

## Precedence probability and prediction intervals

***Citation for published version (APA):***

Chakraborti, S., & Laan, van der, P. (1999). *Precedence probability and prediction intervals*. (Memorandum COSOR; Vol. 9919). Technische Universiteit Eindhoven.

***Document status and date:***

Published: 01/01/1999

***Document Version:***

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

***Please check the document version of this publication:***

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

[Link to publication](#)

***General rights***

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license above, please follow below link for the End User Agreement:

[www.tue.nl/taverne](http://www.tue.nl/taverne)

***Take down policy***

If you believe that this document breaches copyright please contact us at:

[openaccess@tue.nl](mailto:openaccess@tue.nl)

providing details and we will investigate your claim.

Memorandum COSOR 99-19

**Precedence Probability and Prediction Intervals**

S. Chakraborti, P. van der Laan

Revision (with corrections) of Memorandum COSOR 98-12

Eindhoven, December 1999  
The Netherlands

# Precedence Probability and Prediction Intervals<sup>1</sup>

S. Chakraborti  
University of Alabama  
Tuscaloosa, AL  
U.S.A.

and

P. van der Laan<sup>\*</sup>  
Eindhoven University of Technology  
Eindhoven  
The Netherlands

## SUMMARY

Precedence tests are simple yet useful nonparametric tests based on two specified order statistics from two independent random samples or, equivalently, on the count of the number of observations from one of the samples preceding some order statistic of the other sample. The probability that an order statistic from the second sample exceeds an order statistic from the first sample is termed the precedence probability. When the distributions are the same, this probability can be calculated exactly, without any specific knowledge of the underlying common continuous distribution. This fact can be utilized to set up nonparametric prediction intervals in a number of situations. In this paper, prediction intervals are considered for the number of second sample observations that exceed a particular order statistic of the first sample. To aid the user, tables are provided for small sample sizes, where exact calculations are most necessary. The same tables can be used to implement a precedence test for small sample sizes.

*Keywords:* *Distribution-free; Extremes; Exceedance; Nonparametric; Order statistics; Combinatorics.*

---

<sup>1</sup> Revision (with corrections) of Memorandum COSOR 98-12.

<sup>\*</sup>Address for Correspondence: Department of Mathematics and Computing Science, Eindhoven University of Technology, PO Box 513, 5600 MB Eindhoven, The Netherlands. E-mail: p.v.d.laan@tue.nl.

**Acknowledgements:** Support provided in part by NATO Collaborative Research Grant No. CRG920287 and, for the first author, also by a 1998 program II summer research grant from the College of Commerce and Business Administration, The University of Alabama. The authors are grateful to the Editor of *The Statistician* for some useful comments.

## 1. Introduction

Let  $X_{(1)} < X_{(2)} < \dots < X_{(m)}$  be the order statistics of a random sample of size  $m$  from a continuous c.d.f.  $F$  and let  $Y_{(1)} < Y_{(2)} < \dots < Y_{(n)}$  be the order statistics of a second, independent, random sample of size  $n$  from a continuous c.d.f.  $G$ . Consider the probability that the  $j^{\text{th}}$   $Y$ -order statistic exceeds the  $i^{\text{th}}$   $X$ -order statistic,  $\theta = \theta_{ij}(F, G) = P(Y_{(j)} > X_{(i)})$ . The parameter  $\theta$  can be interpreted in several ways. Two such interpretations are: (i) it is the probability that the number of  $Y$  observations that precede  $X_{(i)}$  is at most equal to  $j - 1$  and (ii) it is the probability that the number of  $Y$  observations that exceed  $X_{(i)}$  is at least equal to  $n - j + 1$ . According to the first interpretation,  $\theta$  is termed a “precedence” probability whereas according to the second interpretation,  $\theta$  is referred to as an “exceedance” probability. Both interpretations can be found in the literature as the quantity  $\theta$  arises in various applications. The fields of applications include quality control and reliability where  $\theta$  can be associated with the so-called “warranty time” of a product. In these problems, the underlying probability distributions are often not completely known and frequently can not be assumed to be normal. Thus, a study of the precedence probability, from a distribution-free point of view, is useful.

The study of precedence (and exceedance) goes back to at least the early 1940's. Some of this literature will be referred to later. Nelson (1963) proposed a simple nonparametric test, called the precedence test, for the usual two sample problem  $H_0: F(t) = G(t)$ . Against the one-sided alternative  $H_1: G(t) < F(t)$ , that the  $Y$ 's are stochastically larger than the  $X$ 's, the precedence test rejects  $H_0$  iff, say,  $Y_{(j)} > X_{(i)}$ . Thus the precedence probability  $\theta$  is simply the power of the precedence test.

The concept of precedence (or exceedance) is easy to grasp and is intuitively appealing (one just needs to compare two ordered values from the two samples) to practitioners in many statistical inference problems. Since a precedence test is based on ordered values, in situations (such as life-testing) where data are collected sequentially, such a test can lead to savings in time and resources by allowing an early decision (rejecting  $H_0$  or not) before all the data are collected.

Recently, there has been a resurgence of interest in precedence tests. Nelson (1993) revisited the precedence test. Lin and Sukhatme (1992) studied "best" precedence tests under Lehmann alternatives. Liu (1992) investigated some properties of precedence probabilities, and obtained some results, mainly for the equal sample size case. Chakraborti and van der Laan (1996,1997) provided comprehensive surveys of the area of precedence and precedence-type tests for two- and multi-sample problems, for complete and right-censored data, respectively. Further, van der Laan and Chakraborti (1998) studied "best" precedence tests, based on power, for several types of Lehmann and proportional-hazards alternatives. In this paper the focus is mainly on the precedence probability and in this context the problem of some nonparametric prediction intervals is considered based on exceedance statistics. Necessary formulas and tables are presented so that these can be implemented in practice. In the sequel, an interesting combinatorial identity is obtained.

## 2. Precedence Probability and Prediction Intervals

First note that in general an expression for the precedence probability  $\theta$  can be easily obtained from the distributions of the order statistics  $X_{(i)}$  and  $Y_{(i)}$ . It can be shown (see for example Chakraborti and van der Laan, 1996; hereafter referred to as CV) that  $\theta$  depends on the unknown c.d.fs. only through the so-called "conversion" function  $C(u) = FG^{-1}(u)$ ,  $0 < u < 1$ . Thus  $\theta$  can be calculated explicitly, or at least numerically, when the conversion function is completely specified.

This includes common situations where parametric model assumptions (such as normal or exponential with specified parameter values) are made about  $F$  and  $G$ .

However, when  $F \equiv G$  (for example, under  $H_0$ ),  $C(u) = u$ , and the expression for  $\theta$  reduces to an incomplete beta integral that can be calculated using tables of the incomplete beta function or via the c.d.f. of a binomial distribution. Note that in practice there are situations where the precedence probability  $\theta$ , when  $F \equiv G$ , is important. This is particularly true in problems of prediction. Suppose that a random sample of observations is available from some (continuous) population and based on the sample one wishes to estimate some characteristics of a future sample drawn from the same population. For example, the interest might be to estimate the number of observations  $E$  in the  $Y$ -sample that will exceed some ordered (say the median or the largest) value of the  $X$ -sample. This type of problem is important, for example, in studies of the extremes (in environmental monitoring; hydrology, etc.) and in quality control. For instance, in a production process producing a certain type of light fuses, it might be of interest to estimate, with some degree of confidence, the number of fuses in a future sample that would last longer than say the longest working fuse from the current sample. Such a number or the proportion could be interpreted as one measure of the "quality" of this type of fuses.

Since the number of exceedances  $E$ , is a random variable, an answer to the above problem might be given by a prediction interval. Let  $V_i(E_i)$  denote the number of  $Y$  observations that precede (exceed)  $X_{(i)}$ . The statistic  $V_i$  is called a "precedence" statistic and a test based on  $V_i$  is called a "precedence" test (on the other hand one could just as easily use the "exceedance" statistic  $E_i$ , and could call the resulting test an "exceedance" test). Recall that according to the first interpretation of the precedence probability,  $\theta$  is simply the c.d.f. of  $V_i$  at  $j - 1$ . Also, since  $P(V_i \leq j$

-1) =  $P(E_i \geq n - j + 1)$ , one can consider either the exceedances or the precedences in testing hypotheses or in constructing prediction intervals. See Hahn and Meeker (1991) for some uses of prediction intervals and Vardeman (1992) for a brief introduction.

When  $F \equiv G$ , it has been shown that (see for example, CV)

$$P(V_i = v) = \frac{\binom{i+v-1}{v} \binom{m+n-i-v}{n-v}}{\binom{m+n}{n}}, \quad v = 0, 1, \dots, n; \quad i = 1, 2, \dots, m. \quad (1)$$

**Remark 1** For  $1 \leq a_1 < b_1 \leq m$ , it can be seen that  $P(X_{(a_1)} \leq Y_{(j)} \leq X_{(b_1)}) = P(a_1 \leq W_j \leq b_1 - 1)$ , where  $W_j$  is the number of  $X$ 's preceding  $Y_{(j)}$ . The distribution of  $W_j$  can be obtained from (1) by writing  $m$  for  $n$  and  $j$  for  $i$ . This result gives a nonparametric prediction interval for  $Y_{(j)}$  based on two  $X$ -order statistics. See Fligner and Wolfe (1976,1979) for further details. Note that here and elsewhere, ties can be ignored with probability 1, by the assumed continuity of  $F$  and  $G$ .

**Remark 2** The probability distribution of  $E_i$  (the number of  $Y$ 's that exceed  $X_{(i)}$ ) follows from (1) and is given for completeness

$$P(E_i = e) = \frac{\binom{i+n-e-1}{n-e} \binom{m-i+e}{e}}{\binom{m+n}{n}}, \quad e = 0, 1, \dots, n; \quad i = 1, 2, \dots, m. \quad (2)$$

**Remark 3** In some applications (such as in the analysis of extremes) the probability distribution of  $F_i$ , the number of  $Y$  observations that exceed  $X_{(m-i+1)}$ , the  $i$ th largest (note that  $X_{(i)}$  is the  $i^{\text{th}}$  smallest) of the  $X$ 's, is needed. This exceedance probability is easily obtained from (2) by substituting  $m - i + 1$  for  $i$ . After some simplification the result can be expressed as

$$P(F_i = f) = \frac{i}{m+n} \frac{\binom{m}{i} \binom{n}{f}}{\binom{m+n-1}{i+f-1}}, \quad f = 0, 1, \dots, n; \quad i = 1, 2, \dots, m. \quad (3)$$

These and other related expressions have been obtained by several authors, particularly in the 1950's and the 1960's, using a variety of mathematical-statistical as well as combinatorial techniques. There is the classic paper by Wilks (1942), and other useful references are Gumbel and von Schelling (1950), Epstein (1954) and Rosenbaum (1954).

The distribution of  $E_i$  is computed and presented in Tables 3 and 4 for selected 'small' values of  $m$  and  $n$ :  $m, n = 3(2)15$  and  $i = (m + 1)/2$  and  $i = m$ . Thus, the tables cover exceedances above the median and the largest, respectively. Note that for values of  $m, n$  and  $i$  not covered by the tables, it is not hard to use the explicit formulas given above. First, these tables can be used to implement a precedence test as proposed in Nelson (1963). To illustrate this, for example, suppose  $m = 9, n = 9$  and a size  $\alpha = 0.05$  precedence test is desired at the  $X$ -median, so that  $i = 5$ . Using Table 3, the rejection region can be found as follows. Since  $V_5 = n - E_5$ , where  $E_5$  is the number of  $Y$ 's exceeding  $X_{(5)}$ , from Table 3, we first find the smallest integer  $r$  so that  $P(E_5 \leq r) \geq 0.95$ . This yields  $r = 8$  so that  $P(E_5 \geq 9) \leq 0.05$  and therefore  $P(V_5 \leq 0) \leq 0.05$ . Thus the precedence test has rejection region  $V_5 = 0$ , with an exact size equal to 0.0147. Also, using either of the two interpretations for  $\theta$  with  $j = 1$ , the precedence rejection region can be equivalently expressed in terms of the two order statistics:  $Y_{(1)} > X_{(5)}$ . Secondly, Tables 3 and 4 are useful in the calculation of prediction intervals. This is discussed in the following section.

### 2.1 Prediction intervals

The exact distribution of the exceedance statistic when  $F \equiv G$ , can be used to set up a prediction interval on the number (or the proportion) of future observations that exceed a current order statistic. For example, suppose  $m = 9, n = 7$  and  $i = 5$ , so the interest is in the number of future exceedances in sample of size 7 over the median of a current sample of size 9. The

distribution of  $Y$ -exceedances over the  $X$ -median,  $E_5$ , in this case is found from Table 3 and is given in Table 1 for quick reference.

**Table 1: Distribution of number of exceedances  $E_5$  with  $m=9, n=7$  and  $i=5$**

	0	1	2	3	4	5	6	7
prob	0.02885	0.09178	0.16521	0.21416	0.21416	0.16521	0.09178	0.02885
cuprob	0.02885	0.12063	0.28584	0.50000	0.71412	0.87937	0.97115	1.00000

From Table 1 it is seen that the distribution of  $E_5$  in this case is symmetric and bimodal. The number of future observations that are expected to exceed the current sample median is 3.5. Now, suppose we want a 90% prediction interval on  $E_5$ , the number of  $Y$ -observations exceeding the  $X$ -median. From Table 1, using the cumulative probabilities (cuprob), it is found that the required prediction interval is between 1 and 6, with both endpoints included. This interval is conservative in the sense that the exact confidence coefficient is 0.9423, which is higher than the nominal 0.90. The distribution given in Table 1 also demonstrates the well-known fact that not all typical confidence coefficients might be available for all  $m$ ,  $n$  and  $i$ , owing to the discreteness of the  $E_i$  statistic.

In general, a two-sided prediction interval  $[a, b]$  for  $E_i$ , with confidence coefficient  $1-\alpha$ , can be calculated by solving for two integers  $a$  and  $b$  so that

$$\sum_{e=a}^b P(E_i = e | F \equiv G) \geq 1 - \alpha, \quad (4)$$

where  $P(E_i = e | F \equiv G)$  is given by (2). Note that the coverage of the prediction interval is taken to be greater than or equal to the nominal value  $1 - \alpha$ , owing to the discreteness of  $E_i$ . In case there are several choices for  $a$  and  $b$  satisfying (4), one can take that pair such that the coverage is closest to the nominal value. When  $i$  corresponds to the median of the  $X$ -sample, the distribution

of  $E_i$  is symmetric. In this case, using (4), one can set  $a$  to be the largest integer such that

$$\sum_{e=0}^{a-1} P(E_i = e | F \equiv G) \leq \alpha/2 \text{ and take } b = n-a. \text{ Note also that, the left-hand side of (4) can be}$$

expressed as  $P(Y_{(n-b)} \leq X_{(i)} \leq Y_{(a)})$ .

In some problems only a one-sided prediction interval (or a prediction bound), say of the form  $[0,c]$  is needed. In this case (4) can be easily modified and Tables 3 and 4 can be used to find the intervals.

Now suppose that for the same  $m$  and  $n$ ,  $i = 9$ , so that the interest is in the number of future exceedances over the largest value of the current sample. For this case, the distribution of  $E$  is found from Table 4 and is given in Table 2.

**Table 2: Distribution of number of exceedances  $E_9$  with  $m=9$ ,  $n=7$  and  $i=9$**

	0	1	2	3	4	5	6	7
Prob	0.56250	0.26250	0.11250	0.04327	0.01442	0.00393	0.00079	0.00009
Cuprob	0.56250	0.82500	0.93750	0.98077	0.99519	0.99913	0.99991	1.00000

Thus there is a 11.25% probability that in a future ( $Y$ -) sample of 7, 2 will exceed the largest of the current ( $X$ -) sample of 9 observations from the same continuous population. As it might be expected, this distribution is highly skewed to the right. The probability is over 56% that none of the  $Y$ -sample values will exceed the maximum. Also, note that a 95% two-sided prediction interval for  $E_9$  is the interval from 0 to 3, with both endpoints included.

<<Tables 3 and 4 Here>>

## 2.2 A combinatorial identity

Recall that the precedence probability  $\theta$  is simply the value of the c.d.f. of  $V_i$  at  $j - 1$ . Thus, when  $F \equiv G$ , we have

$$\begin{aligned}
\theta &= P(V_i \leq j-1) \\
&= \sum_{v=0}^{j-1} P(V_i = v) \\
&= \sum_{v=0}^{j-1} \frac{\binom{i+v-1}{v} \binom{m+n-i-v}{n-v}}{\binom{m+n}{n}}.
\end{aligned} \tag{6}$$

On the other hand, when  $F \equiv G$ , the probability  $\theta$  can also be viewed (see for example, van der Laan, 1970; Liu, 1992) as the probability that at least  $i$  of the  $X$ 's are in the first  $i+j-1$  positions out of the total  $m+n$  ordered  $X$ 's and  $Y$ 's, where  $m$   $X$ 's and  $n$   $Y$ 's are drawn from the same distribution. It follows that

$$\theta = \sum_{r=i}^{i+j-1} \frac{\binom{m}{r} \binom{n}{i+j-r-1}}{\binom{m+n}{i+j-1}}. \tag{7}$$

Thus (6) should be equal to (7), and we have the following *combinatorial identity*.

**Lemma**

$$\sum_{r=0}^{j-1} \frac{\binom{i+r-1}{r} \binom{m+n-j-r}{n-r}}{\binom{m+n}{n}} = \sum_{r=i}^{i+j-1} \frac{\binom{m}{r} \binom{n}{i+j-r-1}}{\binom{m+n}{i+j-1}} \quad (8)$$

**Proof**

Note that we can write

$$\sum_{r=i}^{i+j-1} \frac{\binom{m}{r} \binom{n}{i+j-r-1}}{\binom{m+n}{i+j-1}} = \sum_{r=0}^{j-1} \frac{\binom{m}{i+r} \binom{n}{j-1-r}}{\binom{m+n}{i+j-1}},$$

so we need to show that

$$\sum_{r=0}^{j-1} \frac{\binom{i+r-1}{r} \binom{m+n-i-r}{n-r}}{\binom{m+n}{n}} = \sum_{r=0}^{j-1} \frac{\binom{m}{i+r} \binom{n}{j-1-r}}{\binom{m+n}{i+j-1}}.$$

Canceling the common factorials on both sides, it can be seen that the above amounts to showing that

$$\sum_{r=0}^{j-1} \binom{i+j-1}{i+r} \binom{m+n-i-j+1}{m-i-r} - \sum_{r=0}^{j-1} \binom{i+r-1}{i-1} \binom{m+n-i-r}{m-i} = 0. \quad (9)$$

Proof of (9) is given by induction on  $j$ .

For  $j=1$ , (9) is true since

$$\binom{i}{i} \binom{m+n-i}{m-i} - \binom{i-1}{i-1} \binom{m+n-i}{m-i} = 0.$$

To illustrate the general approach, note that for  $j=2$  we have to show that

$$\binom{i+1}{i} \binom{m+n-i-1}{m-i} + \binom{i+1}{i+1} \binom{m+n-i-1}{m-i-1} - \binom{i-1}{i-1} \binom{m+n-i}{m-i} - \binom{i}{i-1} \binom{m+n-i-1}{m-i} = 0. \quad (10)$$

For this, using the well-known identity

$$\binom{M}{s} + \binom{M}{s-1} = \binom{M+1}{s}, \text{ so that } \binom{M}{s} - \binom{M+1}{s} = -\binom{M}{s-1},$$

for any positive integer  $M$  and any non-negative integer  $s=0,1,\dots,M$ , we get,

$$\binom{i+1}{i} \binom{m+n-i-1}{m-i} - \binom{i}{i-1} \binom{m+n-i-1}{m-i} = \binom{m+n-i-1}{m-i} \binom{i}{i},$$

$$\binom{i}{i} \binom{m+n-i-1}{m-i} - \binom{i-1}{i-1} \binom{m+n-i}{m-i} = -\binom{m+n-i-1}{m-i-1},$$

and therefore the L.H.S. of (8) equals

$$-\binom{m+n-i-1}{m-i-1} + \binom{i+1}{i+1} \binom{m+n-i-1}{m-i-1} = 0.$$

Now we suppose the identity (1) is true for fixed  $j$ , then we have to prove that the identity is also true for  $j+1$ . Thus we have to prove

$$\sum_{r=0}^j \binom{i+j}{i+r} \binom{m+n-i-j}{m-i-r} - \sum_{r=0}^j \binom{i+r-1}{i-1} \binom{m+n-i-r}{m-i} = 0 \quad (11)$$

Note that the second sum can be rewritten as

$$\sum_{r=0}^{j-1} \binom{i+r-1}{i-1} \binom{m+n-i-r}{m-i} + \binom{i+j-1}{i-1} \binom{m+n-i-j}{m-i}. \quad (12)$$

By the induction hypothesis, the first sum in (12) is equal to

$$\sum_{r=0}^{j-1} \binom{i+j-1}{i+r} \binom{m+n-i-j+1}{m-i-r},$$

so that the left hand side of (11) equals

$$\sum_{r=0}^j \binom{i+j}{i+r} \binom{m+n-i-j}{m-i-r} - \sum_{r=0}^{j-1} \binom{i+j-1}{i+r} \binom{m+n-i-j+1}{m-i-r} - \binom{j+i-1}{i-1} \binom{m+n-i-j}{m-i}$$

which reduces to

$$\sum_{r=1}^j \binom{i+j}{i+r} \binom{m+n-i-j}{m-i-r} - \sum_{r=0}^{j-1} \binom{i+j-1}{i+r} \binom{m+n-i-j+1}{m-i-r} + \binom{j+i-1}{i-1} \binom{m+n-i-j}{m-i}$$

The last expression can be further simplified as follows

$$= \sum_{r=1}^j \binom{i+j}{i+r} \binom{m+n-i-j}{m-i-r} - \sum_{r=1}^{j-1} \binom{i+j-1}{i+r} \binom{m+n-i-j+1}{m-i-r} - \binom{i+j-1}{i} \binom{m+n-i-j}{m-i-1}$$

$$= \sum_{r=2}^j \binom{i+j}{i+r} \binom{m+n-i-j}{m-i-r} - \sum_{r=1}^{j-1} \binom{i+j-1}{i+r} \binom{m+n-i-j+1}{m-i-r} + \binom{i+j-1}{i+1} \binom{m+n-i-j}{m-i-1}$$

$$= \sum_{r=2}^j \binom{i+j}{i+r} \binom{m+n-i-j}{m-i-r} - \sum_{r=2}^{j-1} \binom{i+j-1}{i+r} \binom{m+n-i-j+1}{m-i-r} - \binom{i+j-1}{i+1} \binom{m+n-i-j}{m-i-2}$$

.....

$$= \sum_{r=j-1}^j \binom{i+j}{i+r} \binom{m+n-i-j}{m-i-r} - \sum_{r=j-1}^{j-1} \binom{i+j-1}{i+r} \binom{m+n-i-j+1}{m-i-r} - \binom{i+j-1}{i+j-2} \binom{m+n-i-j}{m-i-j+1}$$

$$= \sum_{r=j}^j \binom{i+j}{i+r} \binom{m+n-i-j}{m-i-r} - \sum_{r=j-1}^{j-1} \binom{i+j-1}{i+r} \binom{m+n-i-j+1}{m-i-r} + \binom{i+j-1}{i+j-1} \binom{m+n-i-j}{m-i-j+1}$$

$$= \binom{i+j}{i+j} \binom{m+n-i-j}{m-i-j} - \binom{i+j-1}{i+j-1} \binom{m+n-i-j+1}{m-i-j+1} + \binom{i+j-1}{i+j-1} \binom{m+n-i-j}{m-i-j+1}$$

$$= \binom{m+n-i-j}{m-i-j} - \binom{m+n-i-j}{m-i-j} = 0,$$

and the proof is complete.

## References

- Chakraborti, S. and van der Laan, P. (1996): "Precedence tests and confidence bounds for complete data: an overview and some results," *The Statistician*, 45, 351-369.
- Chakraborti, S. and van der Laan, P. (1997): "An overview of precedence tests for censored data," *Biometrical Journal*, 39, 99-116.
- Chakraborti, S. and van der Laan, P. (1999): "Precedence probability, prediction intervals and a combinatorial identity," Memorandum COSOR 99-19, *Eindhoven University of Technology*, Department of Mathematics and Computing Science, Eindhoven, The Netherlands.
- Epstein (1954): "Tables for the distribution of the number of exceedances," *Annals of Mathematical Statistics*, 25, 762-768.
- Fligner, M. A. and Wolfe, D. A. (1976): "Some applications of sample analogs to the probability integral transform and a coverage property," *American Statistician*, 30, 78-85.
- Fligner, M. A. and Wolfe, D. A. (1979): "Methods for obtaining a distribution-free prediction interval for the median of a future sample," *Journal of Quality Technology*, 11, 192-198.
- Gumbel, E. J. and von Schelling, H. (1950): "The distribution of the number of exceedances," *Annals of Mathematical Statistics*, 21, 247-262.
- Hahn, G. J. and Meeker, W. Q. (1991): *Statistical intervals: a guide for practitioners*, John Wiley.
- van der Laan, P. (1970): *Simple distribution-free confidence intervals for a difference in location*, Thesis, *Eindhoven University of Technology*, Eindhoven, The Netherlands.
- van der Laan, P. and Chakraborti, S. (1998): "Precedence tests and Lehmann alternatives," Memorandum COSOR 98-11, *Eindhoven University of Technology*, Department of Mathematics and Computing Science, Eindhoven, The Netherlands..
- Lin and Sukhatme (1992): "On the choice of precedence tests," *Communications in Statistics, Theory and Methods*, 21, 2949-2968.
- Liu (1992): "Precedence probabilities and their applications," *Communications in Statistics, Theory and Methods*, 21, 1667-1682.
- Nelson, L. S. (1963): "Tables for a precedence life test," *Technometrics*, 5, 491-499.
- Nelson, L. S. (1993): "Tests on early failures-the precedence life test," *Journal of Quality Technology*, 25, 140-149.
- Rosenbaum (1954): "Tables for a nonparametric test for location," *Annals of Mathematical*

*Statistics*, 25, 146-150.

Vardeman, S. B. (1992): "What about the other intervals," *American Statistician*, 46, 193-197.

Wilks, S. S. (1942): "Statistical prediction with special reference to the problem of tolerance limits," *Annals of Mathematical Statistics*, 13, 400-409.

**Table 3. Distribution of E for exceedances over the median\***

m=3,i=2													
n=3		n=5		n=7		n=9		n=11		n=13		n=15	
0	0.2	0	0.107	0	0.067	0	0.045455	0	0.032967	0	0.02500000	0	0.01960784
1	0.3	1	0.179	1	0.117	1	0.081818	1	0.060440	1	0.04642857	1	0.03676471
		2	0.214	2	0.150	2	0.109091	2	0.082418	2	0.06428571	2	0.05147059
				3	0.167	3	0.127273	3	0.098901	3	0.07857143	3	0.06372549
						4	0.136364	4	0.109890	4	0.08928571	4	0.07352941
								5	0.115385	5	0.09642857	5	0.08088235
										6	0.10000000	6	0.08578431
												7	0.08823529
m=5,i=3													
n=3		n=5		n=7		n=9		n=11		n=13		n=15	
0	0.17	0	0.083	0	0.046	0	0.028	0	0.017857	0	0.012255	0	0.00877193
1	0.32	1	0.179	1	0.106	1	0.067	1	0.045330	1	0.031863	1	0.02321981
		2	0.238	2	0.159	2	0.108	2	0.075549	2	0.054622	2	0.04063467
				3	0.189	3	0.140	3	0.103022	3	0.077031	3	0.05869453
						4	0.157	4	0.123626	4	0.096289	4	0.07546440
								5	0.134615	5	0.110294	5	0.08939628
										6	0.117647	6	0.09932921
												7	0.10448916
m=7,i=4													
n=3		n=5		n=7		n=9		n=11		n=13		n=15	
0	0.16	0	0.071	0	0.035	0	0.019231	0	0.011438	0	0.007224	0	0.004785
1	0.33	1	0.178	1	0.098	1	0.057692	1	0.035948	1	0.023478	1	0.015949
		2	0.253	2	0.163	2	0.104895	2	0.069130	2	0.046956	2	0.032836
				3	0.204	3	0.146853	3	0.103695	3	0.073787	3	0.053359
						4	0.171329	4	0.131976	4	0.099329	4	0.074702
								5	0.147813	5	0.119195	5	0.093911
										6	0.130031	6	0.108359
												7	0.116099
m=9,i=5													
n=3		n=5		n=7		n=9		n=11		n=13		n=15	
0	0.1590	0	0.062937	0	0.028846	0	0.014706	0	0.008127	0	0.004785	0	0.002964
1	0.3409	1	0.174825	1	0.091783	1	0.050905	1	0.029799	1	0.018294	1	0.011702
		2	0.262238	2	0.165210	2	0.101810	2	0.063854	2	0.041162	2	0.027304
				3	0.214161	3	0.151172	3	0.103150	3	0.070433	3	0.048719
						4	0.181407	4	0.137533	4	0.100619	4	0.073078
								5	0.157538	5	0.125387	5	0.096463
										6	0.139319	6	0.114837
												7	0.124933
m=11,i=6													
n=3		n=5		n=7		n=9		n=11		n=13		n=15	
0	0.153846	0	0.057692	0	0.024887	0	0.011920	0	0.006192	0	0.003432	0	0.002007
1	0.346154	1	0.173077	1	0.087104	1	0.045975	1	0.025542	1	0.014874	1	0.009030
		2	0.269231	2	0.166290	2	0.099024	2	0.059598	2	0.036748	2	0.023288
				3	0.221719	3	0.154037	3	0.102167	3	0.067371	3	0.044851
						4	0.189045	4	0.141462	4	0.101057	4	0.071234
								5	0.165039	5	0.129930	5	0.097947
										6	0.146588	6	0.119713
												7	0.131929
m=13,i=7													
n=3		n=5		n=7		n=9		n=11		n=13		n=15	
0	0.15	0	0.053922	0	0.022136	0	0.010062	0	0.004958	0	0.002609	0	0.001449
1	0.35	1	0.171569	1	0.083437	1	0.042260	1	0.022457	1	0.012494	1	0.007246
		2	0.274510	2	0.166873	2	0.096594	2	0.056143	2	0.033318	2	0.020290
				3	0.227554	3	0.156037	3	0.101057	3	0.064676	3	0.041648
						4	0.195046	4	0.144367	4	0.101057	4	0.069413
								5	0.171019	5	0.133395	5	0.098811
										6	0.152451	6	0.123514
												7	0.137630

m=15,i=8													
n=3		n=5		n=7		n=9		n=11		n=13		n=15	
0	0.147059	0	0.051084	0	0.020124	0	0.008749	0	0.004119	0	0.002070	0	0.001099
1	0.352941	1	0.170279	1	0.080495	1	0.039373	1	0.020137	1	0.010766	1	0.005997
		2	0.278638	2	0.167183	2	0.094495	2	0.053305	2	0.030598	2	0.017991
				3	0.232198	3	0.157491	3	0.099946	3	0.062330	3	0.038981
						4	0.199892	4	0.146588	4	0.100828	4	0.067703
								5	0.175905	5	0.136117	5	0.099298
										6	0.157291	6	0.126556
												7	0.142375

\*  $e = 0, 1, \dots, n$ ; rest of the table entries can be found from  $P(E = e) = P(E = n - e)$  which holds by symmetry. Note that some entries have been cut-off by Excel during formatting.

Table 4: Distribution of E for exceedances above the largest\*

m=3,i=3													
n=3		n=5		n=7		n=9		n=11		n=13		n=15	
0	0.50	0	0.37500	0	0.300000	0	0.250000	0	0.214286	0	0.187500	0	0.166667
1	0.30	1	0.267857	1	0.233333	1	0.204545	1	0.181319	1	0.162500	1	0.147059
2	0.15	2	0.178571	2	0.175000	2	0.163636	2	0.151099	2	0.139286	2	0.128676
3	0.05	3	0.107143	3	0.125000	3	0.127273	3	0.123626	3	0.117857	3	0.111520
		4	0.053571	4	0.083333	4	0.095455	4	0.098901	4	0.098214	4	0.095588
		5	0.017857	5	0.050000	5	0.068182	5	0.076923	5	0.080357	5	0.080882
				6	0.025000	6	0.045455	6	0.057692	6	0.064286	6	0.067402
				7	0.008333	7	0.027273	7	0.041209	7	0.050000	7	0.055147
						8	0.013636	8	0.027473	8	0.037500	8	0.044118
						9	0.004545	9	0.016484	9	0.026786	9	0.034314
								10	0.008242	10	0.017857	10	0.025735
								11	0.002747	11	0.010714	11	0.018382
										12	0.005357	12	0.012255
										13	0.001786	13	0.007353
												14	0.003676
												15	0.001225
m=5,i=5													
n=3		n=5		n=7		n=9		n=11		n=13		n=15	
0	0.625000	0	0.500000	0	0.416667	0	0.357143	0	0.312500	0	0.277778	0	0.25
1	0.267857	1	0.277778	1	0.265152	1	0.247253	1	0.229167	1	0.212418	1	0.197368
2	0.089286	2	0.138889	2	0.159091	2	0.164835	2	0.163690	2	0.159314	2	0.153509
3	0.017857	3	0.059524	3	0.088384	3	0.104895	3	0.113324	3	0.116830	3	0.117389
		4	0.019841	4	0.044192	4	0.062937	4	0.075549	4	0.083450	4	0.088042
		5	0.003968	5	0.018939	5	0.034965	5	0.048077	5	0.057773	5	0.064564
				6	0.006313	6	0.017483	6	0.028846	6	0.038515	6	0.046117
				7	0.001263	7	0.007493	7	0.016026	7	0.024510	7	0.031927
						8	0.002498	8	0.008013	8	0.014706	8	0.021285
						9	0.000500	9	0.003434	9	0.008170	9	0.013545
								10	0.001145	10	0.004085	10	0.008127
								11	0.000229	11	0.001751	11	0.004515
										12	0.000584	12	0.002257
										13	0.000117	13	0.000967
												14	0.000322
												15	6.45E-05
m=7,i=7													
n=3		n=5		n=7		n=9		n=11		n=13		n=15	
0	0.700000	0	0.583333	0	0.500000	0	0.437500	0	0.388889	0	0.350000	0	0.318182
1	0.233333	1	0.265152	1	0.269231	1	0.262500	1	0.251634	1	0.239474	1	0.227273
2	0.058333	2	0.106061	2	0.134615	2	0.150000	2	0.157271	2	0.159649	2	0.159091
3	0.008333	3	0.035354	3	0.061189	3	0.080769	3	0.094363	3	0.103302	3	0.108852
		4	0.008838	4	0.024476	4	0.040385	4	0.053922	4	0.064564	4	0.072568
		5	0.001263	5	0.008159	5	0.018357	5	0.029035	5	0.038738	5	0.046956
				6	0.002040	6	0.007343	6	0.014517	6	0.022136	6	0.029347
				7	0.000291	7	0.002448	7	0.006599	7	0.011192	7	0.017608
						8	0.000612	8	0.00264	8	0.00596	8	0.010062
						9	8.74E-05	9	0.00088	9	0.002709	9	0.005418
								10	0.00022	10	0.001084	10	0.002709
								11	3.14E-05	11	0.000361	11	0.001231
										12	9.03E-05	12	0.000493
										13	1.29E-05	13	0.000164
												14	4.1E-05
												15	5.86E-06
m=9,i=9													
n=3		n=5		n=7		n=9		n=11		n=13		n=15	
0	0.750000	0	0.642857	0	0.562500	0	0.500000	0	0.450000	0	0.409091	0	0.375000
1	0.204545	1	0.247253	1	0.262500	1	0.264706	1	0.260526	1	0.253247	1	0.244565
2	0.040909	2	0.082418	2	0.112500	2	0.132353	2	0.144737	2	0.151948	2	0.155632

3	0.004545	3	0.022478	3	0.043269	3	0.061765	3	0.076625	3	0.087970	3	0.096344
		4	0.004496	4	0.014423	4	0.026471	4	0.038313	4	0.048872	4	0.057806
		5	0.000500	5	0.003934	5	0.010181	5	0.017879	5	0.025874	5	0.033467
				6	0.000787	6	0.003394	6	0.007663	6	0.012937	6	0.018593
				7	8.74E-05	7	0.000926	7	0.002947	7	0.006037	7	0.009843
						8	0.000185	8	0.000982	8	0.002587	8	0.004922
						9	2.06E-05	9	0.000268	9	0.000995	9	0.002297
								10	5.36E-05	10	0.000332	10	0.000984
								11	5.95E-06	11	9.05E-05	11	0.000379
										12	1.81E-05	12	0.000126
										13	2.01E-06	13	3.44E-05
												14	6.88E-06
												15	7.65E-07

m=11,j=11

n=3		n=5		n=7		n=9		n=11		n=13		n=15	
0	0.785714	0	0.687500	0	0.611111	0	0.550000	0	0.5	0	0.458333	0	0.423077
1	0.181319	1	0.229167	1	0.251634	1	0.260526	1	0.261905	1	0.259058	1	0.253846
2	0.030220	2	0.065476	2	0.094363	2	0.115789	2	0.130952	2	0.141304	2	0.148077
3	0.002747	3	0.015110	3	0.031454	3	0.047678	3	0.062030	3	0.074017	3	0.083696
		4	0.002518	4	0.008987	4	0.017879	4	0.027569	4	0.037008	4	0.045652
		5	0.000229	5	0.002074	5	0.00596	5	0.011352	5	0.017530	5	0.023913
				6	0.000346	6	0.001703	6	0.004257	6	0.007791	6	0.011957
				7	3.14E-05	7	0.000393	7	0.001419	7	0.003208	7	0.005664
						8	6.55E-05	8	0.000405	8	0.001203	8	0.002517
						9	5.95E-06	9	9.36E-05	9	0.000401	9	0.001036
								10	1.56E-05	10	0.000115	10	0.000389
								11	1.42E-06	11	2.64E-05	11	0.000130
										12	4.41E-06	12	3.7E-05
										13	4.01E-07	13	8.54E-06
												14	1.42E-06
												15	1.29E-07

m=13,j=13

n=3		n=5		n=7		n=9		n=11		n=13		n=15	
0	0.812500	0	0.722222	0	0.650000	0	0.590909	0	0.541667	0	0.500000	0	0.464286
1	0.162500	1	0.212418	1	0.239474	1	0.253247	1	0.259058	1	0.260000	1	0.257937
2	0.023214	2	0.053105	2	0.079825	2	0.101299	2	0.117754	2	0.130000	2	0.138889
3	0.001786	3	0.010621	3	0.023478	3	0.037321	3	0.050466	3	0.062174	3	0.072222
		4	0.001517	4	0.005869	4	0.012440	4	0.020186	4	0.028261	4	0.036111
		5	0.000117	5	0.001174	5	0.003659	5	0.007437	5	0.012112	5	0.017271
				6	0.000168	6	0.000915	6	0.002479	6	0.004845	6	0.007850
				7	1.29E-05	7	0.000183	7	0.000729	7	0.001785	7	0.003364
						8	2.61E-05	8	0.000182	8	0.000595	8	0.001346
						9	2.01E-06	9	3.65E-05	9	0.000175	9	0.000496
								10	5.21E-06	10	4.37E-05	10	0.000165
								11	4.01E-07	11	8.75E-06	11	4.86E-05
										12	1.25E-06	12	1.22E-05
										13	9.61E-08	13	2.43E-06
												14	3.47E-07
												15	2.67E-08

m=15,j=15

n=3		n=5		n=7		n=9		n=11		n=13		n=15	
0	0.83333	0	0.750000	0	0.681818	0	0.6250000	0	0.576923	0	0.535714	0	0.500000
1	0.14705 <sup>5</sup>	1	0.197368	1	0.227273	1	0.244565	1	0.253846	1	0.257937	1	0.258621
2	0.01838 <sup>9</sup>	2	0.043860	2	0.068182	2	0.088933	2	0.105769	2	0.119048	2	0.129310
3	0.00122 <sup>2</sup>	3	0.007740	3	0.017943	3	0.029644	3	0.041388	3	0.052381	3	0.062261
		4	0.000967	4	0.003987	4	0.008893	4	0.015050	4	0.021825	4	0.028736
		5	6.45E-05	5	0.000704	5	0.002340	5	0.005017	5	0.008540	5	0.012644
				6	8.8E-05	6	0.000520	6	0.001505	6	0.003106	6	0.005268
				7	5.86E-06	7	9.18E-05	7	0.000396	7	0.001035	7	0.002061
						8	1.15E-05	8	8.8E-05	8	0.000311	8	0.000750
						9	7.65E-07	9	1.55E-05	9	8.17E-05	9	0.000250

								10	1.94E-06	10	1.82E-05	10	7.5E-05
								11	1.29E-07	11	3.2E-06	11	1.97E-05
										12	4.01E-07	12	4.38E-06
										13	2.67E-08	13	7.74E-07
												14	9.67E-08
												15	6.45E-09

\* Note that some entries have been cut-off by Excel during formatting.