Effective Equations for Two-Phase Flow with Trapping on the Micro Scale

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1 Introduction

A widely used technique to remove oil from reservoirs is water-drive. Through injection wells water is being pumped into the reservoir, driving the oil to the production wells. The presence of rock heterogeneities in the reservoir generally has an unfavorable effect on the recovery rate. For instance, when preferential paths (high permeability regions) exist from injection to production wells, much oil will be by-passed and consequently the oil recovery rate will be small.

Conversely, when rock heterogeneities occur perpendicular to flow from injection to production wells (so-called cross-bedded or laminated structures) oil may be trapped at the interface from high to low permeability and become inaccessible to flow, leading again to a reduction in recovery rate. This latter case was studied by KORTEKAAS [18], VAN DUIJN et al. [13], and more recently by VAN LINGEN [20] who performed laboratory experiments using a porous column with periodically varying permeability, see Figure 1. In the same context, steady state solutions as well as an averaging procedure were considered by DALE et al. [9], [10].
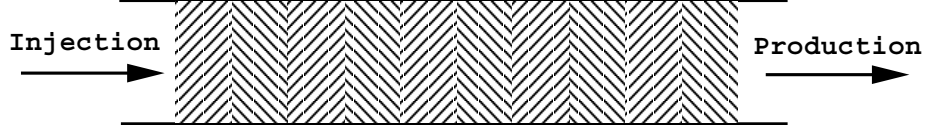
Figure 1: Periodically varying porous medium with high (coarse) and low (fine) permeability layers.

The main purpose of this paper is to derive in a rational way the effective flow equations corresponding to Figure 1, when the ratio of micro scale (periodicity length) and macro scale (column length) is small.

To this end we consider a one dimensional flow of two immiscible and incompressible phases (water being the wetting phase, oil the non-wetting phase) through a heterogeneous porous medium, characterized by a constant porosity Φ and a variable absolute permeability $k = k(x)$. The underlying equations are: mass balance for phases

$$\Phi \frac{\partial S_\alpha}{\partial t} + \frac{\partial q_\alpha}{\partial x} = 0 \quad (\alpha = o, w) ; \tag{1.1}$$

momentum balance for phases (Darcy law)

$$q_\alpha = -k(x) \frac{k_{r\alpha}(S_\alpha)}{\mu_\alpha} \frac{\partial p_\alpha}{\partial x} ; \tag{1.2}$$

and the complementary conditions

$$S_o + S_w = 1 ; \tag{1.3}$$

$$p_o - p_w = p_c(x, S_w). \tag{1.4}$$

Here $S_\alpha, q_\alpha, k_{r\alpha}, \mu_\alpha$ and p_α denote, respectively, the saturation, specific discharge, relative permeability, viscosity and pressure of phase α. Throughout we assume that the phase saturations are normalized, i.e. $0 \leq S_\alpha \leq 1$. Condition (1.3) expresses the presence of two phases only. The phase pressures differ due to interface tension on the pore scale. This is expressed by (1.4), where p_c denotes the induced capillary pressure. In petroleum engineering it is usually described by the Leverett model, see LEVERETT [19] or BEAR [2],

$$p_c(x, S_w) = \sigma \sqrt{\frac{\Phi}{k(x)}} J(S_w), \tag{1.5}$$

where σ denotes the interfacial tension and J the Leverett function. The relative permeabilities $k_{r\alpha} : [0, 1] \rightarrow [0, \infty)$ and the Leverett function $J : (0, 1] \rightarrow [0, \infty)$ are assumed to be smooth generalizations of power law functions (COREY [8], BROOKS & COREY [6]) satisfying the structural properties:

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A_1: $k_{r\alpha}$ strictly increasing in $[0,1]$ with $k_{r\alpha}(0) = 0$;
A_2: $J(0+) = \infty$, $J(1) > 0$ and $J' < 0$ in $(0,1]$,

where the prime denotes differentiation.

Here we explicitly assume $J(1) > 0$. Physically this means that a certain pressure, the capillary entry pressure given by $p_c(x,1)$, has to be exerted on the oil before it can enter a fully water saturated medium.

Equations (1.1) and condition (1.3) imply that the total specific discharge $q := q_o + q_w$ is constant in space. Throughout this paper we consider it constant in time as well. With $q > 0$ given, equations (1.1), (1.2) and conditions (1.3), (1.4) can be combined into a single transport equation for one saturation only. Since we are primary interested in the oil flow, we use the oil saturation for that purpose. In doing so, it is convenient to redefine k_{rw}, p_c and J in terms of S_o. Setting

$$u = S_o \quad (S_w = 1 - u),$$

we now write

$$k_{rw}(u) := \quad k_{rw}(1 - u),$$
$$p_c(x, u) := \quad p_c(x, 1 - u),$$
$$J(u) := \quad J(1 - u).$$

In terms of u assumptions A_{1-2} become

$$\tilde{A}_1 : \begin{cases} k_{rw} \text{ strictly decreasing in } [0,1] \text{ with } k_{rw}(1) = 0; \\ k_{ro} \text{ strictly increasing in } [0,1] \text{ with } k_{ro}(0) = 0; \end{cases}$$

$$\tilde{A}_2 : \quad J(1-) = \infty, \ J(0) > 0 \text{ and } J' > 0 \text{ in } [0,1).$$

Remark 1.1 *In most cases of practical interest the blow-up of J and J' near $u = 1$ is balanced by the behavior of k_{rw} near that point, in the sense that $k_{rw}(u)J'(u) \to 0$ as $u \to 1$. The consequence of this behavior and its possible failure is investigated by* VAN DUIJN & FLORIS *[12]. Though important for the well-posedness of the mathematical formulation, no additional assumptions are required for the purpose of this paper.*

Applying the scalings

$$x := x/L_x, \ t := \frac{t\,q}{\Phi L_x} \text{ and } k := k/K, \tag{1.6}$$

where L_x is a characteristic macroscopic length scale and K a characteristic permeability value, we find for the oil saturation the balance equation

$$\frac{\partial u}{\partial t} + \frac{\partial F}{\partial x} = 0\,, \tag{1.7a}$$

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where

$$F = f(u) - N_c k(x) \lambda(u) \frac{\partial}{\partial x} p_c(x, u). \tag{1.7b}$$

Here

$$f(u) = \frac{k_{ro}(u)}{k_{ro}(u) + M k_{rw}(u)} \tag{1.8}$$

denotes the oil fractional flow function, and

$$\lambda(u) = k_{rw}(u) f(u), \ p_c(x, u) = J(u) / \sqrt{k(x)}. \tag{1.9}$$

The two dimensionless numbers involved are the capillary number N_c and the viscosity ration M. They are given by

$$N_c = \frac{\sigma \sqrt{K \Phi}}{\mu_w q L_x} \text{ and } M = \frac{\mu_o}{\mu_w}. \tag{1.10}$$

Remark 1.2 *(i) Assumptions $\tilde{A}_{1-2}$ imply*

$$f(0) = 0, \ f(1) = 1 \quad and \quad f \ strictly \ increasing \ in \ [0, 1];$$

$$\lambda(0) = \lambda(1) = 0 \quad and \quad \lambda(u) > 0 \ for \ 0 < u < 1.$$

(ii) Depending on the specific application, the value of the capillary number may vary considerably. For instance, adding surfactants or polymers may substantially alter σ or μ_w. Likewise the flow rate q can have different values. Therefore we investigate in Section 2 the consequences of having a moderate and a small value for N_c.

(iii) Petroleum engineers define the capillary number (1.10) in the reciprocal way, i.e. $N_c = \dfrac{\mu_w q L_x}{\sigma \sqrt{K \Phi}}$. Here we do not adopt this convention because we want to emphasize the direct proportionality between the capillary number and the interface tension σ.

Two typical capillary pressures $p_c = p_c(x, u)$ are shown in Figure 2. They relate to fine $(k = k^-)$ and to coarse $(k = k^+)$ material, where $k^- < k^+$.

We consider equation (1.7) in the domain $\Sigma = \mathbb{R}$ and for $t > 0$, subject to the initial condition

$$u(x, 0) = u_0(x) \text{ for } x \in \Sigma. \tag{1.11}$$

When k is constant and $u_0 : \Sigma \to [0, 1]$ is such that $\int_0^{u_0} \lambda(s) J'(s) ds$ is uniformly Lipschitz continuous in Σ, Problem (1.7), (1.11) admits a unique weak solution $u : \Sigma \times [0, \infty) \to [0, 1]$. This follows from the work of ALT & LUCKHAUS [1], VAN DUIJN & YE [14], GILDING [15], [16], or BENILAN & TOURE [3]. This weak solution is smooth whenever $u \in (0, 1)$ and has the usual regularity for

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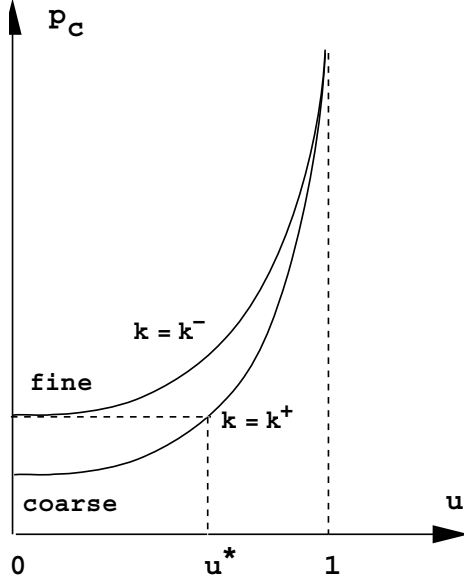
Figure 2: Dimensionless capillary pressure in terms of oil saturation: top (bottom) reflects fine (coarse) material.

degenerate equations across possible free boundaries near $u = 0$ and $u = 1$. When k is piecewise constant, in particular

$$k(x) = \begin{cases} k^+, & x < 0 \\ k^-, & x > 0, \end{cases} \qquad (1.12)$$

equation (1.7) cannot always be interpreted across the interface where k is discontinuous. This is due to a possible discontinuity in capillary pressure. Using a regularization procedure, this was demonstrated by VAN DUIJN et al [13] for equation (1.7), and rigorously proven by BERTSCH et al [5] for a simplified equation. Instead one considers equation (1.7) only in the sub-domains where k is constant, together with matching conditions across k-discontinuities. For k given by (1.12), with $k^- < k^+$, the matching conditions read for all $t > 0$:

$$(i) \qquad [F(t)] = 0; \qquad (1.13)$$

$$(ii) \quad u(0+, t)[p_c(t)] = 0, \quad [p_c(t)] \geq 0, \qquad (1.14)$$

where $[F(t)] = F(0+, t) - F(0-, t)$ and $[p_c(t)]$ likewise. The first condition expresses continuity of flux and is obvious. The second conditions tells us that the capillary pressure is only continuous if both phases are present on both sides of the k-discontinuity. This is to be expected from Darcy law (1.2) since then both phase pressures are continuous. If oil is absent for $x > 0$, i.e. in the fine material, the entry pressure for oil leads to a discontinuous capillary pressure.

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With reference to Figure 2, the pressure condition (1.14) can be formulated as

$$(\text{ii'}) \quad \begin{cases} u(0-,t) < u^* \text{ implies } u(0+,t) = 0 \\[2mm] u(0-,t) \geq u^* \text{ implies } \dfrac{J(u(0-,t))}{\sqrt{k^+}} = \dfrac{J(u(0+,t))}{\sqrt{k^-}}, \end{cases} \tag{1.15}$$

where u^* is uniquely defined by

$$\frac{J(u^*)}{\sqrt{k^+}} = \frac{J(0)}{\sqrt{k^-}}. \tag{1.16}$$

The occurrence of oil trapping at the transition from coarse to fine material is directly explained by conditions (1.13), (1.15). Let k be given by (1.12) and consider a steady state flow ($u = u(x)$) satisfying

$$u(\pm\infty) = 0, \tag{1.17}$$

i.e. injection and production of water, with oil possibly present near $x = 0$. Then, by (1.7),

$$F = \text{ constant } = 0 \text{ on } \mathbb{R}$$

or

$$f(u) \left\{ 1 - N_c \sqrt{k(x)} k_{rw}(u) J'(u) \frac{du}{dx} \right\} = 0 \text{ on } \mathbb{R} \backslash \{0\},$$

with (1.15) at $x = 0$. Since $f(u) > f(0) = 0$ for $u > 0$, we have

$$u = 0 \text{ or } \frac{du}{dx} = \frac{1}{N_c \sqrt{k(x)} k_{rw}(u) J'(u)} > 0 \tag{1.18}$$

for $x \in \mathbb{R} \backslash \{0\}$. Since $u(+\infty) = 0$, we find

$$u(x) = 0 \text{ for all } x > 0 \tag{1.19}$$

and, by (1.15),

$$u(0-) \leq u^*. \tag{1.20}$$

Using (1.20) as initial condition for (1.18) on $(-\infty, 0)$, one easily constructs a family of non-trivial steady states describing the saturation of the trapped oil in the coarse material. The initial condition in an actual displacement process determines which of the steady states is selected. This is discussed by BERTSCH et al [5].

Note that the non-uniqueness results from (1.19). Considering $u(\pm\infty) = \hat{u} \in (0, 1]$, one finds a unique steady state satisfying (1.15) (continuity of pressure) at $x = 0$. Such solutions were considered by YORTSOS & CHANG [25] for smooth k.

We now turn to the problem with micro structure, as indicated in Figure 1, where trapping occurs at all transitions from high to low permeability. As a

result we expect to find a trapping related threshold saturation (irreducible oil saturation) below which the oil becomes immobile. We consider the case of a periodic as well as a random medium.

In Section 2 we assume a periodic micro structure of coarse ($k = k^+$) and fine ($k = k^-$) material, each of length $L_y << L_x$, see Figure 3. This leads to a natural choice of the small expansion parameter $\varepsilon = L_y/L_x$. We outline the
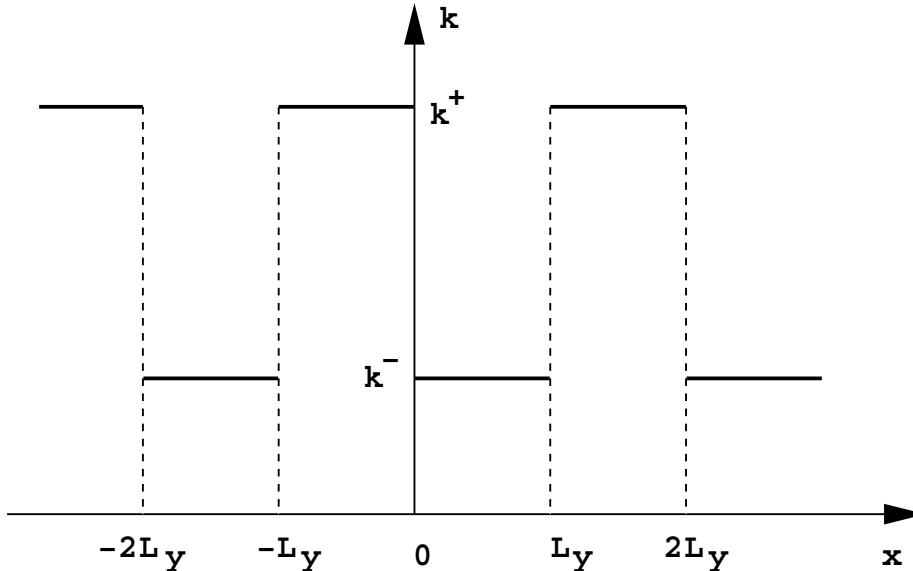
Figure 3: Periodic permeability (unscaled coordinate).

homogenization procedure, study the resulting auxiliary problems and derive the effective (up-scaled or averaged) equations for the limit $\varepsilon \searrow 0$. In doing so, the magnitude of the capillary number N_c is important. We work out two cases:

Capillary limit, $N_c = 0(1)$. This case is relative straightforward because the auxiliary problem only has constant state solutions (compare steady state solutions on $(-\varepsilon, 0)$). As a consequence the effective equation is found explicitly. It is again of convection-diffusion type and it involves weighted harmonic means of the fractional flow and capillary terms. Both convection and diffusion vanish from the equation if the averaged oil saturation drops below $\frac{1}{2}u^*$.

Balance, $N_c = O(\varepsilon)$. This case is much more involved. Now the diffusion term disappears in the homogenization procedure and one is left with a first order conservation law of Buckley-Leverett type. This follows from a detailed study of the auxiliary problem. We show that the upscaled oil-fractional flow function is different from a k-weighted version of f and contains quite surprisingly some elements of the small scale diffusion. Again it vanishes if the averaged oil satu-

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ration drops below a certain value. This irreducible oil saturation is related to a specific solution of the auxiliary problem.

In Section 3 we consider the case of a random micro structure with respect to both the location of the permeability jumps and the value of the permeability. The effective oil flux is obtained only for the capillary limit ($N_c = O(1)$) and involves again the weighted harmonic means of the fractional flow and capillary pressure terms. The homogenized equation has coefficients depending on the realization, but we prove that average saturation, defined by the homogenized parabolic problem, is a deterministic function. Consequently, it is sufficient to solve the effective equation for a single realization.

Section 4 contains some numerical results. There we take power law relative permeabilities and a Brooks-Corey capillary pressure. We compute the effective fractional flow and diffusivity for the capillary limit $N_c = O(1)$, and the effective fractional flow for the balance $N_c = O(\varepsilon)$.

Some concluding remarks are given in Section 5.

DALE et al. [9] studied a similar multi-phase flow problem. They consider steady state flow in a periodic porous column, allowing each periodicity cell to have more sub-layers with different relative permeabilities and Leverett functions. Without using the homogenization ansatz they derive upscaled expressions for the relative permeabilities. In this paper we present a rigorous analysis of the auxiliary problems, resulting in a fairly complete description of the upscaled equations. In particular, the effect of microscopic trapping, as a result of the different entry pressures, is investigated explicitly.

2 Homogenization procedure for periodic layers

A simplified version of Problem (1.7), (1.11), involving a single permeability discontinuity (or trap) only, was studied by BERTSCH et al [5]. They established existence and uniqueness of a solution satisfying the usual porous-media equation regularity away from the trap. In particular the solution is nonnegative and bounded. Moreover the corresponding flux was shown to be continuous in x, for almost all $t > 0$.

In this paper we silently assume the same properties for the saturation and flux in the case of multiple traps at arbitrary distances. In particular $0 \leq u \leq 1$. In our problem we deal with two natural length scales: a macroscopic length scale L_x and a microscopic scale (characteristic length scale of layers) L_y. This disparity in length scales is what provides us with our expansion parameter $\varepsilon = L_y/L_x$. For fixed, but small characteristic layer length L_y the solutions will in general be complicated having a different behavior on the two length scales. Closed-form solutions are unachievable and numerical solutions will be nearly impossible to calculate. It is our object to derive a flow equation at

the macro scale, keeping information about the trapping only through some averaged quantities.

To simplify our considerations we now suppose a periodic structure of the traps being located at the points $\{\varepsilon i : i \in \mathbb{Z}\}$. The corresponding permeability $k^\varepsilon(x)$ is defined by $k^\varepsilon(x) = k(x/\varepsilon)$, where

$$
k = \begin{cases} k^+ & \text{on } (2i - 1, 2i), \\ k^- & \text{on } (2i, 2i + 1). \end{cases} \tag{2.1}
$$

Without loss of generality we assume $0 < k^- < k^+ < \infty$. We distinguish two kinds of matching conditions: one going from k^+ to k^- and vice versa, see also (1.15).

At $x = 2i\varepsilon$ we impose:

$$
\begin{aligned}
&\text{if } u(2i\varepsilon - 0) < u^*, \text{ then } u(2i\varepsilon + 0) = 0; \\
&\text{if } u(2i\varepsilon - 0) \geq u^*, \text{ then } \frac{J(u(2i\varepsilon - 0))}{\sqrt{k^+}} = \frac{J(u(2i\varepsilon + 0))}{\sqrt{k^-}}.
\end{aligned} \tag{2.2}
$$

At $x = (2i + 1)\varepsilon$ we impose:

$$
\begin{aligned}
&\text{if } u((2i + 1)\varepsilon + 0) \geq u^*, \text{ then} \\
&\qquad \frac{J(u((2i + 1)\varepsilon + 0))}{\sqrt{k^+}} = \frac{J(u((2i + 1)\varepsilon - 0))}{\sqrt{k^-}}; \\
&\text{if } u((2i + 1)\varepsilon + 0) < u^*, \text{ then } u((2i + 1)\varepsilon - 0) = 0.
\end{aligned} \tag{2.3}
$$

We now replace k by k^ε in equation (1.7a-b). Clearly this equation holds in the domain $\Sigma^\varepsilon = \mathbb{R} \backslash T_\varepsilon$, where $T_\varepsilon = \varepsilon \bigcup_{i \in \mathbb{Z}} i$. Let u^ε be a solution of (1.7a) satisfying the matching conditions (2.2) and (2.3). Using the uniform L^∞ bound for u^ε, we consider the following two scale asymptotic expansion

$$
u^\varepsilon(x, t) = u^0(x, y, t) + \varepsilon u^1(x, y, t) + \varepsilon^2 u^2(x, y, t) \ldots, \tag{2.4}
$$

where $y = x/\varepsilon$ represents the fast scale. Substituting this expansion into equation (1.7) and equating terms of the same order of ε, gives equations for $u^0, u^1 \ldots$. As established for many linear problems containing periodic non-homogeneities, see for instance, BENSOUSSAN, LIONS & PAPANICOLAOU [4] or SANCHEZ-PALENCIA [23], we expect that

$$
U(x, t) = \frac{1}{2} \int_{-1}^{+1} u^0(x, y, t) dy \tag{2.5}
$$

is the weak limit of u^ε and that $u^0(x, \frac{x}{\varepsilon}, t)$ is the approximation to u^ε in some norm. Proving convergence of the homogenization procedure for nonlinear flow

problems in non-homogeneous geometries poses difficulties as is shown in HOR-
NUNG [17] and MIKELIĆ [21]. Given the nonlinear nature of equation in (1.7)
and matching conditions in (2.2) - (2.3), we shall therefore make no attempt at
proving convergence as $\varepsilon \to 0$. The purpose of this paper is merely to derive
the upscaled equations and to study the corresponding auxiliary problems.

Clearly our results depend strongly on the scaling of the capillary number N_c.
The main cases of interest are $N_c = O(1)$ and $N_c = O(\varepsilon)$. We will deal with
them separately.

2.1   Capillary limit: $N_c = O(1)$

Introducing the oil flux

$$F^\varepsilon = f(u^\varepsilon) - N_c \sqrt{k^\varepsilon(x)} D(u^\varepsilon) \frac{\partial u^\varepsilon}{\partial x}, \tag{2.6}$$

where

$$D(u^\varepsilon) = k_{rw}(u^\varepsilon) f(u^\varepsilon) J'(u_\varepsilon), \tag{2.7}$$

equation (1.7a) becomes

$$\frac{\partial u^\varepsilon}{\partial t} + \frac{\partial F^\varepsilon}{\partial x} = 0 \quad \text{in } \Sigma^\varepsilon \times (0, \infty). \tag{2.8}$$

We now apply expansion (2.4) to F^ε, which gives

$$
\begin{aligned}
F^\varepsilon = \ & -N_c D(u^0) \frac{\partial u^0}{\partial y} \sqrt{k} \varepsilon^{-1} \\
& + f(u^0) - N_c \sqrt{k} \left\{ D(u^0) \left(\frac{\partial u^1}{\partial y} + \frac{\partial u^0}{\partial x} \right) + D'(u^0) u^1 \frac{\partial u^0}{\partial y} \right\} \\
& + \left\{ f'(u^0) u^1 - N_c \sqrt{k} \left[D(u^0) \left(\frac{\partial u^2}{\partial y} + \frac{\partial u^1}{\partial x} \right) \right. \right. \\
& + D'(u^0) u^1 \left(\frac{\partial u^1}{\partial y} + \frac{\partial u^0}{\partial x} \right) \\
& \left. \left. + \left(D''(u^0) \frac{(u^1)^2}{2} + D'(u^0) u^2 \right) \frac{\partial u^0}{\partial y} \right] \right\} \varepsilon + O(\varepsilon^2) \\
=: \ & F^0 \varepsilon^{-1} + F^1 + F^2 \varepsilon + O(\varepsilon^2).
\end{aligned} \tag{2.9}
$$

Using this in (2.8) results in the following equations:

$$\varepsilon^{-2}: \qquad\qquad -N_c \frac{\partial}{\partial y} \left(\sqrt{k} D(u^0) \frac{\partial u^0}{\partial y} \right) = 0,$$

thus, by continuity of F^ε,

$$-N_c \sqrt{k} D(u^0) \frac{\partial u^0}{\partial y} = F^0 = F^0(x, t), \tag{2.10}$$

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which holds for every $x, y, \in \mathbb{R}$ and for all $t > 0$. Note that this observation is expected because of the continuity of the flux.

$$\varepsilon^{-1}: \qquad 0 = \frac{\partial F^0}{\partial x} + \frac{\partial F^1}{\partial y} = \frac{\partial}{\partial x}\left\{-N_c\sqrt{k}D(u^0)\frac{\partial u^0}{\partial y}\right\} +$$

$$\frac{\partial}{\partial y}\left\{f(u^0) - N_c\sqrt{k}\left[D(u^0)\left(\frac{\partial u^1}{\partial y} + \frac{\partial u^0}{\partial x}\right) + D'(u^0)u^1\frac{\partial u^0}{\partial y}\right]\right\} \qquad (2.11)$$

$$\varepsilon^0: \qquad \frac{\partial u^0}{\partial t} + \frac{\partial F^2}{\partial y} + \frac{\partial F^1}{\partial x} = 0. \qquad (2.12)$$

We look for y-periodic solutions of (2.10) satisfying (2.2) and (2.3), with x and t as given parameters. Our goal is to prove that $F^0 = 0$. We argue by contradiction.

Suppose $F^0 < 0$. Let

$$w(y) := J(u^0(y)), \quad \lambda(w) := k_{rw}(J^{-1}(w))f(J^{-1}(w))$$

and

$$\Lambda(w) = \int_{J(0)}^{w} \lambda(s)ds,$$

the last function being strictly increasing. Then (2.10) reads

$$\lambda(w)\sqrt{k}\frac{dw}{dy} = -\frac{F^0}{N_c} =: F > 0.$$

Hence, for $-1 < y < 0$,

$$\Lambda(w(0-)) - \Lambda(w(-1+0)) = \frac{F}{\sqrt{k^+}},$$

giving

$$w(0-) \geq w(-1+0) + \frac{F}{\sqrt{k^+}}\frac{1}{||\lambda||_\infty}. \qquad (2.13)$$

Similarly, for $0 < y < 1$,

$$w(1-0) \geq w(0+) + \frac{F}{\sqrt{k^-}}\frac{1}{||\lambda||_\infty}. \qquad (2.14)$$

Now we apply matching conditions (2.2) and (2.3) in terms of w. First, suppose $w(0-) \leq J(u^*)$. Then $w(0+) = J(0)$ and, by (2.14), $w(1-0) > J(0)$. Hence $w(-1+0) > J(u^*)$ giving - by (2.13) - $w(0-) > J(u^*)$, which contradicts the assumption. Next suppose $w(0-) > J(u^*)$. In this case we obtain $w(0+) =$

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$\sqrt{k^-/k^+}\ w(0-) < w(0-)$. By (2.14) and (2.13) we have

$$w(1-0) \geq \sqrt{\frac{k^-}{k^+}}w(0-) + \frac{F}{\sqrt{k^-}}\frac{1}{||\lambda||_\infty}$$

$$\geq \sqrt{\frac{k^-}{k^+}}w(-1+0) + \frac{F}{||\lambda||_\infty}\left\{\frac{\sqrt{k^-}}{k^+} + \frac{1}{\sqrt{k^-}}\right\}. \tag{2.15}$$

If $w(-1+0) > J(u^*)$, then $w(-1+0) = \sqrt{\frac{k^+}{k^-}}w(1-0)$. Substituting this into (2.15) yields $w(1-0) > w(-1-0)$, which contradicts the periodicity. If $w(-1+0) \leq J(u^*)$, then $w(1-0) = J(0)$, which contradicts (2.14). Hence $F^0 \geq 0$. A similar argument gives $F^0 \leq 0$, implying

$$F^0 = 0.$$

This conclusion allows us to solve equation in (2.10) with the matching conditions. We find

$$u^0(y) = \begin{cases} C > u^* & \text{for } -1 < y < 0, \\ \overline{C} & \text{for } 0 < y < 1, \end{cases} \tag{2.16}$$

where $\overline{C} = J^{-1}\left(\sqrt{\frac{k^-}{k^+}}J(C)\right)$, or

$$u^0(y) = \begin{cases} C \leq u^* & \text{for } -1 < y < 0, \\ 0 & \text{for } 0 < y < 1. \end{cases} \tag{2.17}$$

Now we consider the ε^{-1}-equation (2.11). Since $F^0 = 0$ and the flux is continuous we find

$$F^1 = F^1(x,t).$$

Using (2.16) and (2.17), the local form of F^1 is

$$F^1 = f(C) - N_c\sqrt{k^+}D(C)\left\{\frac{\partial C}{\partial x} + \frac{\partial u^1}{\partial y}\right\}, \tag{2.18}$$

for $-1 < y < 0$, and

$$F^1 = \begin{cases} f(\overline{C}) - N_c\sqrt{k^-}D(\overline{C})\left\{\frac{\partial \overline{C}}{\partial x} + \frac{\partial u^1}{\partial y}\right\} & \text{for } C > u^*, \\ 0 & \text{for } C \leq u^*, \end{cases} \tag{2.19}$$

for $0 < y < 1$. Clearly we only have to consider the non-trivial case $C > u^*$. From (2.18) and (2.19) we deduce

$$\frac{\partial u^1}{\partial y} = \begin{cases} \dfrac{f(C) - F^1}{\sqrt{k^+}N_cD(C)} - \dfrac{\partial C}{\partial x} =: B_1(x,t) & \text{for } -1 < y < 0, \\ \dfrac{f(\overline{C}) - F^1}{\sqrt{k^-}N_cD(\overline{C})} - \dfrac{\partial \overline{C}}{\partial x} =: B_2(x,t) & \text{for } 0 < y < 1. \end{cases}$$

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After integration we observe that $B_1 + B_2 = 0$. Hence we can solve for F^1 to find

$$F^1 = \frac{\dfrac{f(C)}{\sqrt{k^+}D(C)} + \dfrac{f(\overline{C})}{\sqrt{k^-}D(\overline{C})}}{\dfrac{1}{\sqrt{k^+}D(C)} + \dfrac{1}{\sqrt{k^-}D(\overline{C})}} - N_c \frac{\dfrac{\partial C}{\partial x} + \dfrac{\partial \overline{C}}{\partial x}}{\dfrac{1}{\sqrt{k^+}D(C)} + \dfrac{1}{\sqrt{k^-}D(\overline{C})}}.$$

Finally we use the ε^0 - equation in (2.12). Since F^2 is continuous in the fast scale, we find for the averaged oil saturation $U = \frac{1}{2}(C + \overline{C})$ the effective convection-diffusion equation

$$\frac{\partial U}{\partial t} + \frac{\partial}{\partial x}\left\{ \mathcal{F}(U) - N_c \mathcal{D}(U)\frac{\partial U}{\partial x} \right\} = 0, \tag{2.20}$$

where $-\infty < x < \infty$ and $t > 0$. One easily verifies

$$\mathcal{F}(U) = \begin{cases} 0 & \text{for } 0 \leq U \leq \frac{1}{2}u^*, \\ \text{strictly increasing} & \text{for } \frac{1}{2}u^* < U < 1, \\ 1 & \text{for } U = 1, \end{cases}$$

and

$$\mathcal{D}(U) = \begin{cases} 0 & \text{for } 0 \leq U \leq \frac{1}{2}u^*, \\ > 0 & \text{for } \frac{1}{2}u^* < U < 1, \\ 0 & \text{for } U = 1. \end{cases}$$

In Section 4 we show the graphs of $\mathcal{F}$ and $\mathcal{D}$ based on power law relative permeabilities and a Brooks-Corey capillary pressure.

2.2 Balance: $N_c = O(\varepsilon)$

Writing $N_c := N_c\varepsilon$, the oil flux (2.6) becomes

$$F^\varepsilon = f(u^\varepsilon) - N_c\varepsilon\sqrt{k^\varepsilon(x)}D(u^\varepsilon)\frac{\partial u^\varepsilon}{\partial x}. \tag{2.21}$$

Clearly expansion (2.9) changes due to the additional ε factor. It now takes the form

$$\begin{aligned} F^\varepsilon = {}& f(u^0) - N_c\sqrt{k}D(u^0)\frac{\partial u^0}{\partial y} + \Big\{ f'(u^0)u^1 \\ & - N_c\sqrt{k}\left[D(u^0)\left(\frac{\partial u^0}{\partial x} + \frac{\partial u^1}{\partial y} \right) + D'(u^0)u^1\frac{\partial u^0}{\partial y} \right] \Big\}\varepsilon + O(\varepsilon^2) \\ =: {}& F^0 + F^1\varepsilon + O(\varepsilon^2). \end{aligned} \tag{2.22}$$

Using this expansion in (2.8) gives

$$\frac{\partial u^0}{\partial t} + \frac{1}{\varepsilon}\frac{\partial F^0}{\partial y} + \frac{\partial F^0}{\partial x} + \frac{\partial F^1}{\partial y} = O(\varepsilon),$$

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resulting in the equations:

$$\varepsilon^{-1} : \qquad\qquad\qquad \frac{\partial F^0}{\partial y} = 0,$$

or, by the continuity of F^ε,

$$f(u^0) - N_c \sqrt{k} D(u^0) \frac{\partial u^0}{\partial y} = F^0 = F^0(x,t) \qquad (2.23)$$

which holds for every $x, y \in \mathbb{R}$ and for all $t > 0$;

$$\varepsilon^0 : \qquad\qquad\qquad \frac{\partial u^0}{\partial t} + \frac{\partial F^0}{\partial x} + \frac{\partial F^1}{\partial y} = 0. \qquad (2.24)$$

First equation (2.23) needs to be considered. It leads to the following auxiliary problem.

PROBLEM A_u: Given $F \in \mathbb{R}$, find $u : [-1,0) \cup (0,1] \to [0,1]$ satisfying

$$f(u) - N_c \sqrt{k} k_{rw}(u) f(u) J'(u) \frac{du}{dy} = F \text{ in } (-1,0) \cup (0,1) \qquad (2.25)$$

subject to the matching condition $(y = 0)$

$$\begin{cases} \text{if } u(0-) < u^*, & \text{then } u(0+) = 0; \\[2mm] \text{if } u(0-) \geq u^*, & \text{then } \dfrac{J(u(0-))}{\sqrt{k^+}} = \dfrac{J(u(0+))}{\sqrt{k^-}}. \end{cases} \qquad (2.26)$$

and the periodicity condition $(y = \pm 1)$

$$\begin{cases} \text{if } u(-1+0) < u^* & \text{then } u(1-0) = 0; \\[2mm] \text{if } u(-1+0) \geq u^* & \text{then } \dfrac{J(u(-1+0))}{\sqrt{k^+}} = \dfrac{J(u(1-0))}{\sqrt{k^-}}. \end{cases} \qquad (2.27)$$

This problem is considered in detail in the next sections. We prove existence for $0 \leq F \leq 1$ and uniqueness for $F > 0$. Moreover, we show monotone dependence and differentiability of u with respect to F. After that equation (2.24) is averaged over the cell $(-1,0) \cup (0,1)$ to obtain the upscaled (macroscopic) transport equation. This equation turns out to be of Buckley-Leverett type.

2.3 Auxiliary problem

To simplify the analysis we introduce, as in Section 2.1, the function $w = J(u)$ and set
$$\gamma(w) = k_{rw}(J^{-1}(w)) \text{ and } \varphi(w) = f(J^{-1}(w)).$$
In terms of w, the auxiliary problem A_u becomes

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PROBLEM A_w: Given $F \in \mathbb{R}$, find $w : [-1,0) \cup (0,1] \to [J(0), \infty)$ satisfying

$$\varphi(w) \left\{ 1 - N_c \sqrt{k} \gamma(w) \frac{dw}{dy} \right\} = F \text{ in } (-1,0) \cup (0,1) \qquad (2.28)$$

such that (at $y = 0$)

$$\begin{cases} \text{if } w(0-) < J(u^*), & \text{then } w(0+) = J(0); \\[2mm] \text{if } w(0-) \geq J(u^*), & \text{then } w(0+) = \sqrt{\dfrac{k^-}{k^+}} w(0-); \end{cases} \qquad (2.29)$$

and (at $y = \pm 1$)

$$\begin{cases} \text{if } w(-1+0) < J(u^*), & \text{then } w(1-0) = J(0); \\[2mm] \text{if } w(-1+0) \geq J(u^*), & \text{then } w(1-0) = \sqrt{\dfrac{k^-}{k^+}} w(-1+0). \end{cases} \qquad (2.30)$$

We first demonstrate existence and some qualitative properties for $0 = f(0) \leq F \leq f(1) = 1$.

Lemma 2.1 *Let $F > 1$. Then there are no solutions to Problem A_w.*

Proof. Since f is strictly increasing we have

$$\frac{F}{\varphi(w)} - 1 \geq \frac{F}{f(1)} - 1 > 0,$$

and consequently, by (2.28), $dw/dy < 0$ on $(-1,0) \cup (0,1)$. Now suppose $w(0-) < J(u^*)$. Then $w(0+) = J(0)$ and thus $w < J(0)$ on $(0,1)$, contradicting $w \geq J(0)$ from the definition. If $w(0-) \geq J(u^*)$, then clearly $w(-1+0) > w(0-) \geq J(u^*)$ yielding

$$w(0+) = \sqrt{\frac{k^-}{k^+}} w(0-), \quad w(1-0) = \sqrt{\frac{k^-}{k^+}} w(-1+0).$$

This implies $w(1-0) > w(0+)$, contradicting $dw/dy < 0$ on $(0,1)$. $\qquad \square$

Lemma 2.2 *Let $F < 0$. Then there are no solutions to Problem A_w.*

Proof. Equation (2.28) now gives $dw/dy > 0$ on $(-1,0) \cup (0,1)$. Now suppose $w(0-) \leq J(u^*)$. Then $w(0+) = J(0)$ and $w(1-0) > J(0)$. Hence $w(-1+0) > J(u^*)$, contradicting $w(0-) \leq J(u^*)$. Next let $w(0-) > J(u^*)$. Then $w(0+) = \sqrt{\frac{k^-}{k^+}} w(0-)$ and $w(1-0) > w(0+) = \sqrt{\frac{k^-}{k^+}} w(0-) > \sqrt{\frac{k^-}{k^+}} J(u^*) = J(0)$. Thus $w(-1+0) \geq J(u^*)$ and, from the w-monotonicity, $\sqrt{\frac{k^-}{k^+}} w(0-) > \sqrt{\frac{k^-}{k^+}} w(-1+0)$ or $w(0+) > w(1-0)$, contradicting $dw/dy > 0$ on $(0,1)$. $\qquad \square$

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Corollary 2.3 *Let $F = 1$. Then $u = 1$ uniquely solves Problem A_u.*

Proof. We use the u-formulation in Problem A_u. Clearly $u = 1$ is a solution. To show uniqueness, suppose there exists a solution u such that $u(y_0) < 1$ for some $y_0 \in (-1, 0) \cup (0, 1)$. Since $du/dy < 0$ whenever $u < 1$, we have the following two possibilities. Either we have $u < 1$ everywhere and strictly decreasing, or there exists $y_1 < y_0$ such that $u(y_1) = 1$. The first possibility leads to a contradiction using the monotone relations imposed by the matching conditions, since $u(0+) > u(1)$ implies $u(0-) > u(-1)$. The second possibility implies $u(y) = 1$ for all $y \leq y_1$, in particular $u(-1) = 1$, which contradicts the periodicity. $\square$

Lemma 2.4 *Let $F = 0$. Then Problem A_u admits the family of solutions (for all $0 \leq l \leq u^*$):*

$$\phi(u(y)) = \begin{cases} \left[\dfrac{y}{N_c\sqrt{k^+}} + \phi(l)\right]_+ & for \ -1 < y < 0, \\ 0 & for \ 0 < y < 1, \end{cases}$$

where

$$\phi(s) = \int\limits_0^s k_{rw}(v)J'(v)dv.$$

Proof. Equation (2.27) implies that any solution must be a combination of

$$u \equiv 0 \quad \text{and} \quad \frac{d}{dy}\phi(u(y)) = \frac{1}{N_c\sqrt{k}}. \tag{2.31}$$

One immediately deduces that $u(y) = 0$ for $0 < y < 1$ is the only possibility. Any other choice contradicts the periodicity. Then clearly $u(0-) \leq u^*$ and (2.31) provides the required structure. $\square$

Now we consider the case $0 < F < 1$. To understand the structure of the solutions of Problem A_w we first introduce

Definition 2.5 *Given $F \in (0, 1)$, let $\xi_0(F) \in (J(0), \varphi^{-1}(F))$ be the unique root of*

$$\int\limits_{J(0)}^{\xi_0(F)} V(s, F)ds = \frac{1}{N_c\sqrt{k^-}}, \tag{2.32}$$

where $V(\cdot, F) : (J(0), \varphi^{-1}(F)) \cup (\varphi^{-1}(F), \infty) \to \mathbb{R}^+$ is given by

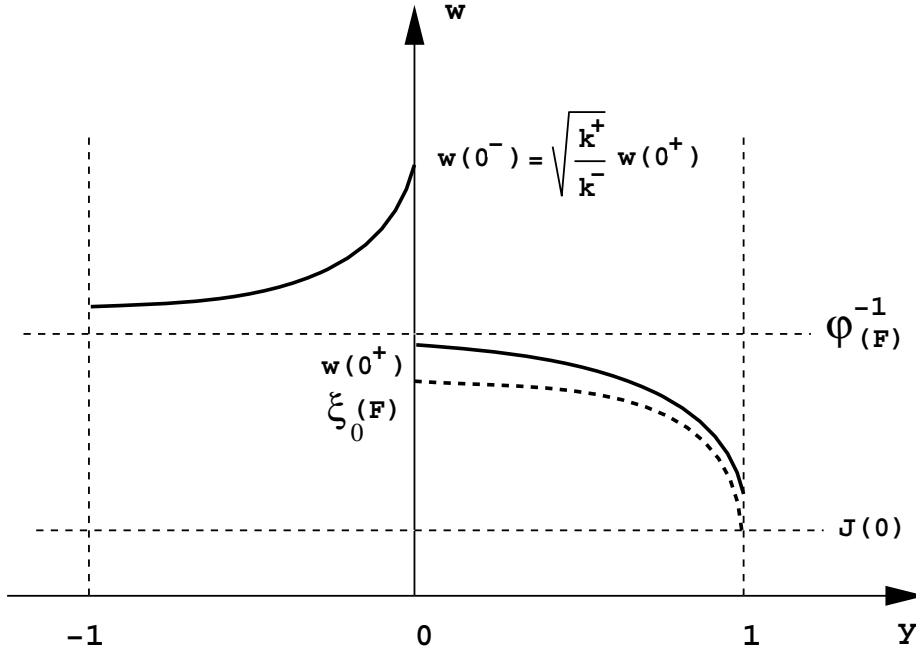$$V(s, F) = \frac{\gamma(s)\varphi(s)}{|F - \varphi(s)|}. \tag{2.33}$$

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Figure 4: Sketch of behavior of solution w.

Clearly $\xi_0(0+) = J(0)$, $\xi_0 \in C^1((0,1))$ and $d\xi_0/dF > 0$ for $F > 0$. We are now in a position to prove the following structure for solutions of Problem A_w, see also Figure 4.

Proposition 2.6 *Let* $0 < F < 1$. *Then any solution of Problem A_w satisfies:*

(i) $\dfrac{dw}{dy} < 0$ *on* $(0,1)$, *with* $\xi_0(F) \le w(0+) < \varphi^{-1}(F)$;

(ii) $\dfrac{dw}{dy} > 0$, $w > \varphi^{-1}(F)$ *on* $(-1,0)$.

Proof. By a uniqueness argument for equation (2.28) we note that either $w \equiv \varphi^{-1}(F)$ or $w \ne \varphi^{-1}(F)$ on the intervals $(-1,0)$ and $(0,1)$. Furthermore $w \lesseqgtr \varphi^{-1}(F)$ implies $dw/dy \lesseqgtr 0$. Using this monotonicity and conditions (2.29), (2.30), the result $w(0+) < \varphi^{-1}(F)$ follows directly, giving $dw/dy < 0$ on $(0,1)$. Integrating equation (2.28) on $(0,1)$ gives

$$\int_{w(1)}^{w(0+)} V(s,F)ds = \frac{1}{N_c\sqrt{k^-}}.$$

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Since $w(1) \geq J(0)$ we find

$$\int\limits_{J(0)}^{w(0+)} V(s,F)ds \geq \frac{1}{N_c\sqrt{k^-}},$$

implying $w(0+) \geq \xi_0(F)$. Since $w(0+) > w(1)$, conditions (2.29), (2.30) give $w(0-) > w(-1)$, proving the second statement of the proposition. $\square$

We shall now demonstrate the solvability for Problem A_w. We start with the simplest case where a solution satisfies $w(1) = J(0)$ and $w(0+) = \xi_0(F)$. By Definition 2.5, such local solutions exist on $(0,1)$. Using again (2.29), (2.30) we find for the left interval

$$w(-1) \leq J(u^*) \text{ and } w(0-) = \sqrt{\frac{k^+}{k^-}}\xi_0(F). \tag{2.34}$$

By (ii) of Proposition 2.6 we need $\varphi^{-1}(F) < J(u^*)$, or

$$F < f(u^*),$$

for such solutions to exist. Integrating (2.28) over $(-1,0)$ and using (2.34) once more yields the nonlinear algebraic equation

$$\int\limits_{w(-1)}^{\sqrt{\frac{k^+}{k^-}}\xi_0(F)} V(s,F)ds = \frac{1}{N_c\sqrt{k^+}}, \tag{2.35}$$

where $\sqrt{\frac{k^+}{k^-}}\xi_0(F) > \sqrt{\frac{k^+}{k^-}}J(0) = J(u^*)$.

If this equation can be solved for $w(-1) \in (\varphi^{-1}(F), J(u^*))$ we have found a solution of Problem A_w satisfying $w(1) = J(0)$. To investigate the solvability we define, for $0 \leq F < f(u^*)$,

$$G(F) = \int\limits_{J(u^*)}^{\sqrt{\frac{k^+}{k^-}}\xi_0(F)} V(s,F)ds. \tag{2.36a}$$

One easily verifies

$$G(0) = 0, \ G(f(u^*)) = \infty \text{ and } dG/dF > 0 \text{ on } (0, f(u^*)).$$

Hence there exists a unique $F^* \in (0, f(u^*))$ such that

$$G(F^*) = \frac{1}{N_c\sqrt{k^+}}. \tag{2.36b}$$

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As a consequence, integral equation (2.35) can be uniquely solved, provided $0 < F \leq F^*$: the left hand side in (2.35) decreases monotonically in $w(-1)$, becomes unbounded when $w(-1) \searrow \varphi(F)$ and attains a value $\leq \dfrac{1}{N_c \sqrt{k^+}}$ when $w(-1) \nearrow J(u^*)$. Thus we have shown, see also Figure 5,

Theorem 2.7 *Let $0 < F \leq F^* < f(u^*)$, where F^* is defined by (2.36b). Further, let $\xi_0(F)$ be given by Definition 2.5. Then Problem A_w admits a solution w satisfying*

$$w(1) = J(0), \; w(0+) = \xi_0(F) \text{ and } w(0-) = \sqrt{\frac{k^+}{k^-}}\xi_0(F).$$
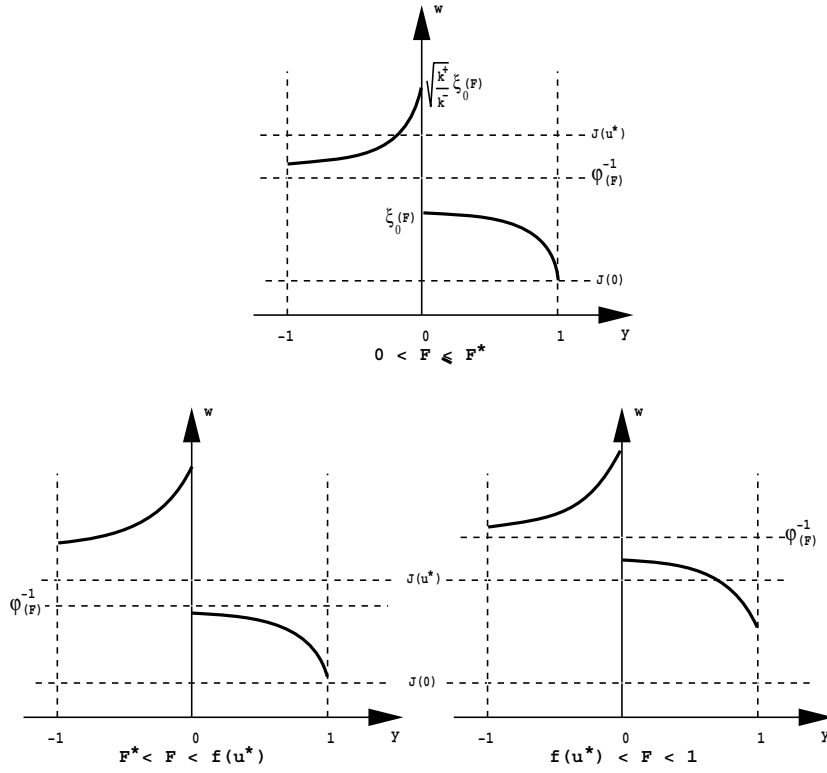

Figure 5: Sketch of behavior of w for the three ranges of F.

Next we consider $F^* < F < 1$. Since now $G(F) > \dfrac{1}{N_c \sqrt{k^+}}$, there are no solutions possible in the class $w(1) = J(0)$. For convenience we introduce

$$\zeta := w(1) \in (b, \varphi^{-1}(F)), \tag{2.37}$$

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where $b = \max\left\{J(0), \sqrt{k^-/k^+}\varphi^{-1}(F)\right\}$ and $z := w(0^+) \in (\zeta, \varphi^{-1}(F))$. Below we construct solutions satisfying $w(1) > b$ and $w(-1) > J(u^*)$. Then the problem of existence for Problem A_w is reduced to the system of algebraic equations (integrating (2.28) on $(-1, 0)$ and $(0, 1)$, and using (2.29), (2.30) and (2.33)):

$$\int_\zeta^z V(s, F)ds = \frac{1}{N_c\sqrt{k^-}}, \qquad (2.38a)$$

$$\int_{\sqrt{\frac{k^+}{k^-}}\zeta}^{\sqrt{\frac{k^+}{k^-}}z} V(s, F)ds = \frac{1}{N_c\sqrt{k^+}}. \qquad (2.38b)$$

To study the solvability of this system we introduce

$$\psi : (b, \varphi^{-1}(F)) \cup \left(\varphi^{-1}(F), \sqrt{\frac{k^+}{k^-}}\varphi^{-1}(F)\right) \to \mathbb{R}$$

by

$$\psi(v) = \begin{cases} \int_b^v V(s, F)ds & \text{for } b < v < \varphi^{-1}(F), \\ \int_v^{\sqrt{\frac{k^+}{k^-}}\varphi^{-1}(F)} V(s, F)ds & \text{for } \varphi^{-1}(F) < v < \sqrt{\frac{k^+}{k^-}}\varphi^{-1}(F). \end{cases}$$

Note that ψ is strictly increasing on $\left(b, \varphi^{-1}(F)\right)$ and strictly decreasing on $\left(\varphi^{-1}(F), \sqrt{\frac{k^+}{k^-}}\varphi^{-1}(F)\right)$, see Figure 6. By the monotonicity of ψ, the function

$$z = z(\zeta) = \psi^{-1}\left(\psi(\zeta) + \frac{1}{N_c\sqrt{k^-}}\right) \qquad (2.39a)$$

is well-defined on $(b, \varphi^{-1}(F))$, satisfying $dz/d\zeta > 0$. Now system (2.38a-b) reduces to study the map $W : (b, \varphi^{-1}(F)) \to \mathbb{R}$, given by

$$W(\zeta) = \psi\left(\sqrt{\frac{k^+}{k^-}}\zeta\right) - \psi\left(\sqrt{\frac{k^+}{k^-}}z\right) - \frac{1}{N_c\sqrt{k^+}}. \qquad (2.39b)$$

We first formulate the theorem.

Theorem 2.8 *For $F^* < F < 1$, there exists a solution to (2.38a-b), i.e. the auxiliary Problem A_w admits a solution.*
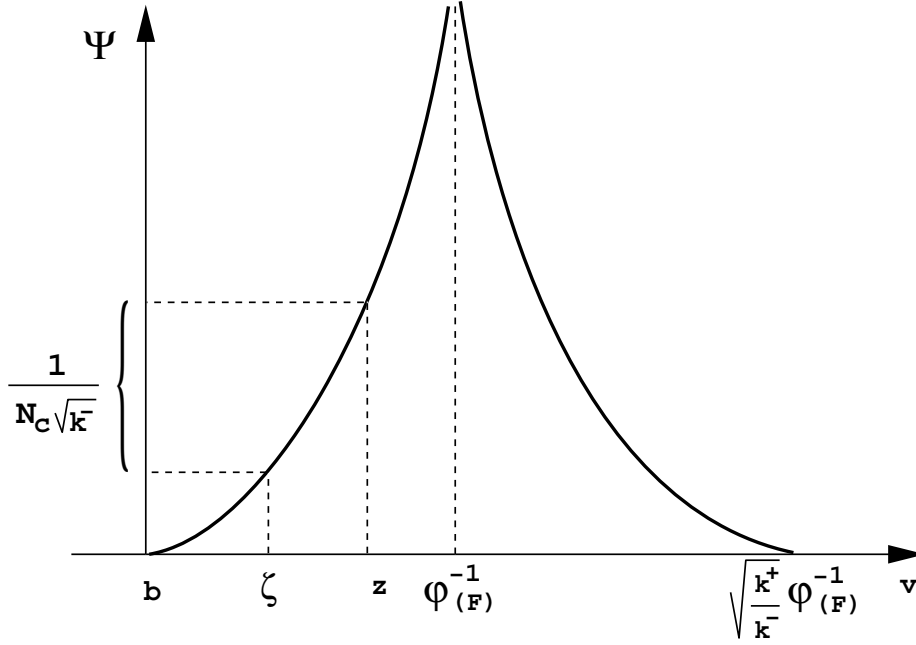
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Figure 6: Sketch of ψ and construction of $z = z(\zeta)$.

Proof. Since $z(\varphi^{-1}(F)-) = \varphi^{-1}(F)$ we have

$$W(\varphi^{-1}(F)-) = -\frac{1}{N_c\sqrt{k^+}} < 0.$$

To investigate the behavior near $\zeta = b$, we use $z > \xi_0(F)$ and consider

$$\int_{\sqrt{\frac{k^+}{k^-}}b}^{\sqrt{\frac{k^+}{k^-}}\xi_0(F)} V(s,F)ds = \begin{cases} +\infty & \text{for } f(u^*) \leq F < 1, \\ > \frac{1}{N_c\sqrt{k^+}} & \text{for } F^* < F < f(u^*). \end{cases}$$

The first follows from $\sqrt{\frac{k^+}{k^-}}b = \varphi^{-1}(F)$ for $F \geq f(u^*)$, the second from $\sqrt{\frac{k^+}{k^-}}b = J(u^*)$ and (2.36a) for $F^* < F < f(u^*)$. As a consequence we find $W(\zeta) > 0$ for ζ close to b. Since W is continuous, the equation $W(\zeta) = 0$ has at least one root, which provides the existence for (2.38a-b). □

2.4 Continuity, monotonicity and uniqueness

To construct the effective equation for U, we need to show that the solution of the auxiliary problem is unique, continuous and monotone in F for $0 <$

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$F \le 1$. The F-dependence is denoted by $u = u(y, F)$, $w = w(y, F)$, or simply $u(F), w(F)$. We treat $F \in (0, F^*)$ and $F \in (F^*, 1)$ first, and then consider the behavior near $F = 0+$, $F = F^*$ and $F = 1$.

$\underline{F \in (0, F^*)}$

Since uniqueness has not yet been demonstrated, we consider here the solution $w(F)$ given by Theorem 2.7. It satisfies

$$\int\limits_{J(0)}^{w(y,F)} V(s, F)ds = \frac{1 - y}{N_c \sqrt{k^-}} \quad \text{for } 0 < y \le 1, \tag{2.40a}$$

$$\int\limits_{w(y,F)}^{\sqrt{\frac{k^+}{k^-}}\xi_0(F)} V(s, F)ds = -\frac{y}{N_c \sqrt{k^+}} \text{for } -1 \le y < 0. \tag{2.40b}$$

The smoothness of ξ_0 and $V(s, \cdot)$ implies $w(y, \cdot) \in C^1((0, F^*))$ for each $y \in [-1, 0) \cup (0, 1]$. Let $\xi(F) = dw/dF$. Differentiating (2.40a) with respect to F yields

$$-\int\limits_{J(0)}^{w(y,F)} \frac{\gamma(s)\varphi(s)}{(F - \varphi(s))^2}ds + V(w(y, F), F)\xi(y, F) = 0.$$

Hence

$$\xi(y, F) > 0 \quad \text{for } 0 < y < 1 \tag{2.41}$$

and

$$\xi(0+, F) = \frac{d\xi_0}{dF} > 0, \; \xi(1, F) = 0.$$

From (2.40b) we find

$$\int\limits_{w(y,F)}^{\sqrt{\frac{k^+}{k^-}}\xi_0(F)} \frac{\gamma(s)\varphi(s)}{(\varphi(s) - F)^2}ds + V\left(\sqrt{\frac{k^+}{k^-}}\xi_0(F), F\right)\sqrt{\frac{k^+}{k^-}}\frac{d\xi_0}{dF}$$

$$= V(w(y, F), F)\xi(y, F),$$

implying

$$\xi(y, F) > 0 \quad \text{for } -1 \le y < 0 \tag{2.42}$$

with

$$\xi(0-, F) = \sqrt{\frac{k^+}{k^-}}\frac{d\xi_0}{dF} > 0.$$

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$\underline{F \in (F^*, 1)}$
Then any solution of Problem A_w satisfies

$$\int\limits_{w(1,F)}^{w(y,F)} V(s,F)ds = \frac{1-y}{N_c\sqrt{k^-}} \quad \text{for } 0 < y \le 1,$$

with $w(1,F) > J(0)$. Hence

$$-\int\limits_{w(1,F)}^{w(y,F)} \frac{\varphi(s)\gamma(s)}{(F-\varphi(s))^2}ds + V(w(y,F),F)\xi(y,F) \tag{2.43}$$

$$= V(w(1,F),F)\xi(1,F),$$

implying the statements:

$$\text{if } \xi(1,F) > 0, \text{ then } \xi(y,F) > 0 \text{ for } 0 < y < 1,$$
$$\text{if } \xi(0+,F) < 0, \text{ then } \xi(1,F) < 0. \tag{2.44a}$$

Similarly we deduce on $(-1, 0)$:

$$\text{if } \xi(0-,F) > 0, \text{ then } \xi(y,F) > 0 \text{ for } -1 < y < 0,$$
$$\text{if } \xi(-1,F) = 0, \text{ then } \xi(0-,F) < 0. \tag{2.44b}$$

The conditions at $y = 0\pm$ and $y = \pm 1$ translate into

$$\xi(0-,F) = \sqrt{\frac{k^+}{k^-}}\xi(0+,F)$$
$$\xi(-1,F) = \sqrt{\frac{k^+}{k^-}}\xi(1,F) \tag{2.45}$$

Next we combine (2.44a) and (2.45). Suppose there exists $\hat{F} \in (F^*, 1)$ such that $\xi(1,\hat{F}) = 0$. Then $\xi(-1,\hat{F}) = 0$, $\xi(0-,\hat{F}) < 0$, $\xi(0+,\hat{F}) < 0$ giving $\xi(1,\hat{F}) < 0$, a contradiction.

Hence either $\xi(1,F) > 0$ or $\xi(1,F) < 0$ for all $F \in (F^*, 1)$. We rule out the second possibility. By (2.45), $\xi(1,F) < 0$ gives $\xi(-1,F) < 0$, implying that $w(-1,F)$ is strictly decreasing in $(F^*, 1)$. However Proposition 2.6 gives $w(-1,F) > \varphi^{-1}(F) \to \infty$ as $F \to 1$, a contradiction. Hence $\xi(1,F) > 0$ and by (2.44a)

$$\xi(y,F) > 0 \text{ for } y \in [-1,0) \cup (0,1]. \tag{2.46}$$

Remark 2.9 *Note that the monotonicity result (2.46) applies to any solution of Problem A_w satisfying $w(1,F) > J(0)$. We use this to show uniqueness for Problem A_w and hence for Problem A_u.*

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Theorem 2.10 *The auxiliary problem (A_u) has a unique solution $u(F)$ for each $F \in (0,1]$. We have*

i) $u(1) = 1;$

ii) $u(F) = J^{-1}(w(F)),$ *where $w(F)$ is given by*

$$\int_{w(1,F)}^{w(y,F)} V(s,F)ds = \frac{1-y}{N_c\sqrt{k^-}} \quad \text{for } 0 < y \leq 1,$$

$$\int_{w(y,F)}^{\sqrt{\frac{k^+}{k^-}}w(0+,F)} V(s,F)ds = -\frac{y}{N_c\sqrt{k^+}} \quad \text{for } -1 \leq y < 0,$$

with $w(1,F) = J(0)$, $w(0+,F) = \xi_0(F)$ for $0 < F \leq F^*$, and $w(1,F) > J(0)$ satisfying $W(w(1,F),F) = 0$ for $F^* < F < 1$.

Proof. In Section 2.3 we have shown that for $F^* < F < 1$ no solutions are possible with $w(1,F) = J(0)$. Furthermore for $0 < F \leq F^*$, Problem A_w is uniquely solvable in the class $w(1,F) = J(0)$. What remains is to rule out solutions satisfying $w(1,F) > J(0)$ for $0 < F \leq F^*$ and to show uniqueness for $F^* < F < 1$ in the class $w(1,F) > J(0)$.

With W given by (2.39b), et us consider the equation

$$W(\zeta(F),F) = 0 \text{ with } \zeta(F) = w(1,F) > J(0).$$

Differentiating with respect to F and denoting $\partial/\partial\zeta$ by a prime gives

$$W'\frac{d\zeta}{dF} + \frac{\partial W}{\partial F} = 0.$$

Since $\partial\zeta/\partial F > 0$, as explained in Remark 2.9, we have

$$W'(\zeta(F),F) < 0 \tag{2.47}$$

whenever $\partial W/\partial F > 0$. The definition of W involves $z = z(\zeta,F)$, given by

$$\psi(z,F) = \psi(\zeta,F) + \frac{1}{N_c\sqrt{k^-}}.$$

Hence

$$\psi'(z,F)\frac{\partial z}{\partial F} = \frac{\partial}{\partial F}(\psi(\zeta,F) - \psi(z,F)),$$

implying $\partial z/\partial F > 0$. Using this we find directly

$$\frac{\partial W}{\partial F} = \frac{\partial}{\partial F}\left(\psi\left(\sqrt{\frac{k^+}{k^-}}\zeta,F\right) - \psi\left(\sqrt{\frac{k^+}{k^-}}z,F\right)\right)$$

$$-\sqrt{\frac{k^+}{k^-}}\psi'\left(\sqrt{\frac{k^+}{k^-}}z,F\right)\frac{\partial z}{\partial F} > 0.$$

Thus (2.47) holds for any solution of (A_w) with $\zeta(F) = w(1, F) > J(0)$.

Next we consider $W(b, F)$. In Section 2.3 we showed $W(b, F) > 0$ for $F > F^*$ and $W(\varphi^{-1}(F), F) = -\frac{1}{N_c \sqrt{k^-}} < 0$. In fact, for $F < f(u^*)$ we have

$$
\begin{aligned}
W(b, F) &= \int_{J(u^*)}^{\sqrt{\frac{k^+}{k^-}} \xi_0(F)} V(s, F) ds - \frac{1}{N_c \sqrt{k^-}} \\
&= G(F) - \frac{1}{N_c \sqrt{k^-}} \quad \text{(see 2.36a)}.
\end{aligned}
\tag{2.48}
$$

Hence

$$
W(b, F) = \begin{cases} > 0 & \text{for } F > F^*, \\ 0 & \text{for } F = F^*, \\ < 0 & \text{for } F < F^*. \end{cases}
$$

Combining these inequalities with (2.47) gives uniqueness for $F > F^*$ and non-existence for $F \le F^*$. $\qquad \square$

Let $\overline{u} : [-1, 0) \cup (0, 1] \to [0, 1]$, defined by (see Lemma 2.4)

$$
\phi(\overline{u}(y)) = \begin{cases} \left[\dfrac{y}{N_c \sqrt{k^+}} + \phi(u^*) \right]_+ & \text{for } -1 \le y < 0, \\ 0 & \text{for } 0 < y \le 1, \end{cases}
$$

denote the maximal solution corresponding to $F = 0$.

We are now in a position to formulate the following continuity and monotonicity results.

Theorem 2.11 *The solution $u(F)$ satisfies:*

(i) $u(\cdot) \in C^1((0, F^*) \cup (F^*, 1))$ *and* $\dfrac{\partial u}{\partial F}(\cdot, F) > 0$ *on* $[-1, 0) \cup (0, 1]$, *except for* $0 < F < F^*$ *where* $\dfrac{\partial u}{\partial F}(1, F) = 0$;

(ii) $\lim\limits_{F \nearrow 1} u(y, F) = 1$;

(iii) $\lim\limits_{F \nearrow F^*} u(y, F) = \lim\limits_{F \searrow F^*} u(y, F) = u(y, F^*)$;

(iv) $\lim\limits_{F \searrow 0} u(y, F) = \overline{u}(y)$.

The convergence in (ii)-(iv) is uniform in the subintervals $[-1, 0)$ and $(0, 1]$.

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Proof. Monotonicity follows directly from the previous results. Therefore we only need to demonstrate the continuity properties (ii)-(iv).

(ii) By Proposition 2.6 we have

$$w(y, F) > \varphi^{-1}(F) \text{ for } -1 \leq y < 0$$

and consequently

$$w(y, F) \geq w(1, F) = \sqrt{\frac{k^-}{k^+}} w(-1, F) > \sqrt{\frac{k^-}{k^+}} \varphi^{-1}(F)$$

for $0 < y \leq 1$ and $F > F^*$. Since $\varphi^{-1}(F) \to \infty$ as $F \nearrow 1$, the uniform convergence of $u(\cdot, F)$ follows.

(iii). The result for $F \nearrow F^*$ is a direct consequence of the continuity of $\xi_0(F)$. To establish the result for $F \searrow F^*$, we consider the function $W(\zeta, F)$ for F near F^* and ζ near $b = J(0)$. Direct computation shows

$$W'(b, F) = -\sqrt{\frac{k^+}{k^-}} \frac{f(u^*) k_{rw}(u^*)}{f(u^*) - F} < 0. \tag{2.49}$$

Since $W(\zeta, F)$ and $W'(\zeta, F)$ are uniformly continuous in $\{(\zeta, F) : b \leq \zeta \leq b + \delta, F^* \leq F \leq F^* + \delta\}$ for δ sufficiently small, we use (2.48) and (2.49) to find

$$\zeta(F) = w(1, F) \searrow J(0) \text{ as } F \searrow F^*.$$

The uniform convergence on both intervals now follows from the $w(y, F)$ expressions in Theorem 2.10.

(iv). The uniform convergence in $(0, 1]$ results from $\xi_0(F) \searrow 0$ as $F \searrow 0$. To establish the result in $[-1, 0)$ we note that monotonicity and boundedness of $u(\cdot, F)$ imply

$$\lim_{F \searrow 0} u(y, F) = \tilde{u}(y), \text{ pointwise in } [-1, 0),$$

with $\tilde{u}(0-) = u^*$. Moreover, since

$$0 < N_c \sqrt{k^+} k_{rw}(u) f(u) J'(u) \frac{du}{dy} = f(u) - F < 1$$

on $[-1, 0)$, $u(\cdot, F)$ is uniformly continuous in F. Hence, by Dini's Theorem, the convergence is uniform in $[-1, 0)$ and $\tilde{u} \in C([-1, 0))$. Let $y_0 \in [-1, 0)$ with $\tilde{u}(y_0) > 0$. For $F > 0$, the integral equation for $u(F)$ can be written as

$$\phi(u(0-, F)) - \phi(u(y, F)) + F \int_{u(y, F)}^{u(0-, F)} \frac{k_{rw}(s) J'(s)}{f(s) - F} ds = -\frac{y}{N_c \sqrt{k^+}}.$$

Let $y = y_0$. Then, for F sufficiently small,

$$0 < F \int_{u(y_0, F)}^{u(0-, F)} \frac{k_{rw}(s) J'(s)}{f(s) - F} ds < F \, Const \int_{\bar{u}(y_0)}^{u(0-, F)} \frac{1}{f(s) - F} ds \to 0$$

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as $F \searrow 0$. Hence

$$\phi(u^*) - \phi(\tilde{u}(y_0)) = -\frac{y}{N_c\sqrt{k^+}},$$

implying $\tilde{u}(y_0) = \overline{u}(y_0)$. $\qquad\square$

2.5   The effective equation

Let $u = u(F)$ denote the unique solution of Problem A_u. As in Section 2.2 we write $F^0 = F^0(x,t)$ and set

$$u^0(x,y,t) = u(y, F^0(x,t))$$

for $x \in \mathbb{R}$, $y \in [-1,0) \cup (0,1]$ and $t > 0$. The equation for the averaged saturation

$$U(x,t) = \frac{1}{2}\int_{-1}^{1} u^0(x,y,t)dy$$

results from (2.24). Integrating this equation in y and using the continuity of $F^1(x,\cdot,t)$ we find

$$\frac{\partial U}{\partial t} + \frac{\partial F^0}{\partial x} = 0 \ \ \text{for} \ \ x \in \mathbb{R}, \ t > 0. \tag{2.50}$$

From here on we drop the superscript and write $F = F^0$. As a consequence of Theorem 2.11 we note that the cell-averaged saturation $U = U(F)$ satisfies

$$U \in C([0,1]) \cap C^1((0,F^*) \cup (F^*,1))$$

with

$$\frac{dU}{dF} > 0 \ \text{on} \ (0,F^*) \cup (F^*,1).$$

Moreover,

$$U(0+) = \overline{U}, \ U(1) = 1,$$

where

$$\overline{U} = \frac{1}{2}\int_{-1}^{0} \overline{u}(y)dy.$$

The continuity and monotonicity allows us to define the inverse $F : [0,1] \to [0,1]$ satisfying, with $F(U^*) = F^*$,

$$F \in C([0,1]) \cap C^1((\overline{U},U^*) \cup (U^*,1))$$

and

$$\frac{dF}{dU} > 0 \ \text{on} \ (\overline{U},U^*) \cup (U^*,1).$$

Further,
$$F(U) = 0 \text{ for } 0 \leq U \leq \overline{U} \text{ and } F(1) = 1.$$

Taking $F = F(U)$ as nonlinear flux function in (2.50), results in an effective equation which is a first order conservation law for U, with $\overline{U}$ as macroscopic irreducible oil saturation.

Under additional (but usual) assumptions on k_{ro}, k_{rw} and J we show that equation (2.50) is of Buckley-Leverett type in the following sense.

Theorem 2.12 *For $\alpha_o, \alpha_w > 1$ and $\beta > 0$, let*
$$\frac{k_{ro}(s)}{s^{\alpha_o}} = O(1), \frac{k_{rw}(s)}{(1-s)^{\alpha_w}} = O(1)$$

and
$$(1-s)^{\beta} J(s) = O(1).$$
Then $F \in C^1([0,1])$ (implying $F'(\overline{U}) = 0$) and $F'(1) = 0$.

Proof. We first consider the behavior near $U = \overline{U}$. Writing equation (2.40a) in terms of $u = J^{-1}(w)$ and differentiating with respect to F yields

$$\frac{\partial u}{\partial F} = \frac{F - f(u)}{k_{rw}(u)f(u)J'(u)} \int\limits_0^u \frac{k_{rw}(s)f(s)J'(s)}{(F - f(s))^2} ds.$$

We now use equation (2.25) twice to rewrite this expression into

$$\frac{\partial u}{\partial F} = -\frac{\partial u}{\partial y} \int\limits_y^1 \frac{1}{F - f(u(s,F))} ds.$$

Next we integrate in y. Setting $U^+(F) = \int_0^1 u(y,F)dy$ and $a(F) = J^{-1}(\xi_0(F))$ we find

$$\frac{dU^+}{dF} = \int\limits_0^1 \frac{a(F) - u(s,F)}{F - f(u(s,F))} ds > \frac{1}{F}(a(F) - U^+(F)).$$

Thus
$$\frac{d}{dF}(F \ U^+(F)) > a(F),$$

implying
$$U^+(F) > \frac{1}{F} \int\limits_0^F a(s)ds \text{ for } 0 < F \leq F^*.$$

Since
$$U(F) > \overline{U} + \frac{1}{2}U^+(F),$$

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we have

$$U(F) > \overline{U} + \frac{1}{2F} \int_0^F a(s)ds \quad \text{for } 0 < F \le F^*. \tag{2.51}$$

We need to estimate $a(F) = u(0+, F)$ from below. For this we use Definition 2.5, i.e.

$$\int_0^{a(F)} \frac{k_{rw}(s)f(s)J'(s)}{F - f(s)}ds = \frac{1}{N_c\sqrt{k^-}},$$

which gives

$$\frac{1}{F - f(a(F))} \int_0^{a(F)} k_{rw}(s)f(s)J'(s)ds > \frac{1}{N_c\sqrt{k^-}},$$

and further

$$0 < F - f(a(F)) < \mathcal{C}\, a(F)f(u(F)) \text{ for } 0 < F < F^*,$$

where $\mathcal{C}$ (here and below) denotes a suitably chosen positive constant.

Now using $f(s)/s^{\alpha_0} = O(1)$ (implied by the asymptotic behavior of k_{ro}) we find, for small F,

$$a(F) > \mathcal{C}F^{1/\alpha_0}.$$

Combining this with (2.51) gives

$$F(U) < \mathcal{C}(U - \overline{U})^{\alpha_0}$$

in a right neighbourhood of $\overline{U}$.

Next we consider the differentiability of $F(U)$ at $U = U^*$. For $F < F^*$ we use (2.40a-b). Differentiating the equations with respect to F and using the continuity of $w(y, F)$ gives directly the existence of $\xi(y, F^*-)$ for each $y \in [-1, 0) \cup (0, 1]$. For $F > F^*$ we first observe that $\xi(1, F)$ is bounded in a right neighbourhood of F^*. This follows from the proof of Theorem 2.11 (iii). Hence, in equation (2.43)

$$V(w(1, F), F)\xi(1, F) \to 0 \text{ as } F \searrow F^*$$

and thus, again using (2.43), $\xi(y, F^*+) = \xi(y, F^*-)$ for $y \in (0, 1)$. A similar argument holds in $(-1, 0)$. As a consequence, F is differentiable at U^*.

To prove $F'(1) = 0$, we construct an upper bound for $U(F)$ near $F = 1$. For $-1 < y < 0$ we have, as in (2.43),

$$\int_{w(y,F)}^{w(0-,F)} \frac{\gamma(s)\varphi(s)}{(\varphi(s) - F)^2}ds \quad +V(w(0-, F), F)\xi(0, F) =$$

$$V(w(y, F), F)\xi(y, F).$$

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Hence

$$\frac{\partial u}{\partial F} > \frac{f(u) - F}{k_{rw}(u)f(u)J'(u)} \int\limits_{u(y,F)}^{u(0-,F)} \frac{k_{rw}(s)f(s)J'(s)}{(f(s) - F)^2}ds,$$

which can be written as

$$\frac{\partial u}{\partial F} > \frac{\partial u}{\partial y} \int\limits_{y}^{0} \frac{1}{f(u(s,F)) - F}ds$$

Consequently, $U^-(F) = \int_{-1}^{0} u(y, F)dy$ satisfies

$$\frac{dU^-}{dF} > \int\limits_{-1}^{0} \frac{u(s,F) - u(-1,F)}{f(u(s,F)) - F}ds > \frac{1}{1 - F}\{U^- - u(-1,F)\}$$

which implies

$$U^-(F) < \frac{1}{1 - F} \int\limits_{F}^{1} u(-1,s)ds. \tag{2.52}$$

Next we estimate $u(-1, F)$ from above near $F = 1$. Since $u(1, F) < f^{-1}(F)$, the periodicity condition implies

$$u(-1, F) < J^{-1}\left(\sqrt{\frac{k^+}{k^-}}J(f^{-1}(F))\right).$$

Using $\dfrac{1 - f(s)}{(1 - s)^{\alpha_w}} = O(1)$ and $(1 - s)^\beta J(s) = O(1)$ we find

$$u(-1, F) < 1 - C(1 - F)^{\frac{1}{\alpha_w}} \text{ near } F = 1.$$

Substituting this estimate into (2.52) and using $U^+(F) < 1$, we deduce

$$F(U) > 1 - C(1 - U)^{\alpha_w}$$

in a left neighbourhood of $U = 1$. □

3   Randomly layered media in the capillary limit

In this section we drop the periodicity assumption and suppose a stationary ergodic geometrical structure. It is characterized by a probability space (Ω, μ), with an ergodic dynamical system $T(x)$, $x \in \mathbb{R}$ (see, e.g. [22] or [11] for details). For a μ-measurable subset $\mathcal{P} \subset \Omega$ we introduce $P = P(\omega) \subset \mathbb{R}$ by

$$P(\omega) = \{x \in \mathbb{R} : T(x)\omega \in \mathcal{P}\}, \tag{3.1}$$

and we call it a random stationary set.

In our application we suppose that $P(\omega)$ has the following form

$$P(\omega) = \bigcup_{i \in \mathbb{Z}} (y_{2i-1}, \ y_{2i}), \tag{3.2}$$

where the random variables $y_i \in \mathbb{R}$ are strictly increasing with respect to i.

A representative example is a Poisson process Π in $\mathbb{R}$ with constant rate $\gamma > 0$. In this case the number of points of Π in an interval $A = (a, b)$ has expectation $\gamma(b - a)$. The number of points of Π in any bounded interval is then finite with probability 1 and Π has no finite limit points. On the other hand the number in $(0, +\infty)$ is infinite so that the points in $(0, +\infty)$ can be written in order as

$$0 < y_1 < y_2 < y_3 < \ldots$$

Similarly the points in $(-\infty, 0)$ can be written in order as

$$\ldots y_{-3} < y_{-2} < y_{-1} < 0.$$

These exhaust the points of Π since the probability that $0 \in \Pi$ is equal to 0. y_n are random variables and the subsequences $\{y_n, n \leq -1\}$ and $\{y_n, n \geq 1\}$ are independent, with the same joint distributions. Furthermore, the random variables $\ell_1 = y_1$, $\ell_n = y_n - y_{n-1}$ $(n \geq 2)$, $\ell_{-1} = -y_{-1}$, $l_{-n} = y_{-n+1} - y_{-n}$ $(n \geq 2)$ are independent and each has probability density $g(y) = \gamma e^{-\gamma|y|}$. The number of points $N(0, t]$ of Π in $(0, t]$ satisfies the law of large numbers

$$\lim_{t \to +\infty} \frac{1}{t} N(0, t] = \gamma \quad \text{with probability } 1.$$

Finally, the process of Poisson is ergodic. Another example are hardcore processes (Gibbs processes, Matérn processes, ...). We construct them from a Poisson point process by eliminating all points having a distance to its neighbors smaller than a prescribed value. They satisfy the mixing property and the ergodicity is assured.

By Birkhoff's Ergodic Theorem there exists a density (fraction of high permeability layers) of P, given by

$$\varphi^+ := \mu(P) = \lim_{N, M \to \infty} \frac{1}{y_{2M+1} - y_{-2N-1}} \sum_{-N}^{M} |y_{2i}(\omega) - y_{2i-1}(\omega)| \tag{3.3}$$

for almost all $\omega \in \Omega$, satisfying

$$0 \leq \varphi^+ \leq 1.$$

The corresponding random permeability is given by

$$k(x, \omega) = k(T(x)\omega) = \begin{cases} k^+(\omega) & \text{if } x \in P \\ k^-(\omega) & \text{if } x \in \mathbb{R} \backslash \overline{P} \end{cases} \tag{3.4}$$

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$k^+(\omega)$ $k^-(\omega)$ $k^+(\omega)$ $k^-(\omega)$ $k^+(\omega)$

$y_{2i-1}(\omega) \quad y_{2i}(\omega) \qquad y_{2i+1}(\omega) \qquad y_{2i+2}(\omega)\, y_{2i+3}(\omega)\, y_{2i+4}(\omega)$
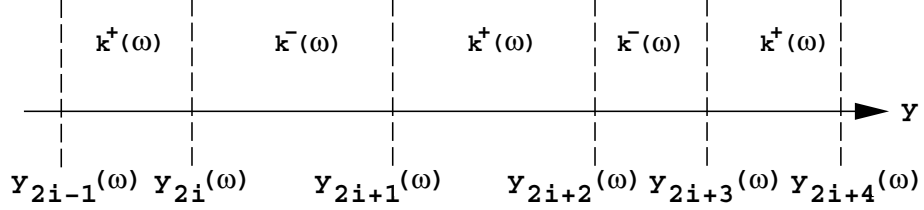
Figure 7: Random distribution of layers

and it is a stationary random variable. Then

$$k^\varepsilon(x,\omega) = k(T(x/\varepsilon)\omega). \tag{3.5}$$

As a consequence $u^* = u^*(\omega)$ through (1.16).

Next we turn to the two scale expansion for the saturation and the flux, adapted to the stochastic case. We write

$$F^\varepsilon = \varepsilon^{-1}F^0 + \varepsilon^0 F^1 + \varepsilon F^2 + \dots , \tag{3.6}$$

where F^k are stationary ergodic random fields and

$$u^\varepsilon = u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + \dots . \tag{3.7}$$

From these expansions and (2.8) we obtain directly

$$\frac{dF^0}{dy} = 0, \text{ implying } F^0 = F^0(x,t), \tag{3.8}$$

with the random variable u^0 satisfying (2.10). We reconsider this equation for a given realization ω, see Figure 7. As before we want to show $F^0 = 0$. Suppose $F^0 < 0$. Introducing w and λ as in Section 2.1 we obtain again

$$\lambda(w)\sqrt{k}\frac{dw}{dy} = -\frac{F^0}{N_c} =: F > 0. \tag{3.9}$$

We argue below that this inequality does not permit us to construct a global non-negative solution satisfying the matching conditions at the interfaces. Suppose $w(y_{2i} - 0) \leq J(u^*)$. By (3.9), implying strict monotonicity of w, we have $w(y_{2i-1} + 0) < J(u^*)$ giving $w(y_{2i-1} - 0) = J(0)$. This contradicts the monotonicity of w in (y_{2i-2}, y_{2i-1}). Next suppose $w(y_{2i} - 0) > J(u^*)$. Then $w(y_{2i} + 0) = \sqrt{\frac{k^-}{k^+}}w(y_{2i}-0)$ and $w(y_{2i+1}-0) \geq w(y_{2i}+0) + \frac{F}{||\lambda||_\infty}\frac{1}{\sqrt{k^-}}|y_{2i+1} - y_{2i}|$. Therefore we have $w(y_{2i+1} + 0) = \sqrt{\frac{k^+}{k^-}}w(y_{2i+1} - 0) \geq w(y_{2i-1} + 0) +$

$\frac{F}{||\lambda||_\infty}\left\{\frac{\sqrt{k^+}}{k^-}|y_{2i+1} - y_{2i}| + \frac{1}{\sqrt{k^+}}|y_{2i} - y_{2i-1}|\right\}.$

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Repeating this reasoning backwards in i shows that w will drop below $J(u^*)$ at the right side of a certain transition, yielding again a contradiction. Hence $F^0 \geq 0$. A similar argument gives $F^0 \leq 0$, so $F^0 = 0$.

This implies for u^0, with $i \in \mathbb{Z}$,

$$u^0(y, \omega) = \begin{cases} C(\omega) > u^* & \text{for } y_{2i-1}(\omega) < y < y_{2i}(\omega) \\ J^{-1}\left(\sqrt{\frac{k^-}{k^+}} J(C(\omega))\right) & \text{for } y_{2i}(\omega) < y < y_{2i+1}(\omega) \end{cases}$$

or

$$u^0(y, \omega) = \begin{cases} C(\omega) \leq u^* & \text{for } y_{2i-1}(\omega) < y < y_{2i}(\omega) \\ 0 & \text{for } y_{2i}(\omega) < y < y_{2i+1}(\omega) \end{cases}.$$

Now consider the ε^{-1}-equation (2.11). Since $F^0 = 0$, the ergodicity of F^1 implies $F^1 = F^1(x, t)$ which is given by

$$F^1 = f(u^0) - N_c \sqrt{k(\omega)} D(u^0) \left(\frac{\partial u^1}{\partial y} + \frac{\partial u^0}{\partial x} \right).$$

Suppose $C(\omega) \leq u^*$. Then $F^1 = 0$ on $(y_{2i-1}(\omega), y_{2i}(\omega))$ implies $F^1 = 0$ for all $x \in \mathbb{R}$ and $t > 0$. If $C(\omega) > u^*$, then we have on $(y_{2i-1}(\omega), y_{2i}(\omega))$

$$\frac{\partial u^1}{\partial y} = \frac{f(C(\omega)) - F^1}{\sqrt{k^+(\omega)} N_c D(C(\omega))} - \frac{\partial C(\omega)}{\partial x} =: B_1(\omega).$$

On $(y_{2i}(\omega), y_{2i+1}(\omega))$ we have

$$\frac{\partial u^1}{\partial y} = \frac{f(\overline{C}(\omega)) - F^1}{\sqrt{k^-(\omega)} D(\overline{C}(\omega))} - \frac{\partial \overline{C}(\omega)}{\partial x} =: B_2(\omega),$$

with $\overline{C}$ as in (2.16). Since $\dfrac{\partial u^1}{\partial y}$ is the local representation of a stationary random variable with zero mean, we have that the mean value of

$$\chi_{\{k=k^+\}} B_1(\omega) + \chi_{\{k=k^-\}} B_2(\omega)$$

is zero. Here χ denotes the characteristic function. Hence

$$\frac{\varphi^+}{\sqrt{k^+(\omega)}} \frac{f(C(\omega)) - F^1}{N_c D(C(\omega))} + \frac{1 - \varphi^+}{\sqrt{k^-(\omega)}} \frac{f(\overline{C}(\omega)) - F^1}{N_c D(\overline{C}(\omega))} =$$

$$\varphi^+ \frac{\partial C(\omega)}{\partial x} + (1 - \varphi^+) \frac{\partial \overline{C}(\omega))}{\partial x}.$$

Solving for F^1 gives (dropping the ω-dependence), for $C(\omega) > u^*$,

$$F^1 = \frac{\dfrac{\varphi^+}{\sqrt{k^+}} \dfrac{f(C)}{D(C)} + \dfrac{1 - \varphi^+}{\sqrt{k^-}} \dfrac{f(\overline{C})}{D(\overline{C})}}{\dfrac{\varphi^+}{\sqrt{k^+}} \dfrac{1}{D(C)} + \dfrac{1 - \varphi^+}{\sqrt{k^-}} \dfrac{1}{D(\overline{C})}} - N_c \frac{\varphi^+ \dfrac{\partial C}{\partial x} + (1 - \varphi^+) \dfrac{\partial \overline{C}}{\partial x}}{\dfrac{\varphi^+}{\sqrt{k^+}} \dfrac{1}{D(C)} + \dfrac{1 - \varphi^+}{\sqrt{k^-}} \dfrac{1}{D(\overline{C})}}. \quad (3.10)$$

Averaging (2.12) and using the ergodicity of F^2 yields the effective transport equation

$$\frac{\partial U}{\partial t} + \frac{\partial F^1}{\partial x} = 0 \text{ for } -\infty < x < \infty, \ t > 0, \quad (3.11)$$

where U denotes the averaged oil saturation

$$U = \varphi^+ C + (1 - \varphi^+)\overline{C}.$$

We note that for $\varphi^+ u^* \geq U > 0$, $F^1 = 0$. In the periodic case $\varphi^+ = \frac{1}{2}$. Hence for each realization ω we obtain an equation of the 'periodic' form (2.20).

We stress that the averaged saturation U is deterministic. This is implied by equation (3.11) and by the deterministic initial condition. Consequently, it suffices to consider only one realization to determine U and its corresponding flux F^1.

Remark 3.1 *In the case of the balance $N_c = O(\varepsilon)$, we could proceed analogously. Now we should solve the problem (2.25) - (2.26) on the real line, for every realization. The periodicity condition (2.27) is replaced by the condition that u takes values between 0 and 1 on $\mathbb{R}$. We note that the matching condition is now posed at every point y_i, $i \in \mathbb{Z}$. Solving the auxiliary problem A_u in the stochastic case is much more complicate than in the periodic case. The analysis of the periodic case was already quite lengthy and in the proofs of Proposition 2.6 and Theorem 2.7 periodicity was essential. Also unboundedness of J complicates proofs. Using arguments from this section we are able to conclude that $F \in [0,1]$, but the complete construction is still an open problem. We expect to consider randomly layered media in the limit $N_c = O(\varepsilon)$ in a future publication.*

4   Numerical results

Since we have no convergence proof, we are going to verify the homogenization procedure numerically for the periodic case. Both the capillary limit and the balance will be considered. We will use the Leverett model with Corey relative permeabilities and Brooks-Corey capillary pressure ([8], [6]). Specifically, the following functions and parameters are used:

$$k_{ro}(u) = u^2, \ k_{rw}(u) = (1-u)^2, \ J(u) = (1-u)^{-\frac{1}{2}}, \ M = 1, \ k^+ = 1, \ k^- = 0.5,$$

with N_c being either 1 or ε, depending on the case.

Tests are done on the interval $(-1, 1)$, i.e $L_x = 1$. For both cases we compute the full problem with a periodic micro-structure as shown in Figure 3; i.e. $k(x) = k^+$ in the coarse layers and $k(x) = k^-$ in the fine layers. The thickness L_y of the layers is related to the number of cells and determines the expansion parameter $\varepsilon = L_y/L_x$. The matching conditions defined in (1.13) and (1.14) or (1.15) are imposed at the interfaces separating the two types of materials. The resulting

solution is averaged on each micro cell consisting of two adjacent layers. This average is compared with the numerical solution of the effective equations.

In the tests we consider a medium originally saturated by oil ($u(x, 0) = 1, x \in (-1, 1)$), with water injection from the left ($u(-1, t) = 0$). At $x = 1$ the Neumann condition is chosen not to affect the flow. For the full problem 80 microstructure cells are considered, implying $\varepsilon = 1/80$.

4.1 Capillary limit ($N_c = O(1)$)

In this case we take $N_c = 1$. Inside each layer of constant permeability we apply a first order explicit discretization scheme with upwind finite volumes. With u_i^n denoting the approximate oil saturation at $t_n = n\tau$ inside the volume centered at $x_i = (i - 1/2)h$ (τ being the time step and h the grid size), the solution at the next time-step follows from

$$u_i^{n+1} = u_i^n - \frac{\tau}{h} \left(F_{i+1/2}^n - F_{i-1/2}^n \right). \tag{4.1}$$

Here $F_{i+1/2}^n$ approximates the flux at $t = t_n$ and $x = x_i + h/2 = ih$, the edge between the volumes centered in x_i and x_{i+1}. Likewise, $F_{i-1/2}^n$ approximates the flux at $t = t_n$ and $x = x_i - h/2 = (i - 1)h$.

If i is such that $x = ih$ lies inside a homogeneous micro-layer, the computation of $F_{i+1/2}^n$ is straightforward. Recalling the notation introduced in (1.8) and (1.9), we have

$$F_{i+1/2}^n = f(u_i^n) - N_c \sqrt{k(x_i)} D \left(\frac{u_i^n + u_{i+1}^n}{2} \right) \frac{u_{i+1}^n - u_i^n}{h}, \tag{4.2}$$

where the permeability $k(x_i)$ is either k^+ or k^-, depending on the type of the material. The diffusion coefficient is given by $D(u) = \lambda(u)J'(u)$ and is calculated at the mean of u_i^n and u_{i+1}^n.

Computing the flux at a position where the permeability and saturation are discontinuous requires more attention. Let us assume that this position is located at $x = ih$, thus separating the control volumes centered in x_i and x_{i+1}. Moreover, let $k(x_i) = k^+$ and $k(x_{i+1}) = k^-$. As in [7] and [13] we introduce two sets of dummy variables at x_i: u_{i+}^n and u_{i-}^n for all $n = 0, 1, 2, \ldots$. They satisfy the pressure condition in (1.15),

$$u_{i-}^n < u^* \text{ implies } u_{i+}^n = 0, \text{ or}$$
$$u_{i-}^n \geq u^* \text{ implies } \frac{J(u_{i-}^n)}{\sqrt{k^+}} = \frac{J(u_{i+}^n)}{\sqrt{k^-}}, \tag{4.3}$$

and they are chosen such that the numerical flux is continuous at $x_{i+1/2}$.

Given a pair $u_{i\pm}^n$ satisfying (4.3), we show below how to obtain $u_{i\pm}^{n+1}$ and $F_{i+1/2}^n$. Once $F_{i+1/2}^n$ is known, we use (4.1) at i and at $i+1$ to determine u_i^{n+1} and u_{i+1}^{n+1}.

In terms of $F_{i+1/2}^n$ we first write

$$
\begin{aligned}
u_{i-}^{n+1} &= u_{i-}^n - \frac{\tau}{h}\left(F_{i+1/2}^n - F_{i-1/2}^n\right), \\
u_{i+}^{n+1} &= u_{i+}^n - \frac{\tau}{h}\left(F_{i+3/2}^n - F_{i+1/2}^n\right),
\end{aligned}
\tag{4.4}
$$

from which we uniquely determine $F_{i+1/2}^n$ and $u_{i\pm}^{n+1}$ satisfying (4.3). To do so we first elliminate $F_{i+1/2}^n$. Summing up the equalities (4.4) yields

$$
u_{i+}^{n+1} = A - u_{i-}^{n+1},
\tag{4.5}
$$

with A defined as

$$
A = u_{i-}^n + u_{i+}^n - \frac{\tau}{h}\left(F_{i+3/2}^n - F_{i-1/2}^n\right).
$$

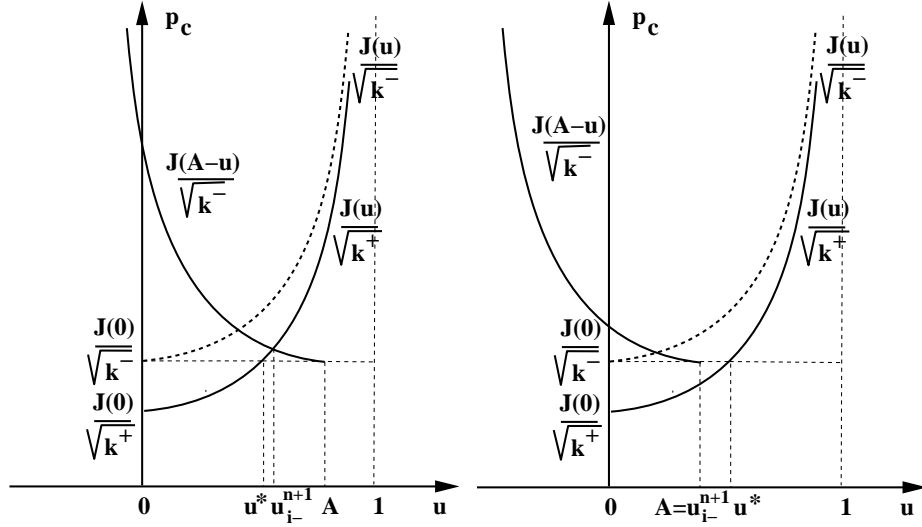Note that the CFL restriction implies $0 \le A \le 2$.

Figure 8: Finding u_{i+}^{n+1}: $A > u^*$ (left) and $A < u^*$ (right).

Now we check which of the two situations in (4.3) applies. Let us assume first that $A \ge u^*$. By contradictiction, since $u_{i\pm}^{n+1}$ satisfy the matching conditions, we get $u_{i+}^{n+1} \ge u^*$. Obviously, $u_{i+}^{n+1} \ge u^*$ also implies $A \ge u^*$. Therefore pressure is continuous in (4.3) iff $A \ge u^*$. In this case (4.5) gives

$$
\frac{J(u_{i+}^{n+1})}{\sqrt{k^-}} = \frac{J(A - u_{i-}^{n+1})}{\sqrt{k^-}} = \frac{J(u_{i-}^{n+1})}{\sqrt{k^+}}.
$$

Monotonicity of J guarantees the existence of a unique $u_{i-}^{n+1} \geq u^*$ satisfying the last equality above (see also Figure 8).

If $A < u^*$, since $u_{i+}^{n+1} \geq 0$, we have $u_{i+}^{n+1} < u^*$ and, due to the matching conditions, $u_{i-}^{n+1} = 0$. As above, $u_{i+}^{n+1} < u^*$ also implies $A < u^*$. Therefore in this case we uniquely obtain $u_{i+}^{n+1} = A < u^*$.

A similar procedure is used at a transition from fine to coarse material. Details are omitted.

As explained in Section 2.1, the effective equation is known explicitly in the capillary limit. Figure 9 shows the effective diffusivity $\mathcal{D}$ and convection $\mathcal{F}$ in terms of the cell averaged oil saturation U. Here we use the relative permeabilities and Leverett function as proposed for this section. This equation is of degenerate
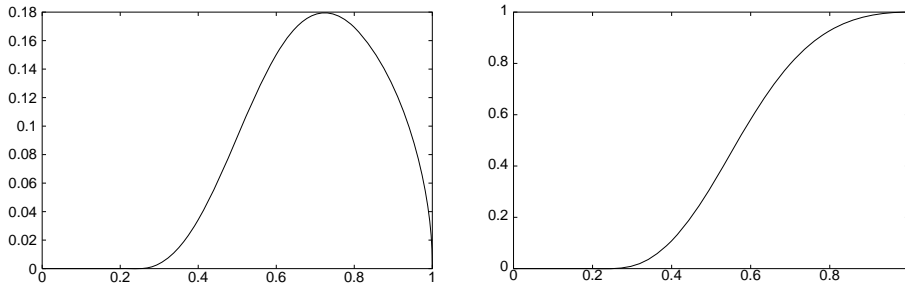

Figure 9: Effective diffusion (left) and convection (right) for the Brooks-Corey model. Note that $\mathcal{D}(U) = \mathcal{F}(U) = 0$ for $0 \leq U \leq 1/2u^* = 0.25$.

parabolic type, since the effective diffusion $\mathcal{D}(U)$ vanishes for $0 \leq U \leq \frac{1}{2}u^*$ and at $U = 1$. Several numerical methods can be applied to such kind of problems. Here we use the explicit upwind scheme (see [24] for the convergence analysis):

$$U_i^{n+1} = U_i^n - \frac{\tau}{h}\left(F_{i+1/2}^n - F_{i-1/2}^n\right),$$
$$F_{i+1/2}^n = \mathcal{F}(U_i^n) - \mathcal{D}\left(\frac{U_i^n + U_{i+1}^n}{2}\right)\frac{U_{i+1}^n - U_i^n}{h}.$$

Figure 10 shows the solution of the effective equation (solid line) and the average of the solution of the full problem (dashed line) at $t = 0.3$ and $t = 0.7$. Since oil is being displaced from the column, both solutions are approaching the macroscopic irreducible oil saturation corresponding to the maximum amount of trapped oil: $U = 1/2u^* = 0.25$. The transition region travels with the same speed in both cases.

The solution of the original problem is shown in Figure 11, together with its average. The leftmost part of the interval is enlarged in the picture on the
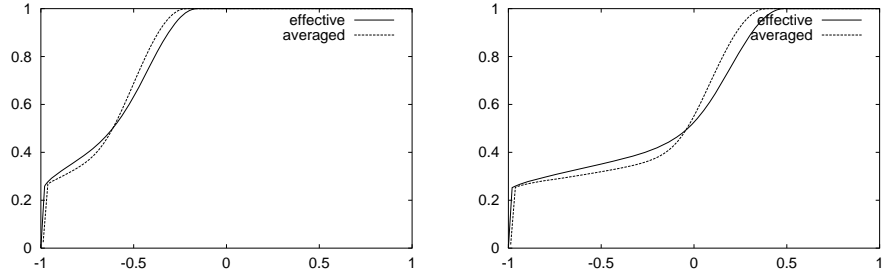
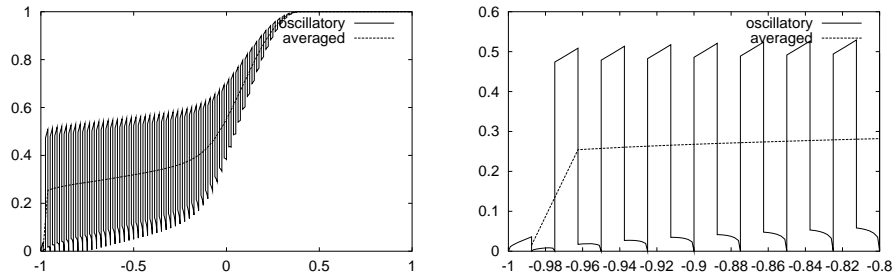Figure 10: Averaged and effective solution oil saturation at $t = 0.3$ and $t = 0.7$.

Figure 11: Full problem; averaged and oscillatory oil saturation at $t = 0.7$, full (left) and zoomed view (right).

right. Note the good agreement with the theoretical results: the profile is highly oscillatory on the macro scale and quite flat within the micro structure. Further note that even though the original problem is of degenerate type, free boundaries seldomly occur inside the homogeneous sub-layers. As a consequence the solution behaves fairly nondegenerate and thus smoothly. Therefore there is no need to consider many points inside the sub-cells. In our computations an interior grid size of $h = L_y/10 \equiv \varepsilon L_x/10$ was sufficient to produce good results. However, since the numerical method is explicit, the time step τ is subject to a CFL condition.

4.2 Balance $\big(N_c = O(\varepsilon)\big)$

With $N_c = \varepsilon$, computations for the full problem are done exactly in the same way as for the capillary limit. However, the effective equation requires more attention. As shown in Section 2.5, this equation is of Buckley-Leverett type, but the fractional flow function is not known explicitly. In this case a table of values for the pairs $(U, \mathcal{F})$ has to be constructed, where $\mathcal{F}$ ranges from 0 to 1. For a given $\mathcal{F}$ value from this table we compute the solution $u(\mathcal{F})$ of the

auxiliary problem (A_u) defined in Section 2.2, and calculate its cell-average as the corresponding U value in the table. For the purpose of this paper we took $\mathcal{F}_i = i\Delta\mathcal{F}$, with $\Delta\mathcal{F} = 10^{-3}$ and $i = \overline{0, 10^3}$. As stated in Lemma 2.4 we take $\mathcal{F}(U) = 0$ for all $U \in [0, \bar{U}]$, $\bar{U}$ being the average of the maximal steady state solution (corresponding to $l = u^*$).

To find accurate solutions of Problem A_u we modify the differential equation through the Kirchhof transform

$$\beta(u) = \int_0^u k_{rw}(v)f(v)J'(v)dv. \tag{4.6}$$

Note that β is strictly increasing and smooth due to the properties of k_{rw}, f and J. In general, the integration cannot be carried out explicitly. Therefore, again we need to construct a table of pairs $(u, \beta(u))$. Here we have applied an adaptive quadrature method.

Thus, instead of solving Problem A_u, we consider the equivalent

PROBLEM A_θ: Given $\mathcal{F}_i$, find $\theta : [-1, 0) \cup (0, 1] \to \mathbb{R}$ satisfying

$$f(\beta^{-1}(\theta)) - N_c\sqrt{k}\frac{d\theta}{dy} = \mathcal{F}_i \text{ in } (-1, 0) \cup (0, 1) \tag{4.7}$$

together with the corresponding matching and periodicity conditions defined in (2.26) and (2.27).

The matching and periodicity conditions can be viewed as boundary conditions for equation (4.7) on the two subintervals. To find a solution $u(\mathcal{F}_i)$ we have applied the following shooting procedure. Choose $\theta(1) \geq 0$ and use this value as initial condition for equation (4.7) on $(0, 1)$. This yields the corresponding $\theta(0+)$ and, by the matching conditions, $\theta(0-)$. Use this value as initial condition for (4.7) on $(-1, 0)$. Then adjust $\theta(1)$ so that $\theta(1)$ and $\theta(-1)$ satisfy the periodicity condition. In carrying out this shooting procedure several technical difficulties had to be resolved. We omit the details in this paper.

Figure 12 shows the effective oil fractional flow function for the specific model considered in this section. Observe that indeed we have recovered Buckley-Leverett model in which the fractional flow has only one inflection point. Note that the theoretical analysis only resulted in $\mathcal{F}'(\overline{U}) = \mathcal{F}'(1) = 0$. No statements about the inflection points could be given. Also note that the upscaled fractional flow contains details of the small scale capillary forces. This effect does not appear explicitly, but it is present due to Problem A_u (or, equivalently, equation (4.7) with the matching and periodicity conditions). Finally notice that the macroscopic irreducible oil saturation $\overline{U}$ for the model considered is much smaller than in the capillary limit case. This is to be expected because the capillary forces are now much smaller ($O(\varepsilon)$).
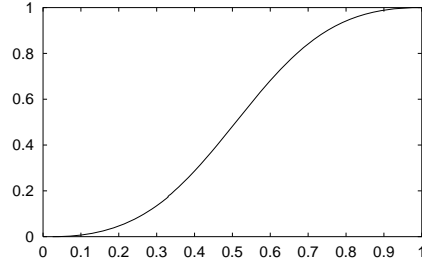
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Figure 12: Effective oil fractional flow for the Brooks-Corey model. Note that $\mathcal{F}(U) = 0$ for $0 \leq U \leq \overline{U} \approx 2.54 \cdot 10^{-2}$.

Once the effective convection is known, the oil saturation equation is solved by the first order explicit upwind scheme

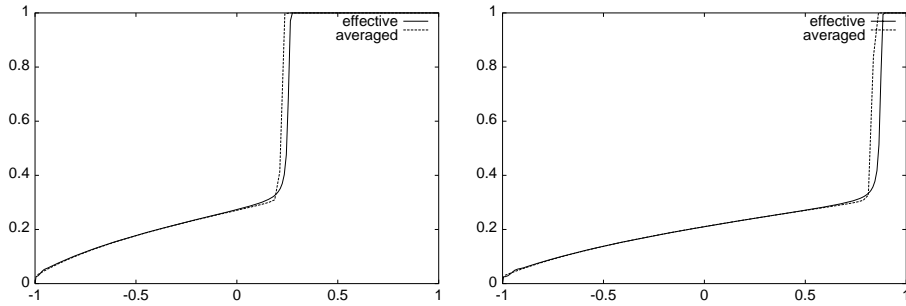$$U_i^{n+1} = U_i^n - \frac{\tau}{h}\left(\mathcal{F}(U_i^n) - \mathcal{F}(U_{i-1})\right).$$

Figure 13: Averaged and effective solution oil saturation at $t = 1.0$ and $t = 1.5$.

Figure 13 shows the solution of the effective equation (solid line) and the average of the solution of the full problem (dashed line) at $t = 1.0$ and $t = 1.5$. For both solutions we see first a rarefaction wave, where the oil saturation approaches the maximum amount of trapped oil, $U = \overline{U}$. This is followed by a shock. Note the good agreement of the two solutions in the rarefaction part. The small difference in the shock location can be explained by numerical errors which occur in the computation of the effective fractional flow function.

The solution of the full problem together with its average is shown in Figure 14, with a zoomed view in the picture on the right. Note the highly oscillatory profile on both scales. Free boundaries and large gradients occur inside almost
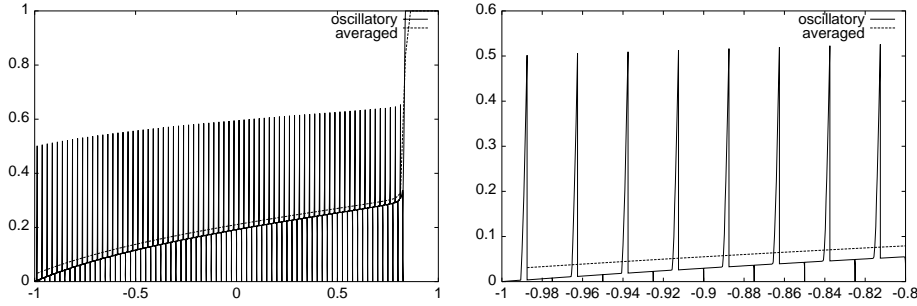
Figure 14: Full problem; averaged and oscillatory oil saturation at $t = 1.5$, full (left) and zoomed view (right).

every homogeneous sub-layer, this being a consequence of the smallness of diffusion ($O(\varepsilon)$). In this case the computational grid has to be quite fine to locate the free boundaries accurately and to compute the macroscopic irreducible oil saturation. In contrast to the capillary limit, we need here many gridpoints inside each sub-layer ($h = L_y/80 \equiv \varepsilon L_x/80$), but under a less restrictive CFL condition.

5 Conclusions

The results of this paper lead to the following conclusions.

- For $N_c = O(1)$ (capillary limit) the effective equation is of degenerate parabolic type. The diffusion and convection vanish up to the macroscopic irreducible oil saturation ($\frac{1}{2}u^*$).

- For $N_c = O(\varepsilon)$ (balance) the upscaled equation is of Buckley-Leverett type, with effects of the local capillary forces in the fractional flow function.

- The macroscopic irreducible oil saturation depends strongly on the value of the capillary number.

- The solution of the auxiliary problem in the capillary limit has two constant states connected by the pressure condition at the interface.

- The solution of the auxiliary problem in the balance is unique and can be classified completely.

- The choice of the characteristic values in (1.6) is important for deciding which of the two cases (capillary limit or balance) applies in a real situation.

- Random layers are considered only in the capillary limit. The effective equation is similar to the periodic one.

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- The method used in this paper can be applied to heterogeneous media in which the porosity, relative permeabilities and Leverett function are periodic as well.

References

[1] H. W. Alt & S. Luckhaus, Quasilinear elliptic-parabolic differential equations, *Math. Z.*, 183(1983), 311 - 341.

[2] J. Bear, Dynamics of fluids in porous media, *American Elsevier, New York*, 1972.

[3] P. Benilan & H. Toure, Sur l'equation generale $u_t = a(., u, \varphi(., u)_x)_x + v$ dans L^1. II: Le probleme d'evolution, *Ann. Inst. Henri Poincare, Anal. Non Lineaire*, 12(1995), 727 - 761.

[4] A. Bensoussan & J. L. Lions & G. Papanicolaou, Asymptotic analysis for periodic structures, Studies in Mathematics and its Applications 5, *North-Holland, Amsterdam, New York, Oxford*, 1978.

[5] M. Bertsch & R. Dal Passo & C. J. van Duijn, Analysis of oil trapping in porous media flow, *to appear*.

[6] R. H. Brooks & A. T. Corey, Hydraulic properties of porous media, Hydrology Papers, *Colorado State University, Fort Collins, Colorado*, 1964.

[7] G. Chavent & J. Jaffret, Mathematical models and finite elements for reservoir simulations, Studies in Mathematics and its Applications 17, *North-Holland, Amsterdam, New York, Oxford*, 1986.

[8] A. T. Corey, The interrelation between gas and oil relative permeabilities, *Producer's Monthly*, 19(1954), 38 - 41.

[9] M. Dale & S. Ekrann & J. Mykkeltveit & G. Virnovsky, Effective relative permeabilities and capillary pressure for one-dimensional heterogeneous media, *Transport in Porous Media*, 26(1997), 229 - 260.

[10] M. Dale, Darcy equations in heterogeneous porous media: interface conditions and uniqueness of periodic solutions, *Working Papers from Stavanger College*, 43(1998).

[11] D. J. Daley, D. Vere-Jones, An introduction to the theory of point processes, Springer Series in Statistics, *Springer-Verlag, New York*, 1988.

[12] C. J. van Duijn & F. J. T. Floris, Mathematical analysis of the influence of power-law fluid rheology on a capillary diffusion zone, *J. Pet. Sc. Eng.*, 7(1992), 215 - 237.

[13] C. J. van Duijn & J. Molenaar & M. J. de Neef, Effects of capillary forces on immiscible two-phase flow in heterogeneous porous media, *Transport in Porous Media*, 21(1995), 71 - 93.

[14] C. J. van Duijn & Q. X. Ye, The flow of two immiscible fluids through a porous medium, *Nonlin. Anal. Theory Meth. Appl.*, 8(1984), 353 - 367.

[15] B. H. Gilding, The occurrence of interfaces in nonlinear diffusion-advection processes, *Arch. Rat. Mech. Anal.*, 100(1988), 243 - 263.

[16] B. H. Gilding, Qualitative mathematical analysis of the Richards equation, *Transport in Porous Media*, 5(1991), 651 - 666.

[17] U. Hornung, ed., Homogenization and porous media, Interdisciplinary Applied Mathematics 6, *Springer-Verlag, New York*, 1997 .

[18] T. F. M. Kortekaas, Water/oil displacement characteristics in crossbedded reservoir zones, *Soc. Pet. Eng. J.*, (1985), 917 - 926.

[19] M. C. Leverett, Capillary behavior in porous solids, *Trans AIME Petr. Eng. Div.*, 142(1941), 152 - 169.

[20] P. van Lingen, Quantification and reduction of capillary entrapment in cross-laminated oil reservoirs, PhD Thesis, *Sub-Faculty of Applied Earth Sciences, Delft University of Technology*, 1998.

[21] A. Mikelić, Homogenization theory and applications to filtration through porous media, in "Filtration in porous media and industrial application", A. Fasano (ed.), Lecture Notes in Mathematics Vol. 1734, *Springer-Verlag, Berlin*, 2000, 127 - 214.

[22] G. C. Papanicolaou, Diffusion in random media, in "Surveys in applied mathematics vol. I", J. B. Keller, D. W. McLaughlin & G. C. Papanicolaou (eds.), *Plenum Press, New York*, 1995, 205 - 254.

[23] E. Sanchez-Palencia, Non-homogeneous media and vibration theory, Lecture Notes in Physics 127, *Springer-Verlag, Berlin Heidelberg New York*, 1980.

[24] M. Watanabe, An approach by difference to the porous medium equation with convection, *Hiroshima Math. J.*, 25(1995), 623 - 645.

[25] Y. C. Yortsos & J. Chang, Capillary effects in steady-state flow in heterogeneous cores, *Transport in Porous Media*, 5(1990), 399 - 420.