and is therefore periodic in the m-variable with period M. From Eq. (4.39) with $\omega_a = \omega_{a,0}$, we find the condition

$$\begin{array}{ll} J|rL & \text{if } MN \text{ is even,} \\ 2J|rL & \text{if } MN \text{ is odd,} \end{array} \tag{4.40}$$

which is a rather strong condition. Thus for a given period MN and a given non-separable lattice Λ generated with lattice generator matrix $\mathbf{\Lambda}$

$$\mathbf{\Lambda} = \boldsymbol{UL} = \frac{1}{DK}\begin{bmatrix} NDK & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -r & D \end{bmatrix},$$

it looks like that the shear operator is only usable if condition (4.40) is fulfilled. However, the same lattice Λ is generated with the matrix $\boldsymbol{L}$ replaced by $\boldsymbol{LK}_0$

$$\boldsymbol{LK}_0 = \begin{bmatrix} 1 & 0 \\ -(r + lD) & D \end{bmatrix},$$

where $\boldsymbol{K}_0$ is the unimodular matrix

$$\boldsymbol{K}_0 = \begin{bmatrix} 1 & 0 \\ -l & 1 \end{bmatrix},$$

with l an arbitrary integer (see Section 2.1). This means that we have more freedom in choosing ω_a if we use the matrix $\boldsymbol{L}\boldsymbol{K}_0$. Put differently, we can choose different modulation slopes for the shear operator to reshear a non-separable lattice Λ into a separable one. With the lattice Λ generated with the help of matrix $\boldsymbol{L}\boldsymbol{K}_0$, the stronger condition (4.40) can be replaced by a weaker condition

$$\exists_{l\in\mathbb{Z}} \quad \begin{array}{ll} J|(r+lD)L & \text{if } MN \text{ is even,} \\ 2J|(r+lD)L & \text{if } MN \text{ is odd,} \end{array} \tag{4.41}$$

i.e., we have to find integers l and l_0, such that

$$\begin{array}{ll} Jl_0 = rL + lDL & \text{if } MN \text{ is even.} \\ 2Jl_0 = rL + lDL & \text{if } MN \text{ is odd.} \end{array} \tag{4.42}$$

These integers l and l_0 exist if and only if

$$\begin{array}{ll} \gcd(J, DL)\big|rL & \text{if } MN \text{ is even,} \\ \gcd(2J, DL)\big|rL & \text{if } MN \text{ is odd.} \end{array} \tag{4.43}$$

Then Eq. (4.42) can be solved by using the Euclidean division algorithm (see [30]), which yields l. The variable ω_a has to be taken equal to $\omega_{a,l} = (r+lD)/2NDK$. Applying the shear operator $\mathcal{Q}_{\omega_{a,l}}$ to shifted and modulated windows $G_{\Lambda;\lfloor m\rfloor_M k}$ [see Eq. (4.20)] where the lattice Λ is generated with the help of the matrix $\boldsymbol{L}\boldsymbol{K}_0$ and using the properties tabulated in Table 4.1 yields

$$\begin{aligned} \mathcal{Q}_{\omega_{a,l}} G_{\Lambda;\lfloor m\rfloor_M k} &= \mathcal{M}^{\lfloor m\rfloor_M}_{-(r+lD)/DK} \mathcal{M}^k_{1/K} \mathcal{Q}_{\omega_{a,l}} \mathcal{T}^m_N G_\Lambda \\ &= e^{-j\pi(r+lD)\lfloor m\rfloor_M^2 N/DK} \mathcal{M}^k_{1/K} \mathcal{T}^m_N \mathcal{Q}_{\omega_{a,l}} G_\Lambda. \end{aligned}$$

So the collection $\{\mathcal{Q}_{\omega_{a,l}} G_{\Lambda;\lfloor m\rfloor_M k} | m = \langle M\rangle, k = \langle K\rangle\}$ establishes a frame on a rectangular lattice. Applying the unitary shear operator $\mathcal{Q}_{\omega_{a,l}}$ to the periodic signal Φ results in the sheared periodic Gabor expansion (4.19), which corresponds to an expansion on a rectangular lattice as well:

$$\begin{aligned} \mathcal{Q}_{\omega_{a,l}} \Phi &= \sum_{m=\langle M\rangle} \sum_{k=\langle K\rangle} \langle \mathcal{Q}_{\omega_{a,l}} \Phi, \mathcal{M}^k_{1/K} \mathcal{T}^m_N \mathcal{Q}_{\omega_{a,l}} \Gamma_\Lambda \rangle \mathcal{M}^k_{1/K} \mathcal{T}^m_N \mathcal{Q}_{\omega_{a,l}} G_\Lambda \\ &= \sum_{m=\langle M\rangle} \sum_{k=\langle K\rangle} \check{A}_{mk} \mathcal{M}^k_{1/K} \mathcal{T}^m_N \mathcal{Q}_{\omega_{a,l}} G_\Lambda, \end{aligned}$$

where

$$\breve{A}_{mk} = A_{mk} e^{-j\pi(r+lD)\lfloor m \rfloor_M^2 N/DK} = \langle \mathcal{Q}_{\omega_{a,l}} \Phi, \mathcal{M}_{1/K}^{k} \mathcal{T}_N^{\lfloor m \rfloor_M} \mathcal{Q}_{\omega_{a,l}} \Gamma_\Lambda \rangle$$

is the array of (modified) Gabor expansion coefficients.

Because of condition (4.43), it is not always possible to reshear a non-separable lattice, which is obtained by shearing the frequency variable, into a separable lattice by this method. It is most likely, that other lattice generator matrices, in particular those which are obtained by shearing the time variable or a combination of the time variable and the frequency variable, yield different conditions. Put differently, the procedure outlined above is limited, but Gabor schemes on the same non-separable lattice but with shifted and modulated windows that are obtained by shearing the time variable or a combination of the time variable and the frequency variable can, conceivably, be transformed into a Gabor scheme on a separable lattice. For this transformation of the non-separable Gabor scheme into a separable one, the Fourier transformation is also needed.

4.5 Non-separable lattices

The periodized Gabor scheme on a non-separable lattice as introduced in Section 4.1 is described by the constants D, J, L, p, q, and r (see Table 4.2). We start this section with the interpretation of these constants used. The size of the period of the signals is limited. Therefore, the number of possible non-separable lattices is limited, too. In this section, we also show how many and which lattices are possible when we consider a periodic Gabor scheme for periodic signals with period MN.

In Section 4.1.1, we found that we have to choose $M = pLD$ with L a positive integer such that condition (4.10) is fulfilled to have a fully periodized Gabor transform and a periodized Gabor expansion [see Eqs. (4.14) and (4.15), respectively] on a non-separable lattice Λ for periodic signals with a period MN. In one period of length MN, there are M time-shifts of the windows of size N. From the lattice generator matrix with the modified Hermite normal form $\boldsymbol{L}''$ [see Eq. (2.10) for the modified Hermite normal], which generates the same lattice Λ,

$$\boldsymbol{U}\boldsymbol{L}'' = \frac{1}{DK} \begin{bmatrix} NDK & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} D & -r' \\ 0 & 1 \end{bmatrix},$$

we see that after each D time-shifts of size N, we have a sample on the time-axis. We shall call this time-shift DN between two points on the time-axis a time-segment. So we have $MN/DN = pL$ time-segments in one period of the signal. From the lattice generator matrix $\boldsymbol{U}\boldsymbol{L}''$, we also see that the smallest modulation size in the frequency domain is $1/DK$. So we can distinguish DK samples in the frequency

Table 4.3: *Summary of the lattice properties.*

M	the number of time-shifts in one signal period.
N	the size of the time-shift.
MN	the length of the signal period.
DN	the size of a time-segment; distance between two points on the time-axis.
$MN/DN = pL$	the number of time-segments.
K	the number frequency-modulations after each time-shift.
$MN/K = qLD$	the size of a frequency-segment; distance between two points on the frequency-axis.
D	the number of samples in one time-frequency rectangle formed by a time-segment and a frequency-segment.
qL	the size of a frequency-modulation.
KM	the number of time-shifts and frequency-modulations, i.e., the number of lattice points in one signal period.
$KM/MN = p/q$	the degree of oversampling.

domain. According to [38], where the Gabor scheme is considered from a group theoretical approach, there are MN samples possible in the frequency domain, i.e., in this approach the smallest frequency-modulation is of size $1/MN$. This is in accordance with our approach, since DK is a divisor of MN ($DKqL = MN$). Then the non-separable lattice Λ is a subgroup of the $\mathbb{Z}_{MN} \times \mathbb{Z}_{MN}$ cyclic group. Instead of DK samples we will also take MN samples. We regard $\mathbb{Z}_{DK} \times \mathbb{Z}_{MN}$ as a subgroup of $\mathbb{Z}_{MN} \times \mathbb{Z}_{MN}$. Note that this convention does not change anything, it is just a matter of convenience. In our approach we write for the smallest frequency-modulation $(qL)(1/MN)$ instead of $1/DK$. In order to deal only with integers, we normalize $1/MN$ to 1. After each time-shift of size N, we have K samples in the frequency domain. Thus the size of the frequency modulations is $MN/K = qLD$, which is also the size of a frequency-segment, the frequency gap between two samples on the frequency-axis. Since r and D are relatively prime, there are D samples in the time-frequency rectangle formed by a time-segment and a frequency-segment of size $DN \times qLD$. In total, there are M time-shifts and for each time-shift we have K modulations. So, in one period, we have MK samples in the time-frequency plane. Consequently, the oversampling is equal to MK/MN, which is indeed equal to p/q. In Table 4.3, we summarized the above mentioned lattice properties.

To illustrate the above mentioned terms, we consider a Gabor scheme on a lattice with matrix $\boldsymbol{L} = \left[\begin{smallmatrix} 1 & 0 \\ -1 & 3 \end{smallmatrix}\right]$, i.e., $r = 1$ and $D = 3$, and with parameters $L = 1$, $p = 3$, $q = 2$, and $J = 2$. The lattice is depicted in Fig. 4.5. In this figure, we show the differences between the terms frequency-modulation, frequency-segment, time-shift and time-segment. In this case the frequency-shift, frequency-segment, time-shift and

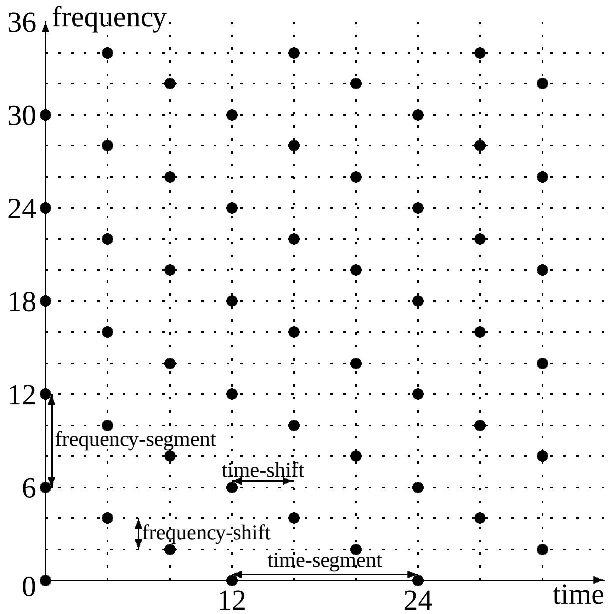

Figure 4.5: *The lattice with parameters* $D = 3$, $r = 1$, $p = 3$, $q = 2$, $J = 2$ *and* $L = 1$.

time-segment are equal to $qL = 2$, $qLD = 6$, $qJ = 4$ and $DN = 12$, respectively.

We have considered the parameters which come along with the Gabor scheme on a non-separable lattice for periodic signals, and now we continue by showing how many and which lattices are possible for these periodic Gabor schemes for periodic signals with period MN. In this context, the matrix $\boldsymbol{L}$ in its Hermite normal form is very important, again. The matrix $\boldsymbol{L}$ has the form [see Eq. (4.3)]

$$\boldsymbol{L} = \begin{bmatrix} 1 & 0 \\ -r & D \end{bmatrix},$$

with $0 \leq r < D$, and where r and D are relatively prime. Since r and D are relatively prime and $0 \leq r < D$, there are $\phi(D)$ different matrices $\boldsymbol{L}$ possible for a given determinant D, where the totient function $\phi(n)$, also called Euler's totient function, is defined as the number of positive integers which are relatively prime to n, where 1 is counted as being relatively prime to all numbers [1]. The period of the signal is $MN = DJLpq$ (see Table 4.2). Thus for a given possible oversampling p/q, the combinations of $DJL = MN/pq$ have to be investigated. Let L be a divisor of MN/pq. Then we can use the divisors of MN/pqL for J and D. Let the integer D be a divisor of MN/pqL. Then there are

$$\sum_{D | \frac{MN}{pqL}} \phi(D)$$

possible non-separable lattices for the given oversampling p/q and integer L. The sum of this series is known (see [1]) and is equal to MN/pqL:

$$\sum_{D|\frac{MN}{pqL}} \phi(D) = MN/pqL. \tag{4.44}$$

Note that the number of possible separable lattices ($D = 1$) for a given oversampling p/q, i.e., the number of combinations of J and L, is equal to the number of divisors of MN/pq. Since the integer L in Eq. (4.44) can be any divisor of the integer MN/pq, there are $\sigma_1(MN/pq)$ possible non-separable lattices for a given oversampling p/q, where $\sigma_k(n)$ is the divisor function. The divisor function $\sigma_k(n)$ is defined as the sum of the kth power of the divisors of an integer n [1]:

$$\sigma_k(n) = \sum_{d|n} d^k.$$

From this, it follows that the total number of non-separable lattices for a given periodic signal with period MN for all possible degrees of oversampling (including the case of critical sampling) is equal to

$$\sum_{d|MN} \sigma_1(d) \quad \text{for all } d = pq, \gcd(p,q) = 1 \text{ and } p/q \geq 1.$$

Note that the number of lattices in the case of undersampling $(p/q < 1)$ is equal to this sum minus $\sigma_1(MN)$. This number $\sigma_1(MN)$ corresponds to the number of possible lattices in the case of critical sampling. Note, moreover, that the number of possible separable lattices is equal to

$$\sum_{d|MN} \sigma_0(d) \quad \text{for all } d = pq, \gcd(p,q) = 1 \text{ and } p/q \geq 1.$$

For periodic signals with period length MN where MN has many divisors, there are many lattices possible. Beautiful numbers in this context are highly composite numbers [43]. A number n is said to be a highly composite number, if the number of divisors of the integer n is larger than the number of divisors of the integer n' for all values n' less than n. Highly composite numbers are, for instance, 24, 36, 48, 60, 120, 180, 240 and 360. Interesting in this context too, are numbers n with a number of divisors greater or equal to the number of divisors of n' for all values n' less than n. For instance, the numbers 24, 30, 36, 48, 60, 72, 84, 90, 96, 108, 120, 168, 180, 240, 336 and 360. In [38], it was already observed that a periodic signal with a period length that contains many divisors allow for many non-separable lattices.

As an example, we consider a Gabor scheme for periodic signals with period $MN = 36$. In Fig. 4.6, we have plotted the number of possible lattices for the

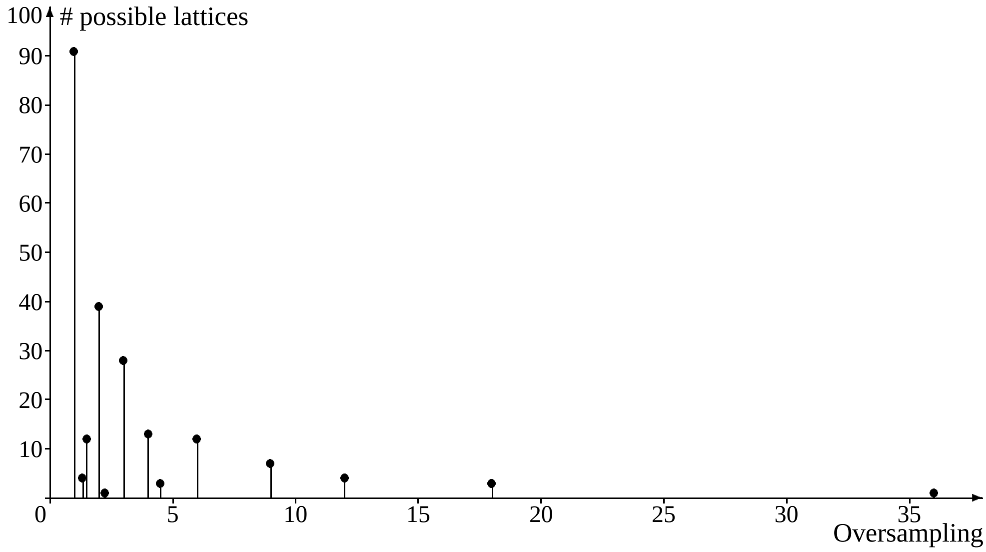

Figure 4.6: *The number of lattices for the possible degrees of oversampling in the case of a periodic signal with period 36.*

possible degrees of oversampling. The possible degrees of oversampling in the case of a periodic signal with period 36 are equal to $1/1$, $4/3$, $3/2$, $2/1$, $9/4$, $3/1$, $4/1$, $9/2$, $6/1$, $9/1$, $12/1$, $18/1$, and $36/1$. In this figure, it can be seen that, for instance, in the case of an oversampling $p/q = 3/2$, the number of possible lattices is equal to 12 [$\sigma(36/6) = 1 + 2 + 3 + 6 = 12$]. There are 218 possible non-separable lattices in the case of oversampling including the case of critical sampling, of which 45 are separable. The case of undersampling included results in $2 \times 218 - \sigma_1(36) = 345$ possible non-separable lattices, of which $2 \times 45 - \sigma_0(36) = 81$ are separable.

4.6 Concluding remarks

In this chapter, the implementation of the non-separable Gabor scheme for discrete-time signals is considered. We analyzed Gabor schemes for signals with infinite support and for periodic signals. The Gabor schemes for signals with infinite support is used in connection with filter banks. However, in practice, signals with infinite support or a very long support are dissected and periodized. For this situation we need a periodic Gabor scheme. For the periodic case, extra conditions with regard to the size of a period have to be fulfilled. This is a difference between the continuous-time (see Chapter 2) and the discrete-time case.

As in the continuous-time case (see Chapter 2), the non-separable lattice is described by means of a lattice generator matrix. Again, the lattice generator matrix is written in the Hermite nomal form, to obtain a shear representation on the shifted and modulated windows, which shear representation then leads to a modification of the rectangular Gabor scheme and results in the Gabor scheme on a non-separable lattice. The lattice generator matrix written in its Hermite normal form also leads to an alternative expression of the shifted and modulated windows. This expression is based on a rectangular lattice, as well; the non-separable lattice is obtained by deleting the shifted and modulated windows of a refined rectangular lattice, which do not belong to the non-separable lattice.

The Zak transformation allows an efficient calculation of the window given the dual window and the array of Gabor expansion coefficients, and allows to reconstruct the signal. The relationship between the periodized non-separable Gabor scheme and the Zak transformation is explored. The results in the non-separable case are very similar to the results in the rectangular case; by using the Fourier transformation and the Zak transformation, the Gabor transform and Gabor's signal expansion can be written as a sum-of-products form, again. The number of elements in the sum-of-products forms not only depends on the oversampling, but on the determinant of the lattice generator matrix, as well.

The non-separable lattice can be obtained by applying a shear operation on a rectangular lattice. It is shown that a non-separable lattice can be resheared into a rectangular lattice by multiplying by quadratic phase terms, which is associated with the shear operation. This technique allows a re-use of algorithms which are designed for the separable case explicitly to calculate the window given the dual window and the array of Gabor expansion coefficients, and to reconstruct the signal. Reshearing a non-separable Gabor scheme into a separable one is possible both for periodic and non-periodic Gabor schemes. However, in the periodic case, additional conditions have to be fulfilled in order to reshear a non-separable Gabor scheme into a separable one.

It is shown how a non-separable Gabor scheme can be implemented with the help of a filter bank. The implementation of a non-separable Gabor scheme looks very similar to the implementation of a separable one. In fact, the non-separable Gabor scheme falls apart in separable Gabor schemes, which can be implemented with a filter bank corresponding to a separable Gabor scheme. Furthermore, the matrix representation of the frame operator is given, which makes it possible to calculate the window and to calculate the frame bounds.

In the case of a non-separable periodic Gabor scheme, the number of possible lattices for a given period length is limited. We showed how many and which lattices are possible for a given period length. Periodic signals with a period length that contains many divisors, allow for the most possible lattices.

Chapter 5

Multi-dimensional non-separable Gabor scheme for discrete-time signals

In this chapter, the concept of the one-dimensional non-separable Gabor scheme for discrete-time signals is extended to the d-dimensional non-separable Gabor scheme for possibly non-separable lattices and possibly non-separated windows. Since this concept is based on the continuous-time case, as presented in Chapter 3, we will frequently refer to Chapter 3. In Section 5.1, we introduce the d-dimensional non-separable Gabor scheme for signals with an infinite support and for periodic signals. In Section 5.2, we show the relationship between the d-dimensional non-separable periodic Gabor scheme and the Zak transformation. It is is shown that the Zak transformation is very useful to calculate the window for a given dual window and to calculate the array of Gabor expansion coefficients, and to reconstruct the signal. We end this chapter by illustrating the ideas as presented in the Sections 5.1 and 5.2 on the basis of the two-dimensional case.

5.1 Gabor's signal expansion on a non-separable lattice

In Chapter 3, we started with the construction of a d-dimensional non-separable Gabor scheme by combining d one-dimensional non-separable Gabor schemes. In the discrete-time case, the d non-separable two-dimensional lattices Λ_i are described by $\boldsymbol{U}_i$ and $\boldsymbol{L}_i$ [cf. Eqs. (4.2) and (4.3)],

$$\boldsymbol{U}_i = \frac{1}{D_i K_i}\begin{bmatrix} N_i D_i K_i & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{L} = \begin{bmatrix} 1 & 0 \\ -r_i & D_i \end{bmatrix},$$

where $i = 0 \ldots D-1$. Then the $2d$-dimensional lattice Λ is described by the following matrices $\boldsymbol{U} \in \mathbb{Q}^{2d\times 2d}$ and $\boldsymbol{L} \in \mathbb{Z}^{2d\times 2d}$ in its Hermite normal form,

$$\boldsymbol{U} = \begin{bmatrix} \boldsymbol{N} & \boldsymbol{0}_d \\ \boldsymbol{0}_d & (\boldsymbol{DK})^{-1} \end{bmatrix} \quad \text{and} \quad \boldsymbol{L} = \begin{bmatrix} \boldsymbol{I}_d & \boldsymbol{0}_d \\ -\boldsymbol{R} & \boldsymbol{D} \end{bmatrix}, \tag{5.1}$$

where $\boldsymbol{N}$, $\boldsymbol{D}$, $\boldsymbol{K}$ and $\boldsymbol{R} \in \mathbb{Z}^{d\times d}$ are diagonal:

$$\begin{aligned} \boldsymbol{N} &= \mathrm{diag}(N_0, \ldots, N_{d-1}), \quad \boldsymbol{D} = \mathrm{diag}(D_0, \ldots, D_{d-1}), \\ \boldsymbol{K} &= \mathrm{diag}(K_0, \ldots, K_{d-1}), \quad \text{and} \quad \boldsymbol{R} = \mathrm{diag}(r_0, \ldots, r_{d-1}). \end{aligned}$$

The matrices $\boldsymbol{I}_d$ and $\mathbf{0}_d \in \mathbb{Z}^{d\times d}$ are the identity matrix and the zero matrix, respectively. Although this lattice Λ is non-separable, we call this lattice 'not completely non-separable', since it is separable in d one-dimensional (non-separable) lattices. In order to have a lattice generator matrix that generates a 'completely' non-separable $2d$-dimensional lattice, the matrix $\boldsymbol{D}$ has to be lower triangular and $\boldsymbol{R}$ has to be non-diagonal. Furthermore, the identity $\boldsymbol{I}_d$ appearing in $\boldsymbol{L}$ in Eq. (5.1) will be replaced by a lower triangular $\boldsymbol{S}$, which can be unified with $\boldsymbol{U}$. Note that the form of the matrix $\boldsymbol{L}$ remains a Hermite normal form after these modifications. In addition, similar as in the continuous-time case (see Section 3.1), the sub-matrix $\boldsymbol{D}$ in $\boldsymbol{U}$ is replaced by $\boldsymbol{C} \in \mathbb{Z}^{d\times d}$, where the ith column vector corresponds to the point on the lattice generated by $\boldsymbol{D}$ on the ith axis and closest to the origin (see Fig. 3.1 on page 60 for an example). Since these points on the axes belong to the lattice generated by $\boldsymbol{D}$, it follows that $\boldsymbol{D}^{-1}\boldsymbol{C}$ is a matrix containing only integers. Note that in the decomposable case ($\boldsymbol{S} = \boldsymbol{I}_d$, $\boldsymbol{R}$ and $\boldsymbol{D}$ are diagonal), the matrix $\boldsymbol{C}$ is equal to $\boldsymbol{D}$. The matrices $\boldsymbol{U}$ and $\boldsymbol{L}$ of the lattice generator matrix $\boldsymbol{\Lambda} = \boldsymbol{U}\boldsymbol{L}$ are now given by [cf. Eq. (3.3)]

$$\boldsymbol{U} = \begin{bmatrix} \boldsymbol{N}\boldsymbol{S} & \mathbf{0}_d \\ \mathbf{0}_d & (\boldsymbol{C}\boldsymbol{K})^{-1} \end{bmatrix} \quad \text{and} \quad \boldsymbol{L} = \begin{bmatrix} \boldsymbol{I}_d & \mathbf{0}_d \\ -\boldsymbol{R} & \boldsymbol{D} \end{bmatrix}, \tag{5.2}$$

respectively. We assume that the integers in one row appearing in $\boldsymbol{L}$ do not have a common divisor. The volume of a cell (a parallelepiped spanned by the column vectors of the matrix $\boldsymbol{\Lambda}$) in the position-frequency space is equal to the determinant of this matrix $\boldsymbol{\Lambda}$. This determinant is equal to $\det(\boldsymbol{N}\boldsymbol{S}(\boldsymbol{C}\boldsymbol{K})^{-1}\boldsymbol{D})$. The equality $\det(\boldsymbol{\Lambda}) = 1$ corresponds to critical sampling, whereas $\det(\boldsymbol{\Lambda}) > 1$ corresponds to undersampling, and $\det(\boldsymbol{\Lambda}) < 1$ corresponds to oversampling. As in the continuous-time case, here the term oversampling is misleading, as well. The set of shifted and modulated windows of an oversampled d-dimensional Gabor scheme, constructed from heavily oversampled and undersampled one-dimensional Gabor schemes, is not complete in $\ell_2(\mathbb{Z}^d)$. Note, moreover, that in the decomposable case ($\boldsymbol{C} = \boldsymbol{D}$, $\boldsymbol{S} = \boldsymbol{I}_d$ and $\boldsymbol{R}$ is diagonal), the expression for the volume of a cell reduces to $\det(\boldsymbol{N}\boldsymbol{K}^{-1})$, which is comparable to the one-dimensional case. In fact, this was the reason why we introduced $\boldsymbol{C}$ (see Section 3.1). The matrix $\boldsymbol{C}$ can be unified with $\boldsymbol{K}$, but then the expression for the oversampling does not reduce to the desired expression $\det(\boldsymbol{N}\boldsymbol{K}^{-1})$. Although the matrix $\boldsymbol{C}$ depends on $\boldsymbol{D}$, the fact that $\boldsymbol{C}$ can be compensated by $\boldsymbol{K}$ and that $\boldsymbol{L}$ is in its Hermite normal form, leads to the observation that we now cover all possible lattices with lattice points on all the axes, i.e., lattice generator matrices that can be can be written as $\boldsymbol{U}\boldsymbol{L}$.

Each point $\underline{\lambda} \in \Lambda$ can be obtained by a matrix-vector product

$$\forall_{\underline{\lambda}\in\Lambda}\exists_{\underline{n}\in\mathbb{Z}^{2d}}\ \ \underline{\lambda} = \boldsymbol{U}\boldsymbol{L}\underline{n},\ \text{with } \boldsymbol{U} \text{ and } \boldsymbol{L} \text{ as defined in (5.2).}$$

By using the sub-matrices $\boldsymbol{\Lambda}_{ik} \in \mathbb{Q}^{d\times d}$ of $\boldsymbol{\Lambda}$,

$$\boldsymbol{\Lambda} = \begin{bmatrix} \boldsymbol{\Lambda}_{11} & \boldsymbol{\Lambda}_{12} \\ \boldsymbol{\Lambda}_{21} & \boldsymbol{\Lambda}_{22} \end{bmatrix},$$

it follows that the shifted and modulated versions $g_{\Lambda;\underline{m}\,\underline{k}}$ of the d-variate window g_Λ are given by

$$\begin{aligned} g_{\Lambda;\underline{m}\,\underline{k}} &= \sigma_{\boldsymbol{\Lambda}\left[\frac{\underline{m}}{\underline{k}}\right]} g_\Lambda = \mathcal{M}_{\boldsymbol{\Lambda}_{21}\underline{m}}\mathcal{M}_{\boldsymbol{\Lambda}_{22}\underline{k}}\mathcal{T}_{\boldsymbol{\Lambda}_{11}\underline{m}}\mathcal{T}_{\boldsymbol{\Lambda}_{12}\underline{k}} g_\Lambda \\ &= \mathcal{M}_{-(\boldsymbol{C}\boldsymbol{K})^{-1}\boldsymbol{R}\underline{m}}\mathcal{M}_{(\boldsymbol{C}\boldsymbol{K})^{-1}\boldsymbol{D}\underline{k}}\mathcal{T}_{\boldsymbol{N}\boldsymbol{S}\underline{m}} g_\Lambda, \end{aligned} \tag{5.3}$$

where $\left[\frac{\underline{m}}{\underline{k}}\right] = \mathrm{col}(\underline{m}, \underline{k})$, and where we used the translation operator $\sigma_{\left[\frac{\underline{\tau}}{\underline{\omega}}\right]}$ defined by

$$\sigma_{\left[\frac{\underline{\tau}}{\underline{\omega}}\right]} = \mathcal{M}_{\underline{\omega}}\mathcal{T}_{\underline{\tau}},$$

with the modulation operator

$$(\mathcal{M}_{\underline{\omega}}\varphi)[\underline{n}] = e^{j2\pi\langle\underline{\omega},\,\underline{n}\rangle}\varphi[\underline{n}] \quad \text{with } \omega \in \mathbb{R}^d,$$

and translation operator

$$(\mathcal{T}_{\underline{\tau}}\varphi)[\underline{n}] = \varphi[\underline{n} - \underline{\tau}] \quad \text{with } \tau \in \mathbb{Z}^d.$$

Note that the shifted and modulated windows $g_{\Lambda;\underline{m}\,\underline{k}}$ are periodic in the frequency variable $\underline{k}$ with respect to the regular partition of $\mathbb{Z}^d$ generated by $\mathrm{part}(\boldsymbol{D}^{-1}\boldsymbol{C}\boldsymbol{K})$ (see Section 3.1 for the definition of part).

The Gabor frame operator $\mathcal{S}_\Lambda$ is defined as

$$\mathcal{S}_\Lambda\varphi = \sum_{\underline{k}\in\mathrm{part}(\boldsymbol{D}^{-1}\boldsymbol{C}\boldsymbol{K})}\ \sum_{\underline{m}\in\mathbb{Z}^d} \langle\varphi, g_{\Lambda;\underline{m}\,\underline{k}}\rangle g_{\Lambda;\underline{m}\,\underline{k}}, \quad \varphi \in \ell_2(\mathbb{Z}^d).$$

Similar to the proof in Section 2.1, it can be shown, by using the properties that are tabulated in Table 5.1, that the Gabor frame operator $\mathcal{S}_\Lambda$ commutes with the translation operator $\sigma_{\boldsymbol{\Lambda}\left[\frac{\underline{m}}{\underline{k}}\right]}$ in the non-separable case, i.e.,

$$\mathcal{S}_\Lambda\sigma_{\boldsymbol{\Lambda}\left[\frac{\underline{m}}{\underline{k}}\right]} = \sigma_{\boldsymbol{\Lambda}\left[\frac{\underline{m}}{\underline{k}}\right]}\mathcal{S}_\Lambda.$$

From this it follows that $\mathcal{S}_\Lambda^{-1}$ also commutes with the time-frequency operator $\sigma_{\boldsymbol{\Lambda}\left[\frac{\underline{m}}{\underline{k}}\right]}$. From $\mathcal{S}_\Lambda^{-1}g_{\Lambda;\underline{m}\,\underline{k}} = \gamma_{\Lambda;\underline{m}\,\underline{k}}$, it follows that the elements of the dual Gabor frame

Table 5.1: *Some properties of the modulation and translation operators.*

$\mathcal{T}_{\underline{\tau}_0}\mathcal{T}_{\underline{\tau}_1} = \mathcal{T}_{\underline{\tau}_1}\mathcal{T}_{\underline{\tau}_0}$
$\mathcal{M}_{\underline{\omega}_0}\mathcal{M}_{\underline{\omega}_1} = \mathcal{M}_{\underline{\omega}_1}\mathcal{M}_{\underline{\omega}_0}$
$\mathcal{M}_{\underline{\omega}}\mathcal{T}_{\underline{\tau}} = e^{j2\pi\langle\underline{\omega},\underline{\tau}\rangle}\mathcal{T}_{\underline{\tau}}\mathcal{M}_{\underline{\omega}}$

$\{\gamma_{\Lambda;\underline{m}\underline{k}}|\underline{m} \in \mathbb{Z}^d, \underline{k} \in \text{part}(\boldsymbol{D}^{-1}\boldsymbol{CK})\}$ are the shifted and modulated versions of the dual window γ_Λ. Thus the d-dimensional non-separable Gabor expansion on a non-separable lattice Λ is given by

$$\varphi = \sum_{\underline{k}\in\text{part}(\boldsymbol{D}^{-1}\boldsymbol{CK})} \sum_{\underline{m}\in\mathbb{Z}^d} a_{\underline{m}\underline{k}} g_{\Lambda;\underline{m}\underline{k}}, \quad \varphi \in \ell_2(\mathbb{Z}^d) \tag{5.4}$$

where the array of Gabor coefficients $\{a_{\underline{m}\underline{k}}\}$ is obtained by the Gabor transform

$$a_{\underline{m}\underline{k}} = \langle \varphi, \gamma_{\Lambda;\underline{m}\underline{k}}\rangle_d.$$

Note that the array $a_{\underline{m}\underline{k}}$ is sheared in the frequency variable $\underline{k}$.

In Section 4.2, we showed that after periodization of the signals, the Zak transformation for periodic signals can be used to calculate the window for a given dual window and to calculate the Gabor expansion coefficients, and to reconstruct the signal. In order to use the Zak transformation in the multi-dimensional case, we need an alternative expression for the shifted and modulated versions of the windows. The procedure to obtain this alternative expression is similar to the procedure as outlined in Section 3.1; we consider the separable lattice Λ_s

$$\Lambda_s = \{\boldsymbol{\Lambda}_s\underline{n}|\underline{n} \in \mathbb{Z}^{2d}\} \quad \text{with} \quad \boldsymbol{\Lambda}_s = \boldsymbol{U} = \begin{bmatrix} \boldsymbol{NS} & \boldsymbol{0}_d \\ \boldsymbol{0}_d & (\boldsymbol{CK})^{-1} \end{bmatrix}.$$

By using the multiplication operator $P_{\boldsymbol{\Lambda}}(\underline{m},\underline{k})$ [see Eq. (3.6)], the shifted and modulated versions of the window $g_{\Lambda;\underline{m}\underline{k}}$ take the form

$$\tilde{g}_{\Lambda_s;\underline{m}\underline{k}} = P_{\boldsymbol{\Lambda}}(\underline{m},\underline{k}) \left(\sigma_{\boldsymbol{\Lambda}_s[\frac{\underline{m}}{\underline{k}}]} g_\Lambda\right) = \tilde{\sigma}_{\boldsymbol{\Lambda}_s[\frac{\underline{m}}{\underline{k}}]} g_\Lambda, \tag{5.5}$$

where

$$\tilde{\sigma}_{\boldsymbol{\Lambda}_s[\frac{\underline{m}}{\underline{k}}]} = P_{\boldsymbol{\Lambda}}(\underline{m},\underline{k}) \left(\sigma_{\boldsymbol{\Lambda}_s[\frac{\underline{m}}{\underline{k}}]}\right).$$

Here we put a tilde on top of $g_{\Lambda_s;\underline{m}\underline{k}}$ and $\sigma_{\boldsymbol{\Lambda}_s[\frac{\underline{m}}{\underline{k}}]}$ to indicate that the multiplication operator $P_{\boldsymbol{\Lambda}}(\underline{m},\underline{k})$ is involved. Note that the shifted and modulated versions $\tilde{g}_{\Lambda_s;\underline{m}\underline{k}}$ are periodic in the frequency variable $\underline{k}$ with respect to the regular partition of $\mathbb{Z}^d$

generated by $\mathrm{part}(\boldsymbol{CK})$. With the modified expression (5.5) for the shifted and modulated versions $\tilde{g}_{\Lambda_S;\underline{m}\,\underline{k}}$ of the window g_Λ, Gabor's signal expansion takes the form

$$\varphi = \sum_{k \in \mathrm{part}(\boldsymbol{CK})} \sum_{m \in \mathbb{Z}^d} \tilde{a}_{mk} \tilde{g}_{\Lambda_S;\underline{m}\,\underline{k}}, \tag{5.6}$$

where

$$\tilde{a}_{mk} = \langle \varphi, \tilde{\gamma}_{\Lambda_S;\underline{m}\,\underline{k}} \rangle_d \tag{5.7}$$

is the array of Gabor expansion coefficients with

$$\tilde{\gamma}_{\Lambda_S;\underline{m}\,\underline{k}} = \tilde{\sigma}_{\boldsymbol{\Lambda}_S \left[\frac{\underline{m}}{\underline{k}} \right]} \gamma_\Lambda.$$

Here $\{\tilde{\gamma}_{\Lambda_S;mk} | \underline{m} \in \mathbb{Z}^d, \underline{k} \in \mathrm{part}(\boldsymbol{CK})\}$ is the dual frame of $\{\tilde{g}_{\Lambda_S;\underline{m}\,\underline{k}} | \underline{m} \in \mathbb{Z}, \underline{k} \in \mathrm{part}(\boldsymbol{CK})\}$. Note that due to the multiplication operator $\boldsymbol{P}_{\boldsymbol{\Lambda}}(\underline{m}, \underline{k})$, the array $\{\tilde{a}_{\underline{m}\,\underline{k}}\}$ of Gabor expansion coefficients contains many zeros.

In this chapter, the Gabor expansion (5.4) based on the shear representation (5.3) for the shifted and modulated windows will not be used explicitly. In the one-dimensional case we used the shear representation for the shifted and modulated windows to reshear a non-separable Gabor scheme into a separable one, by using the shear operator. In the multi-dimensional case this is much more complex. However, conceivably, it can be used to transform a non-separable Gabor scheme into a Gabor scheme with $\boldsymbol{R}$ the zero matrix, as will be elaborated in more detail in Chapter 6. In the next subsection, we will periodize the Gabor expansion (5.6) and the Gabor transform (5.7), which will be used to show the relationship between the d-dimensional non-separable Gabor scheme and the Zak transformation.

5.1.1 Periodization

In order to use the Fourier transformation and the Zak transformation for periodic signals, Gabor's signal expansion (5.6), and the Gabor transform (5.7) have to be periodized. Therefore, we restrict the class of signals φ and dual windows γ_Λ to signals that have a finite support. This condition of finite support implies that the array $\{\tilde{a}_{\underline{m}\,\underline{k}}\}$ has a finite support in the $\underline{m}$-variable for all signals φ in the class. We shall denote the support of the array $\{\tilde{a}_{\underline{m}\,\underline{k}}\}$ in the $\underline{m}$-variable by $\mathcal{H}_{\boldsymbol{M}}$. Here $\mathcal{H}_{\boldsymbol{M}}$ is a subset of $\mathbb{Z}^d$ containing $\det(\boldsymbol{M})$ elements which can generate a regular partition of $\mathbb{Z}^d$ with $\mathrm{part}(\boldsymbol{M})$. For simplicity, we take $\boldsymbol{M} \in \mathbb{Z}^{d \times d}$ diagonal, which leads to a Zak transformation that can be calculated with FFT's (see Section 5.2). In the multi-dimensional case, it is more difficult than in the one-dimensional case to give an explicit condition which should be satisfied in order to be able to use an overlap-add technique [cf. Eq. (4.10)]. This is because the signal φ and the dual window

γ_Λ can have an arbitrary (finite) support. In the one-dimensional case, the support is just an interval. Ultimately, we want to periodize the array $\{\tilde{a}_{\underline{m}\,\underline{k}}\}$ in the $\underline{m}$-variable with respect to the regular partition $\mathbb{Z}^d$ generated by $\mathrm{part}(\boldsymbol{M})$. Therefore, we shall assume that $\boldsymbol{M}$ is taken such that the array $\{\tilde{a}_{\underline{m}\,\underline{k}}\}$ can be identified as one period of the periodized array:

$$\tilde{a}_{\underline{m}\,\underline{k}} = \sum_{\underline{i}\in\mathbb{Z}^d} \tilde{a}_{m+\boldsymbol{M}\underline{i},\underline{k}}, \quad \underline{m} \in \mathcal{H}_{\boldsymbol{M}}, \underline{k} \in \mathrm{part}(\boldsymbol{CK}). \tag{5.8}$$

The period of the signals will be with respect to the regular partition of $\mathbb{Z}^d$ generated by $\mathrm{fund}(\boldsymbol{NSM})$. We shall write capital letters Φ and Γ_Λ to indicate that we deal with the periodized versions of φ and γ_Λ, respectively. So, the periodized version Φ of the signal φ is defined by

$$\Phi[\underline{n}] = \sum_{\underline{i}\in\mathbb{Z}^d} \varphi[\underline{n} + \boldsymbol{NSM}\underline{i}].$$

We define the shifted and modulated versions $\tilde{\Gamma}_{\Lambda_S;mk}$ by

$$\tilde{\Gamma}_{\Lambda_S;mk} = P_{\boldsymbol{\Lambda}}(\underline{m},\underline{k}) \left(\sigma_{\boldsymbol{\Lambda}_S\left[\frac{\underline{m}}{\underline{k}}\right]} \Gamma_\Lambda \right).$$

In order to periodize Gabor's signal expansion (5.6) and the Gabor transform (5.7), the shifted and modulated versions $\tilde{\Gamma}_{\Lambda_S;mk}$ as defined above have to be periodic with $\boldsymbol{NSM}$, as well. For this, the matrix $\boldsymbol{M}$ has to be chosen carefully such that the conditions

$$\tilde{\Gamma}_{\Lambda_S;mk}[n] = \tilde{\Gamma}_{\Lambda_S;mk}[\underline{n} + \boldsymbol{NSM}] \tag{5.9}$$

and (5.8) are fulfilled. From condition (5.9), we find that $\boldsymbol{M}$ has to be chosen such that $\boldsymbol{MS}^T\boldsymbol{N}(\boldsymbol{CK})^{-1}$ is a matrix containing only integers. However, this condition is not enough; similar to the one-dimensional case, we need an additional condition in order to let the translation operator $\tilde{\sigma}_{\boldsymbol{\Lambda}_S\left[\frac{\underline{m}}{\underline{k}}\right]}$ commute with the periodized frame operator $\mathcal{S}_\Lambda$ which is defined by

$$\mathcal{S}_\Lambda \Phi = \sum_{\underline{k}\in\mathrm{part}(\boldsymbol{CK})} \sum_{\underline{m}\in\mathrm{fund}(\boldsymbol{M})} \langle \Phi, \tilde{\Gamma}_{\Lambda_S;\underline{m}\,\underline{k}} \rangle \tilde{\Gamma}_{\Lambda_S;\underline{m}\,\underline{k}}.$$

This additional condition is obtained in a similar way as outlined in Section 4.1.1 and reads

$$\tilde{\Gamma}_{\Lambda_S;\underline{m}\,\underline{k}} = \tilde{\Gamma}_{\Lambda_S;\underline{m}+\boldsymbol{M}\underline{1}_d,\underline{k}},$$

with $\underline{1}_d \in \mathbb{Z}^{d\times 1}$ a vector containing only ones. From this condition we find that $\boldsymbol{M}$ needs also be taken such that $\boldsymbol{D}^{-1}\boldsymbol{R}\boldsymbol{M}$ is a matrix containing only integers. We shall assume that $\boldsymbol{M}$ meets these conditions. Substituting the periodized versions Φ and $\tilde{\Gamma}_{\Lambda_{\rm S};\underline{m}\,\underline{k}}$ into the Gabor transform (5.7) yields the bi-periodic array $\{\tilde{A}_{\underline{m}\,\underline{k}}\}$ which is periodic in the $\underline{m}$-variable and periodic in the $\underline{k}$-variable with respect to the regular partitions of $\mathbb{Z}^d$ generated by $\mathrm{part}(\boldsymbol{M})$ and $\mathrm{part}(\boldsymbol{CK})$, respectively,

$$\tilde{A}_{\underline{m}\,\underline{k}} = \langle \Phi, \tilde{\Gamma}_{\Lambda_{\rm S};\underline{m}\,\underline{k}} \rangle. \tag{5.10}$$

The signal Φ can be reconstructed with the periodized Gabor expansion (5.6)

$$\Phi = \sum_{m\in\mathrm{part}(\boldsymbol{M})} \sum_{k\in\mathrm{part}(\boldsymbol{CK})} \tilde{A}_{\underline{m}\,\underline{k}} \tilde{G}_{\Lambda_{\rm S};\underline{m}\,\underline{k}}. \tag{5.11}$$

On the other hand, Φ is the periodized version of the signal φ

$$\Phi[\underline{n}] = \sum_{\underline{i}\in\mathbb{Z}^d} \varphi[\underline{n} + \boldsymbol{NSM}\underline{i}] = \sum_{\underline{i}\in\mathbb{Z}^d} \sum_{\underline{m}\in\mathbb{Z}^d} \sum_{k\in\mathrm{part}(\boldsymbol{CK})} \tilde{a}_{\underline{m}\,\underline{k}} \tilde{g}_{\Lambda_{\rm S};\underline{m}\,\underline{k}}[\underline{n} + \boldsymbol{NSM}\underline{i}].$$

Unlike the one-dimensional case, it is difficult to give an explicit expression for $\boldsymbol{M}$ for a general dimension d. In Section 5.2, we give an explicit expression for $\boldsymbol{M}$ for the two-dimensional case. In the next section, we show the relationship between the periodic Gabor scheme as presented in this paragraph and the Zak transformation.

Assuming that the set of shifted and modulated windows $\tilde{\Gamma}_{\Lambda_{\rm S};\underline{m}\,\underline{k}}$ constitutes a frame, the relationship between the dual window Γ_Λ and the window G_Λ follows from substituting the periodized Gabor transform (5.10) into Gabor's signal expansion (5.11). In Appendix D.1, it is shown by which manipulations the biorthogonality condition [cf. Eq. (4.21)]

$$\begin{aligned} &\sum_{\underline{m}\in\mathrm{part}(\boldsymbol{M})} e^{j2\pi\underline{k}^T\boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}} G_\Lambda[\underline{n} - \boldsymbol{NS}\underline{m}] \Gamma^*_\Lambda[\underline{n} + \boldsymbol{KC}(\boldsymbol{D}^{-1})^T\underline{k} - \boldsymbol{NS}\underline{m}] \\ &= \frac{\det(\boldsymbol{D})}{\det(\boldsymbol{CK})} \sum_{\underline{u}\in\mathbb{Z}^d} \delta[\underline{k} - \boldsymbol{\Psi M}\underline{u}], \end{aligned} \tag{5.12}$$

where $\boldsymbol{\Psi} = \boldsymbol{D}^T(\boldsymbol{CK})^{-1}\boldsymbol{NS}$, is obtained. Note that $\boldsymbol{\Psi M}$ is a matrix containing only integers. The condition (5.12) should hold for all $\underline{k} \in \mathrm{part}(\boldsymbol{\Psi M})$ and for all $\underline{n} \in \mathrm{part}(\boldsymbol{NSM})$. Thus for a given periodic window Γ_Λ, we have to find a periodic function G_Λ such that this condition (5.12) is fulfilled.

5.2 Discrete Zak transform

In this section, we show the connection between the multi-dimensional non-separable periodic Gabor scheme and the Zak transformation. Similar to the one-dimensional case, we shall use the Zak transformation to calculate the window given the dual window and to calculate the array of Gabor coefficients, and to reconstruct the signal.

The multi-dimensional discrete Zak transform for periodic signals is defined by

$$(\mathcal{Z}\varphi)\,[\underline{n},\underline{\ell};\boldsymbol{N},\boldsymbol{M}] = \sum_{\underline{m}\in\mathrm{part}(\boldsymbol{M})} \Phi[\underline{n}+\boldsymbol{N}\underline{m}]e^{-j2\pi\langle\underline{\ell},\,\boldsymbol{M}^{-1}\underline{m}\rangle}, \tag{5.13}$$

where $\boldsymbol{M} \in \mathbb{Z}^{d\times d}$ is nonsingular. This Zak transformation maps a signal Φ, which is periodic with respect to the regular partition of $\mathbb{Z}^d$ generated by $\mathrm{part}(\boldsymbol{NM})$, to a function $(\mathcal{Z}\varphi)\,[\underline{n},\underline{\ell};\boldsymbol{N},\boldsymbol{M}]$ of $2d$ variables defined on the fundamental parallelepiped

$$\mathcal{R}_{\boldsymbol{N},\boldsymbol{M}} = \{\underline{n},\underline{\ell}|\underline{n}\in\mathrm{fund}(\boldsymbol{N}),\underline{\ell}\in\mathrm{fund}(\boldsymbol{M}^T)\}.$$

The Zak transform $(\mathcal{Z}\varphi)\,[\underline{n},\ell;\boldsymbol{N},\boldsymbol{M}]$ is periodic in the frequency variable $\underline{\ell}$ with respect to the regular partition of $\mathbb{Z}^d$ generated by $\mathrm{part}(\boldsymbol{M}^T)$ and quasi-periodic in the variable $\underline{n}$ with $\mathrm{part}(\boldsymbol{N})$:

$$(\mathcal{Z}\varphi)\,[\underline{n},\underline{\ell}+\boldsymbol{M}^T\underline{k};\boldsymbol{N},\boldsymbol{M}] = (\mathcal{Z}\varphi)\,[\underline{n},\ell;\boldsymbol{N},\boldsymbol{M}]$$

and

$$(\mathcal{Z}\varphi)\,[\underline{n}+\boldsymbol{N}\underline{k},\ell;\boldsymbol{N},\boldsymbol{M}] = (\mathcal{Z}\varphi)\,[\underline{n},\ell;\boldsymbol{N},\boldsymbol{M}]e^{j2\pi\langle\ell,\,\boldsymbol{M}^{-1}\underline{k}\rangle}.$$

The inverse Zak transformation is defined by

$$\Phi[\underline{n}] = \Phi[\underline{n}'+\boldsymbol{N}\underline{m}] = \frac{1}{\det(\boldsymbol{M})}\sum_{\ell\in\mathrm{part}(\boldsymbol{M}^T)} (\mathcal{Z}\varphi)\,[\underline{n}',\underline{\ell},\boldsymbol{N},\boldsymbol{M}]e^{j2\pi\langle\underline{\ell},\,\boldsymbol{M}^{-1}\underline{m}\rangle},$$

where the variable $\underline{n}'$ extends over a region $\mathrm{part}(\boldsymbol{N})$ and $\underline{m}$ over a region $\mathrm{part}(\boldsymbol{M})$.

By using the Zak transformation as defined above [see Eq. (5.13)], the condition (5.12) can be transformed into the following sum-of-products form (see Appendix D.2)

$$\sum_{\underline{v}\in\mathrm{part}(\boldsymbol{V}^T)} g_{\underline{i}\underline{v}}[\underline{n},\underline{\ell}]\gamma^*_{\underline{s}\underline{v}}[\underline{n},\underline{\ell}] = \frac{\det(\boldsymbol{DV})}{\det(\boldsymbol{CK})}\delta[\underline{i}-\underline{s}], \tag{5.14a}$$

where

$$g_{\underline{i}\underline{v}}[\underline{n},\underline{\ell}] = (\mathcal{Z}G_\Lambda)\,[\underline{n}+\boldsymbol{KCD}^{-T}\underline{i},\underline{\ell}-\boldsymbol{M}[\boldsymbol{V}^{-T}\underline{v}+(\boldsymbol{D}^{-1}\boldsymbol{R})^T\underline{i}];\boldsymbol{NS},\boldsymbol{M}] \tag{5.14b}$$

and

$$\gamma_{\underline{i}\,\underline{v}}[\underline{n},\underline{\ell}] = (\mathcal{Z}\Gamma_\Lambda)\,[\underline{n} + \boldsymbol{KCD}^{-T}\underline{i}, \underline{\ell} - \boldsymbol{M}[\boldsymbol{V}^{-T}\underline{v} + (\boldsymbol{D}^{-1}\boldsymbol{R})^T\underline{i}]; \boldsymbol{NS}, \boldsymbol{M}]. \tag{5.14c}$$

Here, the matrix $\boldsymbol{V} \in \mathbb{Z}^{d\times d}$, as introduced in Appendix D.2, is taken such that $\boldsymbol{\Psi V}$, $(\boldsymbol{D}^{-1}\boldsymbol{R})^T\boldsymbol{\Psi V}$ and $\boldsymbol{V}^{-1}\boldsymbol{M}$ are matrices containing only integers (recall $\boldsymbol{\Psi} = \boldsymbol{D}^T(\boldsymbol{CK})^{-1}\boldsymbol{NSV}$). Ideally, this matrix $\boldsymbol{V}$ has the smallest possible determinant; the determinant of $\boldsymbol{V}$ directly influences the number of elements in the sum-of-products form (5.14a). Note that $(\boldsymbol{D}^{-1}\boldsymbol{RM})^T$ is a matrix containing only integers. The vectors $\underline{i}$ and $\underline{v}$ extend over the regions $\mathrm{fund}(\boldsymbol{\Psi V})$ and $\mathrm{part}(\boldsymbol{V}^T)$, and the vectors $\underline{n}$ and $\underline{\ell}$ over regions $\mathrm{part}(\boldsymbol{CK})$ and $\mathrm{part}(\boldsymbol{M})$, respectively. Let the vectors $\underline{i}_0 \cdots \underline{i}_{\det(\boldsymbol{\Psi V})-1}$ and $\underline{v}_0 \cdots \underline{v}_{\det(\boldsymbol{V})-1}$ be the vectors corresponding to the points in the regions $\mathrm{fund}(\boldsymbol{\Psi V})$ and $\mathrm{part}(\boldsymbol{V}^T)$, respectively. Now we combine these functions $g_{\underline{i}\,\underline{v}}$ and $\gamma_{\underline{i}\,\underline{v}}$ into the matrices $\boldsymbol{G}$ and $\boldsymbol{\Gamma}$ of functions with elements

$$\boldsymbol{G}[\underline{n},\underline{\ell}] = \begin{bmatrix} g_{\underline{i}_0\underline{v}_0}[\underline{n},\underline{\ell}] & g_{\underline{i}_0\underline{v}_1}[\underline{n},\underline{\ell}] & \cdots & g_{\underline{i}_0,\underline{v}_{\det(\boldsymbol{V})-1}}[\underline{n},\underline{\ell}] \\ g_{\underline{i}_1\underline{v}_0}[\underline{n},\underline{\ell}] & g_{\underline{i}_1\underline{v}_1}[\underline{n},\underline{\ell}] & \cdots & g_{\underline{i}_1,\underline{v}_{\det(\boldsymbol{V})-1}}[\underline{n},\underline{\ell}] \\ \vdots & \vdots & \cdots & \vdots \\ g_{\underline{i}_{\det(\boldsymbol{\Psi V})-1},\underline{v}_0}[\underline{n},\underline{\ell}] & g_{\underline{i}_{\det(\boldsymbol{\Psi V})-1},\underline{v}_1}[\underline{n},\underline{\ell}] & \cdots & g_{\underline{i}_{\det(\boldsymbol{\Psi V})-1},\underline{v}_{\det(\boldsymbol{V})-1}}[\underline{n},\underline{\ell}] \end{bmatrix}$$

and

$$\boldsymbol{\Gamma}[\underline{n},\underline{\ell}] = \begin{bmatrix} \gamma_{\underline{i}_0\underline{v}_0}[\underline{n},\underline{\ell}] & \gamma_{\underline{i}_0\underline{v}_1}[\underline{n},\underline{\ell}] & \cdots & \gamma_{\underline{i}_0,\underline{v}_{\det(\boldsymbol{V})-1}}[\underline{n},\underline{\ell}] \\ \gamma_{\underline{i}_1\underline{v}_0}[\underline{n},\underline{\ell}] & \gamma_{\underline{i}_1\underline{v}_1}[\underline{n},\underline{\ell}] & \cdots & \gamma_{\underline{i}_1,\underline{v}_{\det(\boldsymbol{V})-1}}[\underline{n},\underline{\ell}] \\ \vdots & \vdots & \cdots & \vdots \\ \gamma_{\underline{i}_{\det(\boldsymbol{\Psi V})-1},\underline{v}_0}[\underline{n},\underline{\ell}] & \gamma_{\underline{i}_{\det(\boldsymbol{\Psi V})-1},\underline{v}_1}[\underline{n},\underline{\ell}] & \cdots & \gamma_{\underline{i}_{\det(\boldsymbol{\Psi V})-1},\underline{v}_{\det(\boldsymbol{V})-1}}[\underline{n},\underline{\ell}] \end{bmatrix},$$

respectively. With the help of these matrices $\boldsymbol{G}$ and $\boldsymbol{\Gamma}$, Eq. (5.14a) can now be expressed in the matrix product

$$\boldsymbol{G\Gamma}^* = \frac{\det(\boldsymbol{DV})}{\det(\boldsymbol{CK})}\boldsymbol{I}_{\det(\boldsymbol{\Psi V})}. \tag{5.15}$$

Note that, similar to the one-dimensional case, the matrix $\boldsymbol{\Gamma}^*$ is not a square matrix in the case of oversampling and does not have an inverse, but in general has a (non-unique) left inverse. The optimum solution in the sense of minimum ℓ_2-norm can be found with the help of the generalized (Moore-Penrose) inverse $\boldsymbol{\Gamma}^\dagger$ [see Eq. (2.24)]. The optimum solution $\boldsymbol{G}_{opt}$ then reads

$$\boldsymbol{G}_{opt} = \frac{\det(\boldsymbol{DV})}{\det(\boldsymbol{CK})}(\boldsymbol{\Gamma}^\dagger)^*,$$

which corresponds to the minimum ℓ_2-norm window $G_{\Lambda;opt}$. So, given Γ_Λ, we determine the matrix $\boldsymbol{\Gamma}$ by means of the Zak transformation (5.13). From that we can determine the matrix $\boldsymbol{G}_{opt}$. Then with the aid of $\boldsymbol{G}_{opt}$ we can determine the Zak transform $(\mathcal{Z}G_\Lambda)\,[\underline{n},\underline{\ell};\boldsymbol{NS},\boldsymbol{M}]$, and finally from that, the window $G_{\Lambda;opt}$. In Section 5.3, as an example, we calculate some windows $G_{\Lambda;opt}$ in the case of a non-separable lattice and a non-separable window Γ_Λ for the two-dimensional case by using the method as outlined above.

Suppose the signal Φ has the Gabor expansion (5.11). Then, by using the $2d$-dimensional Fourier expansion $\mathcal{F}_{dis}^{(2d)}$, defined as

$$(\mathcal{F}_{dis}^{(2d)}\tilde{A})[\underline{n},\underline{\ell};\boldsymbol{CK},\boldsymbol{M}] = \sum_{\underline{m}\in\text{part}(\boldsymbol{M})}\sum_{\underline{k}\in\text{part}(\boldsymbol{CK})}\tilde{A}_{\underline{m}\,\underline{k}}e^{-j2\pi(\underline{\ell}^T\boldsymbol{M}^{-1}\underline{m}\,-\,\underline{n}^T(\boldsymbol{CK})^{-1}\underline{k})}, \quad (5.16)$$

with inverse

$$\tilde{A}_{\underline{m}\,\underline{k}} = \frac{1}{\det(\boldsymbol{CKM})}\sum_{\underline{n}\in\text{part}((\boldsymbol{CK})^T)}\sum_{\underline{\ell}\in\text{part}(\boldsymbol{M}^T)}(\mathcal{F}_{dis}^{(2d)}\tilde{A})[\underline{n},\underline{\ell};\boldsymbol{CK},\boldsymbol{M}] \times e^{j2\pi(\underline{\ell}^T\boldsymbol{M}^{-1}\underline{m}\,-\,\underline{n}^T(\boldsymbol{CK})^{-1}\underline{k})},$$

it can be shown (see Appendix D.3) that the periodized Gabor transform (5.10) can be transformed into a sum-of-products form

$$(\mathcal{F}_{dis}^{(2d)}\tilde{A})[\underline{n},\underline{\ell}+\boldsymbol{MV}^{-T}\underline{v};\boldsymbol{CK},\boldsymbol{M}] = \det(\boldsymbol{CK})\sum_{\underline{i}\in\text{part}(\boldsymbol{\Psi V})}\gamma^*_{\underline{i}\,\underline{v}}[\underline{n},\underline{\ell}]\varphi_i[\underline{n},\underline{\ell}], \quad (5.17)$$

where

$$\varphi_i[\underline{n},\underline{\ell}] = (\mathcal{Z}\Phi)\,[\underline{n}+\boldsymbol{KCD}^{-T}\underline{i},\underline{\ell}-\boldsymbol{M}(\boldsymbol{D}^{-1}\boldsymbol{R})^T\underline{i};\boldsymbol{NSV},\boldsymbol{V}^{-1}\boldsymbol{M}],$$

and where the vectors $\underline{v}$, $\underline{n}$ and $\underline{\ell}$ extend over regions $\text{part}(\boldsymbol{V}^T)$, $\text{part}(\boldsymbol{CK})$ and $\text{part}(\boldsymbol{MV}^{-T})$, respectively. The Fourier expansion $(\mathcal{F}_{dis}^{(2d)}\tilde{A})$ is completely determined by the functions

$$a_{\underline{v}}[\underline{n},\underline{\ell}] = (\mathcal{F}_{dis}^{(2d)}\tilde{A})[\underline{n},\underline{\ell}+\boldsymbol{MV}^{-T}\underline{v},\boldsymbol{CK},\boldsymbol{M}].$$

The functions $a_{\underline{v}}$ can be combined into a column vector $\underline{a}$ of functions

$$\underline{a}[\underline{n},\underline{\ell}] = (a_{\underline{v}_0}[\underline{n},\underline{\ell}],a_{\underline{v}_1}[\underline{n},\underline{\ell}],\dots,a_{\underline{v}_{\det(\boldsymbol{V})-1}}[\underline{n},\underline{\ell})^T, \quad (5.18)$$

where $\underline{v}_0, \ldots, \underline{v}_{\det(\boldsymbol{V})-1}$ are the vectors in the region $\mathrm{part}(\boldsymbol{V}^T)$. Likewise, the functions $\varphi_{\underline{i}}$ can be combined into a column vector $\underline{\phi}$ of functions with elements

$$\underline{\phi}[\underline{n}, \underline{\ell}] = (\varphi_{\underline{i}_0}[\underline{n}, \underline{\ell}], \varphi_{\underline{i}_1}[\underline{n}, \underline{\ell}], \ldots, \varphi_{\underline{i}_{\det(\boldsymbol{\Psi V})-1}}[\underline{n}, \underline{\ell}])^T, \tag{5.19}$$

where $\underline{i}_0 \ldots \underline{i}_{\det(\boldsymbol{\Psi V})-1}$ are the vectors in a region $\mathrm{part}(\boldsymbol{\Psi V})$. With the help of the vectors $\underline{a}$ and $\underline{\phi}$ [see Eqs. (5.18) and (5.19), respectively], Eq. (5.17) can now be expressed in the matrix-vector product

$$\underline{a} = \det(\boldsymbol{CK})\, \boldsymbol{\Gamma}^* \underline{\phi}. \tag{5.20}$$

The relation (5.15) applied to an arbitrary vector $\underline{\phi}$ leads to the condition

$$\boldsymbol{G\Gamma}^* \underline{\phi} = \frac{\det(\boldsymbol{DV})}{\det(\boldsymbol{CK})}\, \underline{\phi}.$$

Substitution of Eq. (5.20) into the previous expression yields

$$\underline{\phi} = \frac{1}{\det(\boldsymbol{DV})}\, \boldsymbol{G}\underline{a}.$$

The result in the multi-dimensional case, looks very similar to the result obtained in the one-dimensional case (see section 4.2). Again, the sum-of-products forms can be written in matrix-matrix and matrix-vector products. The optimal window $G_{\Lambda;opt}$ in the sense of minimum ℓ_2-norm can be found with the Moore-Penrose inverse $\boldsymbol{\Gamma}^\dagger$ and the Zak transformation, the array of Gabor coefficients $\tilde{A}_{\underline{m}\,\underline{k}}$ can be found with the help of the Zak transformation and the Fourier transformation, and the reconstruction of the signal Φ (and therefore φ) can be obtained by the Zak transformation and Fourier transformation.

5.3 Two-dimensional non-separable Gabor scheme

To illustrate the concepts as presented in Sections 5.1 and 5.2, we consider the two-dimensional non-separable periodic Gabor scheme in more detail in this section. In Section 5.1, we found that the diagonal matrix $\boldsymbol{M}$ has to meet some conditions in order to periodize the Gabor scheme. In this section, we give explicit expressions for the two elements of this diagonal matrix $\boldsymbol{M}$. Furthermore, as an example, we calculate the windows G_Λ for a given separated dual window Γ_Λ and a non-separated dual window Γ_Λ, in both cases for a non-separable lattice Λ. The two-dimensional Gabor scheme can be used for image analysis, image compression, texture analysis and segmentation. For these applications, we refer to [28, 34, 36, 42, 46, 61, 62, 70, 77, 80–82].

In the two-dimensional case, the four-dimensional lattice Λ is generated by the lattice generator matrix $\boldsymbol{\Lambda} = \boldsymbol{UL}$ [see Eq. (5.2)], where

$$\boldsymbol{U} = \begin{bmatrix} \boldsymbol{NS} & \boldsymbol{0} \\ \boldsymbol{0} & (\boldsymbol{CK})^{-1} \end{bmatrix} \quad \text{and} \quad \boldsymbol{L} = \begin{bmatrix} \boldsymbol{I}_2 & \boldsymbol{0} \\ -\boldsymbol{R} & \boldsymbol{D} \end{bmatrix},$$

with

$$\boldsymbol{N} = \begin{bmatrix} N_0 & 0 \\ 0 & N_1 \end{bmatrix}, \quad \boldsymbol{S} = \begin{bmatrix} 1 & 0 \\ L_{21} & L_{22} \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} \frac{L_{33}L_{44}}{\gcd(L_{43}, L_{44})} & 0 \\ 0 & L_{44} \end{bmatrix},$$

$$\boldsymbol{K} = \begin{bmatrix} K_0 & 0 \\ 0 & K_1 \end{bmatrix}, \quad \boldsymbol{R} = \begin{bmatrix} L_{31} & L_{32} \\ L_{41} & L_{42} \end{bmatrix}, \quad \boldsymbol{D} = \begin{bmatrix} L_{33} & 0 \\ L_{43} & L_{44} \end{bmatrix}.$$

Note that

$$\boldsymbol{D}^{-1}\boldsymbol{C} = \frac{1}{L_{33}L_{44}} \begin{bmatrix} L_{44} & 0 \\ -L_{43} & L_{33} \end{bmatrix} \begin{bmatrix} \frac{L_{33}L_{44}}{\gcd(L_{43}, L_{44})} & 0 \\ 0 & L_{44} \end{bmatrix} = \begin{bmatrix} \frac{L_{44}}{\gcd(L_{43}, L_{44})} & 0 \\ \frac{-L_{43}}{\gcd(L_{43}, L_{44})} & 1 \end{bmatrix}$$

is a matrix containing only integers. Furthermore, we assume that the integers L_{21} and L_{22} are relatively prime, the integers L_{31}, L_{32} and L_{33} are relatively prime, and the integers L_{41}, L_{42}, L_{43} and L_{44} are relatively prime. In Section 5.1, we found that the diagonal matrix $\boldsymbol{M} = \operatorname{diag}(M_0, M_1)$ has to be chosen such that the matrices $\boldsymbol{C}^{-1}\boldsymbol{K}^{-1}\boldsymbol{NSM}$ and $\boldsymbol{D}^{-1}\boldsymbol{RM}$ are matrices containing only integers. In addition, $\boldsymbol{M}$ has also to fulfill condition (5.8). From

$$\boldsymbol{D}^{-1}\boldsymbol{RM} = \frac{1}{L_{33}L_{44}} \begin{bmatrix} L_{31}L_{44} & L_{32}L_{44} \\ L_{33}L_{41} - L_{31}L_{43} & L_{33}L_{42} - L_{32}L_{43} \end{bmatrix} \boldsymbol{M}$$

and

$$\boldsymbol{C}^{-1}\boldsymbol{K}^{-1}\boldsymbol{NSM} = \frac{1}{L_{33}L_{44}K_0K_1} \begin{bmatrix} \gcd(L_{43}, L_{44})K_1N_0 & 0 \\ L_{21}L_{33}K_0N_1 & L_{22}L_{33}K_0N_1 \end{bmatrix} \boldsymbol{M},$$

it follows that the integer M_0 is equal to

$$\begin{aligned} M_0 = \operatorname{lcm}\Bigg(& \frac{L_{33}}{\gcd(L_{31}, L_{33})}, \frac{L_{33}L_{44}}{\gcd(L_{33}L_{41} - L_{31}L_{43}, L_{33}L_{44})}, \\ & \frac{L_{33}L_{44}K_0}{\gcd(\gcd(L_{43}, L_{44})N_0, L_{33}L_{44}K_0)}, \frac{L_{44}K_1}{\gcd(L_{21}N_1, L_{44}K_1)} \Bigg) L_0 \quad (5.21) \\ = {} & \frac{L_{33}L_{44}K_0K_1}{c_0} L_0, \quad (5.22) \end{aligned}$$

with

$$c_0 = \gcd(L_{33}L_{44}K_0K_1, L_{21}L_{33}K_0N_1, \gcd(L_{43}, L_{44})K_1N_0, (L_{33}L_{41} - L_{31}L_{43})K_0K_1, L_{31}L_{44}K_0K_1).$$

The integer M_1 is equal to

$$M_1 = \mathrm{lcm}\left(\frac{L_{33}}{\gcd(L_{32}, L_{33})}, \frac{L_{33}L_{44}}{\gcd(L_{33}L_{42} - L_{32}L_{43}, L_{33}L_{44})}, \frac{L_{44}K_1}{\gcd(L_{22}N_1, L_{44}K_1)}\right)L_1 = \frac{L_{33}L_{44}K_1}{c_1}L_1, \tag{5.23}$$

with

$$c_1 = \gcd(L_{33}L_{44}K_1, L_{22}L_{33}N_1, (L_{33}L_{42} - L_{32}L_{43})K_1, L_{32}L_{44}K_1).$$

Here L_0 and L_1 are integers such that condition (5.8) is fulfilled. Note that these expressions for M_0 and M_1 are more complicated than in the one-dimensional case where we found $M = pLD$. This is due to the facts that r and D are relatively prime and $\boldsymbol{L}$ is much simpler in the one-dimensional case.

As an example, we calculate the corresponding optimal windows $G_{\Lambda;opt}$, by using the Zak transformation as outlined in Section 5.2, for a given truncated separated and non-separated Gaussian dual window Γ_Λ in the case of a non-separable lattice Λ. In the first example, we consider the (separated) truncated Gaussian dual window γ_Λ

$$\gamma_\Lambda[\underline{n}] = \begin{cases} e^{-\pi \underline{n}^T \underline{n}/16} & \text{if } n_0 \in \{-6 \dots 6\}, n_1 \in \{-6 \dots 6\}, \\ 0 & \text{otherwise,} \end{cases}$$

where the lattice Λ is generated by the lattice generator matrix $\boldsymbol{\Lambda} = \boldsymbol{U}\boldsymbol{L}$ with

$$\boldsymbol{L} = \left[\begin{array}{cc|cc} 1 & 0 & & \\ 0 & 1 & & \\ \hline -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{array}\right].$$

From this matrix it follows that $\boldsymbol{D} = \boldsymbol{C} = \mathrm{diag}(2, 2)$ and $\boldsymbol{R} = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$. We take $K_0 = K_1 = 4$, $N_0 = N_1 = 3$, $\boldsymbol{S} = \boldsymbol{I}_2$, and $L_0 = L_1 = 1$, i.e.,

$$\boldsymbol{U} = \left[\begin{array}{cc|cc} 3 & 0 & & \\ 0 & 3 & & \\ \hline & & 1/8 & 0 \\ & & 0 & 1/8 \end{array}\right],$$

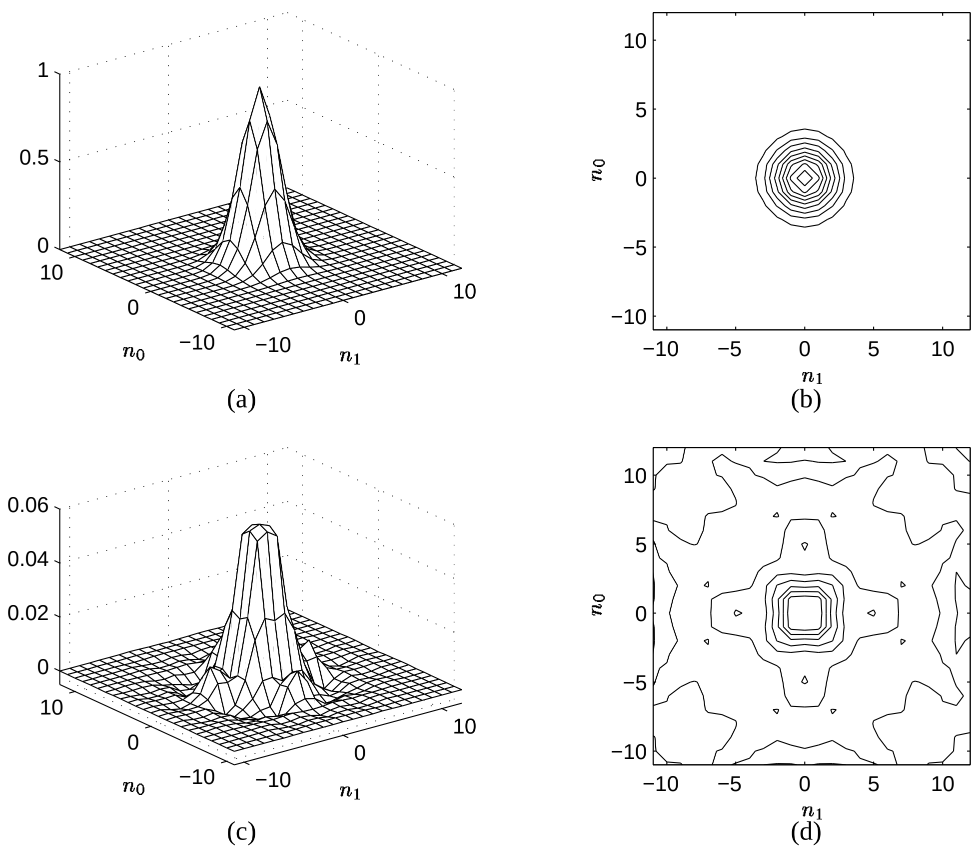

Figure 5.1: *(a) The Gaussian dual window Γ_Λ, (b) the contour plot of the Gaussian dual window Γ_Λ, (c) the window $G_{\Lambda;opt}$, and (d) the contour plot of the window $G_{\Lambda;opt}$.*

from which it follows that $M_0 = M_1 = 8$ [see Eqs. (5.22) and (5.23)]. The oversampling factor is equal to $1/\det(\boldsymbol{NS}(\boldsymbol{CK})^{-1}\boldsymbol{D}) = 16/9$. We shall assume that the support of the signal φ is such that condition (5.8) is fulfilled. The dual window Γ_Λ is periodic with respect to the regular partition of $\mathbb{Z}^2$ generated by $\mathrm{fund}(\boldsymbol{NM}) = \mathrm{fund}(\mathrm{diag}(24, 24))$. The dual window Γ_Λ is depicted in Fig. 5.1a and its contour plot is depicted in Fig. 5.1b. The (real valued) optimal window $G_{\Lambda;opt}$ and its contour plot are depicted in Fig. 5.1c and Fig. 5.1d, respectively.

In the last example, we construct the dual window Γ_Λ from a non-separated elliptical Gaussian h

$$h[\underline{n}] = e^{-\pi\langle \underline{n}, \boldsymbol{E}\underline{n}\rangle/1225}, \quad \text{with} \quad \boldsymbol{E} = \begin{bmatrix} 37 & -12 \\ -12 & 37 \end{bmatrix}.$$

The eigenvectors of $\boldsymbol{E}$ are $(1,1)/\sqrt{2}$ and $(-1,1)/\sqrt{2}$ with corresponding eigenvalues $1/25$ and $1/49$, respectively. We take a truncated version of the non-separated Gaussian h for the dual window γ_Λ

$$\gamma_\Lambda[\underline{n}] = \begin{cases} h[\underline{n}] & \text{if } h[\underline{n}] \geq 10^{-6}, \\ 0 & \text{otherwise.} \end{cases}$$

The non-separable lattice Λ is generated with the help of the matrix $\boldsymbol{L}$ of the form

$$\boldsymbol{L} = \left[\begin{array}{cc|cc} 1 & 0 & & \\ 0 & 1 & & \\ \hline 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 2 \end{array}\right],$$

i.e., $\boldsymbol{D} = \left[\begin{smallmatrix} 1 & 0 \\ -1 & 2 \end{smallmatrix}\right]$, $\boldsymbol{R} = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $\boldsymbol{C} = \mathrm{diag}(2,2)$. We take $K_0 = K_1 = 4$, $N_0 = N_1 = 3$, $\boldsymbol{S} = \left[\begin{smallmatrix} 1 & 0 \\ -1 & 2 \end{smallmatrix}\right]$, and $L_0 = L_1 = 2$, i.e.,

$$\boldsymbol{U} = \left[\begin{array}{cc|cc} 3 & 0 & & \\ -3 & 6 & & \\ \hline & & 1/8 & 0 \\ & & 0 & 1/8 \end{array}\right],$$

from which it follows that $M_0 = 16$ and $M_1 = 8$ [see Eqs. (5.22) and (5.23)]. The oversampling factor is equal to $1/\det(\boldsymbol{NS}(\boldsymbol{CK})^{-1}\boldsymbol{D}) = 16/9$. We shall assume that the support of φ is such that condition (5.8) is fulfilled. The dual window Γ_Λ is periodic with respect to the regular partition of $\mathbb{Z}^2$ generated by $\mathrm{fund}(\boldsymbol{NSM}) = \mathrm{fund}\left(\left[\begin{smallmatrix} 48 & 0 \\ -48 & 48 \end{smallmatrix}\right]\right)$. The dual window Γ_Λ and its contour plot are depicted in Fig. 5.2a and Fig. 5.2b, respectively. The (real valued) optimal window $G_{\Lambda;opt}$ and its contour plot are depicted in Fig. 5.2c and Fig. 5.2d, respectively.

5.4 Concluding remarks

In this chapter, the one-dimensional non-separable Gabor scheme for discrete-time signals is extended to the multi-dimensional non-separable Gabor scheme for possibly non-separable lattices and possibly non-separated windows. The extension to the multi-dimensional non-separable case is based on the one-dimensional case as presented in Chapter 4 and the multi-dimensional case as presented in Chapter 3; the non-separable lattice is obtained by the lattice generator matrix $\boldsymbol{\Lambda} = \boldsymbol{UL}$. The lattice generator matrix $\boldsymbol{\Lambda}$ leads to a shear representation of the shifted and modulated windows. This shear representation is not used explicitly; however, conceivably, it can be used to reshear a non-separable Gabor scheme into a Gabor scheme where the matrix $\boldsymbol{R}$ is a zero matrix, as will be elaborated in more detail in the last chapter.

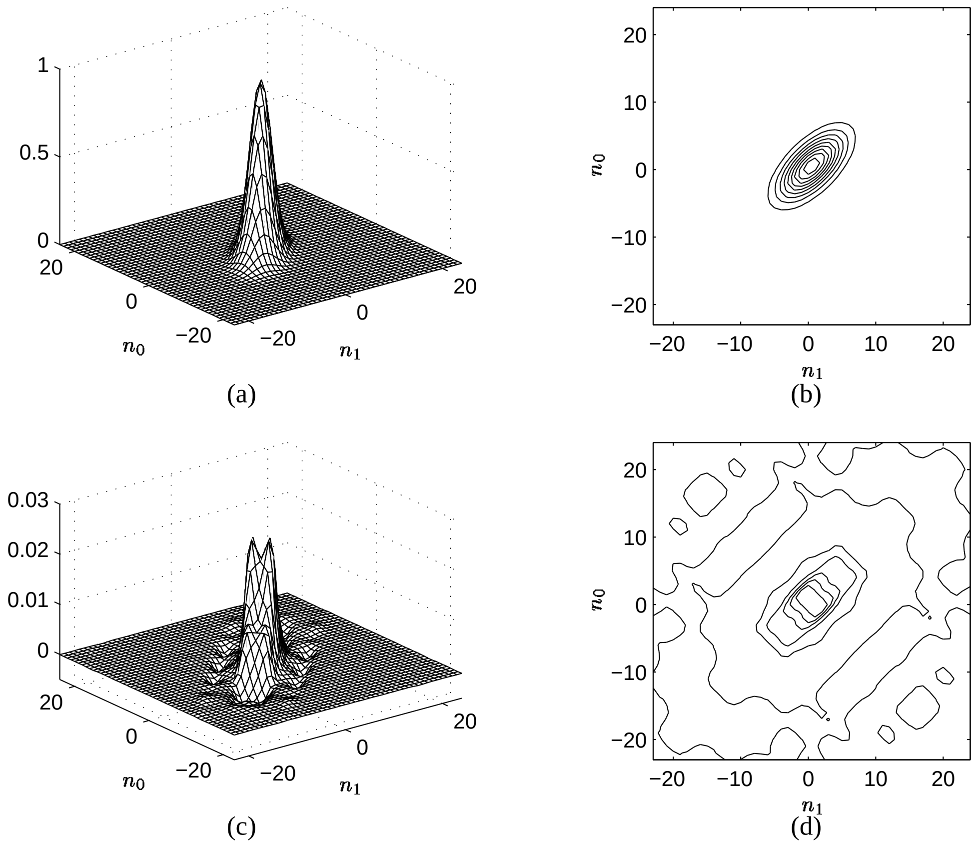

Figure 5.2: *(a) The Gaussian dual window Γ_Λ, (b) the contour plot of the Gaussian dual window Γ_Λ, (c) the window $G_{\Lambda;opt}$, and (d) the contour plot of the window $G_{\Lambda;opt}$.*

The connection between the multi-dimensional non-separable periodic Gabor scheme and the Zak transformation is shown. In order to show this connection, an alternative expression for the shifted and modulated windows is derived, similarly to the one-dimensional case; a separable lattice that refines the non-separable lattice, the inverse of the matrix $\boldsymbol{L}$, and the Poisson summation formula lead to the alternative expression. A special case, the two-dimensional Gabor scheme, is elaborated in more detail.

Chapter 6

Summary and conclusions

Gabor proposed to expand a signal into a series of properly shifted and modulated versions of a window. The Gabor expansion with a Gaussian window can be seen as a tiling of the time-frequency plane with circles on a rectangular (separable) lattice. A hexagonal or quincunx tiling of the time-frequency plane with these circles yields a higher packing density. Therefore it is expected that a quincunx Gabor scheme with the same oversampling is less sensitive to disturbance than the rectangular Gabor scheme. Put differently, it is expected that the set of shifted and modulated versions of the window corresponding to a quincunx lattice yields a tighter frame. To confirm this expectation, the ratios of the frame bounds in the case of a quincunx lattice and a rectangular lattice are compared. We showed that the frame in the quincunx case is indeed tighter. This motivated us to consider Gabor schemes on general non-separable lattices.

We considered non-separable Gabor schemes for continuous-time and discrete-time multi-dimensional signals. In the discrete-time case, we considered both the periodic Gabor scheme and the non-periodic Gabor scheme. The one-dimensional as well as the multi-dimensional case are discussed, where the one-dimensional Gabor schemes are treated for illustrative purposes. The separable lattice is extended to the non-separable lattice in a structured way; the lattice is described with the help of the lattice generator matrix $\boldsymbol{\Lambda}$, which can be factorized in a diagonal matrix $\boldsymbol{U}$ and a matrix $\boldsymbol{L}$ in the Hermite normal form. This lattice generator matrix provides a shear representation on the shifted and modulated windows, which shear representation then leads to a modification of the rectangular Gabor scheme and results in the Gabor scheme on a non-separable lattice.

The Zak transformation plays a central role in this thesis. The Zak transformation has proved its value in connection with the separable Gabor scheme. It can be used to calculate the dual window of a given window and the Gabor expansion coefficients, and to reconstruct the signal. We showed that the Zak transformation can also be used in the case of non-separable Gabor schemes. In order to show the connection between the Zak transformation and the non-separable Gabor scheme, we used an alternative expression for the shifted and modulated versions of the windows. This alternative expression follows from the separable lattice that refines the non-separable lattice. The set of shifted and modulated versions of the window, which corresponds to the non-separable lattice, is obtained by masking the separable lattice; a shifted and modulated version of the window in this separable lattice is multiplied by one if it belongs to the non-separable lattice and is multiplied by zero otherwise. The alternative expression provides a method to exploit the known expressions for the separable case within the scope of the Zak transformation. By using the Fourier transformation and the Zak transformation, we obtain sum-of-products forms in the case of a non-separable lattice, similar to the separable case. However, unlike the separable case, the number of elements in the sum-of-products forms now not only depends on the oversampling, but depends on the determinant of the matrix $\boldsymbol{L}$, as well.

The separable Gabor scheme for discrete-time one-dimensional signals can be implemented with the help of a uniform DFT filter bank, i.e., a filter bank where the filters are modulated versions of a prototype filter. We showed that a non-separable lattice is the union of $D = \det(\boldsymbol{L})$ separable lattices. As a consequence, the non-separable Gabor scheme can be implemented with D uniform DFT filter banks in parallel, where each filter bank has a different prototype filter. The complexity of this implementation of a filter bank corresponding to a non-separable Gabor scheme turns out to be comparable with the implementation of the separable one with the same oversampling.

In the last decade, many algorithms have been designed to calculate the dual window of a given window and to calculate the array of Gabor expansion coefficients, and to reconstruct the signal, in the case of a separable lattice. We showed that a one-dimensional non-separable Gabor scheme can be resheared into a separable Gabor scheme by multiplications by the quadratic phase term that can be associated with the shear. As a result, algorithms that are designed for the separable case can be re-used in the non-separable case to calculate the dual window of a given window and to calculate the array of Gabor expansion coefficients, and to reconstruct the signal. To use the shear operator in the case of the periodic Gabor scheme, an additional condition has to be fulfilled.

Another way to obtain a non-separable lattice is via a scaled rotation operation on the separable lattice. It is shown in the continuous-time case, that the fractional Fourier transform, which can be seen as a rotation in the time-frequency plane, translates a non-separable Gabor scheme into a separable one. As a consequence, the fractional Fourier transformation also provides a method to re-use algorithms that are designed for the separable case, explicitly.

In the case of a periodic Gabor scheme for discrete-time signals, the number of possible lattices is limited. We showed how many and which lattices are possible for a given period length. Periodic Gabor schemes for signals with a period length

Table 6.1: *Two properties of the shear operator.*

$\mathcal{Q}_{\boldsymbol{X}}\mathcal{T}_{\underline{\tau}} = e^{-j\langle \boldsymbol{X}\underline{\tau},\underline{\tau}\rangle}\mathcal{M}_{(\boldsymbol{X}+\boldsymbol{X}^T)\underline{\tau}}\mathcal{T}_{\underline{\tau}}\mathcal{Q}_{\boldsymbol{X}}$
$\mathcal{Q}_{\boldsymbol{X}}\mathcal{M}_{\underline{\omega}} = \mathcal{M}_{\underline{\omega}}\mathcal{Q}_{\boldsymbol{X}}$

having many divisors generate a large number of lattices.

6.1 Discussion

In the Sections 2.4 and 4.4, we showed that, in the one-dimensional case, a non-separable lattice can be resheared into a separable lattice. As a result, methods for the separable case to calculate the dual window and the array of Gabor expansion coefficients, and to reconstruct the signal, can be re-used in the non-separable case. In the multi-dimensional case it is much more complex to reshear a non-separable lattice into a separable lattice. In this case, the shear operator $\mathcal{Q}_{\boldsymbol{X}}$ should be defined by [cf. Eq. (2.35)]

$$(\mathcal{Q}_{\boldsymbol{X}}\varphi)(\underline{t}) = e^{j\langle \boldsymbol{X}\underline{t},\underline{t}\rangle}\varphi(\underline{t}), \quad \boldsymbol{X} \in \mathbb{R}^{d\times d} \quad \text{and symmetric}, \quad \varphi \in L_2(\mathbb{R}^d).$$

Applying the shear operator $\mathcal{Q}_{\boldsymbol{X}}$ to $g_{\Lambda;\underline{m}\underline{k}}$,

$$g_{\Lambda;\underline{m}\underline{k}} = \mathcal{M}_{-\boldsymbol{B\Omega C}^{-1}\boldsymbol{R}\underline{m}}\mathcal{M}_{\boldsymbol{B\Omega C}^{-1}\boldsymbol{D}\underline{k}}\mathcal{T}_{\boldsymbol{ATS}\underline{m}}g_\Lambda,$$

and using the properties that are tabulated in Table 6.1 yields

$$\mathcal{Q}_{\boldsymbol{X}}g_{\Lambda;\underline{m}\underline{k}} = e^{-j\langle \boldsymbol{X}\boldsymbol{H}_0\underline{m},\boldsymbol{H}_0\underline{m}\rangle}\mathcal{M}_{-\boldsymbol{H}_1\boldsymbol{R}\underline{m}}\mathcal{M}_{\boldsymbol{H}_1\boldsymbol{D}\underline{k}}\mathcal{M}_{2\boldsymbol{X}\boldsymbol{H}_0\underline{m}}\mathcal{T}_{\boldsymbol{H}_0\underline{m}}\mathcal{Q}_{\boldsymbol{X}}g_\Lambda,$$

where $\boldsymbol{H}_0 = \boldsymbol{ATS}$ and $\boldsymbol{H}_1 = \boldsymbol{B\Omega C}^{-1}$. From this expression, we see that we have to take $\boldsymbol{X} = \frac{1}{2}\boldsymbol{H}_1\boldsymbol{R}\boldsymbol{H}_0^{-1}$, in order to reshear a non-separable lattice Λ into a less non-separable lattice. Thus the matrix $\boldsymbol{H}_1\boldsymbol{R}\boldsymbol{H}_0^{-1}$ has to be symmetric. Let us, for simplicity, consider the two-dimensional case. Then with

$$\boldsymbol{R} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}, \quad \boldsymbol{H}_0 = \begin{bmatrix} \alpha_0 T_0 & 0 \\ \alpha_1 T_1 S_{21} & \alpha_1 T_1 S_{22} \end{bmatrix},$$

$$\boldsymbol{H}_1 = \text{diag}(\beta_0\Omega_0/C_{11}, \beta_1\Omega_1/C_{22}), \quad \text{and} \quad \boldsymbol{D} = \begin{bmatrix} D_{11} & 0 \\ D_{21} & D_{22}, \end{bmatrix}$$

we find from $(\boldsymbol{H}_1\boldsymbol{R}\boldsymbol{H}_0^{-1})^T = \boldsymbol{H}_1\boldsymbol{R}\boldsymbol{H}_0^{-1}$ the condition (recall $\boldsymbol{\Omega T} = 2\pi\boldsymbol{I}_2$)

$$R_{21}S_{22} - R_{22}S_{21} = \frac{\alpha_0\beta_0C_{22}}{\alpha_1\beta_1C_{11}}R_{12}. \tag{6.1}$$

However, the same lattice is generated if we add a matrix $\boldsymbol{DZ}$, with the integer matrix

$$\boldsymbol{Z} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix},$$

to the matrix $\boldsymbol{R}$. Then condition (6.1) becomes

$$\begin{aligned} &(R_{21} + D_{21}Z_{11} + D_{22}Z_{21})S_{22} - (R_{22} + D_{21}Z_{12} + D_{22}Z_{22})S_{21} = \\ &\frac{\beta_0\alpha_0 C_{22}}{\beta_1\alpha_1 C_{11}}(R_{12} + D_{11}Z_{12}), \end{aligned} \tag{6.2}$$

which cannot always be satisfied. Note that in the decomposable case ($\boldsymbol{R}$ is diagonal, $\boldsymbol{S} = \boldsymbol{I}_2$, and $\boldsymbol{D} = \boldsymbol{C}$ is diagonal), this condition can always be fulfilled ($Z_{12} = Z_{21} = 0$). For example, consider the lattice that is generated by the lattice generator matrix

$$\boldsymbol{UL} = \begin{bmatrix} \boldsymbol{H}_0 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{H}_1 \end{bmatrix} \begin{bmatrix} \boldsymbol{I}_d & \boldsymbol{0} \\ -\boldsymbol{R} & \boldsymbol{D} \end{bmatrix},$$

with

$$\boldsymbol{H}_0 = \operatorname{diag}(\alpha_0 T_0, \alpha_1 T_1), \quad \boldsymbol{H}_1 = \operatorname{diag}(\beta_0 \Omega_0/2, \beta_1 \Omega_1/2),$$

$$\boldsymbol{D} = 2\boldsymbol{I}_2, \quad \text{and} \quad \boldsymbol{R} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Then condition (6.2) becomes

$$(1 + 2Z_{21}) = \frac{\beta_0\alpha_0}{\beta_1\alpha_1}(1 + 2Z_{12}).$$

From this expression, we see that the possibility to reshear this non-separable lattice into a separable lattice heavily depends on the ratio $\beta_0\alpha_0/\beta_1\alpha_1$.

Without further details, we remark that the d-dimensional Fourier transformation in combination with a modified Hermite normal form [cf. Eq. (2.9) in the one-dimensional case] leads to a different condition. A general solution to reshear a d-dimensional non-separable Gabor scheme into a separable one is difficult to find. The discrete-time periodic case is even more difficult, since then additional conditions have to be fulfilled (cf. the one-dimensional case in Section 4.4).

Appendix A

Appendix to Chapter 2

A.1 Derivation of bi-orthogonality condition (2.19)

We assume that the set $\{g_{\Lambda;mk}|m,k \in \mathbb{Z}\}$ is a frame. In Gabor's signal expansion (2.13), we substitute from the Gabor transform (2.14)

$$\varphi(t) = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \varphi(t')\gamma^*(t'-m\alpha T)e^{jmr\beta\Omega t'/D}e^{-jk\beta\Omega t'}\,dt'\right] \times g(t-m\alpha T)e^{-jmr\beta\Omega t/D}e^{jk\beta\Omega t}.$$

Rearranging factors

$$\varphi(t) = \int_{-\infty}^{\infty} \varphi(t')\left[\sum_{m=-\infty}^{\infty} \gamma^*(t'-m\alpha T)g(t-m\alpha T)e^{jmr\beta\Omega(t'-t)/D}\right] \times \left[\sum_{k=-\infty}^{\infty} e^{-jk\beta\Omega(t'-t)}\right] dt'$$

and replacing the sum of exponentials by a sum of Dirac functions yields

$$\varphi(t) = \int_{-\infty}^{\infty} \varphi(t')\left[\sum_{m=-\infty}^{\infty} \gamma^*(t'-m\alpha T)g(t-m\alpha T)e^{jmr\beta\Omega(t'-t)/D}\right] \times \left[\frac{T}{\beta}\sum_{k=-\infty}^{\infty} \delta\left(t'-t+k\frac{T}{\beta}\right)\right] dt'.$$

Rearranging the factors again

$$\varphi(t) = \frac{T}{\beta}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} \delta\left(t'-t+k\frac{T}{\beta}\right)\varphi(t') \times \left[\sum_{m=-\infty}^{\infty} \gamma^*(t'-m\alpha T)g(t-m\alpha T)e^{jmr\beta\Omega(t'-t)/D}\right] dt'$$

and evaluating the integral yields

$$\varphi(t) = \frac{T}{\beta}\sum_{k=-\infty}^{\infty}\varphi\left(t - k\frac{T}{\beta}\right) \times \left[\sum_{m=-\infty}^{\infty} e^{-j2\pi mkr/D}\gamma^{*}\left(t - k\frac{T}{\beta} - m\alpha T\right) g(t - m\alpha T)\right].$$

This relationship holds for any signal $\varphi \in L_2(\mathbb{R})$ if and only if the $k = 0$ term in the summation over k is the only non-vanishing term, which immediately leads to the bi-orthogonality condition

$$\sum_{m=-\infty}^{\infty} e^{-j2\pi mkr/D}\gamma^{*}\left(t - k\frac{T}{\beta} - m\alpha T\right) g\left(t - m\alpha T\right) = \frac{\beta}{T}\delta[k],$$

where $\delta[k]$ is a Kronecker delta, with $\delta[0] = 1$ and $\delta[k] = 0$ for $k \neq 0$. This condition should hold for all $t \in \mathbb{R}$ and all $k \in \mathbb{Z}$.

A.2 Derivation of sum-of-products form (2.21)

The bi-orthogonal condition looks like

$$\sum_{m=-\infty}^{\infty} e^{-j2\pi mkr/D}\gamma^{*}\left(t - k\frac{T}{\beta} - m\alpha T\right) g\left(t - m\alpha T\right) = \frac{\beta}{T}\delta[k],$$

which should hold for all $t \in \mathbb{R}$ and all $k \in \mathbb{Z}$. In order to separate m and k in the exponent, we introduce the integer $f = D/\gcd(D, q)$. Replacing k with $\ell f q - i$, with $\ell \in \mathbb{Z}$ and $i = 0 \ldots fq - 1$, and using the identity $q = p\alpha\beta$ yields (recall that r and D are relatively prime)

$$\sum_{m=-\infty}^{\infty} e^{-j2\pi m(\ell f q - i)r/D}\gamma^{*}\left(t - (\ell f p + m)\alpha T + i\frac{T}{\beta}\right) g\left(t - m\alpha T\right) = \frac{\beta}{T}\delta[\ell f q - i].$$

Multiplying both sides by $\exp(j\ell f p z)$ with $z \in \mathbb{R}$ and summing over $\ell \in \mathbb{Z}$ yields

$$\sum_{m=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} e^{j2\pi mir/D}\gamma^{*}\left(t - (\ell f p + m)\alpha T + i\frac{T}{\beta}\right) g\left(t - m\alpha T\right) e^{j\ell f p z} = \frac{\beta}{T}\sum_{\ell=-\infty}^{\infty}\delta[\ell f q - i]e^{j\ell f p z}.$$

Using the property of the Kronecker delta (recall that $i = 0 \dots fq-1$)

$$\frac{\beta}{T}\sum_{\ell=-\infty}^{\infty}\delta[\ell fq - i]e^{j\ell fpz} = \frac{\beta}{T}\delta[i]$$

and the Poisson summation formula

$$\frac{1}{fp}\sum_{k=\langle fp\rangle} e^{j2\pi k\ell/fp} = \sum_{k=-\infty}^{\infty}\delta[\ell - kfp]$$

results in

$$\sum_{m=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} e^{j2\pi mir/D} \gamma^*\left(t-(\ell+m)\alpha T + i\frac{T}{\beta}\right) g\left(t-m\alpha T\right) e^{j\ell z}\sum_{k=\langle fp\rangle} e^{j2\pi k\ell/fp} = \frac{fp\beta}{T}\delta[i].$$

Rearranging terms

$$\sum_{k=\langle fp\rangle}\sum_{m=-\infty}^{\infty} e^{j2\pi mir/D} g\left(t-m\alpha T\right) e^{-j2\pi km/fp} e^{-jmz} \times \sum_{\ell=-\infty}^{\infty}\gamma^*\left(t-(\ell+m)\alpha T + i\frac{T}{\beta}\right) e^{j(\ell+m)z} e^{j2\pi k(\ell+m)/fp} = \frac{fp\beta}{T}\delta[i],$$

replacing $\ell + m$ with $-\ell$

$$\sum_{k=\langle fp\rangle}\sum_{m=-\infty}^{\infty} g\left(t-m\alpha T\right) e^{-j2\pi m(k/fp - ir/D)} e^{-jmz} \sum_{\ell=-\infty}^{\infty}\gamma^*\left(t+\ell\alpha T + i\frac{T}{\beta}\right) e^{-j\ell z} e^{-j2\pi k\ell/fp} = \frac{fp\beta}{T}\delta[i],$$

and changing the sign of m yields

$$\sum_{k=\langle fp\rangle}\sum_{m=-\infty}^{\infty} g\left(t+m\alpha T\right) e^{j2\pi m(k/fp - ir/D)} e^{jmz} \sum_{\ell=-\infty}^{\infty}\gamma^*\left(t+\ell\alpha T + i\frac{T}{\beta}\right) e^{-j\ell z} e^{-j2\pi k\ell/fp} = \frac{fp\beta}{T}\delta[i].$$

Replacing z with $-2\pi(y - nr/D)$,

$$\sum_{k=\langle fp\rangle} \sum_{m=-\infty}^{\infty} g\left(t + m\alpha T\right) e^{-j2\pi m(y - k/fp + (i-n)r/D)}$$
$$\times \sum_{\ell=-\infty}^{\infty} \gamma^{*}\left(t + \ell\alpha T + i\frac{T}{\beta}\right) e^{j2\pi\ell(y - k/fp - nr/D)} = \frac{fp\beta}{T}\delta[i],$$

and replacing i with $-i + n$ with $n = 0 \ldots fq - 1$ results in

$$\sum_{k=\langle fp\rangle} \sum_{m=-\infty}^{\infty} g\left(t + m\alpha T\right) e^{-j2\pi m(y - k/fp - ir/D)}$$
$$\times \sum_{\ell=-\infty}^{\infty} \gamma^{*}\left(t + \ell\alpha T + (n-i)\frac{T}{\beta}\right) e^{j2\pi\ell(y - k/fp - nr/D)} = \frac{fp\beta}{T}\delta[i-n].$$

Finally, replacing t with $(Dx + i)\frac{T}{\beta}$ results in the sum-of-products form

$$\sum_{k=\langle fp\rangle} \sum_{m=-\infty}^{\infty} g\left((Dx + i)\frac{T}{\beta} + m\alpha T\right) e^{-j2\pi m(y - k/fp - ir/D)}$$
$$\times \sum_{\ell=-\infty}^{\infty} \gamma^{*}\left((Dx + n)\frac{T}{\beta} + \ell\alpha T\right) e^{j2\pi\ell(y - k/fp - nr/D)} = \frac{fp\beta}{T}\delta[i-n].$$

A.3 Derivation of sum-of-products form (2.25)

In the Fourier transformed array $(\mathcal{F}^{(2)}\tilde{a})(x,y)$, we substitute from the Gabor transform (2.14)

$$(\mathcal{F}^{(2)}\tilde{a})(x,y) = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} e^{-j2\pi(my - kx)}$$
$$\times \left[\int \varphi(t)\frac{1}{D}\sum_{\ell=\langle D\rangle} e^{-j2\pi\ell(rm+k)/D}\gamma^{*}(t - m\alpha T)e^{-jk\beta\Omega t/D}\, dt\right],$$

and rearranging factors yields

$$(\mathcal{F}^{(2)}\tilde{a})(x,y) = \frac{1}{D}\sum_{m=-\infty}^{\infty} e^{-j2\pi my} \sum_{\ell=\langle D\rangle} e^{-j2\pi\ell rm/D}$$
$$\times \left[\int \varphi(t)\gamma^{*}(t - m\alpha T)\left\{\sum_{k=-\infty}^{\infty} e^{-jk\beta\Omega\left(t + \ell\frac{T}{\beta} - xD\frac{T}{\beta}\right)/D}\right\} dt\right].$$

Replacing the sum of exponentials by a sum of Dirac functions

$$(\mathcal{F}^{(2)}\tilde{a})(x,y) = \frac{1}{D}\sum_{m=-\infty}^{\infty} e^{-j2\pi my} \sum_{\ell=\langle D\rangle} e^{-j2\pi \ell rm/D}$$
$$\times \left[\int \varphi(t)\gamma^*(t-m\alpha T)\left\{\sum_{k=-\infty}^{\infty}\frac{DT}{\beta}\delta\left(t+\ell\frac{T}{\beta}-xD\frac{T}{\beta}-kD\frac{T}{\beta}\right)\right\}dt\right]$$

and rearranging factors again yields

$$(\mathcal{F}^{(2)}\tilde{a})(x,y) = \frac{T}{\beta}\sum_{m=-\infty}^{\infty} e^{-j2\pi my} \sum_{\ell=\langle D\rangle} e^{-j2\pi \ell rm/D}$$
$$\left[\sum_{k=-\infty}^{\infty}\int \varphi(t)\gamma^*(t-m\alpha T)\delta\left(t+\ell\frac{T}{\beta}-xD\frac{T}{\beta}-kD\frac{T}{\beta}\right)dt\right].$$

Evaluating the integral and replacing ℓ in the exponential by $\ell - kD$ (note that $\exp(-j2\pi(\ell-kD)rm/D) = \exp(-j2\pi\ell rm/D)$ for all $k \in \mathbb{Z}$) yields

$$(\mathcal{F}^{(2)}\tilde{a})(x,y) = \frac{T}{\beta}\sum_{m=-\infty}^{\infty} e^{-j2\pi my}\left[\sum_{k=-\infty}^{\infty}\sum_{\ell=\langle D\rangle} e^{-j2\pi(\ell-kD)rm/D}\right.$$
$$\left.\times\,\varphi\left(xD\frac{T}{\beta}-(\ell-kD)\frac{T}{\beta}\right)\gamma^*\left(xD\frac{T}{\beta}-(\ell-kD)\frac{T}{\beta}-m\alpha T\right)\right].$$

Replacing the double summation over k and ℓ by a single summation over k and using the identity $\beta = q/p\alpha$ yields

$$(\mathcal{F}^{(2)}\tilde{a})(x,y) = \frac{T}{\beta}\sum_{m=-\infty}^{\infty} e^{-j2\pi my}\left[\sum_{k=-\infty}^{\infty} e^{-j2\pi mkr/D}\right.$$
$$\left.\times\,\varphi\left(xD\frac{T}{\beta}-k\frac{p}{q}\alpha T\right)\gamma^*\left(xD\frac{T}{\beta}-k\frac{p}{q}\alpha T-m\alpha T\right)\right].$$

Replacing the summation over k by a double summation through the substitution $k = -fq\ell - i$, where $\ell \in \mathbb{Z}$ and i extends over an interval of length fq,

$$(\mathcal{F}^{(2)}\tilde{a})(x,y) = \frac{T}{\beta}\sum_{m=-\infty}^{\infty} e^{-j2\pi my}\left[\sum_{\ell=-\infty}^{\infty}\sum_{i=\langle fq\rangle} e^{j2\pi mir/D}\right.$$
$$\left.\times\,\varphi\left((xD+i)\frac{T}{\beta}+\ell fp\alpha T\right)\gamma^*\left((xD+i)\frac{T}{\beta}+(\ell fp-m)\alpha T\right)\right]$$

and rearranging factors again yields

$$(\mathcal{F}^{(2)}\tilde{a})(x,y) = \frac{T}{\beta}\sum_{i=\langle fq\rangle}\left[\sum_{\ell=-\infty}^{\infty}\varphi\left((xD+i)\frac{T}{\beta}+\ell\alpha fpT\right)e^{-j2\pi\ell fp(y-ir/D)}\right]$$
$$\times\left[\sum_{m=-\infty}^{\infty}\gamma\left((xD+i)\frac{T}{\beta}+(\ell fp-m)\alpha T\right)e^{-j2\pi(\ell fp-m)(y-ir/D)}\right]^{*}.$$

Substitution k for $\ell fp - m$,

$$(\mathcal{F}^{(2)}\tilde{a})(x,y) = \frac{T}{\beta}\sum_{i=\langle fq\rangle}\left[\sum_{\ell=-\infty}^{\infty}\varphi\left((xD+i)\frac{T}{\beta}+\ell\alpha fpT\right)e^{-j2\pi\ell fp(y-ir/D)}\right]$$
$$\times\left[\sum_{k=-\infty}^{\infty}\gamma\left((xD+i)\frac{T}{\beta}+k\alpha T\right)e^{-j2\pi k(y-ir/D)}\right]^{*},$$

and using the definition of the Zak transformation (1.6) leads to

$$(\mathcal{F}^{(2)}\tilde{a})(x,y) = \frac{T}{\beta}\sum_{i=\langle fq\rangle}(\mathcal{Z}\varphi)\left((Dx+i)\frac{T}{\beta},\left[y-\frac{ir}{D}\right]\frac{\Omega}{\alpha};\alpha fpT\right)$$
$$\times(\mathcal{Z}\gamma)^{*}\left((Dx+i)\frac{T}{\beta},\left[y-\frac{ir}{D}\right]\frac{\Omega}{\alpha};\alpha T\right).$$

Finally, replacing y by $y - k/fp$ and using the periodicity of the Zak transform $(\mathcal{Z}\varphi)\,(DxT/\beta, y\Omega/\alpha; fp\alpha T)$ leads to the result

$$(\mathcal{F}^{(2)}\tilde{a})\left(x,y-\frac{k}{fp}\right) = \frac{T}{\beta}\sum_{i=\langle fq\rangle}(\mathcal{Z}\varphi)\left((Dx+i)\frac{T}{\beta},\left[y-\frac{ir}{D}\right]\frac{\Omega}{\alpha};\alpha fpT\right)$$
$$\times(\mathcal{Z}\gamma)^{*}\left((Dx+i)\frac{T}{\beta},\left[y-\frac{k}{fp}-\frac{ir}{D}\right]\frac{\Omega}{\alpha};\alpha T\right).$$

A.4 Derivation of dual window (2.46)

We use the inversion formula of the Zak transformation (1.7) to calculate the sheared window $\mathcal{Q}_{\omega_{a,0}}\gamma$ [see Eq. (2.44)]

$$2^{\frac{1}{4}}v(t/\alpha T)^{2}(\mathcal{Q}_{\omega_{a,0}}\gamma)(t+m\alpha T) = \frac{1}{\pi\theta_2\theta_3\theta_4}\int_{\frac{2\pi}{\alpha T}}\left[\sum_{m'=-\infty}^{\infty}(-1)^{m'}c_{|m'|}\right.$$
$$\left.\times\, e^{-j2\pi m'[\alpha^2+j(r/D)](t/\alpha T)}e^{-jm'\omega\alpha T}\right]e^{jm\omega\alpha T}\,d\omega,$$

and rearranging factors yields

$$2^{\frac{1}{4}}v^{(t/\alpha T)^2}(\mathcal{Q}_{\omega_{a,0}}\gamma)(t+m\alpha T)=\frac{1}{\pi\theta_2\theta_3\theta_4}$$
$$\times\sum_{m'=-\infty}^{\infty}(-1)^{m'}c_{|m'|}e^{2\pi m'[\alpha^2+j(r/D)](t/\alpha T)}\left[\int_{\frac{2\pi}{\alpha T}}e^{-j(m'-m)\omega\alpha T}\,d\omega\right].$$

Since the collection $\{\exp(jk\omega\alpha T)|k\in\mathbb{Z}\}$ is an orthonormal basis in the Hilbert space of αT periodic functions, this expression is equivalent to

$$2^{\frac{1}{4}}v^{(t/\alpha T)^2}(\mathcal{Q}_{\omega_{a,0}}\gamma)(t+m\alpha T)=\frac{2}{\theta_2\theta_3\theta_4\alpha T}(-1)^m c_{|m|}v^{-2m(t/\alpha T)},$$

with $v=\exp(-\pi(\alpha^2+jr/D))$. Substituting $\tau=t+m_t\alpha T$ with m_t an integer such that $-\frac{1}{2}\alpha T<t<\frac{1}{2}\alpha T$,

$$\alpha T2^{\frac{1}{4}}v^{[(\tau-m_t\alpha T)/\alpha T]^2}(\mathcal{Q}_{\omega_{a,0}}\gamma)(\tau)=$$
$$\frac{2}{\theta_2\theta_3\theta_4}(-1)^{m_t}c_{|m_t|}v^{-2m_t[(\tau-m_t\alpha T)/\alpha T]},$$

and multiplying both sides by $v^{2\tau m_t/\alpha T-m_t^2}$ and using the expression for c_{m_t} [see Eq. (2.45)] yields

$$\alpha T2^{\frac{1}{4}}v^{(\tau/\alpha T)^2}(\mathcal{Q}_{\omega_{a,0}}\gamma)(\tau)=$$
$$\frac{2}{\theta_2\theta_3\theta_4}(-1)^{m_t}\sum_{n=0}^{\infty}(-1)^n\left[v^{(n+\frac{1}{2})(2|m_t|+n+\frac{1}{2})}\right]v^{m_t^2}.$$

From this expression we see that $\mathcal{Q}_{\omega_{a,0}}\gamma$ is an even function, i.e., $(\mathcal{Q}_{\omega_{a,0}}\gamma)(\tau)=(\mathcal{Q}_{\omega_{a,0}}\gamma)(-\tau)$. So by redefining the integer m_t as an integer such that $(m_t-\frac{1}{2}\alpha T)<|t|<(m_t+\frac{1}{2}\alpha T)$, with $-\frac{1}{2}\alpha T<t<\frac{1}{2}\alpha T$, we get the equivalent expression

$$\alpha T2^{\frac{1}{4}}v^{(\tau/\alpha T)^2}(\mathcal{Q}_{\omega_{a,0}}\gamma)(\tau)=$$
$$\frac{2}{\theta_2\theta_3\theta_4}(-1)^{m_t}\sum_{n=0}^{\infty}(-1)^n\left[v^{(n+\frac{1}{2})(2m_t+n+\frac{1}{2})}\right]v^{m_t^2}.$$

Rearranging factors,

$$\alpha T2^{\frac{1}{4}}v^{(\tau/\alpha T)^2}(\mathcal{Q}_{\omega_{a,0}}\gamma)(\tau)=\frac{2}{\theta_2\theta_3\theta_4}\sum_{n=0}^{\infty}(-1)^{n+m}\left[v^{(n+m_t+\frac{1}{2})^2}\right],$$

and replacing n by $-m_t + u$ results in

$$\alpha T 2^{\frac{1}{4}} v^{(\tau/\alpha T)^2} (\mathcal{Q}_{\omega_{a,0}}\gamma)(\tau) = \frac{2}{\theta_2\theta_3\theta_4} \sum_{u=m_t}^{\infty} (-1)^u \left[v^{(u+\frac{1}{2})^2} \right].$$

And finally, multiplying both sides by $2^{-\frac{1}{4}} v^{-(\tau/\alpha T)^2} \exp(-jr\beta\Omega\tau^2/2\alpha T D)/\alpha T$ results in

$$\gamma(\tau) = \frac{2^{3/4}}{\theta_2\theta_3\theta_4\alpha T} e^{-jr\beta\Omega\tau^2/2\alpha T D} v^{-(\tau/\alpha T)^2} \sum_{u=m_t}^{\infty} (-1)^u \left[v^{(u+\frac{1}{2})^2} \right],$$

where m_t is an integer such that $(m_t - \frac{1}{2}\alpha T) < |t| < (m_t + \frac{1}{2}\alpha T)$ and $-\frac{1}{2}\alpha T < t < \frac{1}{2}\alpha T$.

Appendix B

Appendix to Chapter 3

B.1 Derivation of bi-orthogonality condition (3.10)

We assume that the set $\{g_{\Lambda;\underline{m}\underline{k}} | \underline{m}, \underline{k} \in \mathbb{Z}^d\}$ is a frame. In Gabor's expansion (3.8), we substitute from the Gabor transform (3.9)

$$\begin{aligned}\varphi(\underline{t}) = & \frac{1}{\det(\boldsymbol{D})} \sum_{\underline{m}\in\mathbb{Z}^d} \sum_{\underline{k}\in\mathbb{Z}^d} \sum_{\underline{i}\in\mathrm{part}(\boldsymbol{D}^T)} e^{j2\pi \underline{i}^T \boldsymbol{D}^{-1}[\boldsymbol{R}\ \ \boldsymbol{I}_d]\left[\begin{smallmatrix}\underline{m}\\ \underline{k}\end{smallmatrix}\right]} \\ & \times \int_{-\infty}^{\infty} \varphi(\underline{t}')\gamma^*(\underline{t}' - \boldsymbol{ATS}\underline{m}) e^{-j\underline{t}'^T \boldsymbol{B\Omega C}^{-1}\underline{k}}\, d\underline{t}' \\ & \times g(\underline{t} - \boldsymbol{ATS}\underline{m}) e^{j\underline{t}^T \boldsymbol{B\Omega C}^{-1}\underline{k}}.\end{aligned}$$

Rearranging factors,

$$\begin{aligned}\varphi(\underline{t}) = & \frac{1}{\det(\boldsymbol{D})} \sum_{\underline{m}\in\mathbb{Z}^d} \sum_{\underline{i}\in\mathrm{part}(\boldsymbol{D}^T)} e^{j2\pi \underline{i}^T \boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}} \\ & \times \int_{-\infty}^{\infty} \varphi(\underline{t}')\gamma^*(\underline{t}' - \boldsymbol{ATS}\underline{m}) g(\underline{t} - \boldsymbol{ATS}\underline{m}) \\ & \times \left[\sum_{\underline{k}\in\mathbb{Z}^d} e^{-j(\underline{t}' - \underline{t} - \boldsymbol{B}^{-1}\boldsymbol{TCD}^{-T}\underline{i})^T \boldsymbol{B\Omega C}^{-1}\underline{k}}\right] d\underline{t}',\end{aligned}$$

and replacing the sum of exponentials by a sum of Dirac functions yields

$$\begin{aligned}\varphi(\underline{t}) =& \frac{1}{\det(\boldsymbol{D})} \sum_{\underline{m}\in\mathbb{Z}^d} \sum_{\underline{i}\in\text{part}(\boldsymbol{D}^T)} e^{j2\pi \underline{i}^T \boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}} \\ &\times \int_{-\infty}^{\infty} \varphi(\underline{t}')\gamma^*(\underline{t}' - \boldsymbol{ATS}\underline{m}) g(\underline{t} - \boldsymbol{ATS}\underline{m}) \\ &\times \left[\det(\boldsymbol{B}^{-1}\boldsymbol{TC}) \sum_{\underline{k}\in\mathbb{Z}^d} \delta\left(\underline{t}' - \underline{t} - \boldsymbol{B}^{-1}\boldsymbol{TCD}^{-T}\underline{i} - \boldsymbol{B}^{-1}\boldsymbol{TC}\underline{k}\right)\right] d\underline{t}'.\end{aligned}$$

Rearranging factors again results in

$$\begin{aligned}\varphi(\underline{t}) =& \frac{\det(\boldsymbol{CT})}{\det(\boldsymbol{BD})} \sum_{\underline{m}\in\mathbb{Z}^d} g(\underline{t} - \boldsymbol{ATS}\underline{m}) \sum_{\underline{k}\in\mathbb{Z}^d} \sum_{\underline{i}\in\text{part}(\boldsymbol{D}^T)} e^{j2\pi \underline{i}^T \boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}} \\ &\times \int_{-\infty}^{\infty} \delta\left(\underline{t}' - t - \boldsymbol{B}^{-1}\boldsymbol{TCD}^{-T}\underline{i} - \boldsymbol{B}^{-1}\boldsymbol{TC}\underline{k}\right) \varphi(\underline{t}')\gamma^*(\underline{t}' - \boldsymbol{ATS}\underline{m})\, d\underline{t}'.\end{aligned}$$

Evaluating the integral and replacing $\underline{i}$ in the exponential by $\underline{i} + \boldsymbol{D}^T\underline{k}$ (note that $\exp(j2\pi\underline{i}^T\boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}) = \exp(j2\pi(\underline{i} + \boldsymbol{D}^T\underline{k})^T\boldsymbol{D}^{-1}\boldsymbol{R}\underline{m})$ for all $\underline{k} \in \mathbb{Z}^d$) yields

$$\begin{aligned}\varphi(\underline{t}) =& \frac{\det(\boldsymbol{CT})}{\det(\boldsymbol{BD})} \sum_{\underline{m}\in\mathbb{Z}^d} g(\underline{t} - \boldsymbol{ATS}\underline{m}) \sum_{\underline{k}\in\mathbb{Z}^d} \sum_{\underline{i}\in\text{part}(\boldsymbol{D}^T)} e^{j2\pi (\underline{i} + \boldsymbol{D}^T\underline{k})^T \boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}} \\ &\times \varphi\left(\underline{t} + \boldsymbol{B}^{-1}\boldsymbol{TCD}^{-T}[\underline{i} + \boldsymbol{D}^T\underline{k}]\right) \\ &\times \gamma^*\left(\underline{t} + \boldsymbol{B}^{-1}\boldsymbol{TCD}^{-T}[\underline{i} + \boldsymbol{D}^T\underline{k}] - \boldsymbol{ATS}\underline{m}\right).\end{aligned}$$

Replacing the double summation over $\underline{k}$ and $\underline{i}$ by a single summation over $\underline{k}$,

$$\begin{aligned}\varphi(\underline{t}) =& \frac{\det(\boldsymbol{CT})}{\det(\boldsymbol{BD})} \sum_{\underline{m}\in\mathbb{Z}^d} g(\underline{t} - \boldsymbol{ATS}\underline{m}) \sum_{\underline{k}\in\mathbb{Z}^d} e^{j2\pi \underline{k}^T \boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}} \\ &\times \varphi\left(\underline{t} + \boldsymbol{B}^{-1}\boldsymbol{TCD}^{-T}\underline{k}\right) \gamma^*\left(\underline{t} + \boldsymbol{B}^{-1}\boldsymbol{TCD}^{-T}\underline{k} - \boldsymbol{ATS}\underline{m}\right),\end{aligned}$$

and rearranging factors again yields

$$\begin{aligned}\varphi(\underline{t}) =& \frac{\det(\boldsymbol{CT})}{\det(\boldsymbol{BD})} \sum_{\underline{k}\in\mathbb{Z}^d} \varphi\left(\underline{t} + \boldsymbol{B}^{-1}\boldsymbol{TCD}^{-T}\underline{k}\right) \sum_{\underline{m}\in\mathbb{Z}^d} e^{j2\pi \underline{k}^T \boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}} \\ &\times g(\underline{t} - \boldsymbol{ATS}\underline{m})\gamma^*\left(\underline{t} + \boldsymbol{B}^{-1}\boldsymbol{TCD}^{-T}\underline{k} - \boldsymbol{ATS}\underline{m}\right).\end{aligned}$$

This relationship holds for any signal $\varphi \in L_2(\mathbb{R}^d)$ if and only if the $\underline{k} = \underline{0}$ term in the summation over $\underline{k}$ is the only non-vanishing term, which immediately leads to the bi-orthogonality condition (with the sign of $\underline{k}$ changed)

$$\sum_{\underline{m}\in\mathbb{Z}^d} e^{-j2\pi\underline{k}^T\boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}} g(\underline{t} - \boldsymbol{ATS}\underline{m})\gamma^*\left(\underline{t} - \boldsymbol{B}^{-1}\boldsymbol{TCD}^{-T}\underline{k} - \boldsymbol{ATS}\underline{m}\right)$$
$$= \frac{\det(\boldsymbol{BD})}{\det(\boldsymbol{CT})}\delta[\underline{k}].$$

This condition should hold for all $\underline{t} \in \mathbb{R}^d$ and all $\underline{k} \in \mathbb{Z}^d$.

B.2 Derivation of sum-of-products form (3.12a)

The bi-orthogonality condition looks like

$$\sum_{\underline{m}\in\mathbb{Z}^d} e^{-j2\pi\underline{k}^T\boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}} g(\underline{t} - \boldsymbol{ATS}\underline{m})\gamma^*\left(\underline{t} - \boldsymbol{B}^{-1}\boldsymbol{TCD}^{-T}\underline{k} - \boldsymbol{ATS}\underline{m}\right)$$
$$= \frac{\det(\boldsymbol{BD})}{\det(\boldsymbol{CT})}\delta[\underline{k}],$$

which should hold for all $\underline{t} \in \mathbb{R}^d$ and all $\underline{k} \in \mathbb{Z}^d$. We replace $\underline{k}$ by $\boldsymbol{\Psi V}\underline{\ell} - \underline{i}$, with $\boldsymbol{\Psi} = \boldsymbol{D}^T\boldsymbol{C}^{-1}\boldsymbol{ABS}$, and where $\underline{\ell} \in \mathbb{Z}^d$ and $\underline{i} \in \text{fund}(\boldsymbol{\Psi V})$. Here $\boldsymbol{V}$ is a matrix containing only integers, such that $\boldsymbol{\Psi V}$ and $\boldsymbol{R}^T\boldsymbol{C}^{-1}\boldsymbol{ABSV}$ are matrices containing only integers. Compare the matrix $\boldsymbol{V}$ with fp in the one-dimensional case. In fact, in the decomposable case ($\boldsymbol{R}$ a diagonal matrix, $\boldsymbol{C} = \boldsymbol{D}$ a diagonal matrix and $\boldsymbol{S} = \boldsymbol{I}_2$), $\boldsymbol{V}$ reduces to a diagonal matrix with integers such as fp on the diagonal. Thus, replacing $\underline{k}$ by $\boldsymbol{\Psi V}\underline{\ell} - \underline{i}$ yields

$$\sum_{\underline{m}\in\mathbb{Z}^d} e^{j2\pi\underline{i}^T\boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}}\gamma^*\left(\underline{t} + \boldsymbol{B}^{-1}\boldsymbol{TCD}^{-T}\underline{i} - \boldsymbol{ATS}[\boldsymbol{V}\underline{\ell} + \underline{m}]\right)$$
$$\times\, g(\underline{t} - \boldsymbol{ATS}\underline{m}) = \frac{\det(\boldsymbol{BD})}{\det(\boldsymbol{CT})}\delta[\boldsymbol{\Psi V}\underline{\ell} - \underline{i}].$$

Recall that $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{T}$ are diagonal matrices, and therefore we have $\boldsymbol{B}^{-1}\boldsymbol{TABS} = \boldsymbol{ATS}$. Multiplying both sides by $\exp(j2\pi\underline{z}^T\boldsymbol{V}\underline{\ell})$ and summing over $\underline{\ell}$ yields

$$\sum_{\underline{m}\in\mathbb{Z}^d}\sum_{\underline{\ell}\in\mathbb{Z}^d} e^{j2\pi\underline{i}^T\boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}}\gamma^*\left(\underline{t} + \boldsymbol{B}^{-1}\boldsymbol{TCD}^{-T}\underline{i} - \boldsymbol{ATS}[\boldsymbol{V}\underline{\ell} + \underline{m}]\right)$$
$$\times\, g(\underline{t} - \boldsymbol{ATS}\underline{m})e^{j2\pi\underline{z}^T\boldsymbol{V}\underline{\ell}} = \frac{\det(\boldsymbol{BD})}{\det(\boldsymbol{CT})}\sum_{\underline{\ell}\in\mathbb{Z}^d}\delta[\boldsymbol{\Psi V}\underline{\ell} - \underline{i}]e^{j2\pi\underline{z}^T\boldsymbol{V}\underline{\ell}}.$$

Since the variable $\underline{i}$ extends over a region $\text{fund}(\boldsymbol{\Psi V})$, this expression reduces to

$$\sum_{\underline{m}\in\mathbb{Z}^d}\sum_{\underline{\ell}\in\mathbb{Z}^d} e^{j2\pi\underline{i}^T\boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}}\gamma^*\left(\underline{t}+\boldsymbol{B}^{-1}\boldsymbol{TCD}^{-T}\underline{i}-\boldsymbol{ATS}[\boldsymbol{V}\underline{\ell}+\underline{m}]\right)$$
$$\times\, g(\underline{t}-\boldsymbol{ATS}\underline{m})e^{j2\pi\underline{z}^T\boldsymbol{V}\underline{\ell}} = \frac{\det(\boldsymbol{BD})}{\det(\boldsymbol{CT})}\delta[\underline{i}].$$

Using the Poisson summation formula

$$\sum_{\underline{v}\in\text{part}(\boldsymbol{V}^T)} e^{j2\pi\underline{v}^T\boldsymbol{V}^{-1}\underline{\ell}} = \det(\boldsymbol{V})\sum_{\underline{v}\in\mathbb{Z}^d}\delta[\underline{\ell}-\boldsymbol{V}\underline{v}]$$

results in

$$\sum_{\underline{m}\in\mathbb{Z}^d}\sum_{\underline{\ell}\in\mathbb{Z}^d} e^{j2\pi\underline{i}^T\boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}}\gamma^*\left(\underline{t}+\boldsymbol{B}^{-1}\boldsymbol{TCD}^{-T}\underline{i}-\boldsymbol{ATS}[\underline{\ell}+\underline{m}]\right)$$
$$\times\, g(\underline{t}-\boldsymbol{ATS}\underline{m})e^{j2\pi\underline{z}^T\underline{\ell}}\sum_{\underline{v}\in\text{part}(\boldsymbol{V}^T)} e^{j2\pi\underline{v}^T\boldsymbol{V}^{-1}\underline{\ell}} = \frac{\det(\boldsymbol{BDV})}{\det(\boldsymbol{CT})}\delta[\underline{i}].$$

Rearranging factors,

$$\sum_{\underline{v}\in\text{part}(\boldsymbol{V}^T)}\sum_{\underline{m}\in\mathbb{Z}^d} g(\underline{t}-\boldsymbol{ATS}\underline{m})e^{j2\pi\underline{i}^T\boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}}\sum_{\underline{\ell}\in\mathbb{Z}^d} e^{j2\pi\underline{v}^T\boldsymbol{V}^{-1}\underline{\ell}}e^{j2\pi\underline{z}^T\underline{\ell}}$$
$$\times\,\gamma^*\left(\underline{t}+\boldsymbol{B}^{-1}\boldsymbol{TCD}^{-T}\underline{i}-\boldsymbol{ATS}[\underline{\ell}+\underline{m}]\right) = \frac{\det(\boldsymbol{BDV})}{\det(\boldsymbol{CT})}\delta[\underline{i}],$$

and replacing $\underline{\ell}$ by $-(\underline{k}+\underline{m})$, followed by a change of sign of $\underline{m}$ yields

$$\sum_{\underline{v}\in\text{part}(\boldsymbol{V}^T)}\sum_{\underline{m}\in\mathbb{Z}^d} g(\underline{t}+\boldsymbol{ATS}\underline{m})e^{-j2\pi\underline{i}^T\boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}}\sum_{\underline{k}\in\mathbb{Z}^d} e^{-j2\pi\underline{v}^T\boldsymbol{V}^{-1}(\underline{k}-\underline{m})}$$
$$\times\, e^{-j2\pi\underline{z}^T(\underline{k}-\underline{m})}\gamma^*\left(\underline{t}+\boldsymbol{B}^{-1}\boldsymbol{TCD}^{-T}\underline{i}+\boldsymbol{ATS}\underline{k}\right) = \frac{\det(\boldsymbol{BDV})}{\det(\boldsymbol{CT})}\delta[\underline{i}].$$

Rearranging factors again,

$$\sum_{\underline{v}\in\text{part}(\boldsymbol{V}^T)}\sum_{\underline{m}\in\mathbb{Z}^d} g(\underline{t}+\boldsymbol{ATS}\underline{m})e^{-j2\pi\underline{i}^T\boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}}e^{j2\pi\underline{z}^T\underline{m}}e^{j2\pi\underline{v}^T\boldsymbol{V}^{-1}\underline{m}}$$
$$\times\sum_{\underline{k}\in\mathbb{Z}^d}\gamma^*\left(\underline{t}+\boldsymbol{B}^{-1}\boldsymbol{TCD}^{-T}\underline{i}+\boldsymbol{ATS}\underline{k}\right)e^{-j2\pi\underline{z}^T\underline{k}}e^{-j2\pi\underline{v}^T\boldsymbol{V}^{-1}\underline{k}}$$
$$= \frac{\det(\boldsymbol{BDV})}{\det(\boldsymbol{CT})}\delta[\underline{i}],$$

and replacing $\underline{i}$ by $-\underline{i}+\underline{n}$ results in

$$\sum_{\underline{v}\in\text{part}(\boldsymbol{V}^T)}\sum_{\underline{m}\in\mathbb{Z}^d} g(\underline{t}+\boldsymbol{ATS}\underline{m})e^{j2\pi(\underline{i}-\underline{n})^T\boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}}e^{j2\pi\underline{z}^T\underline{m}}e^{j2\pi\underline{v}^T\boldsymbol{V}^{-1}\underline{m}}$$
$$\times\sum_{\underline{k}\in\mathbb{Z}^d}\gamma^*\left(\underline{t}-\boldsymbol{B}^{-1}\boldsymbol{TCD}^{-T}(\underline{i}-\underline{n})+\boldsymbol{ATS}\underline{k}\right)e^{-j2\pi\underline{z}^T\underline{k}}e^{-j2\pi\underline{v}^T\boldsymbol{V}^{-1}\underline{k}}$$
$$=\frac{\det(\boldsymbol{BDV})}{\det(\boldsymbol{CT})}\delta[\underline{i}-\underline{n}].$$

Replacing $\underline{z}$ by $-(\underline{y}-(\boldsymbol{D}^{-1}\boldsymbol{R})^T\underline{n})$,

$$\sum_{\underline{v}\in\text{part}(\boldsymbol{V}^T)}\sum_{\underline{m}\in\mathbb{Z}^d} g(\underline{t}+\boldsymbol{ATS}\underline{m})e^{-j2\pi(\underline{y}-(\boldsymbol{D}^{-1}\boldsymbol{R})^T\underline{i}-\boldsymbol{V}^{-T}\underline{v})^T\underline{m}}$$
$$\times\sum_{\underline{k}\in\mathbb{Z}^d}\gamma^*\left(\underline{t}-\boldsymbol{B}^{-1}\boldsymbol{TCD}^{-T}(\underline{i}-\underline{n})+\boldsymbol{ATS}\underline{k}\right)$$
$$\times e^{j2\pi(\underline{y}-(\boldsymbol{D}^{-1}\boldsymbol{R})^T\underline{n}-\boldsymbol{V}^{-T}\underline{v})^T\underline{k}}=\frac{\det(\boldsymbol{BDV})}{\det(\boldsymbol{CT})}\delta[\underline{i}-\underline{n}],$$

and replacing $\underline{t}$ by $\boldsymbol{B}^{-1}\boldsymbol{T}(\boldsymbol{C}\underline{x}+\boldsymbol{CD}^{-T}\underline{i})$ and using the Zak transformation (3.11) yields

$$\sum_{\underline{v}\in\text{part}(\boldsymbol{V}^T)} g_{\underline{i}\underline{v}}(\underline{x},\underline{y})\gamma^*_{\underline{n}\underline{v}}(\underline{x},\underline{y})=\frac{\det(\boldsymbol{BDV})}{\det(\boldsymbol{CT})}\delta[\underline{i}-\underline{n}],$$

with

$$g_{\underline{i}\underline{v}}(\underline{x},\underline{y})=$$
$$(\mathcal{Z}g)\left(\boldsymbol{B}^{-1}\boldsymbol{T}(\boldsymbol{C}\underline{x}+\boldsymbol{CD}^{-T}\underline{i}),2\pi(\boldsymbol{ATS})^{-T}(\underline{y}-\boldsymbol{V}^{-T}\underline{v}-(\boldsymbol{D}^{-1}\boldsymbol{R})^T\underline{i});\boldsymbol{ATS}\right)$$

and

$$\gamma_{\underline{i}\underline{v}}(\underline{x},\underline{y})=$$
$$(\mathcal{Z}\gamma)\left(\boldsymbol{B}^{-1}\boldsymbol{T}(\boldsymbol{C}\underline{x}+\boldsymbol{CD}^{-T}\underline{i}),2\pi(\boldsymbol{ATS})^{-T}(\underline{y}-\boldsymbol{V}^{-T}\underline{v}-(\boldsymbol{D}^{-1}\boldsymbol{R})^T\underline{i});\boldsymbol{ATS}\right).$$

B.3 Derivation of sum-of-products form (3.15a)

In the Fourier transformed array $(\mathcal{F}^{(2d)}\tilde{a})(\underline{x},\underline{y})$ [see Eq. (3.14)], we substitute from the Gabor transform (3.9)

$$(\mathcal{F}^{(2d)}\tilde{a})(\underline{x},\underline{y}) = \sum_{\underline{m}\in\mathbb{Z}^d}\sum_{\underline{k}\in\mathbb{Z}^d} P^*_{\mathbf{\Lambda}}(\underline{m},\underline{k}) \times \int_{-\infty}^{\infty} \varphi(\underline{t})\gamma^*(\underline{t}-\boldsymbol{ATS}\underline{m})e^{-j\underline{t}^T\boldsymbol{B\Omega C}^{-1}\underline{k}}\,d\underline{t}\; e^{-j2\pi(\underline{y}^T\underline{m}-\underline{x}^T\underline{k})}.$$

Using the expression (3.6) for the multiplication operator $P_{\mathbf{\Lambda}}(\underline{m},\underline{k})$

$$(\mathcal{F}^{(2d)}\tilde{a})(\underline{x},\underline{y}) = \frac{1}{\det(\boldsymbol{D})}\sum_{\underline{m}\in\mathbb{Z}^d}\sum_{\underline{k}\in\mathbb{Z}^d}\sum_{\underline{i}\in\mathrm{part}(\boldsymbol{D}^T)} e^{-j2\pi\underline{i}^T\boldsymbol{D}^{-1}[\boldsymbol{R}\;\;\boldsymbol{I}_d]\left[\frac{\underline{m}}{\underline{k}}\right]} \times \int_{-\infty}^{\infty} \varphi(\underline{t})\gamma^*(\underline{t}-\boldsymbol{ATS}\underline{m})e^{-j\underline{t}^T\boldsymbol{B\Omega C}^{-1}\underline{k}}\,d\underline{t}\; e^{-j2\pi(\underline{y}^T\underline{m}-\underline{x}^T\underline{k})}$$

and rearranging factors yields

$$(\mathcal{F}^{(2d)}\tilde{a})(\underline{x},\underline{y}) = \frac{1}{\det(\boldsymbol{D})}\sum_{\underline{m}\in\mathbb{Z}^d} e^{-j2\pi\underline{y}^T\underline{m}} \sum_{\underline{i}\in\mathrm{part}(\boldsymbol{D}^T)} e^{-j2\pi\underline{i}^T\boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}} \times \int_{-\infty}^{\infty} \varphi(\underline{t})\gamma^*(\underline{t}-\boldsymbol{ATS}\underline{m}) \times \left[\sum_{\underline{k}\in\mathbb{Z}^d} e^{-j(\underline{t}-\boldsymbol{B}^{-1}\boldsymbol{TC}[\underline{x}-\boldsymbol{D}^{-T}\underline{i}])^T\boldsymbol{B\Omega C}^{-1}\underline{k}}\right] d\underline{t}\,.$$

Replacing the sum of exponentials by a sum of Dirac functions,

$$(\mathcal{F}^{(2d)}\tilde{a})(\underline{x},\underline{y}) = \frac{\det(\boldsymbol{CT})}{\det(\boldsymbol{BD})}\sum_{\underline{m}\in\mathbb{Z}^d} e^{-j2\pi\underline{y}^T\underline{m}} \sum_{\underline{i}\in\mathrm{part}(\boldsymbol{D}^T)} e^{-j2\pi\underline{i}^T\boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}} \times \int_{-\infty}^{\infty} \varphi(\underline{t})\gamma^*(\underline{t}-\boldsymbol{ATS}\underline{m}) \times \left[\sum_{\underline{k}\in\mathbb{Z}^d} \delta\left(\underline{t}-\boldsymbol{B}^{-1}\boldsymbol{TC}[\underline{x}-\boldsymbol{D}^{-T}\underline{i}+\underline{k}]\right)\right] d\underline{t},$$

and rearranging factors again yields

$$(\mathcal{F}^{(2d)}\tilde{a})(\underline{x},\underline{y}) = \frac{\det(\boldsymbol{CT})}{\det(\boldsymbol{BD})} \sum_{\underline{m}\in\mathbb{Z}^d} e^{-j2\pi\underline{y}^T\underline{m}} \sum_{\underline{i}\in\text{part}(\boldsymbol{D}^T)} e^{-j2\pi\underline{i}^T\boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}} \times \sum_{\underline{k}\in\mathbb{Z}^d} \int_{-\infty}^{\infty} \delta\left(\underline{t} - \boldsymbol{B}^{-1}\boldsymbol{TC}[\underline{x} - \boldsymbol{D}^{-T}\underline{i} + \underline{k}]\right) \varphi(\underline{t})\gamma^*(\underline{t} - \boldsymbol{ATS}\underline{m})\, d\underline{t}.$$

Evaluating the integral and replacing $\underline{i}$ in the exponential by $\underline{i} - \boldsymbol{D}^T\underline{k}$ (note that $\exp(-j2\pi\underline{i}^T\boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}) = \exp(-j2\pi(\underline{i} - \boldsymbol{D}^T\underline{k})^T\boldsymbol{D}^{-1}\boldsymbol{R}\underline{m})$ for all $\underline{k} \in \mathbb{Z}^d$) yields

$$\begin{aligned}(\mathcal{F}^{(2d)}\tilde{a})(\underline{x},\underline{y}) = \frac{\det(\boldsymbol{CT})}{\det(\boldsymbol{BD})} &\sum_{\underline{m}\in\mathbb{Z}^d} e^{-j2\pi\underline{y}^T\underline{m}} \sum_{\underline{k}\in\mathbb{Z}^d} \\ &\times \sum_{\underline{i}\in\text{part}(\boldsymbol{D}^T)} e^{-j2\pi(\underline{i} - \boldsymbol{D}^T\underline{k})^T\boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}} \\ &\times \varphi(\boldsymbol{B}^{-1}\boldsymbol{TC}\underline{x} - \boldsymbol{B}^{-1}\boldsymbol{TCD}^{-T}[\underline{i} - \boldsymbol{D}^T\underline{k}]) \\ &\times \gamma^*(\boldsymbol{B}^{-1}\boldsymbol{TC}\underline{x} - \boldsymbol{B}^{-1}\boldsymbol{TCD}^{-T}[\underline{i} - \boldsymbol{D}^T\underline{k}] - \boldsymbol{ATS}\underline{m}).\end{aligned}$$

Replacing the double summation over $\underline{k}$ and $\underline{i}$ by a single summation over $-\underline{k}$,

$$\begin{aligned}(\mathcal{F}^{(2d)}\tilde{a})(\underline{x},\underline{y}) = \frac{\det(\boldsymbol{CT})}{\det(\boldsymbol{BD})} &\sum_{\underline{m}\in\mathbb{Z}^d} e^{-j2\pi\underline{y}^T\underline{m}} \sum_{\underline{k}\in\mathbb{Z}^d} e^{j2\pi\underline{k}^T\boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}} \\ &\times \varphi(\boldsymbol{B}^{-1}\boldsymbol{TC}\underline{x} + \boldsymbol{B}^{-1}\boldsymbol{TCD}^{-T}\underline{k}) \\ &\times \gamma^*(\boldsymbol{B}^{-1}\boldsymbol{TC}\underline{x} + \boldsymbol{B}^{-1}\boldsymbol{TCD}^{-T}\underline{k} - \boldsymbol{ATS}\underline{m}),\end{aligned}$$

and substitution of $\underline{k}$ by $\boldsymbol{\Psi V}\underline{\ell} + \underline{i}$ with $\underline{\ell} \in \mathbb{Z}^d$ and $\underline{i} \in \text{fund}(\boldsymbol{\Psi V})$, results in (recall that $\boldsymbol{\Psi} = \boldsymbol{D}^T\boldsymbol{C}^{-1}\boldsymbol{ABS}$)

$$\begin{aligned}(\mathcal{F}^{(2d)}\tilde{a})(\underline{x},\underline{y}) = \frac{\det(\boldsymbol{CT})}{\det(\boldsymbol{BD})} &\sum_{\underline{m}\in\mathbb{Z}^d} e^{-j2\pi\underline{y}^T\underline{m}} \sum_{\underline{\ell}\in\mathbb{Z}^d} \sum_{\underline{i}\in\text{fund}(\boldsymbol{\Psi V})} e^{j2\pi\underline{i}^T\boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}} \\ &\times \varphi(\boldsymbol{B}^{-1}\boldsymbol{TC}\underline{x} + \boldsymbol{B}^{-1}\boldsymbol{TCD}^{-T}\underline{i} + \boldsymbol{ATSV}\underline{\ell}) \\ &\times \gamma^*(\boldsymbol{B}^{-1}\boldsymbol{TC}\underline{x} + \boldsymbol{B}^{-1}\boldsymbol{TCD}^{-T}\underline{i} + \boldsymbol{ATS}(\boldsymbol{V}\underline{\ell} - \underline{m})).\end{aligned}$$

Rearranging factors again,

$$(\mathcal{F}^{(2d)}\tilde{a})(\underline{x},\underline{y}) = \frac{\det(\boldsymbol{CT})}{\det(\boldsymbol{BD})} \sum_{\underline{i}\in\text{fund}(\boldsymbol{\Psi V})}$$
$$\times \sum_{\underline{\ell}\in\mathbb{Z}^d} \varphi(\boldsymbol{B}^{-1}\boldsymbol{TC}\underline{x} + \boldsymbol{B}^{-1}\boldsymbol{TCD}^{-T}\underline{i} + \boldsymbol{ATSV}\underline{\ell})$$
$$\times \sum_{\underline{m}\in\mathbb{Z}^d} \gamma^*(\boldsymbol{B}^{-1}\boldsymbol{TC}\underline{x} + \boldsymbol{B}^{-1}\boldsymbol{TCD}^{-T}\underline{i} + \boldsymbol{ATS}(\boldsymbol{V}\underline{\ell} - \underline{m}))$$
$$\times\, e^{-j2\pi\underline{y}^T\underline{m}} e^{j2\pi\underline{i}^T\boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}},$$

and replacing $\underline{m}$ by $\boldsymbol{V}\underline{\ell} - \underline{k}$ yields

$$(\mathcal{F}^{(2d)}\tilde{a})(\underline{x},\underline{y}) = \frac{\det(\boldsymbol{CT})}{\det(\boldsymbol{BD})} \sum_{\underline{i}\in\text{fund}(\boldsymbol{\Psi V})}$$
$$\times \sum_{\underline{\ell}\in\mathbb{Z}^d} \varphi(\boldsymbol{B}^{-1}\boldsymbol{TC}\underline{x} + \boldsymbol{B}^{-1}\boldsymbol{TCD}^{-T}\underline{i} + \boldsymbol{ATSV}\underline{\ell})$$
$$\times \sum_{\underline{k}\in\mathbb{Z}^d} \gamma^*(\boldsymbol{B}^{-1}\boldsymbol{TC}\underline{x} + \boldsymbol{B}^{-1}\boldsymbol{TCD}^{-T}\underline{i} + \boldsymbol{ATS}\underline{k})$$
$$\times\, e^{-j2\pi\underline{y}^T(\boldsymbol{V}\underline{\ell} - \underline{k})} e^{j2\pi\underline{i}^T\boldsymbol{D}^{-1}\boldsymbol{R}(\boldsymbol{V}\underline{\ell} - \underline{k})}.$$

Rearranging factors and using the Zak transformation (3.11) results in

$$(\mathcal{F}^{(2d)}\tilde{a})(\underline{x},\underline{y}) = \frac{\det(\boldsymbol{CT})}{\det(\boldsymbol{BD})} \sum_{\underline{i}\in\text{fund}(\boldsymbol{\Psi V})} \varphi_{\underline{i}}(\underline{x},\underline{y})$$
$$\times\, (\mathcal{Z}\gamma)^* \left(\boldsymbol{B}^{-1}\boldsymbol{T}(\boldsymbol{C}\underline{x} + \boldsymbol{CD}^{-T}\underline{i}), 2\pi(\boldsymbol{ATS})^{-T}(\underline{y} - (\boldsymbol{D}^{-1}\boldsymbol{R})^T\underline{i}); \boldsymbol{ATS}\right),$$

with

$$\varphi_{\underline{i}}(\underline{x},\underline{y}) = (\mathcal{Z}\varphi)\left(\boldsymbol{B}^{-1}\boldsymbol{T}(\boldsymbol{C}\underline{x} + \boldsymbol{CD}^{-T}\underline{i}), 2\pi(\boldsymbol{ATS})^{-T}(\underline{y} - (\boldsymbol{D}^{-1}\boldsymbol{R})^T\underline{i}); \boldsymbol{ATSV}\right).$$

Replacing $\underline{y}$ by $\underline{y} - \boldsymbol{V}^{-T}\underline{v}$ and using the periodicity property of the Zak transform $\varphi_{\underline{i}}(\underline{x},\underline{y})$, leads to the result

$$\overline{a}(\underline{x}, \underline{y} - \boldsymbol{V}^{-T}\underline{v}) = \frac{\det(\boldsymbol{CT})}{\det(\boldsymbol{BD})} \sum_{\underline{i}\in\text{fund}(\boldsymbol{\Psi V})} \varphi_{\underline{i}}(\underline{x},\underline{y})\gamma^*_{\underline{i}\,\underline{v}}(\underline{x},\underline{y}),$$

where

$$\gamma_{\underline{i}\,\underline{v}}(\underline{x},\underline{y}) =$$
$$(\mathcal{Z}\gamma)\left(\boldsymbol{B}^{-1}\boldsymbol{T}(\boldsymbol{C}\underline{x} + \boldsymbol{CD}^{-T}\underline{i}), 2\pi(\boldsymbol{ATS})^{-T}(\underline{y} - (\boldsymbol{D}^{-1}\boldsymbol{R})^T\underline{i} - \boldsymbol{V}^{-T}\underline{v}); \boldsymbol{ATS}\right).$$

Appendix C

Appendix to Chapter 4

C.1 Derivation of bi-orthogonality condition (4.21)

We assume that the set $\{G_{\Lambda;\lfloor m \rfloor_M k} | m = \langle M \rangle, k = \langle K \rangle\}$ is a frame. In Gabor's expansion (4.15), we substitute from the Gabor transform (4.18)

$$\begin{aligned}\Phi[n] = &\sum_{m=\langle M \rangle} \sum_{k=\langle K \rangle} \sum_{\ell=\langle MN \rangle} \Phi[\ell]\Gamma^*[\ell - mN] e^{j2\pi \lfloor m \rfloor_M \ell / DK} e^{-j2\pi k \ell / K} \\ &\times G[n - mN] e^{-j2\pi \lfloor m \rfloor_M n / DK} e^{j2\pi k n / K}.\end{aligned}$$

Rearranging factors,

$$\begin{aligned}\Phi[n] = &\sum_{m=\langle M \rangle} \sum_{\ell=\langle MN \rangle} \Phi[\ell]\Gamma^*[\ell - mN] G[n - mN] e^{j2\pi \lfloor m \rfloor_M (\ell - n) / DK} \\ &\times \left(\sum_{k=\langle K \rangle} e^{-j2\pi k(\ell - n)/K} \right),\end{aligned}$$

and replacing the sum of exponentials by a sum of Kronecker delta functions yields

$$\begin{aligned}\Phi[n] = &\sum_{m=\langle M \rangle} \sum_{\ell=\langle MN \rangle} \Phi[\ell]\Gamma^*[\ell - mN] G[n - mN] e^{j2\pi \lfloor m \rfloor_M (\ell - n) / DK} \\ &\times \left(K \sum_{k=-\infty}^{\infty} \delta[\ell - n + kK] \right).\end{aligned}$$

Rearranging factors again,

$$\begin{aligned}\Phi[n] = &K \sum_{m=\langle M \rangle} \sum_{k=-\infty}^{\infty} \sum_{\ell=\langle MN \rangle} \delta[\ell - n + kK] \\ &\times \Phi[\ell]\Gamma^*[\ell - mN] G[n - mN] e^{j2\pi \lfloor m \rfloor_M (\ell - n) / DK},\end{aligned}$$

and evaluating the summation over ℓ yields

$$\Phi[n] = K \sum_{m=\langle M\rangle} \sum_{k=\langle MN/K\rangle} \Phi[n-kK]\Gamma^*[n-kK-mN]G[n-mN]e^{-j2\pi\lfloor m\rfloor_M k/D}.$$

Note that $MN/K = pLDN/K = pLDqJ/pJ = qLD$ is an integer. Note, moreover, that since $D|M$, we have $\exp(-j2\pi\lfloor m\rfloor_M k/D) = \exp(-j2\pi mk/D)$ for all integers m. After a rearrangement of factors we get

$$\Phi[n] = K \sum_{k=\langle qLD\rangle} \Phi[n-kK] \sum_{m=\langle M\rangle} e^{-j2\pi mkr/D}\Gamma^*[n-kK-mN]G[n-mN].$$

This relationship holds for any periodized signal $\Phi[n]$ if and only if the $\lfloor k\rfloor_{qLD} = 0$ term in the summation over k is the only non-vanishing term, which immediately leads to the bi-orthogonality condition

$$K \sum_{m=\langle M\rangle} e^{-j2\pi mkr/D}\Gamma^*[n-kK-mN]G[n-mN] = \sum_{\ell=-\infty}^{\infty} \delta[k-\ell qLD],$$

where k and n extend over intervals of length qLD and MN, respectively. The same bi-orthogonality condition can be obtained by using the expression (4.7) of the shifted and modulated versions of the windows.

C.2 Derivation of sum-of-products form (4.23)

The relationship between the periodized windows G and Γ is given by

$$K \sum_{m=\langle M\rangle} e^{-j2\pi mkr/D}\Gamma^*[n-kK-mN]G[n-mN] = \sum_{\ell=-\infty}^{\infty} \delta[k-\ell qLD],$$

where the variables k and n extend over intervals of length qLD and MN, respectively. In order to separate the variables m and k in the exponent, the integer $f = D/\gcd(D,q)$ is used, again. Recall that r and D are relatively prime. Replacing k with $sfq-i$, where the variable s extends over an interval of length $\gcd(D,q)L$ and $i = 0\ldots fq-1$,

$$K \sum_{m=\langle M\rangle} e^{-j2\pi m(sfq-i)r/D}\Gamma^*[n-(sfq-i)K-mN)G[n-mN]$$
$$= \sum_{\ell=-\infty}^{\infty} \delta[sfq-i-\ell qLD],$$

multiplying both sides by $\exp(j2\pi sfpz)$, with $z \in \mathbb{R}$, and summing over the variable $s = \langle \gcd(D,q)L \rangle$ yields

$$K \sum_{m=\langle M \rangle} \sum_{s=\langle \gcd(D,q)L \rangle} e^{j2\pi mir/D} \Gamma^*[n-(sfq-i)K-mN]G[n-mN] \\ \times e^{j2\pi sfpz} = \sum_{s=\langle \gcd(D,q)L \rangle} \sum_{\ell=-\infty}^{\infty} \delta[sfq-i-\ell qLD] e^{j2\pi sfpz}. \quad \text{(C.1)}$$

First we simplify the expression on the right-hand side,

$$\sum_{s=\langle \gcd(D,q)L \rangle} \sum_{\ell=-\infty}^{\infty} \delta[(s-\ell LD/f)fq-i] e^{j2\pi sfpz}.$$

Using the Poisson summation formula

$$\sum_{k=\langle \gcd(D,q)L \rangle} e^{j2\pi \ell k/\gcd(D,q)L} = \gcd(D,q)L \sum_{k=-\infty}^{\infty} \delta[\ell - \gcd(D,q)Lk] \quad \text{(C.2)}$$

yields the expression

$$\frac{1}{\gcd(D,q)L} \sum_{s=\langle \gcd(D,q)L \rangle} \sum_{\ell=-\infty}^{\infty} \delta[(s-\ell)fq-i] e^{j2\pi sfpz} \\ \times \sum_{k=\langle \gcd(D,q)L \rangle} e^{j2\pi k\ell/\gcd(D,q)L}.$$

Replacing ℓ with $s-u$,

$$\frac{1}{\gcd(D,q)L} \sum_{s=\langle \gcd(D,q)L \rangle} \sum_{u=-\infty}^{\infty} \delta[ufq-i] e^{j2\pi sfpz} \\ \times \sum_{k=\langle \gcd(D,q)L \rangle} e^{j2\pi k(s-u)/\gcd(D,q)L},$$

and rearranging factors results in

$$\frac{1}{\gcd(D,q)L} \sum_{s=\langle \gcd(D,q)L \rangle} e^{j2\pi sfpz} \sum_{k=\langle \gcd(D,q)L \rangle} e^{j2\pi ks/\gcd(D,q)L} \\ \times \left(\sum_{u=-\infty}^{\infty} \delta[ufq-i] e^{-j2\pi uk/\gcd(D,q)L} \right).$$

Evaluating the last term (recall $i = 0 \ldots fq-1$),

$$\frac{1}{\gcd(D,q)L}\delta[i] \sum_{s=\langle \gcd(D,q)L\rangle} e^{j2\pi sfpz} \sum_{k=\langle \gcd(D,q)L\rangle} e^{j2\pi ks/\gcd(D,q)L},$$

and using the Poisson summation formula (C.2) finally results in the simplified expression (recall that $M = pLD$)

$$\delta[i] \sum_{s=\langle 1\rangle} e^{j2\pi sMz}.$$

Substitution of this expression into Eq. (C.1) and using the identity $qK = pN$,

$$K \sum_{m=\langle M\rangle} \sum_{s=\langle \gcd(D,q)L\rangle} e^{j2\pi mir/D} \Gamma^*[n-(sfp+m)N+iK]G[n-mN]e^{j2\pi sfpz}$$
$$= \delta[i] \sum_{s=\langle 1\rangle} e^{j2\pi sMz},$$

and using the Poisson summation formula

$$\sum_{\ell=\langle fp\rangle} e^{j2\pi s\ell/fp} = fp \sum_{\ell=-\infty}^{\infty} \delta[s-fp\ell]$$

yields

$$\sum_{m=\langle M\rangle} \sum_{s=\langle M\rangle} e^{j2\pi mir/D} \Gamma^*[n-(s+m)N+iK]G[n-mN]e^{j2\pi sz}$$
$$\times \sum_{\ell=\langle fp\rangle} e^{j2\pi s\ell/fp} = \frac{fp}{K}\delta[i] \sum_{s=\langle 1\rangle} e^{j2\pi sMz}.$$

Rearranging factors again,

$$\sum_{\ell=\langle fp\rangle} \sum_{m=\langle M\rangle} G[n-mN]e^{j2\pi mir/D} e^{-j2\pi mz} e^{-j2\pi m\ell/fp}$$
$$\times \sum_{s=\langle M\rangle} \Gamma^*[n-(s+m)N+iK)e^{j2\pi(s+m)z} e^{j2\pi(s+m)\ell/fp}$$
$$= \frac{fp}{K}\delta[i] \sum_{s=\langle 1\rangle} e^{j2\pi sMz},$$

and replacing s on the left-hand side with $-k-m$ yields

$$\sum_{\ell=\langle fp\rangle}\sum_{m=\langle M\rangle} G[n-mN]e^{j2\pi mir/D}e^{-j2\pi mz}e^{-j2\pi m\ell/fp}$$
$$\times\sum_{k=\langle M\rangle}\Gamma^*[n+kN+iK]e^{-j2\pi kz}e^{-j2\pi k\ell/fp}=\frac{fp}{K}\delta[i]\sum_{s=\langle 1\rangle}e^{j2\pi sMz}.$$

Replacing i with $s-i$, with $s=0\ldots fq-1$,

$$\sum_{\ell=\langle fp\rangle}\sum_{m=\langle M\rangle} G[n-mN]e^{-j2\pi mz}e^{-j2\pi m(\ell/fp+ir/D-sr/D)}$$
$$\times\sum_{k=\langle M\rangle}\Gamma^*[n+sK-iK+kN]e^{-j2\pi kz}e^{-j2\pi k\ell/fp}=\frac{fp}{K}\delta[i-s]\sum_{s=\langle 1\rangle}e^{j2\pi sMz},$$

and replacing z with $-(u-srM/D)/M)$ yields

$$\sum_{\ell=\langle fp\rangle}\sum_{m=\langle M\rangle} G[n-mN]e^{j2\pi m(u-\ell M/fp-irM/D)/M}$$
$$\times\sum_{k=\langle M\rangle}\Gamma^*[n+sK-iK+kN]e^{j2\pi k(u-\ell M/fp-srM/D)/M}=\frac{fp}{K}\delta[i-s].$$

Replacing n with $n+iK$ and changing the sign of m results in the sum-of-products form

$$\sum_{\ell=\langle fp\rangle}\sum_{m=\langle M\rangle} G[n+iK+mN]e^{-j2\pi m(u-\ell M/fp-irM/D)/M}$$
$$\times\sum_{k=\langle M\rangle}\Gamma^*[n+sK+kN]e^{j2\pi k(u-\ell M/fp-srM/D)/M}=\frac{fp}{K}\delta[i-s].$$

Using the Zak transformation (1.15) finally leads to the result

$$\sum_{u=\langle fp\rangle}(\mathcal{Z}g)[n+iK,\ell-uM/fp-irM/D;N,M]$$
$$\times(\mathcal{Z}\gamma)^*[n+sK,\ell-uM/fp-srM/D;N,M]=\frac{fp}{K}\delta[i-s].$$

C.3 Derivation of sum-of-products form (4.26)

In the Fourier expansion (4.25) $(\mathcal{F}_{dis}^{(2)}\tilde{A})[n,\ell;DK,M]$, we substitute from the Gabor transform (4.18),

$$(\mathcal{F}_{dis}^{(2)}\tilde{A})[n,\ell;DK,M] = \sum_{m=<M>}\sum_{k=<DK>} e^{-j2\pi(m\ell/M - kn/DK)}$$
$$\times\left(\frac{1}{D}\sum_{s=\langle D\rangle} e^{-j2\pi s(rm+k)/D}\sum_{\ell'=<MN>}\Phi[\ell']\Gamma^*[\ell'-mN]e^{-j2\pi k\ell'/DK}\right),$$

and rearranging factors yields

$$(\mathcal{F}_{dis}^{(2)}\tilde{A})[n,\ell;DK,M] = \frac{1}{D}\sum_{m=<M>} e^{-j2\pi m\ell/M}\sum_{s=\langle D\rangle} e^{-j2\pi srm/D}$$
$$\times\sum_{\ell'=<MN>}\Phi[\ell']\Gamma^*[\ell'-mN]\left(\sum_{k=<DK>} e^{-j2\pi k(\ell'-n+sK)/DK}\right).$$

Replacing the sum of exponentials in the last term with a sum of Kronecker deltas,

$$(\mathcal{F}_{dis}^{(2)}\tilde{A})[n,\ell;DK,M] = \frac{1}{D}\sum_{m=<M>} e^{-j2\pi m\ell/M}\sum_{s=\langle D\rangle} e^{-j2\pi srm/D}$$
$$\times\sum_{\ell'=<MN>}\Phi[\ell']\Gamma^*[\ell'-mN]\left(DK\sum_{k=-\infty}^{\infty}\delta[\ell'-n+sK-kDK]\right),$$

and rearranging factors again yields

$$(\mathcal{F}_{dis}^{(2)}\tilde{A})[n,\ell;DK,M] = K\sum_{m=<M>} e^{-j2\pi m\ell/M}\sum_{s=\langle D\rangle} e^{-j2\pi srm/D}$$
$$\times\sum_{k=-\infty}^{\infty}\sum_{\ell'=<MN>}\delta[\ell'-n+sK-kDK]\Phi[\ell']\Gamma^*[\ell'-mN].$$

Evaluating the sum over ℓ' yields

$$(\mathcal{F}_{dis}^{(2)}\tilde{A})[n,\ell;DK,M] = K\sum_{m=<M>} e^{-j2\pi m\ell/M}\sum_{s=\langle D\rangle} e^{-j2\pi srm/D}$$
$$\times\sum_{k=\langle MN/DK\rangle}\Phi[n+(kD-s)K]\Gamma^*[n+(kD-s)K-mN].$$

Note that $MN/DK = pLDN/DK = qL$ is an integer. Rearranging factors and replacing $-s$ in the exponential by $kD - s$ (note that $\exp(j2\pi(kD-s)rm/D) = \exp(-j2\pi srm/D)$ for all integers k) yields

$$(\mathcal{F}_{dis}^{(2)}\tilde{A})[n,\ell;DK,M] = K \sum_{m=<M>} e^{-j2\pi m\ell/M} \sum_{k=\langle MN/DK\rangle} \sum_{s=\langle D\rangle}$$
$$\times\, e^{j2\pi r(kD-s)m/D}\Phi[n+(kD-s)K]\Gamma^*[n+(kD-s)K-mN].$$

Replacing the double summation over k and s by a single summation over k, and using the identity $pN = qK$ yields

$$(\mathcal{F}_{dis}^{(2)}\tilde{A})[n,\ell;DK,M] = K \sum_{m=<M>} e^{-j2\pi m\ell/M}$$
$$\times \sum_{k=\langle MN/K\rangle} e^{j2\pi mkr/D}\Phi[n+kK]\Gamma^*[n+(kp/q-m)N].$$

Replacing k with $kfq+i$, where $k = \langle M/fp\rangle$ and $i = 0\ldots fq-1$, results in

$$(\mathcal{F}_{dis}^{(2)}\tilde{A})[n,\ell;DK,M] = K \sum_{m=<M>} e^{-j2\pi m\ell/M}$$
$$\times \sum_{i=0}^{fq-1} \sum_{k=\langle M/fp\rangle} e^{j2\pi mir/D}\Phi[n+kfqK+iK]\Gamma^*[n+(kfp-m)N+iK].$$

Rearranging factors,

$$(\mathcal{F}_{dis}^{(2)}\tilde{A})[n,\ell;DK,M] = K \sum_{i=0}^{fq-1} \sum_{k=\langle M/fp\rangle} \sum_{m=<M>} e^{-j2\pi m\ell/M}$$
$$\times\, e^{j2\pi mir/D}\Phi[n+kfqK+iK]\Gamma^*[n+(kfp-m)N+iK],$$

and replacing m with $kfp - m'$ yields

$$(\mathcal{F}_{dis}^{(2)}\tilde{A})[n,\ell;DK,M] = K \sum_{i=0}^{fq-1} \sum_{k=\langle M/fp\rangle} \sum_{m'=<M>} e^{-j2\pi(kfp-m')\ell/M}$$
$$\times\, e^{j2\pi(kfp-m')ir/D}\Phi[n+kfqK+iK]\Gamma^*[n+m'N+iK].$$

Rearranging factors again,

$$(\mathcal{F}_{dis}^{(2)}\tilde{A})[n,\ell;DK,M] = K\sum_{i=0}^{fq-1}\sum_{m'=<M>}\Gamma^*[n+m'N+iK]$$
$$\times e^{j2\pi m'(\ell-irM/D)/M}\sum_{k=\langle M/fp\rangle}\Phi[n+kfqK+iK]e^{-j2\pi kfp(\ell-irM/D)/M},$$

and using the discrete Zak transformation (1.15) results in

$$(\mathcal{F}_{dis}^{(2)}\tilde{A})[n,\ell;DK,M] = K\sum_{i=0}^{fq-1}(\mathcal{Z}\gamma)^*[n+iK,\ell-irM/D;N,M]$$
$$\times(\mathcal{Z}\varphi)[n+iK,\ell-irM;fpN,M/fp].$$

Finally, replacing ℓ with $\ell - sM/fp$ and using the periodicity property of the discrete Zak transform $(\mathcal{Z}\varphi)\,[n+iK,\ell-irM/D;fpN,M/fp]$ lead to the result

$$\overline{a}[n,\ell-sM/fp;DK,M] = K\sum_{i=0}^{fq-1}(\mathcal{Z}\gamma)^*[n+iK,\ell-sM/fp-irM/D,N,M]$$
$$\times(\mathcal{Z}\varphi)[n+iK,\ell-irM;fpN,M/fp].$$

Appendix D

Appendix to Chapter 5

D.1 Derivation of bi-orthogonality condition (5.12)

We assume that the set $\{\tilde{G}_{\Lambda_S;\underline{m}\underline{k}} | \underline{m} \in \mathrm{part}(\boldsymbol{M}), \underline{k} \in \mathrm{part}(\boldsymbol{CK})\}$ is a frame. In the Gabor expansion (5.11), we substitute from the Gabor transform (5.10)

$$\begin{aligned}\Phi[\underline{n}] = &\sum_{\underline{m}\in\mathrm{part}(\boldsymbol{M})} \sum_{\underline{k}\in\mathrm{part}(\boldsymbol{CK})} P_{\boldsymbol{\Lambda}}(\underline{m}, \underline{k}) \sum_{\underline{\ell}\in\mathrm{part}(\boldsymbol{NSM})} \Phi[\underline{\ell}]\Gamma^*[\underline{\ell} - \boldsymbol{NS}\underline{m}] \\ &\times e^{-j2\pi\underline{\ell}^T(\boldsymbol{CK})^{-1}\underline{k}} G[\underline{n} - \boldsymbol{NS}\underline{m}] e^{j2\pi\underline{n}^T(\boldsymbol{CK})^{-1}\underline{k}}.\end{aligned}$$

Using the expression (3.6) for the multiplication operator $P_{\boldsymbol{\Lambda}}(\underline{m}, \underline{k})$ and rearranging factors yields

$$\begin{aligned}\Phi[\underline{n}] = &\frac{1}{\det(\boldsymbol{D})} \sum_{\underline{m}\in\mathrm{part}(\boldsymbol{M})} \sum_{\underline{i}\in\mathrm{part}(\boldsymbol{D}^T)} e^{j2\pi\underline{i}^T\boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}} \sum_{\underline{\ell}\in\mathrm{part}(\boldsymbol{NSM})} \Phi[\underline{\ell}]\Gamma^*[\underline{\ell} - \boldsymbol{NS}\underline{m}] \\ &\times G[\underline{n} - \boldsymbol{NS}\underline{m}] \sum_{\underline{k}\in\mathrm{part}(\boldsymbol{CK})} e^{j2\pi\underline{i}^T\boldsymbol{D}^{-1}\underline{k}} e^{-j2\pi(\underline{\ell} - \underline{n})^T(\boldsymbol{CK})^{-1}\underline{k}}.\end{aligned}$$

Combining the exponentials in the last sum (recall that $\boldsymbol{K}$ and $\boldsymbol{C}$ are diagonal matrices),

$$\begin{aligned}\Phi[\underline{n}] = &\frac{1}{\det(\boldsymbol{D})} \sum_{\underline{m}\in\mathrm{part}(\boldsymbol{M})} \sum_{\underline{i}\in\mathrm{part}(\boldsymbol{D}^T)} e^{j2\pi\underline{i}^T\boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}} \sum_{\underline{\ell}\in\mathrm{part}(\boldsymbol{NSM})} \Phi[\underline{\ell}]\Gamma^*[\underline{\ell} - \boldsymbol{NS}\underline{m}] \\ &\times G[\underline{n} - \boldsymbol{NS}\underline{m}] \sum_{\underline{k}\in\mathrm{part}(\boldsymbol{CK})} e^{-j2\pi(\underline{\ell} - \underline{n} - \boldsymbol{KCD}^{-T}\underline{i})^T(\boldsymbol{CK})^{-1}\underline{k}},\end{aligned}$$

and replacing the last sum of exponentials by sums of Dirac functions yields

$$\begin{aligned}\Phi[\underline{n}] =& \frac{1}{\det(\boldsymbol{D})} \sum_{\underline{m}\in\text{part}(\boldsymbol{M})} \sum_{\underline{i}\in\text{part}(\boldsymbol{D}^T)} e^{j2\pi\underline{i}^T\boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}} \sum_{\underline{\ell}\in\text{part}(\boldsymbol{NSM})} \Phi[\underline{\ell}]\Gamma^*[\underline{\ell} - \boldsymbol{NS}\underline{m}] \\ &\times G[\underline{n} - \boldsymbol{NS}\underline{m}] \sum_{\underline{k}\in\mathbb{Z}^d} \det(\boldsymbol{CK})\delta[\underline{\ell} - \underline{n} - \boldsymbol{KCD}^{-T}\underline{i} + \boldsymbol{KC}\underline{k}].\end{aligned}$$

Rearranging factors again,

$$\begin{aligned}\Phi[\underline{n}] =& \frac{\det(\boldsymbol{CK})}{\det(\boldsymbol{D})} \sum_{\underline{m}\in\text{part}(\boldsymbol{M})} \sum_{\underline{i}\in\text{part}(\boldsymbol{D}^T)} e^{j2\pi\underline{i}^T\boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}} G[\underline{n} - \boldsymbol{NS}\underline{m}] \\ &\times \sum_{\underline{k}\in\mathbb{Z}^d} \sum_{\underline{\ell}\in\text{part}(\boldsymbol{NSM})} \delta[\underline{\ell} - \underline{n} - \boldsymbol{KCD}^{-T}\underline{i} + \boldsymbol{KC}\underline{k}]\Phi[\underline{\ell}]\Gamma^*[\underline{\ell} - \boldsymbol{NS}\underline{m}],\end{aligned}$$

and evaluating the sum over $\underline{\ell}$ and $\underline{k}$, and replacing $\underline{i}$, which appears in the exponent, by $i - \boldsymbol{D}^T\underline{k}$ (note that $\exp(j2\pi\underline{i}^T\boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}) = \exp(j2\pi(\underline{i} - \boldsymbol{D}^T\underline{k})\boldsymbol{R}\underline{m})$ for all $\underline{k} \in \mathbb{Z}^d$) results in

$$\begin{aligned}\Phi[\underline{n}] =& \frac{\det(\boldsymbol{CK})}{\det(\boldsymbol{D})} \sum_{\underline{m}\in\text{part}(\boldsymbol{M})} \sum_{\underline{i}\in\text{part}(\boldsymbol{D}^T)} e^{j2\pi(\underline{i} - \boldsymbol{D}^T\underline{k})^T\boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}} G[\underline{n} - \boldsymbol{NS}\underline{m}] \\ &\times \sum_{\underline{k}\in\text{part}((\boldsymbol{KC})^{-1}\boldsymbol{NSM})} \Phi[\underline{n} + \boldsymbol{KCD}^{-T}(\underline{i} - \boldsymbol{D}^T\underline{k})] \\ &\times \Gamma^*[\underline{n} + \boldsymbol{KCD}^{-T}(\underline{i} - \boldsymbol{D}^T\underline{k}) - \boldsymbol{NS}\underline{m}].\end{aligned}$$

Rearranging factors again,

$$\begin{aligned}\Phi[\underline{n}] =& \frac{\det(\boldsymbol{CK})}{\det(\boldsymbol{D})} \sum_{\underline{m}\in\text{part}(\boldsymbol{M})} G[\underline{n} - \boldsymbol{NS}\underline{m}] \sum_{\underline{k}\in\text{part}((\boldsymbol{KC})^{-1}\boldsymbol{NSM})} \\ &\times \sum_{\underline{i}\in\text{part}(\boldsymbol{D}^T)} e^{j2\pi(\underline{i} - \boldsymbol{D}^T\underline{k})^T\boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}} \Phi[\underline{n} + \boldsymbol{KCD}^{-T}(\underline{i} - \boldsymbol{D}^T\underline{k})] \\ &\times \Gamma^*[\underline{n} + \boldsymbol{KCD}^{-T}(\underline{i} - \boldsymbol{D}^T\underline{k}) - \boldsymbol{NS}\underline{m}],\end{aligned}$$

and replacing the summation over $\underline{k}$ together with the summation over $\underline{i}$ by a summation over $\underline{\ell}$ yields

$$\begin{aligned}\Phi[\underline{n}] =& \frac{\det(\boldsymbol{CK})}{\det(\boldsymbol{D})} \sum_{\underline{m}\in\text{part}(\boldsymbol{M})} G[\underline{n} - \boldsymbol{NS}\underline{m}] \sum_{\underline{\ell}\in\text{part}(\boldsymbol{\Psi M})} e^{j2\pi\underline{\ell}^T\boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}} \\ &\times \Phi[\underline{n} + \boldsymbol{KCD}^{-T}\underline{\ell}]\Gamma^*[\underline{n} + \boldsymbol{KCD}^{-T}\underline{\ell} - \boldsymbol{NS}\underline{m}].\end{aligned}$$

Recall that $\boldsymbol{\Psi} = \boldsymbol{D}^T(\boldsymbol{KC})^{-1}\boldsymbol{NS}$. Rearranging factors yields

$$\Phi[\underline{n}] = \frac{\det(\boldsymbol{CK})}{\det(\boldsymbol{D})} \sum_{\underline{\ell} \in \mathrm{part}(\boldsymbol{\Psi M})} \Phi[\underline{n} + \boldsymbol{KCD}^{-T}\underline{\ell}] \times \sum_{\underline{m} \in \mathrm{part}(\boldsymbol{M})} e^{j2\pi \underline{\ell}^T \boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}} G[\underline{n} - \boldsymbol{NS}\underline{m}]\Gamma^*[\underline{n} + \boldsymbol{KCD}^{-T}\underline{\ell} - \boldsymbol{NS}\underline{m}].$$

For all $\underline{\ell} \in \mathrm{part}(\boldsymbol{\Psi M})$, we can write $\underline{\ell} = \underline{u} + \boldsymbol{\Psi M}\underline{v}$ with $\underline{u} \in \mathrm{fund}(\boldsymbol{\Psi M})$ and $\underline{v} \in \mathbb{Z}^d$. Then this relationship holds for any arbitrary signal Φ if and only if the $\underline{u} = \underline{0}$ in the summation over $\underline{\ell} = \underline{u} + \boldsymbol{\Psi M}\underline{v}$ is the only non-vanishing term, which finally leads to the condition

$$\sum_{\underline{m} \in \mathrm{part}(\boldsymbol{M})} e^{j2\pi \underline{\ell}^T \boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}} G[\underline{n} - \boldsymbol{NS}\underline{m}]\Gamma^*[\underline{n} + \boldsymbol{KCD}^{-T}\underline{\ell} - \boldsymbol{NS}\underline{m}] = \frac{\det(\boldsymbol{D})}{\det(\boldsymbol{CK})} \sum_{\underline{u} \in \mathbb{Z}^d} \delta[\underline{\ell} - \boldsymbol{\Psi M}\underline{u}].$$

This condition should hold for all $\underline{\ell} \in \mathrm{part}(\boldsymbol{\Psi M})$ and all $\underline{n} \in \mathrm{part}(\boldsymbol{NSM})$.

D.2 Derivation of sum-of-products form (5.14)

The relationship between the periodized windows G and Γ looks like

$$\sum_{\underline{m} \in \mathrm{part}(\boldsymbol{M})} e^{j2\pi \underline{k}^T \boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}} G[\underline{n} - \boldsymbol{NS}\underline{m}]\Gamma^*[\underline{n} - \boldsymbol{KCD}^{-T}\underline{k} - \boldsymbol{NS}\underline{m}] = \frac{\det(\boldsymbol{D})}{\det(\boldsymbol{CK})} \sum_{\underline{u} \in \mathbb{Z}^d} \delta[\underline{k} - \boldsymbol{\Psi M}\underline{u}],$$

where the variable $\underline{k}$ extends over a region $\mathrm{part}(\boldsymbol{\Psi M})$. We replace the variable $\underline{k}$ by $\boldsymbol{\Psi V}\underline{\ell} + \underline{i}$ with $\underline{\ell} \in \mathrm{part}(\boldsymbol{V}^{-1}\boldsymbol{M})$ and $\underline{i} \in \mathrm{fund}(\boldsymbol{\Psi V})$. Here $\boldsymbol{V}$ is a matrix such that $\boldsymbol{\Psi V}$, $(\boldsymbol{D}^{-1}\boldsymbol{R})^T\boldsymbol{\Psi V}$ and $\boldsymbol{V}^{-1}\boldsymbol{M}$ are matrices containing only integers. Compare the matrix $\boldsymbol{V}$ with fp in the one-dimensional case. In fact, in the decomposable case ($\boldsymbol{R}$ a diagonal matrix, $\boldsymbol{C} = \boldsymbol{D}$ a diagonal matrix and $\boldsymbol{S} = \boldsymbol{I}_2$), $\boldsymbol{V}$ reduces to a diagonal matrix with integers such as fp on the diagonal. Thus, replacing $\underline{k}$ by $\boldsymbol{\Psi V}\underline{\ell} + \underline{i}$,

$$\sum_{\underline{m} \in \mathrm{part}(\boldsymbol{M})} e^{-j2\pi \underline{i}^T \boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}} G[\underline{n} - \boldsymbol{NS}\underline{m}]\Gamma^*[\underline{n} + \boldsymbol{NS}(\boldsymbol{V}\underline{\ell} - \underline{m}) - \boldsymbol{KCD}^{-T}\underline{i}] = \frac{\det(\boldsymbol{D})}{\det(\boldsymbol{CK})} \sum_{\underline{u} \in \mathbb{Z}^d} \delta[\boldsymbol{\Psi V}(\underline{\ell} - \boldsymbol{V}^{-1}\boldsymbol{M}\underline{u}) - \underline{i}],$$

multiplying both sides by $\exp(j2\pi\underline{z}^T\boldsymbol{V}\underline{\ell})$, and summing over $\underline{\ell} = \text{part}(\boldsymbol{V}^{-1}\boldsymbol{M})$ yields

$$\begin{aligned}
&\sum_{\underline{m}\in\text{part}(\boldsymbol{M})}\ \sum_{\underline{\ell}\in\text{part}(\boldsymbol{V}^{-1}\boldsymbol{M})} e^{-j2\pi\underline{i}^T\boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}} G[\underline{n}-\boldsymbol{N}\boldsymbol{S}\underline{m}] \\
&\times \Gamma^*[\underline{n}+\boldsymbol{N}\boldsymbol{S}(\boldsymbol{V}\underline{\ell}-\underline{m}) - \boldsymbol{K}\boldsymbol{C}\boldsymbol{D}^{-T}\underline{i}]e^{j2\pi\underline{z}^T\boldsymbol{V}\underline{\ell}} \\
&= \frac{\det(\boldsymbol{D})}{\det(\boldsymbol{C}\boldsymbol{K})} \sum_{\underline{\ell}\in\text{part}(\boldsymbol{V}^{-1}\boldsymbol{M})}\ \sum_{\underline{u}\in\mathbb{Z}^d} \delta[\boldsymbol{\Psi}\boldsymbol{V}(\underline{\ell}-\boldsymbol{V}^{-1}\boldsymbol{M}\underline{u}) - \underline{i}]e^{j2\pi\underline{z}^T\boldsymbol{V}\underline{\ell}}. \qquad \text{(D.1)}
\end{aligned}$$

First we simplify the expression on the right-hand side; using the Poisson summation formula

$$\sum_{\underline{k}\in\text{part}((\boldsymbol{V}^{-1}\boldsymbol{M})^T)} e^{j2\pi\underline{k}^T\boldsymbol{M}^{-1}\boldsymbol{V}\underline{u}} = \det(\boldsymbol{V}^{-1}\boldsymbol{M}) \sum_{\underline{k}\in\mathbb{Z}^d} \delta[\underline{u} - \boldsymbol{V}^{-1}\boldsymbol{M}\underline{k}] \qquad \text{(D.2)}$$

yields

$$\begin{aligned}
&\frac{\det(\boldsymbol{D}\boldsymbol{V})}{\det(\boldsymbol{C}\boldsymbol{K}\boldsymbol{M})} \sum_{\underline{\ell}\in\text{part}(\boldsymbol{V}^{-1}\boldsymbol{M})}\ \sum_{\underline{u}\in\mathbb{Z}^d} \delta[\boldsymbol{\Psi}\boldsymbol{V}(\underline{\ell}-\underline{u}) - \underline{i}]e^{j2\pi\underline{z}^T\boldsymbol{V}\underline{\ell}} \\
&\times \sum_{\underline{k}\,\text{part}((\boldsymbol{V}^{-1}\boldsymbol{M})^T)} e^{j2\pi\underline{k}^T\boldsymbol{M}^{-1}\boldsymbol{V}\underline{u}}.
\end{aligned}$$

Replacing $\underline{u}$ by $\underline{\ell} - \underline{u}'$,

$$\begin{aligned}
&\frac{\det(\boldsymbol{D}\boldsymbol{V})}{\det(\boldsymbol{C}\boldsymbol{K}\boldsymbol{M})} \sum_{\underline{\ell}\in\text{part}(\boldsymbol{V}^{-1}\boldsymbol{M})}\ \sum_{\underline{u}'\in\mathbb{Z}^d} \delta[\boldsymbol{\Psi}\boldsymbol{V}\underline{u}' - \underline{i}]e^{j2\pi\underline{z}^T\boldsymbol{V}\underline{\ell}} \\
&\sum_{\underline{k}\,\text{part}((\boldsymbol{V}^{-1}\boldsymbol{M})^T)} e^{j2\pi\underline{k}^T\boldsymbol{M}^{-1}\boldsymbol{V}(\underline{\ell}-\underline{u}')},
\end{aligned}$$

and rearranging factors results in

$$\begin{aligned}
&\frac{\det(\boldsymbol{D}\boldsymbol{V})}{\det(\boldsymbol{C}\boldsymbol{K}\boldsymbol{M})} \sum_{\underline{\ell}\in\text{part}(\boldsymbol{V}^{-1}\boldsymbol{M})} e^{j2\pi\underline{z}^T\boldsymbol{V}\underline{\ell}} \sum_{\underline{k}\,\text{part}((\boldsymbol{V}^{-1}\boldsymbol{M})^T)} e^{j2\pi\underline{k}^T\boldsymbol{M}^{-1}\boldsymbol{V}\underline{\ell}} \\
&\times \left(\sum_{\underline{u}'\in\mathbb{Z}^d} \delta[\boldsymbol{\Psi}\boldsymbol{V}\underline{u}' - \underline{i}]e^{-j2\pi\underline{k}^T\boldsymbol{M}^{-1}\boldsymbol{V}\underline{u}'} \right).
\end{aligned}$$

Since $\underline{i}$ lies in $\text{fund}(\boldsymbol{\Psi V})$, evaluating the last term yields

$$\frac{\det(\boldsymbol{DV})}{\det(\boldsymbol{CKM})}\delta[\underline{i}] \sum_{\underline{\ell}\in\text{part}(\boldsymbol{V}^{-1}\boldsymbol{M})} e^{j2\pi\underline{z}^T\boldsymbol{V}\underline{\ell}} \sum_{\underline{k}\,\text{part}((\boldsymbol{V}^{-1}\boldsymbol{M})^T)} e^{j2\pi\underline{k}^T\boldsymbol{M}^{-1}\boldsymbol{V}\underline{\ell}}.$$

Using the Poisson summation formula (D.2) finally leads to the simplified expression

$$\frac{\det(\boldsymbol{D})}{\det(\boldsymbol{CK})}\delta[\underline{i}] \sum_{\underline{\ell}\in\text{part}(\boldsymbol{I}_d)} e^{j2\pi\underline{z}^T\boldsymbol{M}\underline{\ell}}.$$

Replacing the term on the right-hand side in Eq. (D.1) by the previous expression,

$$\sum_{\underline{m}\in\text{part}(\boldsymbol{M})} \sum_{\underline{\ell}\in\text{part}(\boldsymbol{V}^{-1}\boldsymbol{M})} e^{-j2\pi\underline{i}^T\boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}}\Gamma^*[\underline{n}+\boldsymbol{NS}(\boldsymbol{V}\underline{\ell}-\underline{m})-\boldsymbol{KCD}^{-T}\underline{i}]$$
$$\times\, G[\underline{n}-\boldsymbol{NS}\underline{m}]e^{j2\pi\underline{z}^T\boldsymbol{V}\underline{\ell}} = \frac{\det(\boldsymbol{D})}{\det(\boldsymbol{CK})}\delta[\underline{i}] \sum_{\underline{\ell}\in\text{part}(\boldsymbol{I}_d)} e^{j2\pi\underline{z}^T\boldsymbol{M}\underline{\ell}},$$

and using the Poisson formula

$$\sum_{\underline{v}\in\text{part}(\boldsymbol{V}^T)} e^{j2\pi\underline{v}^T\boldsymbol{V}^{-1}\underline{\ell}} = \det(\boldsymbol{V}) \sum_{\underline{v}\in\mathbb{Z}^d} \delta[\underline{\ell}-\boldsymbol{V}\underline{v}]$$

yields

$$\sum_{\underline{m}\in\text{part}(\boldsymbol{M})} \sum_{\underline{\ell}\in\text{part}(\boldsymbol{M})} e^{-j2\pi\underline{i}^T\boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}}\Gamma^*[\underline{n}+\boldsymbol{NS}(\underline{\ell}-\underline{m})-\boldsymbol{KCD}^{-T}\underline{i}]$$
$$\times\, G[\underline{n}-\boldsymbol{NS}\underline{m}]e^{j2\pi\underline{z}^T\underline{\ell}} \sum_{\underline{v}\in\text{part}(\boldsymbol{V}^T)} e^{j2\pi\underline{v}^T\boldsymbol{V}^{-1}\underline{\ell}} = \frac{\det(\boldsymbol{DV})}{\det(\boldsymbol{CK})}\delta[\underline{i}] \sum_{\underline{\ell}\in\text{part}(\boldsymbol{I}_d)} e^{j2\pi\underline{z}^T\boldsymbol{M}\underline{\ell}}.$$

Rearranging factors,

$$\sum_{\underline{v}\in\text{part}(\boldsymbol{V}^T)} \sum_{\underline{m}\in\text{part}(\boldsymbol{M})} e^{-j2\pi\underline{i}^T\boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}}G[\underline{n}-\boldsymbol{NS}\underline{m}]$$
$$\times \sum_{\underline{\ell}\in\text{part}(\boldsymbol{M})} \Gamma^*[\underline{n}+\boldsymbol{NS}(\underline{\ell}-\underline{m})-\boldsymbol{KCD}^{-T}\underline{i}]e^{j2\pi\underline{z}^T\underline{\ell}}e^{j2\pi\underline{v}^T\boldsymbol{V}^{-1}\underline{\ell}}$$
$$= \frac{\det(\boldsymbol{DV})}{\det(\boldsymbol{CK})}\delta[\underline{i}] \sum_{\underline{\ell}\in\text{part}(\boldsymbol{I}_d)} e^{j2\pi\underline{z}^T\boldsymbol{M}\underline{\ell}},$$

and replacing $\underline{\ell}$ by $\underline{k}+\underline{m}$ yields

$$\sum_{\underline{v}\in\text{part}(\boldsymbol{V}^T)}\sum_{\underline{m}\in\text{part}(\boldsymbol{M})} e^{-j2\pi\underline{i}^T\boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}}G[\underline{n}-\boldsymbol{NS}\underline{m}]$$
$$\times\sum_{\underline{m}\in\text{part}(\boldsymbol{M})}\Gamma^*[\underline{n}-\boldsymbol{KCD}^{-T}\underline{i}+\boldsymbol{NS}\underline{k}]e^{j2\pi\underline{z}^T(\underline{k}+\underline{m})}e^{j2\pi\underline{v}^T\boldsymbol{V}^{-1}(\underline{k}+\underline{m})}$$
$$=\frac{\det(\boldsymbol{DV})}{\det(\boldsymbol{CK})}\delta[\underline{i}]\sum_{\underline{\ell}\in\text{part}(\boldsymbol{I}_d)}e^{j2\pi\underline{z}^T\boldsymbol{M}\underline{\ell}}.$$

Replacing $\underline{i}$ by $\underline{i}-\underline{\ell}$ and changing the sign of $\underline{m}$,

$$\sum_{\underline{v}\in\text{part}(\boldsymbol{V}^T)}\sum_{\underline{m}\in\text{part}(\boldsymbol{M})} e^{j2\pi(\underline{i}-\underline{\ell})^T\boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}}G[\underline{n}+\boldsymbol{NS}\underline{m}]$$
$$\times\sum_{\underline{m}\in\text{part}(\boldsymbol{M})}\Gamma^*[\underline{n}-\boldsymbol{KCD}^{-T}(\underline{i}-\underline{\ell})+\boldsymbol{NS}\underline{k}]e^{j2\pi\underline{z}^T(\underline{k}-\underline{m})}e^{j2\pi\underline{v}^T\boldsymbol{V}^{-1}(\underline{k}-\underline{m})}$$
$$=\frac{\det(\boldsymbol{DV})}{\det(\boldsymbol{CK})}\delta[\underline{i}-\underline{\ell}]\sum_{\underline{\ell}\in\text{part}(\boldsymbol{I}_d)}e^{j2\pi\underline{z}^T\boldsymbol{M}\underline{\ell}},$$

and replacing $\underline{n}$ by $\underline{n}+\boldsymbol{KCD}^{-T}\underline{i}$ and rearranging terms results in

$$\sum_{\underline{v}\in\text{part}(\boldsymbol{V}^T)}\sum_{\underline{m}\in\text{part}(\boldsymbol{M})} G[\underline{n}+\boldsymbol{KCD}^{-T}\underline{i}+\boldsymbol{NS}\underline{m}]e^{j2\pi(\underline{i}-\underline{\ell})^T\boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}}$$
$$\times e^{-j2\pi\underline{z}^T\underline{m}}e^{-j2\pi\underline{v}^T\boldsymbol{V}^{-1}\underline{m}}\sum_{\underline{m}\in\text{part}(\boldsymbol{M})}\Gamma^*[\underline{n}+\boldsymbol{KCD}^{-T}\underline{\ell}+\boldsymbol{NS}\underline{k}]e^{j2\pi\underline{z}^T\underline{k}}$$
$$\times e^{j2\pi\underline{v}^T\boldsymbol{V}^{-1}\underline{k}}=\frac{\det(\boldsymbol{DV})}{\det(\boldsymbol{CK})}\delta[\underline{i}-\underline{\ell}]\sum_{\underline{\ell}\in\text{part}(\boldsymbol{I}_d)}e^{-j2\pi\underline{z}^T\boldsymbol{M}\underline{\ell}}.$$

Replacing z by $\boldsymbol{M}^{-1}(\underline{u}-\boldsymbol{M}(\boldsymbol{D}^{-1}\boldsymbol{R})^T\underline{\ell})$ and using the Zak transformation (5.13) finally leads to the sum-of-products form

$$\sum_{\underline{v}\in\text{part}(\boldsymbol{V}^T)}(\mathcal{Z}g)\,[\underline{n}+\boldsymbol{KCD}^{-T}\underline{i},\underline{u}-\boldsymbol{M}[\boldsymbol{V}^{-T}\underline{v}+(\boldsymbol{D}^{-1}\boldsymbol{R})^T\underline{i}];\boldsymbol{NS},\boldsymbol{M}]$$
$$\times(\mathcal{Z}\gamma)^*\,[\underline{n}+\boldsymbol{KCD}^{-T}\underline{\ell},\underline{u}-\boldsymbol{M}[\boldsymbol{V}^{-T}\underline{v}+(\boldsymbol{D}^{-1}\boldsymbol{R})^T\underline{\ell}];\boldsymbol{NS},\boldsymbol{M}]$$
$$=\frac{\det(\boldsymbol{DV})}{\det(\boldsymbol{CK})}\delta[\underline{i}-\underline{\ell}].$$

D.3 Derivation of sum-of-products form (5.17)

In the Fourier expansion of the periodized array $(\mathcal{F}_{dis}^{(2d)}\tilde{A})[\underline{n},\underline{\ell};\boldsymbol{CK},\boldsymbol{M}]$ of Gabor expansion coefficients [see Eq. (5.16)], we substitute from the Gabor transform (5.10)

$$(\mathcal{F}_{dis}^{(2d)}\tilde{A})[\underline{n},\underline{\ell};\boldsymbol{CK},\boldsymbol{M}] = \sum_{\underline{m}\in\text{part}(\boldsymbol{M})}\sum_{\underline{k}\in\text{part}(\boldsymbol{CK})} P_{\boldsymbol{\Lambda}}(\underline{m},\underline{k}) \sum_{\underline{u}\in\text{part}(\boldsymbol{NSM})} \Phi[\underline{u}]$$
$$\times\, \Gamma^*[\underline{u}-\boldsymbol{NS}\underline{m}]e^{-j2\pi\underline{u}^T(\boldsymbol{CK})^{-1}\underline{k}}e^{-j2\pi(\underline{\ell}^T\boldsymbol{M}^{-1}\underline{m}\,-\,\underline{n}^T(\boldsymbol{CK})^{-1}\underline{k})}.$$

Using the expression Eq. (5.5) for the multiplication operator $P_{\boldsymbol{\Lambda}}(\underline{m},\underline{k})$ and rearranging factors,

$$(\mathcal{F}_{dis}^{(2d)}\tilde{A})[\underline{n},\underline{\ell};\boldsymbol{CK},\boldsymbol{M}] = \sum_{\underline{m}\in\text{part}(\boldsymbol{M})}\sum_{\underline{i}\in\text{part}(\boldsymbol{D}^T)} e^{j2\pi\underline{i}^T\boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}} \sum_{\underline{u}\in\text{part}(\boldsymbol{NSM})} \Phi[\underline{u}]$$
$$\times\, \Gamma^*[\underline{u}-\boldsymbol{NS}\underline{m}]e^{-j2\pi\underline{\ell}^T\boldsymbol{M}^{-1}\underline{m}}$$
$$\times \left(\sum_{\underline{k}\in\text{part}(\boldsymbol{CK})} e^{-j2\pi(\underline{u}-\underline{n}-\boldsymbol{KCD}^{-T}\underline{i})^T(\boldsymbol{CK})^{-1}\underline{k}}\right),$$

and replacing the last summation of exponentials by a sum of Dirac functions results in

$$(\mathcal{F}_{dis}^{(2d)}\tilde{A})[\underline{n},\underline{\ell};\boldsymbol{CK},\boldsymbol{M}] = \det(\boldsymbol{CK}) \sum_{\underline{m}\in\text{part}(\boldsymbol{M})}\sum_{\underline{i}\in\text{part}(\boldsymbol{D}^T)} e^{j2\pi\underline{i}^T\boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}}$$
$$\times \sum_{\underline{u}\in\text{part}(\boldsymbol{NSM})} \Phi[\underline{u}]\Gamma^*[\underline{u}-\boldsymbol{NS}\underline{m}]e^{-j2\pi\underline{\ell}^T\boldsymbol{M}^{-1}\underline{m}}$$
$$\times \left(\sum_{\underline{k}\in\mathbb{Z}^d} \delta[\underline{u}-\underline{n}-\boldsymbol{KCD}^{-T}\underline{i}-\boldsymbol{KC}\underline{k}]\right).$$

Rearranging factors,

$$(\mathcal{F}_{dis}^{(2d)}\tilde{A})[\underline{n},\underline{\ell};\boldsymbol{CK},\boldsymbol{M}] = \sum_{\underline{m}\in\text{part}(\boldsymbol{M})}\sum_{\underline{i}\in\text{part}(\boldsymbol{D}^T)} e^{j2\pi\underline{i}^T\boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}}e^{-j2\pi\underline{\ell}^T\boldsymbol{M}^{-1}\underline{m}}$$
$$\times \det(\boldsymbol{CK})\sum_{\underline{k}\in\mathbb{Z}^d}\sum_{\underline{u}\in\text{part}(\boldsymbol{NSM})} \delta[\underline{u}-\underline{n}-\boldsymbol{KCD}^{-T}\underline{i}-\boldsymbol{KC}\underline{k}]\Phi[\underline{u}]\Gamma^*[\underline{u}-\boldsymbol{NS}\underline{m}],$$

and evaluating the sum over $\underline{u}$ and $\underline{k}$ yields

$$(\mathcal{F}_{dis}^{(2d)}\tilde{A})[\underline{n},\underline{\ell};\boldsymbol{CK},\boldsymbol{M}] = \det(\boldsymbol{CK}) \sum_{\underline{m}\in\text{part}(\boldsymbol{M})} \sum_{\underline{i}\in\text{part}(\boldsymbol{D}^T)} e^{j2\pi\underline{i}^T\boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}}$$
$$\times\, e^{-j2\pi\underline{\ell}^T\boldsymbol{M}^{-1}\underline{m}} \sum_{\underline{u}\in((\boldsymbol{KC})^{-1}\boldsymbol{NSM})} \Phi[\underline{n}+\boldsymbol{KCD}^{-T}(\underline{i}+\boldsymbol{D}^T\underline{u})]$$
$$\times\, \Gamma^*[\underline{n}+\boldsymbol{KCD}^{-T}(\underline{i}+\boldsymbol{D}^T\underline{u})-\boldsymbol{NS}\underline{m}].$$

Rearranging factors and replacing $\underline{i}$, which appears in the exponent, by $\underline{i}+\boldsymbol{D}^T\underline{u}$ (note that $\exp(j2\pi(\underline{i}+\boldsymbol{D}^T\underline{u})^T\boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}) = \exp(j2\pi\underline{i}^T\boldsymbol{D}^{-1}\boldsymbol{R}\underline{m})$ for all $\underline{u}\in\mathbb{Z}^d$) yields

$$(\mathcal{F}_{dis}^{(2d)}\tilde{A})[\underline{n},\underline{\ell};\boldsymbol{CK},\boldsymbol{M}] = \det(\boldsymbol{CK}) \sum_{\underline{m}\in\text{part}(\boldsymbol{M})} e^{-j2\pi\underline{\ell}^T\boldsymbol{M}^{-1}\underline{m}}$$
$$\times \sum_{\underline{u}\in((\boldsymbol{KC})^{-1}\boldsymbol{NSM})} \sum_{\underline{i}\in\text{part}(\boldsymbol{D}^T)} e^{j2\pi(\underline{i}+\boldsymbol{D}^T\underline{u})^T\boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}}$$
$$\times\, \Phi[\underline{n}+\boldsymbol{KCD}^{-T}(\underline{i}+\boldsymbol{D}^T\underline{u})]\Gamma^*[\underline{n}+\boldsymbol{KCD}^{-T}(\underline{i}+\boldsymbol{D}^T\underline{u})-\boldsymbol{NS}\underline{m}].$$

Replacing the double summation over $\underline{u}$ and $\underline{i}$ by a summation over $\underline{k}$ yields

$$(\mathcal{F}_{dis}^{(2d)}\tilde{A})[\underline{n},\underline{\ell};\boldsymbol{CK},\boldsymbol{M}] = \det(\boldsymbol{CK}) \sum_{\underline{m}\in\text{part}(\boldsymbol{M})} e^{-j2\pi\underline{\ell}^T\boldsymbol{M}^{-1}\underline{m}}$$
$$\times \sum_{\underline{k}\in\text{part}(\boldsymbol{\Psi M})} e^{j2\pi\underline{k}^T\boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}}\Phi[\underline{n}+\boldsymbol{KCD}^{-T}\underline{k}]\Gamma^*[\underline{n}+\boldsymbol{KCD}^{-T}\underline{k}-\boldsymbol{NS}\underline{m}].$$

Recall that $\boldsymbol{\Psi} = \boldsymbol{D}^T(\boldsymbol{CK})^{-1}\boldsymbol{NS}$. Replacing the vector $\underline{k}$ by $\boldsymbol{\Psi V}\underline{u}+\underline{i}$ with $\underline{u}\in\text{part}(\boldsymbol{V}^{-1}\boldsymbol{M})$ and $\underline{i}\in\text{fund}(\boldsymbol{\Psi V})$ yields

$$(\mathcal{F}_{dis}^{(2d)}\tilde{A})[\underline{n},\underline{\ell};\boldsymbol{CK},\boldsymbol{M}] = \det(\boldsymbol{CK}) \sum_{\underline{m}\in\text{part}(\boldsymbol{M})} e^{-j2\pi\underline{\ell}^T\boldsymbol{M}^{-1}\underline{m}}$$
$$\times \sum_{\underline{u}\in\text{part}(\boldsymbol{V}^{-1}\boldsymbol{M})} \sum_{\underline{i}\in\text{part}(\boldsymbol{\Psi V})} e^{j2\pi\underline{i}^T\boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}}\Phi[\underline{n}+\boldsymbol{KCD}^{-T}\underline{i}+\boldsymbol{NSV}\underline{u}]$$
$$\times\, \Gamma^*[\underline{n}+\boldsymbol{KCD}^{-T}\underline{i}+\boldsymbol{NS}(\boldsymbol{V}\underline{u}-\underline{m})].$$

Rearranging factors,

$$(\mathcal{F}_{dis}^{(2d)}\tilde{A})[\underline{n},\underline{\ell};\boldsymbol{CK},\boldsymbol{M}] = \det(\boldsymbol{CK})$$
$$\times \sum_{\underline{u}\in\text{part}(\boldsymbol{V}^{-1}\boldsymbol{M})} \sum_{\underline{i}\in\text{part}(\boldsymbol{\Psi V})} \Phi[\underline{n}+\boldsymbol{KCD}^{-T}\underline{i}+\boldsymbol{NSV}\underline{u}]$$
$$\times \sum_{\underline{m}\in\text{part}(\boldsymbol{M})} \Gamma^*[\underline{n}+\boldsymbol{KCD}^{-T}\underline{i}+\boldsymbol{NS}(\boldsymbol{V}\underline{u}-\underline{m})]e^{-j2\pi\underline{\ell}^T\boldsymbol{M}^{-1}\underline{m}}e^{j2\pi\underline{i}^T\boldsymbol{D}^{-1}\boldsymbol{R}\underline{m}},$$

and replacing $\underline{m}$ by $\boldsymbol{V}\underline{u}-\underline{k}$, with $\underline{k}\in\text{part}(\boldsymbol{M})$, yields

$$(\mathcal{F}_{dis}^{(2d)}\tilde{A})[\underline{n},\underline{\ell};\boldsymbol{CK},\boldsymbol{M}] = \det(\boldsymbol{CK})$$
$$\times \sum_{\underline{u}\in\text{part}(\boldsymbol{V}^{-1}\boldsymbol{M})} \sum_{\underline{i}\in\text{part}(\boldsymbol{\Psi V})} \Phi[\underline{n}+\boldsymbol{KCD}^{-T}\underline{i}+\boldsymbol{NSV}\underline{u}]$$
$$\times \sum_{\underline{k}\in\text{part}(\boldsymbol{M})} \Gamma^*[\underline{n}+\boldsymbol{KCD}^{-T}\underline{i}+\boldsymbol{NS}\underline{k}]e^{-j2\pi\underline{\ell}^T\boldsymbol{M}^{-1}(\boldsymbol{V}\underline{u}-\underline{k})}e^{j2\pi\underline{i}^T\boldsymbol{D}^{-1}\boldsymbol{R}(\boldsymbol{V}\underline{u}-\underline{k})}.$$

Rearranging factors and using the Zak transformation (5.13) yields

$$(\mathcal{F}_{dis}^{(2d)}\tilde{A})[\underline{n},\underline{\ell};\boldsymbol{CK},\boldsymbol{M}] = \det(\boldsymbol{CK})$$
$$\times \sum_{\underline{i}\in\text{part}(\boldsymbol{\Psi V})} \sum_{\underline{k}\in\text{part}(\boldsymbol{M})} \Gamma^*[\underline{n}+\boldsymbol{KCD}^{-T}\underline{i}+\boldsymbol{NS}\underline{k}]e^{j2\pi(\underline{\ell}-\boldsymbol{M}(\boldsymbol{D}^{-1}\boldsymbol{R})^T\underline{i})^T\boldsymbol{M}^{-1}\underline{k}}$$
$$\times\, \varphi_{\underline{i}}[\underline{n},\ell],$$

with

$$\varphi_{\underline{i}}[\underline{n},\ell] = (\mathcal{Z}\varphi)\,[\underline{n}+\boldsymbol{KCD}^{-T}\underline{i},\underline{\ell}-\boldsymbol{M}(\boldsymbol{D}^{-1}\boldsymbol{R})^T\underline{i};\boldsymbol{NSV},\boldsymbol{V}^{-1}\boldsymbol{M}].$$

Replacing $\underline{\ell}$ by $\underline{\ell}+\boldsymbol{MV}^{-T}\underline{v}$, and using the Zak transformation (5.13), results in the sum-of-products form

$$(\mathcal{F}_{dis}^{(2d)}\tilde{A})[\underline{n},\underline{\ell}+\boldsymbol{MV}^{-T}\underline{v};\boldsymbol{CK},\boldsymbol{M}] = \det(\boldsymbol{CK}) \sum_{\underline{i}\in\text{part}(\boldsymbol{\Psi V})} \gamma_{\underline{i}\underline{v}}^*[\underline{n},\underline{\ell}]\varphi_{\underline{i}}[\underline{n},\underline{\ell}]$$

with

$$\gamma_{\underline{i}\underline{v}}[\underline{n},\underline{\ell}] = (\mathcal{Z}\gamma)\,[\underline{n}+\boldsymbol{KCD}^{-T}\underline{i},\underline{\ell}-\boldsymbol{M}[\boldsymbol{V}^{-T}\underline{v}+(\boldsymbol{D}^{-1}\boldsymbol{R})^T\underline{i}];\boldsymbol{NS},\boldsymbol{M}].$$

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Samenvatting

De Gabor-ontwikkeling beschrijft een signaal als een superpositie van verschoven en gemoduleerde versies van een vensterfunctie. Als vensterfunctie is destijds door Gabor bij de presentatie van de Gabor-ontwikkeling een Gaussische vensterfunctie gekozen, omdat deze optimaal is met betrekking tot de localisatie in het tijd-frequentie domein. De Gabor coëfficiënten kunnen berekend worden door inproducten van het signaal met een duale vensterfunctie te nemen. Deze Gabor coëfficiënten kunnen geïnterpreteerd worden als samples van het gevensterde Fourier getransformeerde signaal gebaseerd op deze duale vensterfunctie. Het gevolg hiervan is dat als deze duale vensterfunctie goed gelocaliseerd is in het tijd-frequentie domein, de coëfficiënten van de Gabor-ontwikkeling tijd-frequentie informatie van het signaal vertonen.

Gabor beschouwde het separabel (vierkant) geval, d.w.z., het Gabor schema waarvan de set van verschoven en gemoduleerde versies van de vensterfunctie correspondeert met een separabel grid in het tijd-frequentie domein. In dit proefschrift wordt aangetoond dat een niet-separabel (hexagonaal) Gabor schema met een Gaussische vensterfunctie beter is in de zin van stabiliteit dan een separabel Gabor schema. Het is aannemelijk dat andere niet-separabele Gabor schema's dan het hexagonale Gabor schema met betrekking tot andere type vensterfuncties een hogere stabiliteit opleveren dan een separabele Gabor schema. Niet-separabele multi-dimensionale Gabor schema's voor tijd-continue en tijd-discrete signalen is het onderwerp van dit proefschrift. Het een-dimensionale Gabor schema wordt apart behandeld voor illustratieve doeleinden.

De Zak transformatie speelt een belangrijke rol in dit proefschrift. De Zak transformatie heeft zijn waarde bewezen voor het separabele Gabor schema. Deze kan gebruikt worden om de duale vensterfunctie voor een gegeven vensterfunctie te berekenen, om de Gabor coëfficiënten te berekenen, en om het signaal te reconstrueren. In dit proefschrift wordt aangetoond dat de Zak transformatie ook gebruikt kan worden voor niet-separabele Gabor schema's.

Niet-separabele grids kunnen verkregen worden door, bijvoorbeeld, een rotatie of een 'shear' van een rechthoekig grid. Een rotatie in het tijd-frequentie domein kan geassocieerd worden met de fractionele Fourier getransformeerde en vermenigvuldiging met kwadratische fasetermen kan geassocieerd worden met de 'shear'. In dit proefschrift wordt aangetoond hoe deze operaties gebruikt kunnen worden om algo-

ritmen die ontwikkeld zijn voor separabele Gabor schema's te hergebruiken in het niet-separabele geval.

Het separabele Gabor schema kan geïmplementeerd worden in de vorm van een filterbank. Er wordt aangetoond dat ook het niet-separabele Gabor schema geïmplementeerd kan worden in de vorm van een filterbank.

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Arno van Leest
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Curriculum Vitae

Adriaan Johan van Leest was born in ’s-Hertogenbosch, the Netherlands, on February 22, 1973. In 1991, he finished secondary school (atheneum-B) at the Titus Brandsma lyceum, Oss. He received the ’ingenieur’ degree in Electrical Engineering from the Eindhoven University of Technology, Eindhoven, the Netherlands, in 1996. His master report was entitled: ‘Subband coding using IIR filters’. From October 1996 to January 2001 he was with the Signal Processing System (SPS) group in the Dept. of Electrical Engineering of the Eindhoven University of Technology, where he carried out his Ph.D. research on Gabor schemes. In April and May 1999, he was a guest researcher at Numerical Harmonic Analysis Group (NUHAG), University Vienna, Austria.

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