

# Computational evidence for Deligne's conjecture regarding exceptional Lie groups

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## Computational evidence for Deligne's conjecture regarding exceptional Lie groups

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**Abstract.** For  $j \leq 4$ , we obtain, for all exceptional groups, a uniform decomposition of the  $j$ th tensor power of the adjoint representation, in agreement with the conjectures of Deligne [1].

### *Évidence pour une conjecture de Deligne sur les groupes de Lie exceptionnels*

**Résumé.** Pour  $j \leq 4$ , on obtient pour tous les groupes exceptionnels une décomposition uniforme de la puissance tensorielle  $j$ -ième de la représentation adjointe, en accord avec les conjectures de Deligne [1].

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### *Version française abrégée*

Soit  $\mathcal{G}^0$  le groupe déployé adjoint de type  $A_1, A_2, G_2, D_4, F_4, E_6, E_7$ , ou  $E_8$  et soit  $\Gamma$  le groupe des automorphismes de son diagramme de Dynkin. Le groupe  $\Gamma$  est cyclique d'ordre 2 pour  $A_2$  et  $E_6$ , isomorphe à  $S_3$  pour  $D_4$  et trivial pour les autres types. Soit  $\mathcal{G}$  le produit semi-direct de  $\Gamma$  par  $\mathcal{G}^0$  (vu comme groupe algébrique sur  $\mathbb{Q}$ ) et soit  $\mathfrak{g}$  son algèbre de Lie.

Pour les 5 types exceptionnels, Vogel a observé que  $\wedge^2 \mathfrak{g}$  se décompose en la somme de  $\mathfrak{g}$  et d'une représentation irréductible  $X$ , tandis que  $S^2 \mathfrak{g}$  se décompose en  $1 + Y + Z$  avec  $Y$  et  $Z$  irréductibles. Utilisant que les seuls éléments invariants de  $S^4 \mathfrak{g}$  sont les multiples du carré du Casimir, il montre que les dimensions de  $\mathfrak{g}, X, Y, Z$  et les scalaires par lesquels agit l'opérateur de Casimir sont donnés par des fonctions rationnelles d'un paramètre  $a$ . Cette structure persiste pour tous les groupes considérés ci-dessus, avec  $a = 1/2, 1/3, 1/4, 1/6, 1/9, 1/12, 1/18, 1/30$ .

Deligne a demandé (pour tous ces groupes sauf  $D_4$ ) si un tel phénomène persistait pour les puissances tensorielles supérieures de la représentation adjointe. Dans cette Note, nous donnons les résultats de calculs fait à l'aide du programme LiE [2]. Nous trouvons que pour chaque groupe  $\mathcal{G}$ , il existe des représentations virtuelles  $X_i$  ( $2 \leq i \leq 4$ ),  $Y_i$  ( $2 \leq i \leq 4$ ),  $Y_i^*$  ( $2 \leq i \leq 4$ ),  $A, C, C^*$ ,

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Note présentée par Pierre DELIGNE.

$D, D^*, E, F, F^*, G, G^*, H, H^*, I, I^*$ , et  $J$ , irréductibles, nulles ou l'opposée d'une représentation irréductible, en termes desquelles (et des représentations triviales et adjointes) les représentations  $\mathfrak{g}^{\otimes 3}$  et  $\mathfrak{g}^{\otimes 4}$ , et plus précisément les pléthismes de  $\mathfrak{g}$  de degré  $\leq 4$  se décomposent de façon uniforme (en « constituants »). Avec les notations précédentes, on a  $X_2 = X, Y_2 = Y, Y_2^* = Z$ . La décomposition des carrés extérieurs et symétriques des constituants  $X_2, Y_2$  et  $Y_2^*$  de  $\mathfrak{g}^{\otimes 2}$ , celle des produits tensoriels (constituant de  $\mathfrak{g}^{\otimes 2}$ )  $\otimes$  (constituant de  $\mathfrak{g}^{\otimes 2}$ ) et celle des produits tensoriels  $\mathfrak{g} \otimes$  (constituant de  $\mathfrak{g}^{\otimes 3}$ ) est également uniforme. Si on regarde  $*$  comme une involution laissant fixe  $1, \mathfrak{g}, X_i, A, E$  et  $J$ , toutes ces formules de décomposition sont stables par l'involution  $*$ .

Sur chacun des modules considérés, l'opérateur de Casimir agit par un scalaire, donné par une forme linéaire en  $a$  dépendant du module. L'involution  $*$  échange  $a$  et  $a^* := -1/6 - a$ .

Soit  $j = 2, 3$  ou  $4$ . Chacune des formules de décomposition relative à  $\mathfrak{g}^{\otimes j}$  fournit une relation linéaire entre dimensions de modules. Des relations additionnelles sont obtenues en considérant la trace de l'opérateur de Casimir : utiliser (1) et (2) de [1]. Supposant connues les dimensions des constituants de  $\mathfrak{g}^{\otimes i}$  pour  $i < j$ , on obtient ainsi suffisamment de relations  $\mathbb{Q}(a)$ -linéaires pour déterminer uniquement des fonctions rationnelles en  $a$  donnant les dimensions des modules pour  $a = 1/2, 1/3, 1/4, 1/6, 1/9, 1/12, 1/18, 1/30$ , selon le type. Par construction, les formules obtenues sont stables par l'involution  $*$ , agissant sur le module et sur  $a$ . Miracle : les numérateurs et dénominateurs des fonctions rationnelles obtenues sont des produits de facteurs linéaires rationnels.

Let  $\mathcal{G}^0$  denote the split adjoint connected Lie group of type  $A_1, A_2, G_2, D_4, F_4, E_6, E_7$  or  $E_8$ , and let  $\Gamma$  be the automorphism group of its Dynkin diagram. Then  $\Gamma$  is cyclic of order 2 for  $E_6$  and  $A_2$ ,  $\Gamma \cong S_3$  or  $D_4$ , and  $\Gamma$  is trivial for the other types. We extend  $\mathcal{G}^0$  by  $\Gamma$  to form the algebraic group  $\mathcal{G}$  over  $\mathbb{Q}$  and write  $\mathfrak{g}$  for its Lie algebra.

It was observed by Vogel (at least for the 5 exceptional groups  $\mathcal{G}$ ) that the  $\mathcal{G}$ -modules  $\bigwedge^2 \mathfrak{g}$  and  $\text{Sym}^2 \mathfrak{g}$  can be decomposed into irreducibles in a uniform way, and that the dimension of the constituents and the values by which the Casimir operator is acting on them are given by rational functions of a parameter  $a$ . For the types  $A_1, A_2, G_2, D_4, F_4, E_6, E_7, E_8$ , we have  $a = 1/2, 1/3, 1/4, 1/6, 1/9, 1/12, 1/18, 1/30$ .

Deligne asked (for all 8 groups except  $D_4$ ) whether this could be extended to higher tensor powers of  $\mathfrak{g}$ . In this Note we give the results of computations done using the program LiE [2]. It turns out that for each group we can find virtual representations  $X_i (2 \leq i \leq 4), Y_i (2 \leq i \leq 4), Y_i^* (2 \leq i \leq 4), A, C, C^*, D, D^*, E, F, F^*, G, G^*, H, H^*, I, I^*$ , and  $J$  such that the pattern extends in a nice way to  $\mathfrak{g}^{\otimes 3}$  and  $\mathfrak{g}^{\otimes 4}$ . These representations are either irreducible, 0, or the negative of an irreducible. Here  $*$  can be thought of as an involution keeping  $X_i, A, E$ , and  $J$  fixed.

We now give the decomposition formulae. Using notation of [1] for plethysms we find:

$$\begin{aligned} [(2)] \mathfrak{g} &= \text{Sym}^2 \mathfrak{g} = 1 + Y_2 + Y_2^*, \\ [(1, 1)] \mathfrak{g} &= \mathfrak{g} + X_2. \\ [(3)] \mathfrak{g} &= \mathfrak{g} + X_2 + A + Y_3 + Y_3^*, \\ [(2, 1)] \mathfrak{g} &= 2\mathfrak{g} + X_2 + Y_2 + Y_2^* + A + C + C^*, \\ [(1, 1, 1)] \mathfrak{g} &= 1 + X_2 + Y_2 + Y_2^* + X_3. \end{aligned}$$

$$\begin{aligned}
 [(4)]\mathfrak{g} &= 1 + 2Y_2 + 2Y_2^* + C + C^* + X_3 + D + D^* + J + Y_4 + Y_4^*, \\
 [(3, 1)]\mathfrak{g} &= 2\mathfrak{g} + 3X_2 + 2Y_2 + 2Y_2^* + 3A + 2C + 2C^* + X_3 + Y_3 + Y_3^* \\
 &\quad + D + D^* + E + F + F^* + G + G^*, \\
 [(2, 2)]\mathfrak{g} &= 2 \cdot 1 + X_2 + 3Y_2 + 3Y_2^* + A + C + C^* + 2X_3 + D + D^* + E + H + H^* + J, \\
 [(2, 1, 1)]\mathfrak{g} &= 3\mathfrak{g} + 4X_2 + Y_2 + Y_2^* + 3A + 2C + 2C^* + X_3 + Y_3 + Y_3^* + E + F + F^* + I + I^*, \\
 [(1, 1, 1, 1)]\mathfrak{g} &= \mathfrak{g} + Y_2 + Y_2^* + A + C + C^* + X_3 + X_4. \\
 \mathfrak{g} \otimes X_2 &= \mathfrak{g} + X_2 + Y_2 + Y_2^* + A + C + C^* + X_3, \\
 \mathfrak{g} \otimes Y_2 &= \mathfrak{g} + X_2 + Y_2 + A + C + Y_3. \\
 \mathfrak{g} \otimes A &= X_2 + Y_2 + Y_2^* + 2A + C + C^* + X_3 + D + D^* + E + F + F^* + J, \\
 \mathfrak{g} \otimes C &= X_2 + Y_2 + A + 2C + X_3 + Y_3 + D + E + F + G + H + I, \\
 \mathfrak{g} \otimes X_3 &= X_2 + A + C + C^* + X_3 + E + F + F^* + I + I^* + X_4, \\
 \mathfrak{g} \otimes Y_3 &= Y_2 + C + Y_3 + D + G + Y_4. \\
 [(2)]X_2 &= 1 + 2Y_2 + 2Y_2^* + A + C + C^* + 2X_3 + D + D^* + E + H + H^* + J + X_4, \\
 [(1, 1)]X_2 &= \mathfrak{g} + 2X_2 + 2A + C + C^* + Y_3 + Y_3^* + E + F + F^* + I + I^*, \\
 X_2 \otimes Y_2 &= \mathfrak{g} + 2X_2 + Y_2 + Y_2^* + 2A + 2C + C^* + X_3 + Y_3 + D + E + F + F^* + G + I, \\
 [(2)]Y_2 &= 1 + 2Y_2 + Y_2^* + C + X_3 + D + H + J + Y_4, \\
 [(1, 1)]Y_2 &= \mathfrak{g} + X_2 + A + C + Y_3 + F + G, \\
 Y_2 \otimes Y_2^* &= X_2 + Y_2 + Y_2^* + A + C + C^* + X_3 + D + D^* + E.
 \end{aligned}$$

The starred versions of these formulae hold was well. Warning: the decomposition formulae for the plethysms in  $\mathfrak{g}^{\otimes 4}$  have only been checked on  $\mathcal{G}^0$ .

The following two tables describe the irreducible (virtual) constituents, in terms of their highest weights. For  $A_2.2$  and  $E_6.2$ , an irreducible representation  $V$  is either induced by an irreducible representation  $W$  of  $\mathcal{G}^0$  (non-isomorphic to its transformed by the non-trivial element  $\sigma$  of  $\Gamma$ ), or remains irreducible when restricted to  $\mathcal{G}^0$ . In the latter case, it is determined by its restriction to  $\mathcal{G}^0$  and by the sign  $\pm$  by which  $\sigma$  acts on the highest weight subspace. Notation:  $\lambda^0$  (resp.  $\lambda^\pm$ ), for  $\lambda$  the highest weight of  $W$  (resp.  $V$ ). For  $D_4 \cdot S_3$ , the possibilities are: (a) The restriction to  $\mathcal{G}^0$  is twice an irreducible  $W$ . On the highest weight subspace,  $S_3$  acts by its 2-dimensional representation. (b) The restriction to  $\mathcal{G}^0$  is irreducible;  $S_3$  acts trivially, or by the sign character, on the highest weight subspace. (c) The representation is induced from an irreducible representation  $W$  of a subgroup  $D_4.2$ . The representation  $W$  has an irreducible restriction to  $\mathcal{G}^0$  and is described as for  $A_2.2$  and  $E_6.2$ . (d) The representation is induced from an irreducible representation of  $\mathcal{G}^0$ . Notation:  $2\lambda$ ,  $\lambda^\pm$ ,  $\lambda^{0\pm}$ ,  $\lambda^0$ .

	$A_1$	$A_2$	$G_2$	$D_4.S_3$	$F_4$
$\mathfrak{g}$	[2]	[1, 1] <sup>-</sup>	[0, 1]	[0, 1, 0, 0] <sup>+</sup>	[1, 0, 0, 0]
$X_2$	0	[0, 3] <sup>0</sup>	[3, 0]	[1, 0, 1, 1] <sup>+</sup>	[0, 1, 0, 0]
$Y_2$	[4]	[2, 2] <sup>+</sup>	[0, 2]	[0, 2, 0, 0] <sup>+</sup>	[2, 0, 0, 0]
$Y_2^*$	0	[1, 1] <sup>+</sup>	[2, 0]	[0, 0, 0, 2] <sup>0+</sup>	[0, 0, 0, 2]
$A$	0	[2, 2] <sup>-</sup>	[2, 1]	[0, 1, 0, 2] <sup>0+</sup>	[1, 0, 0, 2]
$C$	0	[1, 4] <sup>0</sup>	[3, 1]	[1, 1, 1, 1] <sup>+</sup>	[1, 1, 0, 0]
$C^*$	-[2]	0	[1, 1]	2[1, 0, 1, 1]	[0, 0, 1, 1]
$X_3$	-[4]	0	[4, 0]	[0, 0, 2, 2] <sup>0+</sup>	[0, 0, 2, 0]
$Y_3$	[6]	[3, 3] <sup>-</sup>	[0, 3]	[0, 3, 0, 0] <sup>+</sup>	[3, 0, 0, 0]
$Y_3^*$	0	[0, 0] <sup>-</sup>	[1, 0]	2[0, 1, 0, 0]	[0, 0, 1, 0]
$D$	0	[3, 3] <sup>+</sup>	[2, 2]	[0, 2, 0, 2] <sup>0+</sup>	[2, 0, 0, 2]
$D^*$	[2]	0	0	2[0, 2, 0, 0]	[1, 0, 1, 0]
$E$	0	0	[1, 2]	2[1, 1, 1, 1]	[1, 0, 1, 1]
$F$	0	0	[5, 0]	[1, 0, 1, 3] <sup>0+</sup>	[0, 1, 0, 2]
$F^*$	0	[-2, 2] <sup>-</sup>	0	[0, 0, 2, 2] <sup>0-</sup>	[0, 0, 1, 2]
$G$	0	[2, 5] <sup>0</sup>	[3, 2]	[1, 2, 1, 1] <sup>+</sup>	[2, 1, 0, 0]
$G^*$	0	[-0, 0] <sup>-</sup>	0	[0, 0, 0, 2] <sup>0-</sup>	[1, 0, 0, 1]
$H$	0	[0, 6] <sup>0</sup>	[6, 0]	[2, 0, 2, 2] <sup>+</sup>	[0, 2, 0, 0]
$H^*$	-[0]	[-1, 1] <sup>+</sup>	0	[1, 0, 1, 1] <sup>-</sup>	[0, 0, 0, 3]
$I$	-[6]	0	[4, 1]	[0, 1, 2, 2] <sup>0+</sup>	[1, 0, 2, 0]
$I^*$	0	[-0, 3] <sup>0</sup>	0	[0, 1, 0, 2] <sup>0-</sup>	[0, 1, 0, 1]
$J$	0	0	0	[0, 0, 0, 4] <sup>0+</sup>	[0, 0, 0, 4]
$X_4$	0	[-1, 4] <sup>0</sup>	0	[1, 0, 1, 3] <sup>0-</sup>	[0, 0, 2, 1]
$Y_4$	[8]	[4, 4] <sup>+</sup>	[0, 4]	[0, 4, 0, 0]	[4, 0, 0, 0]
$Y_4^*$	0	0	0	2[0, 0, 0, 0]	[0, 0, 0, 1]

The weight  $\lambda$  is a linear combination  $\sum a_i \omega_i$  of fundamental weights. The fundamental weights are numbered as in the tables of Bourbaki, and  $\lambda$  is described as the sequence of the coefficients  $a_i$ .

For  $V$  an irreducible  $\mathcal{G}$ -module, let  $\gamma(V)$  denote the scalar by which the Casimir operator acts on  $V$ . For each of the 25 modules above, this value has been computed. It turns out that for each module this scalar is a linear expression in  $a$ . If we scale the Casimir operator such that  $\gamma(\mathfrak{g}) = 1$ , we find:

$V$	$\gamma(V)$	$V$	$\gamma(V)$
$\mathfrak{g}$	1	$E$	21/6
$X_n$	$n$	$F$	22/6 + 2a
$Y_n$	$n + (n^2 - n)a$	$G$	4 + 8a
$A$	16/6	$H$	4 + 6a
$C$	3 + 3a	$I$	4 + 4a
$D$	22/6 + 4a	$J$	20/6

With  $a^* = -1/6 - a$  these expressions remain valid after applying the involution  $*$ . This determines the value  $\gamma(V)$  for the starred modules  $V$  missing from the above table.

Let  $j \in \{2, 3, 4\}$ . Each of the decomposition formulae belonging to  $\mathfrak{g}^{\otimes j}$  gives a linear relation between the dimensions of the modules, provided the dimensions of the modules occurring in  $\mathfrak{g}^{\otimes i}$  with  $i < j$  are given. Additional relations were obtained by computing the trace of the Casimir operator, where we used (1) and (2) of [1]. In this way we obtain sufficiently many linear equations over  $\mathbb{Q}(a)$  to solve for the dimensions of the constituents occurring in  $\mathfrak{g}^{\otimes j}$  but not before. This procedure leads to a unique solution. Thus we find rational functions in  $a$  giving the dimensions of our modules for the special values  $a = 1/2, 1/3, 1/4, 1/6, 1/9, 1/12, 1/18, 1/30$ . Miraculously, for all these rational functions both numerator and denominator factor in  $\mathbb{Q}[a]$  as a product of linear

	$E_{6.2}$	$E_7$	$E_8$
$\mathfrak{g}$ .....	$[0, 1, 0, 0, 0, 0]^+$	$[1, 0, 0, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0, 0, 0, 1]$
$X_2$ .....	$[0, 0, 0, 1, 0, 0]^+$	$[0, 0, 1, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0, 0, 1, 0]$
$Y_2$ .....	$[0, 2, 0, 0, 0, 0]^+$	$[2, 0, 0, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0, 0, 0, 2]$
$Y_2^*$ .....	$[1, 0, 0, 0, 0, 1]^+$	$[0, 0, 0, 0, 0, 1, 0]$	$[1, 0, 0, 0, 0, 0, 0, 0]$
$A$ .....	$[1, 1, 0, 0, 0, 1]^+$	$[1, 0, 0, 0, 0, 1, 0]$	$[1, 0, 0, 0, 0, 0, 0, 1]$
$C$ .....	$[0, 1, 0, 1, 0, 0]^+$	$[1, 0, 1, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0, 0, 1, 1]$
$C^*$ .....	$[0, 0, 0, 0, 1, 1]^0$	$[0, 1, 0, 0, 0, 0, 1]$	$[0, 1, 0, 0, 0, 0, 0, 0]$
$X_3$ .....	$[0, 0, 1, 0, 1, 0]^+$	$[0, 0, 0, 1, 0, 0, 0]$	$[0, 0, 0, 0, 0, 1, 0, 0]$
$Y_3$ .....	$[0, 3, 0, 0, 0, 0]^+$	$[3, 0, 0, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0, 0, 0, 3]$
$Y_3^*$ .....	$[1, 0, 0, 0, 0, 1]^+$	$[0, 0, 0, 0, 0, 0, 2]$	0
$D$ .....	$[1, 2, 0, 0, 0, 1]^+$	$[2, 0, 0, 0, 0, 1, 0]$	$[1, 0, 0, 0, 0, 0, 0, 2]$
$D^*$ .....	$[1, 1, 0, 0, 0, 1]^+$	$[1, 0, 0, 0, 0, 0, 2]$	0
$E$ .....	$[0, 1, 0, 0, 1, 1]^0$	$[1, 1, 0, 0, 0, 0, 1]$	$[0, 1, 0, 0, 0, 0, 0, 1]$
$F$ .....	$[1, 0, 0, 1, 0, 1]^+$	$[0, 0, 1, 0, 0, 1, 0]$	$[1, 0, 0, 0, 0, 0, 1, 0]$
$F^*$ .....	$[0, 0, 1, 0, 0, 2]^0$	$[0, 0, 0, 0, 1, 0, 1]$	$[0, 0, 1, 0, 0, 0, 0, 0]$
$G$ .....	$[0, 2, 0, 1, 0, 0]^+$	$[2, 0, 1, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0, 0, 1, 2]$
$G^*$ .....	$[0, 0, 0, 1, 0, 0]^+$	0	$-[0, 1, 0, 0, 0, 0, 0, 0]$
$H$ .....	$[0, 0, 0, 2, 0, 0]^+$	$[0, 0, 2, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0, 0, 2, 0]$
$H^*$ .....	$[0, 0, 0, 0, 0, 3]^0$	0	0
$I$ .....	$[0, 1, 1, 0, 1, 0]^+$	$[1, 0, 0, 1, 0, 0, 0]$	$[0, 0, 0, 0, 0, 1, 0, 1]$
$I^*$ .....	$[0, 0, 1, 0, 1, 0]^+$	$[0, 2, 0, 0, 0, 0, 0]$	0
$J$ .....	$[2, 0, 0, 0, 0, 2]^+$	$[0, 0, 0, 0, 0, 2, 0]$	$[2, 0, 0, 0, 0, 0, 0, 0]$
$X_4$ .....	$[0, 0, 2, 0, 0, 1]^0$	$[0, 1, 0, 0, 1, 0, 0]$	$[0, 0, 0, 0, 1, 0, 0, 0]$
$Y_4$ .....	$[0, 4, 0, 0, 0, 0]^+$	$[4, 0, 0, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0, 0, 0, 4]$
$Y_4^*$ .....	$[0, 1, 0, 0, 0, 0]^+$	0	$-[1, 0, 0, 0, 0, 0, 0, 0]$

factors. The formulae are given in the following table, where  $\lambda = -6a$ . As all formulae are stable under \* we omit the starred modules (observe that  $\lambda^* = 1 - \lambda$ ).

$$\dim \mathfrak{g} = -2 \frac{(\lambda + 5)(\lambda - 6)}{\lambda(\lambda - 1)},$$

$$\dim X_2 = 5 \frac{(\lambda + 5)(\lambda - 6)(\lambda + 3)(\lambda - 4)}{\lambda^2(\lambda - 1)^2},$$

$$\dim Y_2 = -90 \frac{(\lambda - 4)(\lambda + 5)}{\lambda^2(\lambda - 1)(2\lambda - 1)},$$

$$\dim A = -27 \frac{(\lambda - 4)(\lambda - 5)(\lambda - 6)(\lambda + 5)(\lambda + 4)(\lambda + 3)}{\lambda^2(3\lambda - 1)(3\lambda - 2)(\lambda - 1)^2},$$

$$\dim C = 640 \frac{(\lambda - 3)(\lambda - 5)(\lambda + 5)(\lambda + 3)}{\lambda^3(\lambda - 1)(2\lambda - 1)(3\lambda - 2)},$$

$$\dim X_3 = -10 \frac{(\lambda - 3)(\lambda - 5)(\lambda - 6)(\lambda + 5)(\lambda + 4)(\lambda + 2)}{(\lambda - 1)^3 \lambda^3},$$

$$\dim Y_3 = -10 \frac{(\lambda - 4)(\lambda - 5)(\lambda - 6)(5\lambda - 6)(\lambda + 5)}{\lambda^3(3\lambda - 1)(2\lambda - 1)(\lambda - 1)^2},$$

$$\dim D = -90 \frac{(\lambda - 3)(\lambda - 6)(5\lambda - 6)(2\lambda - 5)(\lambda + 5)(\lambda + 4)(\lambda + 3)}{\lambda^3(4\lambda - 1)(2\lambda - 1)^2(\lambda - 1)^3},$$

$$\begin{aligned} \dim E &= 512 \frac{(\lambda-3)(\lambda-4)(\lambda-5)(\lambda-6)(\lambda+5)(\lambda+4)(\lambda+3)(\lambda+2)}{\lambda^2(4\lambda-1)(4\lambda-3)(2\lambda-1)^2(\lambda-1)^2}, \\ \dim F &= -810 \frac{(\lambda-4)(\lambda-6)(2\lambda-5)(\lambda+5)(\lambda+3)(\lambda+2)}{\lambda^3(3\lambda-1)(2\lambda-1)^2(\lambda-1)^2}, \\ \dim G &= 45 \frac{(\lambda-4)(\lambda-5)(\lambda-6)(3\lambda-4)(2\lambda-5)(\lambda+5)(\lambda+3)}{\lambda^4(3\lambda-1)(2\lambda-1)^2(\lambda-1)^2}, \\ \dim H &= -10 \frac{(\lambda-5)(\lambda-6)(3\lambda-4)(5\lambda-6)(2\lambda-5)(\lambda+5)(\lambda+4)(\lambda+3)}{\lambda^4(3\lambda-2)(2\lambda-1)^2(\lambda-1)^3}, \\ \dim I &= -405 \frac{(\lambda-4)(\lambda-5)(\lambda+5)(\lambda+4)(2\lambda-5)(\lambda+2)(5\lambda-6)}{\lambda^4(4\lambda-3)(2\lambda-1)(3\lambda-2)(\lambda-1)^2}, \\ \dim J &= 81 \frac{(\lambda-3)(\lambda-4)(\lambda-6)(2\lambda+3)(2\lambda-5)(\lambda+5)(\lambda+3)(\lambda+2)}{\lambda^2(3\lambda-1)(3\lambda-2)(2\lambda-1)^2(\lambda-1)^2}, \\ \dim X_4 &= 5 \frac{(\lambda-4)(\lambda-5)(\lambda-6)(2\lambda+3)(2\lambda-5)(\lambda+5)(\lambda+4)(\lambda+3)}{\lambda^4(\lambda-1)^4}, \\ \dim Y_4 &= -5 \frac{(\lambda-3)(\lambda-4)(\lambda-5)(\lambda-6)(3\lambda-4)(2\lambda-5)(\lambda+5)(7\lambda-6)}{\lambda^4(4\lambda-1)(3\lambda-1)(2\lambda-1)(3\lambda-2)(\lambda-1)^2}. \end{aligned}$$

The only rational values of  $\lambda$  for which all those dimension formulae (including those obtained by replacing  $\lambda$  by  $1-\lambda$ ) take integer values with  $\dim \mathfrak{g} \geq 0$  are those corresponding to  $a$  or  $a^*$  equal to  $5/6, 2/3, 1/2, 1/3, 1/4, 1/6, 1/9, 1/12, 1/18, 1/24, 1/30$ . The value  $a = 5/6$  corresponds to the trivial group (noted by Deligne): all dimension formulae take values 1, 0, or  $-1$  and when the dimension is  $\pm 1$ , the Casimir is 0. Of the remaining values, only  $2/3$  and  $1/24$  are not accounted for by a group in our list. For  $a = 2/3$ , all dimension formulae take values 1, 0, or  $-1$ , with the value 1 for  $\dim \mathfrak{g}$ . For  $a = 1/24$ , the dimension formula gives  $\dim \mathfrak{g} = 190$ .

The rational values of  $\lambda$  for which  $\dim \mathfrak{g}$  is negative and all dimension formulae take integer values can all be found by applying the involution  $a \mapsto -a/(12a+1)$  (which corresponds to  $\lambda \mapsto \lambda/(2\lambda-1)$ ) to the values of  $a$  above, excluding  $1/12$  and  $1/6$ . This involution transforms  $\dim \mathfrak{g}$  into  $-244 - \dim \mathfrak{g}$ . The values for  $\lambda$  obtained in this way from  $1/12$  and  $1/6$  (namely  $1/4$  and  $1/3$ ) are poles of some of the dimension formulae.

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### References

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