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Abstract

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and $a_1, \ldots, a_{d'}$ an algebraic basis of $\mathfrak{g}$. Further let A_i denote the generators of left translations, acting on the L_p-spaces $L_p(G\,;dg)$ formed with left Haar measure dg, in the directions a_i. We consider second-order operators

$$H = -\sum_{i,j=1}^{d'} A_i\, c_{ij}\, A_j + \sum_{i=1}^{d'} (c_i\, A_i + A_i\, c_i') + c_0\, I$$

corresponding to a quadratic form with real measurable coefficients c_{ij} and complex c_i, c_i', $c_0 \in L_\infty$. The matrix $C = (c_{ij})$ of principal coefficients, which is not necessarily symmetric, is assumed to satisfy the subellipticity condition

$$\Re C = 2^{-1}\Big(C + C^*\Big) \geq \mu I > 0$$

uniformly over G.

We prove that H generates a strongly continuous holomorphic semigroup S on L_2 with a kernel K which satisfies Gaussian bounds

$$|K_z(g\,;h)| \leq a\,|z|^{-D'/2} e^{\omega|z|} e^{-b(|gh^{-1}|')^2 |z|^{-1}}$$

for $g, h \in G$ and z in a subsector $\Lambda(\theta)$ of the sector of holomorphy. Moreover, the kernel is Hölder continuous and there is a $\nu \in \langle 0, 1\rangle$ such that for all $\kappa > 0$ one has estimates

$$|K_z(k^{-1}g\,;l^{-1}h) - K_z(g\,;h)| \leq a\,|z|^{-D'/2} e^{\omega|z|} \left(\frac{|k|' + |l|'}{|z|^{1/2} + |gh^{-1}|'}\right)^{\nu} e^{-b(|gh^{-1}|')^2 |z|^{-1}}$$

for $g, h, k, l \in G$ and z in the subsector with $|k|' + |l|' \leq \kappa\,|z|^{1/2} + 2^{-1}|gh^{-1}|'$.

Moreover, if all the coefficients of H are real-valued then

$$K_t(g\,;h) \geq a'\, t^{-D'/2} e^{-\omega' t} e^{-b'(|gh^{-1}|')^2 t^{-1}}$$

for some $a', b' > 0$ and $\omega' \geq 0$ uniformly for $g, h \in G$ and $t > 0$.

1 Introduction

We continue the analysis of the semigroup kernels associated with second-order subelliptic operators with variable coefficients acting on the L_p-spaces over a d-dimensional Lie group G begun in [ElR3]. In the latter paper we considered operators

$$H = -\sum_{i,j=1}^{d'} A_i c_{ij} A_j + \sum_{i=1}^{d'} (c_i A_i + A_i c'_i) + c_0 \tag{1}$$

with complex coefficients c_{ij}, c_i, c'_i, $c_0 \in L_\infty$ where $A_i = dL(a_i)$ denotes the generator of left translations L, acting on one of the classical function spaces, in the direction a_i of the Lie algebra $\mathfrak{g}$ of G, and $a_1, \ldots, a_{d'}$ is an algebraic basis of $\mathfrak{g}$. Subellipticity corresponds to the condition

$$\Re C = 2^{-1}(C + C^*) \geq \mu I > 0 \quad ,$$

in the sense of $d' \times d'$-matrices, uniformly over G. In this paper we analyze operators for which the principal coefficients c_{ij} are real-valued but the lower-order coefficients can still be complex. Moreover, we do not assume that the matrix $C = (c_{ij})$ of principal coefficients is hermitian.

Our main result establishes Hölder continuity estimates comparable to the classic results of Morrey [Mor], Nash [Nas] and De Giorgi [Gio] for strongly elliptic operators with real measurable coefficients on $\mathbf{R}^d$ and Gaussian upper bounds of the type first obtained by Aronson [Aro]. In addition we derive Gaussian lower bounds on the kernels associated with operators for which all coefficients are real. The proofs are again a combination of parabolic and elliptic techniques based on a mixture of the methods introduced by Nash [Nas] and De Giorgi [Gio] and influenced by the exposition of Giaquinta [Gia] and recent work of Auscher [Aus]. In particular De Giorgi estimates are combined with arguments involving Morrey–Campanato spaces.

Throughout the paper we adopt the notation and definitions of [ElR3]. In particular the operator H formally given by (1) is defined by form methods as a closed sectorial operator on the L_2-space, $L_2(G\,;dg)$, over G formed with respect to left Haar measure. Then H is the generator of a strongly continuous semigroup S on L_2 with a holomorphic extension to a sector $\Lambda(\theta) = \{z \in \mathbf{C}\backslash\{0\} : |\arg z| < \theta\}$ in the complex plane with $\theta \geq \theta_C = \operatorname{arccot}(\gamma_C/\mu_C)$ where

$$\mu_C = \sup\{\, \mu : 2^{-1}(C + C^*) \geq \mu I \,\} \quad ,$$
$$\gamma_C = \inf\{\, \gamma : \gamma I \geq (2i)^{-1}\Big(C - C^*\Big) \geq -\gamma I \,\} \quad .$$

(The value of μ_C corresponds to the ellipticity constant.) Moreover, S has a distribution kernel $K_t \in \mathcal{D}'(G \times G)$ such that

$$(\psi, S_t\varphi) = \int_G dg\, \overline{\psi(g)} \int_G d\hat{h}\, K_t(g\,;h)\, \varphi(h)$$

for all $\varphi, \psi \in C_c^\infty(G)$ and $t > 0$ where $d\hat{h}$ denotes right Haar measure. We will prove that the kernel has a holomorphic extension K_z to a sector $\Lambda(\theta) \supseteq \Lambda(\theta_C)$ satisfying the semigroup property

$$K_{z_1+z_2}(g\,;h) = \int_G d\hat{k}\, K_{z_1}(g\,;k)\, K_{z_2}(k\,;h) \tag{2}$$

for all $z_1, z_2 \in \Lambda(\theta_C)$. In addition, we derive bounds on the kernel in terms of the right invariant distance $d'(\,\cdot\,;\,\cdot\,)$, the control distance, canonically associated with the algebraic basis $a_1, \ldots, a_{d'}$ (see, for example [Rob] Sections IV.2 and IV.4c). This distance has the characterization

$$d'(g\,;h) = \sup\{\, |\psi(g) - \psi(h)| \,:\, \psi \in C_c^\infty(G)\,,\, \sum_{i=1}^{d'} |(A_i\psi)|^2 \leq 1,\; \psi \text{ real}\,\}$$

(see, for example, [Rob], Lemma IV.2.3, or [ElR2], Lemma 4.2). Other parameters which enter the estimates are the subelliptic modulus $g \mapsto |g|' = d'(g\,;e)$, where e is the identity of G, and the local dimension D', i.e., the integer for which the left Haar measure $|B'(g\,;r)|$ of the ball $B'(g\,;r) = \{\, h \in G \,:\, d'(g\,;h) < r\,\}$ satisfies bounds

$$c^{-1} r^{D'} \leq |B'(e\,;r)| \leq c\, r^{D'} \tag{3}$$

for some $c > 0$ and all $r \leq 1$.

Theorem 1.1 *Let H be a subelliptic operator of the form* (1) *with real measurable principal coefficients c_{ij} and complex measurable lower order coefficients c_i, c_i' and c_0. Then, for each $\theta \in \langle 0, \theta_C\rangle$, there exist $a, b > 0$ and $\omega \geq 0$ such that the kernel K satisfies*

$$|K_z(g\,;h)| \leq a\,|z|^{-D'/2} e^{\omega|z|} e^{-b(|gh^{-1}|')^2|z|^{-1}} \tag{4}$$

uniformly for $g, h \in G$ and $z \in \Lambda(\theta)$. The kernel has the semigroup property (2) *and is Hölder continuous. In particular there is a $\nu \in \langle 0,1\rangle$ such that for each $\theta \in \langle 0, \theta_C\rangle$ and $\kappa > 0$ there exist $a, b > 0$ and $\omega \geq 0$ such that*

$$|K_z(k^{-1}g\,;l^{-1}h) - K_z(g\,;h)| \leq a\,|z|^{-D'/2} e^{\omega|z|} \left(\frac{|k|' + |l|'}{|z|^{1/2} + |gh^{-1}|'}\right)^{\nu} e^{-b(|gh^{-1}|')^2|z|^{-1}} \tag{5}$$

for all $g, h, k, l \in G$ and $z \in \Lambda(\theta)$ with $|k|' + |l|' \leq \kappa\,|z|^{1/2} + 2^{-1}|gh^{-1}|'$.

Moreover, if all the coefficients of H are real-valued then

$$K_t(g\,;h) \geq a'\,t^{-D'/2} e^{-\omega' t} e^{-b'(|gh^{-1}|')^2 t^{-1}} \tag{6}$$

for some $a', b' > 0$ and $\omega' \geq 0$ uniformly for $g, h \in G$ and $t > 0$.

The theorem gives a satisfactory improvement of earlier results [BrR], [SaS], for second-order subelliptic operators on Lie groups, insofar it removes all unnecessary regularity, symmetry and reality restrictions on the coefficients and is valid for all groups, modular or unimodular, polynomial or exponential. Bounds such as (4) and (5) are well known for subelliptic operators of all orders if the coefficients are smooth (see, for example, [ElR1]). It is also known that positivity and bounds such as (6) are only possible for second-order operators with real coefficients (see [ABR] and [Rob]). Note that the theorem establishes the bounds (4) and (5) throughout the sector $\Lambda(\theta_C)$ and consequently if the matrix of principal coefficients $C = (c_{ij})$ is symmetric the results are valid in the open right half-plane. In fact the bounds (4) follow for complex z from their real z counterparts together with the L_2-holomorphy of the semigroup S (see [Dav], Theorem 3.4.8) and similar reasoning could be applied to the continuity estimates (5). These bounds allow one to extend the semigroup S to a holomorphic semigroup on all the L_p-spaces with a holomorphy sector containing the p-independent subsector $\Lambda(\theta_C)$.

The proof of the upper bounds and the Hölder continuity uses the strategy developed in [ElR3] following ideas of [Aus]. It suffices to prove De Giorgi estimates for the principal part of the operator. These estimates are established by a refinement of the Nash–De Giorgi techniques based on the general reasoning given in Chapter 5 of Giaquinta's book [Gia].

For the proof of the lower bounds we use the strategy of [BrR] (see also [Str]). First one replaces H by a sequence of subelliptic operators H_n obtained by regularizing the coefficients of H. Secondly, one establishes kernel bounds and continuity estimates uniformly for the regularizing sequence. Thirdly, one deduces the required result by a limit over the regularizing sequence. The regularization and limiting processes are described in detail in [ElR1] and [ElR3]. Hence we concentrate on deriving the uniform bounds.

2 De Giorgi estimates

The key to bounds on the semigroup kernel is the derivation of De Giorgi estimates for weak solutions of the elliptic equation $H\varphi = 0$ on small balls $B'(g\,;r) = \{\, h \in G : d'(g\,;h) < r \,\}$. In [ElR3] these estimates were derived for operators with complex uniformly continuous coefficients by a series of L_2-estimates based on the Poincaré and Caccioppoli inequalities. In the present context we need more refined L_p-estimates with $p \in [1,2\rangle$. A principal new ingredient is the Poincaré–Sobolev inequality.

First, for each bounded open subset $\Omega \subseteq G$, introduce the spaces

$$H'_{p;1}(\Omega) = \{\varphi \in L_p(\Omega\,;dg) : A_i\varphi \in L_p(\Omega\,;dg) \text{ for all } i \in \{1,\ldots,d'\}\} \quad ,$$

where $A_i\varphi$ denotes the distributional derivative in $\mathcal{D}'(\Omega)$. We use the notation $\nabla'\varphi = (A_1\varphi,\ldots,A_{d'}\varphi)$ and equip the spaces with the norms $\varphi \mapsto (\|\varphi\|_{p,\Omega}^p + \|\nabla'\varphi\|_{p,\Omega}^p)^{1/p}$ where

$$\|\varphi\|_{p,\Omega} = \Big(\int_\Omega dh\,|\varphi(h)|^p\Big)^{1/p}$$

and

$$\|\nabla'\varphi\|_{p,\Omega} = \Big(\int_\Omega dh\,\Big(\sum_{i=1}^{d'} |(A_i\varphi)(h)|^2\Big)^{p/2}\Big)^{1/p} \quad .$$

Moreover, for $\varphi \in L_{1,\mathrm{loc}}$ we denote by $\langle\varphi\rangle_\Omega$ the average of φ over Ω. Finally, if $\Omega = B'(g\,;r)$ we simplify notation by setting $\|\cdot\|_{p,g,r} = \|\cdot\|_{p,\Omega}$ and when $g = e$ we drop the e, e.g., $B'(r) = B'(e\,;r)$, $\|\nabla'\varphi\|_{2,r} = \|\nabla'\varphi\|_{2,e,r}$ etc..

Proposition 2.1 *Let $p \in [1, D'\rangle$ and $1/q = 1/p - 1/D'$. Then there exist $c_p > 0$ and $R_p \in \langle 0,1]$ such that*

$$\|\varphi - \langle\varphi\rangle_{g,r}\|_{q,g,r} \le c_p\,\|\nabla'\varphi\|_{p,g,r} \tag{7}$$

for all $g \in G$, $r \in \langle 0, R_p]$, $\varepsilon > 0$ and $\varphi \in H'_{q;1}(B'(g\,;r(1+\varepsilon)))$.

Proof The proof of the estimates (7) for $g = e$ and $\varphi \in C^\infty(\overline{B'(r)})$ follows from [FLW], Theorem 1. Their extension to $\varphi \in H'_{q;1}(B'(g\,;r(1+\varepsilon)))$ then follows the proof of Proposition 2.1 in [ElR3]. □

Our aim is to establish that the principal part H_P of the subelliptic operator H, and the principal part H_P^* of its adjoint H^*, satisfy De Giorgi estimates. These estimates then yield the upper bounds and continuity estimates for the semigroup kernel stated in Theorem 1.1. Since these results were already established for dimension $d = 1$ in Section 5 of [ElR3] and dimension $d = 2$ in [ElR4], even if the principal coefficients are complex, it suffices to assume $d \geq 3$. In particular $D' \geq 2$.

The De Giorgi estimates for $G = \mathbf{R}^d$ are established in Section 5.1 in [Gia]. The proof contains two technical ingredients. The first is the classical Sobolev–Poincaré inequality on balls of radius R and Proposition 2.1 provides the Lie group replacement. The second is the existence of a sequence of cut-off functions η with $\eta = 1$ on the ball of radius r, with $\operatorname{supp} \eta$ contained in the ball of radius R and with $\|\nabla'\eta\|_\infty \leq c(R-r)^{-1}$ for all small $r < R$, where c is independent of r and R. It is particularly important that these cut-off functions exist for all r arbitrarily close to R. Unfortunately we are unable to prove the existence of the η with respect to the balls $B'(r)$ and $B'(R)$ when r is close to R. Nevertheless one does have cut-off functions with respect to surrogate balls formed by exponentiating rectangles in the Lie algebra (see [ElR3], Lemma 2.4). It is, however, unclear whether these latter balls admit Sobolev–Poincaré inequalities. Despite these difficulties we will show that one can make estimates as in [Gia] by starting with the surrogate balls and then changing to the balls $B'(R)$.

For $r > 0$ let $\widetilde{B}_r$ be the subsets of $\mathfrak{g}$ as defined by (12) in [ElR3]. Then there exists a $c > 1$ such that

$$\exp \widetilde{B}_{c^{-1}r} \subset B'(r) \subset \exp \widetilde{B}_{cr} \tag{8}$$

uniformly for all sufficiently small r. In the sequel we need the following lemma.

Lemma 2.2 *There exist $R_0 \in \langle 0, 1]$, $b_1 > 0$ and, for all $r, R \in \langle 0, R_0]$ with $r < R$, an $\eta_{r,R} \in C_c^\infty(\exp \widetilde{B}_R)$ with $0 \leq \eta_{r,R} \leq 1$, $\eta_{r,R} = 1$ on $\exp \widetilde{B}_r$ such that $\|A_i \eta_{r,R}\|_\infty \leq b_1(R-r)^{-1}$ for all for all $i \in \{1, \ldots, d'\}$.*

Proof This follows immediately from Lemma 2.4 of [ElR3]. □

For $M, \mu > 0$ let $\mathcal{E}_{r,P}^{\mathrm{div}}(\mu, M)$ be the set of all pure second-order subelliptic operators

$$H = -\sum_{i,j=1}^{d'} A_i \, c_{ij} \, A_j$$

with real-valued coefficients c_{ij} such that $\mu_C \geq \mu$ and $\|C\|_\infty = \sup_{g \in G} \|C(g)\| \leq M$, where $\|C(g)\|$ denotes the l_2-norm of the matrix $C(g) = (c_{ij}(g))$. We aim to prove estimates uniform for all operators in $\mathcal{E}_{r,P}^{\mathrm{div}}(\mu, M)$. We first prove a version of the Caccioppoli inequalities, as in [ElR3], Lemma 2.5, but now for the balls $\exp \widetilde{B}_r$ instead of $B'(r)$.

Let R_0 be as in Lemma 2.2. We may assume that (8) is valid for all $r \in \langle 0, R_0]$.

Lemma 2.3 *For all $M, \mu > 0$ there exists a $b_2 > 0$ such that*

$$\int_{\exp \widetilde{B}_r} |\nabla'((\varphi - k)^+)|^2 \leq b_2 (R-r)^{-2} \int_{\exp \widetilde{B}_R} |(\varphi - k)^+|^2$$

uniformly for all $H \in \mathcal{E}_{r,P}^{\mathrm{div}}(\mu, M)$, all $r, R \in \mathbf{R}$ with $0 < r < R \leq R_0$, all $k \in \mathbf{R}$ and all real-valued $\varphi \in H'_{2;1}(\exp \widetilde{B}_R)$ satisfying $H\varphi = 0$ weakly on $\exp \widetilde{B}_R$.

Proof We may assume that $k = 0$, since one also has $H(\varphi - k) = 0$ weakly on $\exp \widetilde{B}_R$. Next, let $\eta_{r,R}$ and b_1 be as in Lemma 2.2. Note that $\varphi^+ \in H'_{2;1}(\exp \widetilde{B}_R)$ and $A_i(\varphi^+) = 1_{[\varphi>0]} A_i\varphi$ by an argument similar to the proof of Lemma 7.6 in [GiT] but using the estimates underlying the proof of Lemma 2.4 in [ElR1]. Therefore $A_i(\varphi^+)\, A_j(\varphi^+) = 1_{[\varphi>0]} A_i\varphi\, A_j\varphi = A_i(\varphi^+)\, A_j\varphi$ and $\varphi^+ A_i\varphi = \varphi^+ A_i(\varphi^+)$. Then

$$\mu_C\, \|\eta_{r,R}\nabla'(\varphi^+)\|_2^2 \leq (\eta_{r,R}\nabla'(\varphi^+), C\eta_{r,R}\nabla'(\varphi^+))$$
$$= (\eta_{r,R}^2\nabla'(\varphi^+), C\nabla'\varphi) = \sum_{i,j=1}^{d'} ([\eta_{r,R}^2, A_i](\varphi^+), c_{ij} A_j\varphi)$$

where the last step uses $H\varphi = 0$ weakly on $\exp \widetilde{B}_R$. But $[\eta_{r,R}^2, A_i] = -2\eta_{r,R}(A_i\eta_{r,R})$ and hence

$$\mu_C\, \|\eta_{r,R}\nabla'(\varphi^+)\|_2^2 \leq -2((\nabla'\eta_{r,R})(\varphi^+), C\eta_{r,R}\nabla'\varphi)$$
$$= -2((\nabla'\eta_{r,R})\varphi^+, C\eta_{r,R}\nabla'(\varphi^+))$$
$$\leq 2\,\|C\|_\infty\, \|(\nabla'\eta_{r,R})\varphi^+\|_2\, \|\eta_{r,R}\nabla'(\varphi^+)\|_2 \quad .$$

Therefore

$$\int_{\exp \widetilde{B}_r} |\nabla'((\varphi)^+)|^2 \leq \|\eta_{r,R}\nabla'(\varphi^+)\|_2^2 \leq 4\,\mu_C^{-2}\|C\|_\infty^2\, \|(\nabla'\eta_{r,R})\varphi^+\|_2^2$$
$$\leq 4\, b_1^2\, \mu_C^{-2}\|C\|_\infty^2\, (R-r)^{-2} \int_{\exp \widetilde{B}_R} |\varphi^+|^2 \quad ,$$

by the estimates of Lemma 2.2. □

Set $\theta = 2^{-1} + (4^{-1} + 2(D')^{-1})^{1/2} > 1$. Then $\theta^2 - \theta - 2(D')^{-1} = 0$. Moreover, if $k \in \mathbf{R}$, $0 < r \leq R < \infty$ and $\varphi \in H'_{2;1}(\exp \widetilde{B}_R)$ set

$$\widetilde{A}(k,r) = [\varphi > k] \cap \exp \widetilde{B}_r \quad .$$

It will be clear from the context which φ is used.

Proposition 2.4 *For all $M, \mu > 0$ there exists a $b_3 > 0$ such that*

$$\operatorname*{ess\,sup}_{g\in \exp \widetilde{B}_{R/2}} \varphi(g) \leq k + b_3\Big(R^{-D'} \int_{\widetilde{A}(k,R)} |\varphi - k|^2\Big)^{1/2} \Big(R^{-D'}|\widetilde{A}(k,R)|\Big)^{(\theta-1)/2}$$

uniformly for all $H \in \mathcal{E}^{\mathrm{div}}_{r,P}(\mu, M)$, $R \in \langle 0, R_0]$, $k \in \mathbf{R}$ and real-valued $\varphi \in H'_{2;1}(\exp \widetilde{B}_R)$ satisfying $H\varphi = 0$ weakly on $\exp \widetilde{B}_R$.

Proof We follow the spirit of the proof of Theorem 5.1 in [Gia]. Let $\eta_{r,R}$ and b_1 be as in Lemma 2.2 and b_2 be as in Lemma 2.3. Let $H \in \mathcal{E}^{\mathrm{div}}_{r,P}(\mu, M)$, $S \in \langle 0, R_0]$ and $\varphi \in H'_{2;1}(\exp \widetilde{B}_S)$ be a real-valued function satisfying $H\varphi = 0$ weakly on $\exp \widetilde{B}_S$. Then for all $0 < r < R \leq S$ and $k \in \mathbf{R}$ one has

$$\int_{\exp \widetilde{B}_{(r+R)/2}} |\nabla'(\eta_{r,(r+R)/2}(\varphi - k)^+)|^2 \leq 2\int_{\exp \widetilde{B}_{(r+R)/2}} |\nabla'\eta_{r,(r+R)/2}|^2\, |(\varphi - k)^+)|^2$$
$$+ 2\int_{\exp \widetilde{B}_{(r+R)/2}} |\eta_{r,(r+R)/2}|^2\, |\nabla'((\varphi - k)^+)|^2$$
$$\leq (8d'b_1^2 + 2b_2)(R-r)^{-2} \int_{\exp \widetilde{B}_R} |(\varphi - k)^+|^2 \quad .$$

Next, by the subelliptic Sobolev inequalities for Lie groups, [Rob], Theorem IV.5.6, there exists a $b > 0$ such that

$$\Big(\int_G |\psi|^{2D'/(D'-2)}\Big)^{(D'-2)/D'} \leq b\int_G |\nabla'\psi|^2 + b\int_G |\psi|^2$$

uniformly for all $\psi \in L'_{2;1}(G)$. Therefore

$$\begin{aligned}
&\Big(\int_{\exp\widetilde{B}_r} |(\varphi-k)^+|^{2D'/(D'-2)}\Big)^{(D'-2)/D'} \\
&\qquad\leq \Big(\int_G |\eta_{r,(r+R)/2}(\varphi-k)^+|^{2D'/(D'-2)}\Big)^{(D'-2)/D'} \\
&\qquad\leq b\int_G |\nabla'(\eta_{r,(r+R)/2}(\varphi-k)^+)|^2 + b\int_G |\eta_{r,(r+R)/2}(\varphi-k)^+|^2 \\
&\qquad= b\int_{\exp\widetilde{B}_{(r+R)/2}} |\nabla'(\eta_{r,(r+R)/2}(\varphi-k)^+)|^2 + b\int_{\exp\widetilde{B}_{(r+R)/2}} |\eta_{r,(r+R)/2}(\varphi-k)^+|^2 \\
&\qquad\leq b'(R-r)^{-2}\int_{\exp\widetilde{B}_R} |(\varphi-k)^+|^2 \quad ,
\end{aligned}$$

where $b' = b(8d'b_1^2 + 2b_2 + R_0^2)$. Hence one deduces that

$$\begin{aligned}
\int_{\widetilde{A}(k,r)} |\varphi-k|^2 &= \int_{\widetilde{A}(k,r)} |(\varphi-k)^+|^2 \leq |\widetilde{A}(k,r)|^{2/D'}\Big(\int_{\widetilde{A}(k,r)} |(\varphi-k)^+|^{2D'/(D'-2)}\Big)^{(D'-2)/D'} \\
&\leq b'(R-r)^{-2}|\widetilde{A}(k,r)|^{2/D'}\int_{\exp\widetilde{B}_R} |(\varphi-k)^+|^2 \\
&\leq b'(R-r)^{-2}|\widetilde{A}(k,R)|^{2/D'}\int_{\widetilde{A}(k,R)} |\varphi-k|^2
\end{aligned}$$

by the Hölder inequality.

Now let $h \in \mathbf{R}$ and suppose $k < h$. Then $\widetilde{A}(h,r) \subseteq \widetilde{A}(k,r)$ and $|\varphi-h|^2 \leq |\varphi-k|^2$ on $\widetilde{A}(h,r)$. So

$$\int_{\widetilde{A}(h,r)} |\varphi-h|^2 \leq \int_{\widetilde{A}(h,r)} |\varphi-k|^2 \leq \int_{\widetilde{A}(k,r)} |\varphi-k|^2$$

and the functions $l \mapsto \int_{\widetilde{A}(l,r)} |\varphi-l|^2$ and $l \mapsto |\widetilde{A}(l,r)|$ are decreasing. In particular,

$$\int_{\widetilde{A}(h,r)} |\varphi-h|^2 \leq \int_{\widetilde{A}(k,r)} |\varphi-k|^2 \leq b'(R-r)^{-2}|\widetilde{A}(k,R)|^{2/D'}\int_{\widetilde{A}(k,R)} |\varphi-k|^2 \quad . \tag{9}$$

Moreover,

$$\begin{aligned}
|\widetilde{A}(h,r)| &= |h-k|^{-2}\int_{\widetilde{A}(h,r)} |h-k|^2 \leq |h-k|^{-2}\int_{\widetilde{A}(h,r)} |\varphi-k|^2 \\
&\leq |h-k|^{-2}\int_{\widetilde{A}(k,R)} |\varphi-k|^2 \quad .
\end{aligned} \tag{10}$$

Next define $\Phi\colon \mathbf{R}\times\langle 0,S] \to \mathbf{R}$ by

$$\Phi(h,r) = |\widetilde{A}(h,r)|\Big(\int_{\widetilde{A}(h,r)} |\varphi-h|^2\Big)^{D'\theta/2} \quad .$$

Then it follows from (9) and (10) that

$$\Phi(h,r) \leq (b')^{D'\theta/2}(R-r)^{-D'\theta}|h-k|^{-2}\Phi(k,R)^{\theta} \tag{11}$$

uniformly for all $0 < r < R \leq S$ and $h, k \in \mathbf{R}$ with $k < h$, where we have used $\theta^2 - \theta - 2(D')^{-1} = 0$. Moreover, the function $h \mapsto \Phi(h,r)$ is decreasing for all r and the function $r \mapsto \Phi(h,r)$ is increasing for all h.

Finally, let $k \in \mathbf{R}$. Set

$$s = 2^{(\theta-1)^{-1}\theta(1+D'\theta/2)}(b')^{D'\theta/4}(2^{-1}S)^{-D'\theta/2}\Phi(k,S)^{(\theta-1)/2} \quad .$$

For $n \in \mathbf{N}_0$ set $k_n = k + s - 2^{-n}s$ and $r_n = 2^{-1}S + 2^{-(n+1)}S$. Then it follows by induction from (11) that

$$\Phi(k_n, r_n) \leq 2^{-(\theta-1)^{-1}(2+D'\theta)n}\,\Phi(k,S)$$

for all $n \in \mathbf{N}_0$. So by monotonicity one obtains

$$0 \leq \Phi(k+s, 2^{-1}S) \leq \Phi(k_n, r_n) \leq 2^{-(\theta-1)^{-1}(2+D'\theta)n}\,\Phi(k,S)$$

for all $n \in \mathbf{N}$ and hence $\Phi(k+s, 2^{-1}S) = 0$. Thus either $|\widetilde{A}(k+s, 2^{-1}S)| = 0$ or $\int_{\widetilde{A}(k+s,2^{-1}S)} |\varphi - (k+s)|^2 = 0$. But $\varphi - (k+s) > 0$ on $\widetilde{A}(k+s, 2^{-1}S)$. Therefore, in both cases, it follows that $|\widetilde{A}(k+s, 2^{-1}S)| = 0$. Hence

$$\operatorname*{ess\,sup}_{g \in \exp \widetilde{B}_{S/2}} \varphi(g) \leq k + s \quad .$$

Since

$$S^{-D'\theta/2}\,\Phi(k,S)^{(\theta-1)/2} = \Big(S^{-D'}\int_{\widetilde{A}(k,S)} |\varphi - k|^2\Big)^{1/2}\Big(S^{-D'}|\widetilde{A}(k,S)|\Big)^{(\theta-1)/2}$$

the proposition follows. □

Next we turn the bounds of the previous proposition, which involve the balls $\exp \widetilde{B}_r$ into bounds which involve the "normal" balls $B'(r)$. If $k \in \mathbf{R}$, $0 < r \leq R < \infty$ and $\varphi \in H'_{2;1}(\exp \widetilde{B}_R)$ set

$$A(k,r) = [\varphi > k] \cap B'(r) \quad .$$

Proposition 2.5 *There is a $\sigma \in \langle 0,1 \rangle$ such that for all $M, \mu > 0$ there exists a $b_3 > 0$ such that*

$$\operatorname*{ess\,sup}_{g \in B'(\sigma R)} \varphi(g) \leq k + b_3\Big(R^{-D'}\int_{A(k,R)} |\varphi - k|^2\Big)^{1/2}\Big(R^{-D'}|A(k,R)|\Big)^{(\theta-1)/2}$$

uniformly for all $H \in \mathcal{E}^{\mathrm{div}}_{r,P}(\mu, M)$, $R \in \langle 0, R_0]$, $k \in \mathbf{R}$ and all real-valued $\varphi \in H'_{2;1}(B'(R))$ satisfying $H\varphi = 0$ weakly on $B'(R)$.

Proof This follows immediately from Proposition 2.4 and the inclusions (8), if one takes $\sigma = 2^{-1}c^{-2}$ with c as in (8). □

Let σ be as in Proposition 2.5.

Corollary 2.6 *For all $M, \mu > 0$ there exists a $b_4 > 0$ such that*

$$\operatorname*{ess\,sup}_{g \in B'(\sigma R)} |\varphi(g)| \leq b_4 \Big(R^{-D'} \int_{B'(R)} |\varphi|^2 \Big)^{1/2}$$

uniformly for all $H \in \mathcal{E}^{\rm div}_{r,P}(\mu, M)$, $R \in \langle 0, R_0]$, $k \in \mathbf{R}$ and all real-valued $\varphi \in H'_{2;1}(B'(R))$ satisfying $H\varphi = 0$ weakly on $B'(R)$.

Proof Let b_3 be as in Proposition 2.5. Then applying Proposition 2.5 with $k = 0$ one deduces that

$$\operatorname*{ess\,sup}_{g \in B'(\sigma R)} \varphi(g) \leq b_3 \Big(R^{-D'} \int_{B'(R)} |\varphi|^2 \Big)^{1/2} \Big(R^{-D'} |B'(R)| \Big)^{(\theta - 1)/2} \quad .$$

But these estimates are also valid for φ replaced by $-\varphi$. Now the corollary follows from the volume estimates (3). □

Lemma 2.3 also has a weaker version for the balls $B'(R)$. Note that we can take the same σ as in Proposition 2.5 in the next lemma but this is not essential, although it is convenient.

Lemma 2.7 *For all $M, \mu > 0$ there exists a $b_5 > 0$ such that*

$$\int_{B'(\sigma r)} |\nabla'((\varphi - k)^+)|^2 \leq b_5 R^{-2} \int_{B'(R)} |(\varphi - k)^+|^2$$

uniformly for all $H \in \mathcal{E}^{\rm div}_{r,P}(\mu, M)$, $0 < r \leq R \leq R_0$, $k \in \mathbf{R}$ and all real-valued $\varphi \in H'_{2;1}(B'(R))$ satisfying $H\varphi = 0$ weakly on $B'(R)$.

Before we can prove the De Giorgi estimates for subelliptic operators with real measurable coefficients we need one more technical lemma.

Lemma 2.8 *There exist $S \in \langle 0, R_0]$ and $\beta > 0$ such that for all $M, \mu > 0$ there exists a $b_6 > 0$ such that*

$$R^{-D'} |A(k_n, R)| \leq b_6 n^{-\beta}$$

uniformly for all $H \in \mathcal{E}^{\rm div}_{r,P}(\mu, M)$, $R \in \langle 0, \sigma^2 S]$, $n \in \mathbf{N}_0$ and real-valued $\varphi \in H'_{2;1}(B'(\sigma^{-2} R))$ satisfying $H\varphi = 0$ weakly on $B'(\sigma^{-2}R)$ and $|A(k_0, R)| \leq 2^{-1} |B'(R)|$ where

$$k_n = \operatorname*{ess\,sup}_{g \in B'(\sigma^{-1} R)} \varphi(g) - 2^{-(n+1)} \Big(\operatorname*{ess\,sup}_{g \in B'(\sigma^{-1} R)} \varphi(g) - \operatorname*{ess\,inf}_{g \in B'(\sigma^{-1} R)} \varphi(g) \Big) \quad .$$

Proof Note that the essential suprema and infima are finite by Corollary 2.6. If $\operatorname{ess\,sup}_{g \in B'(\sigma^{-1} R)} \varphi(g) = \operatorname{ess\,inf}_{g \in B'(\sigma^{-1} R)} \varphi(g)$ then $|A(k_n, R)| = 0$ and the lemma is trivial. So we may assume that $\operatorname{ess\,sup}_{g \in B'(\sigma^{-1} R)} \varphi(g) \neq \operatorname{ess\,inf}_{g \in B'(\sigma^{-1} R)} \varphi(g)$.

Let $h > k \geq k_0$. Set $v = \varphi \wedge h - \varphi \wedge k \in H'_{2;1}(B'(\sigma^{-1} R))$. Then $|B'(R) \cap [v \neq 0]| = |A(k, R)| \leq |A(k_0, R)| \leq 2^{-1} |B'(R)|$.

Now fix $p \in [1, 2\rangle$ and set $q = pD'/(D' - p)$ and $S = R_0 \wedge R_p$, where R_p is as in Proposition 2.1. Since $D' \geq 2$ it follows that $q \in \langle 2, \infty \rangle$ and $1/q = 1/p - 1/D'$. Then it follows from the Sobolev–Poincaré inequality of Proposition 2.1 that one has bounds

$$\|v - \langle v \rangle_R\|_{q,R} \leq c_p \, \|\nabla' \psi\|_{p,R} \quad . \tag{12}$$

But by the Hölder inequality one obtains

$$\begin{aligned}\|\langle v\rangle_R\|_{q,R} &= |B'(R)|^{-1+1/q}\int_{B'(R)\cap[v\neq 0]}|v|\\ &\le |B'(R)|^{-1+1/q}|B'(R)\cap[v\neq 0]|^{1-1/q}\|v\|_{q,R}\\ &\le |B'(R)|^{-1+1/q}(2^{-1}|B'(R)|)^{1-1/q}\|v\|_{q,R} = 2^{-1+1/q}\|v\|_{q,R} \quad .\end{aligned}$$

So by the triangle inequality, and (12), one deduces that

$$\|v\|_{q,R} \le c_p\,\|\nabla'\psi\|_{p,R} + 2^{-1+1/q}\|v\|_{q,R}$$

and hence

$$R^{-D'/q}\|v\|_{q,R} \le b\,R\,R^{-D'/p}\|\nabla'\psi\|_{p,R} \quad ,$$

where $b = (1-2^{-1+1/q})^{-1}c_p$. Using the definition of v one then deduces that

$$\begin{aligned}&R^{-D'}|h-k|^q\,|A(h,R)|\\ &\quad = R^{-D'}\int_{A(h,R)}|v|^q \le R^{-D'}\int_{B'(R)}|v|^q\\ &\quad \le b^q\Big(R^p\,R^{-D'}\int_{B'(R)}|\nabla' v|^p\Big)^{q/p} = b^q\Big(R^{-D'}\int_{A(k,R)\backslash A(h,R)}(R\,|\nabla'\varphi|)^p\Big)^{q/p}\\ &\quad \le b^q\Big(\Big(R^{-D'}|A(k,R)\backslash A(h,R)|\Big)^{(2-p)/2}\Big(R^{-D'}\int_{A(k,R)\backslash A(h,R)}(R\,|\nabla'\varphi|)^2\Big)^{p/2}\Big)^{q/p}\\ &\quad \le b^q\Big(R^{-D'}|A(k,R)\backslash A(h,R)|\Big)^{q(2-p)/(2p)}\Big(R^{-D'}\int_{A(k,R)}R^2|\nabla'\varphi|^2\Big)^{q/2} \quad , \qquad (13)\end{aligned}$$

where we have used the Hölder inequality. But by the Caccioppoli inequality, Lemma 2.7,

$$\begin{aligned}R^{-D'}\int_{A(k,R)}R^2|\nabla'\varphi|^2 &= R^{-D'}\int_{B'(R)}R^2|\nabla'((\varphi-k)^+)|^2\\ &\le b_5\,R^{-D'}\int_{B'(\sigma^{-1}R)}|(\varphi-k)^+|^2\\ &\le b_5\,R^{-D'}\int_{B'(\sigma^{-1}R)}|(M(\sigma^{-1}R)-k)^+|^2 \le b_5\,c\,\sigma^{-D'}(M(\sigma^{-1}R)-k)^2 \quad ,\end{aligned}$$

where $M(\sigma^{-1}R) = \operatorname{ess\,sup}_{g\in B'(\sigma^{-1}R)}\varphi(g)$ and c is as in (3). Together with (13) this gives

$$|h-k|^{2p/(2-p)}\Big(R^{-D'}|A(k_i,R)|\Big)^{\alpha} \le b'\,R^{-D'}|A(k,R)\backslash A(h,R)|\,\Big(M(\sigma^{-1}R)-k\Big)^{2p/(2-p)} \quad ,$$

where $\alpha = 2p(q(2-p))^{-1}$ and $b' = (b^2 b_5 c\sigma^{-D'})^{p/(2-p)}$. Next apply these estimates with $h = k_i$ and $k = k_{i-1}$, where $i \in \mathbf{N}$. Then

$$\Big(R^{-D'}|A(k_i,R)|\Big)^{\alpha} \le 2^{2p/(2-p)}b'R^{-D'}\Big(|A(k_{i-1},R)| - |A(k_i,R)|\Big) \quad .$$

Thus one obtains

$$\begin{aligned}n\Big(R^{-D'}|A(k_n,R)|\Big)^{\alpha} &\le \sum_{i=1}^{n}\Big(R^{-D'}|A(k_i,R)|\Big)^{\alpha}\\ &\le 2^{2p/(2-p)}b'R^{-D'}\Big(|A(k_0,R)| - |A(k_n,R)|\Big)\\ &\le 2^{2p/(2-p)}b'R^{-D'}|A(k_0,R)| \le 2^{2p/(2-p)}b'\,c\end{aligned}$$

for all $n \in \mathbf{N}$ where we have used $|A(k_0, R)| \leq |B'(R)| \leq c\,R^{D'}$. Therefore

$$R^{-D'}|A(k_n, R)| \leq (2^{2p/(2-p)}b'\,c)^{\beta}n^{-\beta}$$

with $\beta = 1/\alpha$ and the proof of the lemma is complete. □

At this point we can state and prove the uniform De Giorgi estimates which we require.

Proposition 2.9 *For all $M, \mu > 0$ there exist $\nu > 0$ and $c_{DG} > 0$ such that*

$$\int_{B'(g;r)} |\nabla'\varphi|^2 \leq c_{DG}(r/R)^{D'-2+2\nu} \int_{B'(g;R)} |\nabla'\varphi|^2$$

uniformly for all $H \in \mathcal{E}^{\rm div}_{r,P}(\mu, M)$, $r, R \in \mathbf{R}$ with $0 < r \leq R \leq 1$, $g \in G$ and all (complex-valued) $\varphi \in H'_{2;1}(B'(R))$ satisfying $H\varphi = 0$ weakly on $B'(g\,;R)$.

Proof As in the proofs of Propositions 3.3 and 3.4 in [ElR3] it suffices to prove the estimates for $g = e$, $R \in \langle 0, S \wedge R_N]$, $r \leq \sigma^2 R$ and $\varphi \in H'_{2;1}(B'(2R))$ satisfying $H\varphi = 0$ weakly on $B'(2R)$ and $\langle\varphi\rangle_R = 0$, where R_N is the constant in the Poincaré inequality (11) of Proposition 2.1 in [ElR3] and S is as in Lemma 2.8. Moreover, since H is a real operator, it suffices to prove the inequality for real-valued φ.

Let b_3, b_4, b_5, b_6 and β be as in Proposition 2.5, Corollary 2.6 and Lemmas 2.7 and 2.8. Moreover, let c be as in (3) and c_N be as in the Neumann-type Poincaré inequality of Proposition 2.1 in [ElR3]. Let R and φ be as above. For all $r \in \langle 0, \sigma R]$ the function φ is essentially bounded on $B'(r)$ by Corollary 2.6 and we set

$$m(r) = \operatorname*{ess\,inf}_{g \in B'(r)} \varphi(g) \quad , \quad M(r) = \operatorname*{ess\,sup}_{g \in B'(r)} \varphi(g) \quad .$$

Now suppose $r \in \langle 0, \sigma^2 R]$. Set $k_0 = 2^{-1}\Big(M(\sigma^{-1}r) + m(\sigma^{-1}r)\Big)$. We may assume that $|A(k_0, r)| \leq 2^{-1}|B'(r)|$, otherwise replace φ by $-\varphi$. Next, for all $n \in \mathbf{N}$ we set $k_n = M(\sigma^{-1}r) - 2^{-(n+1)}\Big(M(\sigma^{-1}r) - m(\sigma^{-1}r)\Big)$. Then it follows from Proposition 2.5 that

$$\begin{aligned} M(\sigma r) &\leq k_n + b_3\Big(r^{-D'}\int_{A(k_n,r)} |M(r) - k_n|^2\Big)^{1/2}\Big(r^{-D'}|A(k_n, r)|\Big)^{(\theta-1)/2} \\ &\leq k_n + b_3 c^{1/2}\Big(M(\sigma^{-1}r) - k_n\Big)(b_6 n^{-\beta})^{(\theta-1)/2} \end{aligned}$$

uniformly for all $n \in \mathbf{N}$, where Lemma 2.8 is used in the last inequality. Next fix $N \in \mathbf{N}$ such that

$$b_3 c^{1/2}(b_6 N^{-\beta})^{(\theta-1)/2} \leq 2^{-1} \quad .$$

Note that N depends only on M and μ. Then

$$\begin{aligned} M(\sigma r) &\leq M(\sigma^{-1}r) - 2^{-(N+1)}\Big(M(\sigma^{-1}r) - m(\sigma^{-1}r)\Big) + 2^{-1}2^{-(N+1)}\Big(M(\sigma^{-1}r) - m(\sigma^{-1}r)\Big) \\ &= M(\sigma^{-1}r) - 2^{-(N+2)}\Big(M(\sigma^{-1}r) - m(\sigma^{-1}r)\Big) \end{aligned}$$

and hence

$$\begin{aligned} M(\sigma r) - m(\sigma r) &\leq M(\sigma^{-1}r) - m(\sigma^{-1}r) - 2^{-(N+2)}\Big(M(\sigma^{-1}r) - m(\sigma^{-1}r)\Big) \\ &= (1 - 2^{-(N+2)})\Big(M(\sigma^{-1}r) - m(\sigma^{-1}r)\Big) \quad . \end{aligned}$$

This is valid for all $r \in \langle 0, \sigma^2 R]$. Therefore one deduces by induction that

$$M(\sigma^{2n+1}r) - m(\sigma^{2n+1}r) \le (1 - 2^{-(N+2)})^n \Big(M(\sigma R) - m(\sigma R)\Big)$$

for all $n \in \mathbf{N}$ and

$$M(r) - m(r) \le a(r/R)^\nu \Big(M(\sigma R) - m(\sigma R)\Big)$$

for all $r \in \langle 0, \sigma R]$, where $a = (1 - 2^{-(N+2)})^{-3/2}$ and $\nu = (2\log\sigma)^{-1}\log(1 - 2^{-(N+2)}) > 0$. Finally, applying Lemma 2.7 to φ with $k = \langle\varphi\rangle_{\sigma^{-1}r}$ and to $-\varphi$ with $k = -\langle\varphi\rangle_{\sigma^{-1}r}$ one deduces that for all $r \in \langle 0, \sigma^2 R]$

$$\begin{aligned}
\int_{B'(r)} |\nabla'\varphi|^2 &\le b_5 \sigma^2 r^{-2} \int_{B'(\sigma^{-1}r)} |\varphi - \langle\varphi\rangle_{\sigma^{-1}r}|^2 \\
&\le b_5 \sigma^2 r^{-2} \int_{B'(\sigma^{-1}r)} |M(\sigma^{-1}r) - m(\sigma^{-1}r)|^2 \\
&\le a^2 b_5 c \sigma^{-D'+2} r^{D'-2} (\sigma^{-1}r/R)^{2\nu} \Big(M(\sigma R) - m(\sigma R)\Big)^2 \\
&\le a^2 b_5 c \sigma^{-D'+2-2\nu} r^{D'+2\nu-2} R^{-2\nu} \Big(2 \operatorname*{ess\,sup}_{g \in B'(\sigma R)} |\varphi(g)|\Big)^2 \\
&\le 4a^2 b_4^2 b_5 c \sigma^{-D'+2-2\nu} r^{D'+2\nu-2} R^{-(D'+2\nu)} \int_{B'(R)} |\varphi|^2 \\
&\le 4a^2 b_4^2 b_5 c c_N \sigma^{-D'+2-2\nu} (r/R)^{D'+2\nu-2} \int_{B'(R)} |\nabla'\varphi|^2 \quad ,
\end{aligned}$$

where we have used $\langle\varphi\rangle_R = 0$ and the Neumann-type Poincaré inequality in the last step. This completes the proof of the De Giorgi estimates for operators with real measurable coefficients. □

3 Kernel estimates

In this section we prove the estimates for the semigroup kernel stated in Theorem 1.1.

The upper bounds and continuity estimates were already established for dimension one in Section 5 of [ElR3] and for dimension two in [ElR4]. In fact these low-dimensional results do not require the principal coefficients of H to be real-valued. If, however, $D' \geq 2$ then the bounds and continuity are a direct corollary of Theorem 4.1 of [ElR3] and the De Giorgi estimates of Proposition 2.9. Note that this latter proposition can be applied both to H and its adjoint H^*.

It remains to prove the Gaussian lower bounds in the last statement of Theorem 1.1. There is a large literature on lower bounds of semigroup kernels associated with real second-order elliptic operators. (References directly relevant to the current paper can be found in the books [Dav], [Rob], [VSC].) Most of this work is based on the arguments of Nash [Nas] for pure second-order operators on $\mathbf{R}^d$ and the subsequent discussion is largely composed of arguments already contained in the literature. The only essential new feature arises from the lack of symmetry of the principal coefficients. Previous Lie group results also require some smoothness of the coefficients or place restrictions on the growth properties of the group. Therefore we have to rearrange the reasoning to take care of these difficulties.

First we reduce to the case of smooth coefficients by regularization following the ideas of [BrR], [Str], [ElR3]. If $\tau_n \in C_c^\infty(G)$, $\tau_n \geq 0$, $\|\tau_n\|_1 = 1$ is an approximation to the identity and $c \in L_\infty$ the regularization $c^{(n)}$ is defined by

$$c^{(n)}(g) = (R(\tau_n)c)(g) = \int_G dh\, \tau_n(h)\, c(gh) \quad ,$$

where R is the right regular representation. Then let H_n denote the operators constructed from H by regularization of the coefficients as in [ElR3], Section 2.4, i.e., one replaces the c_{ij} in H by $c_{ij}^{(n)}$, etc.. Since $\tau_n \geq 0$ and $\|\tau_n\|_1 = 1$ it follows that $\mu_{C^{(n)}} \geq \mu_C$, $\|C^{(n)}\|_\infty \leq \|C\|_\infty$, $\|c^{(n)}\|_\infty \leq \|c\|_\infty$ and $\|c_0^{(n)}\|_\infty \leq \|c_0\|_\infty$ (see (13) and (14) in [ElR3]). Moreover, if $K^{(n)}$ denotes the kernel of the semigroup generated by H_n it follows from the uniform bounds of Proposition 2.9 together with Propositions 4.5 and 2.8 of [ElR3] that $\lim_{n\to\infty} K_t^{(n)} = K_t$ uniformly on compacta of $G \times G$. Thus we can effectively assume the coefficients of H are smooth as long as the final estimates are independent of the smoothness. For operators with smooth coefficients it follows from [ElR1] Corollary 3.5 that the kernel belongs to $C_{b;\infty}(G \times G)$.

Next we observe that the kernel K_t associated with an operator H with all real coefficients is pointwise positive, i.e., $K_t(g\,;h) \geq 0$ for all $g, h \in G$. If the principal coefficients are symmetric this is a standard result but it also follows for non-symmetric c_{ij} by the dispersivity of H (see [ABR], Proposition 2.7). It is not a priori evident, however, that K_t is strictly positive and this complicates the ensuing arguments. To avoid this difficulty we replace K_t by $K_t^{(\delta)} = K_t + \delta$ with $\delta \in \langle 0, 1]$ and eventually take the limit $\delta \to 0$. Now we turn to Nash's arguments. The first important step is to deduce local lower bounds on the kernel.

The basic ingredient in Nash's method is an L_1-lower bound

$$\inf_{t\in\langle 0,1]} \inf_{g\in G} \int_G d\hat{h}\, K_t(g\,;h) \geq c > 0 \quad . \tag{14}$$

This is straightforward for pure second-order operators since

$$\int_G d\hat{h}\, K_t(g\,;h) = 1 \quad .$$

The lower-order terms present a difficulty, however, which can be circumvented by an argument of Stroock [Str] which we apply to the modified kernels $K^{(\delta)}$.

Choose a real $\chi \in C^\infty(G)$ such that $\int_G d\hat{h}\, e^{-\chi(h)} = 1$ and $A_i\chi \in L_\infty$ for all $i \in \{1, \ldots, d'\}$. A function with these properties can be constructed by regularization of the modulus $g \mapsto |g|'$ with a C^∞-function. (See, for example, [Rob], pages 201–202.) Next fix $g \in G$ and define ψ by setting $\psi(h) = \chi(hg^{-1})$. Then ψ satisfies the properties required of χ and in addition $\|A_i\psi\|_\infty = \|A_i\chi\|_\infty$. Further define k_t by setting $k_t(h) = K_t^{(\delta)}(g\,;h) = K_t(g\,;h) + \delta$. Next for each $\gamma \in \langle 0, 1]$ introduce $H_\gamma \colon \langle 0, \infty\rangle \to \langle 0, \infty\rangle$ by

$$H_\gamma(t) = \int_G d\hat{h}\, e^{-\psi(h)} k_t^\gamma(h) \quad ,$$

where $k_t^\gamma(h) = (k_t(h))^\gamma$. Note that the factor $e^{-\psi}$ is necessary since $k_t^\gamma \geq \delta^\gamma$. Then

$$H_\gamma'(t) = -\gamma \int_G d\hat{h}\, e^{-\psi(h)} k_t^{\gamma-1}(h)\, (\widetilde{H}_h K_t)(g\,;h)$$

where $\widetilde{H}_h$ denotes the adjoint of H with respect to right Haar measure acting on the h variable. (Note that $k_t^{\gamma-1}$ is bounded for $\gamma \in \langle 0,1\rangle$ because $k_t \geq \delta$. This together with the boundedness of $h \mapsto (\widetilde{H}_h K_t)(g\,;h)$ ensures that the integral exists.) Using the explicit form of $\widetilde{H}$ we can now integrate by parts, i.e., evaluate $\widetilde{H}$ in terms of the associated form. This operation causes no difficulty because we are assuming the coefficients are smooth and hence the kernel is a C^∞-function over $G \times G$. One finds

$$H'_\gamma(t) = \gamma(1-\gamma)\sum_{i,j=1}^{d'}\int_G d\hat{h}\, e^{-\psi(h)} k_t^{\gamma-2}(h)\,(A_i k_t)(h)\, c_{ji}(h)\,(A_j k_t)(h)$$
$$+\gamma\sum_{i,j=1}^{d'}\int_G d\hat{h}\, e^{-\psi(h)} k_t^{\gamma-1}(h)\,(A_i\psi)(h)\, c_{ji}(h)\,(A_j k_t)(h) + R$$

where the first terms on the right hand side denote the contribution to the differential equation of the terms in $\widetilde{H}$ with two derivatives and R indicates the contribution of the lower order terms. Note that we have identified $A_j k_t$ with the derivative in the j-th direction, with respect to the second variable, of K_t. Simple rearrangement then gives

$$H'_\gamma(t) = 4\gamma^{-1}(1-\gamma)\sum_{i,j=1}^{d'}\int_G d\hat{h}\, e^{-\psi(h)}(A_i k_t^{\gamma/2})(h)\, c_{ji}(h)\,(A_j k_t^{\gamma/2})(h)$$
$$+2\sum_{i,j=1}^{d'}\int_G d\hat{h}\, e^{-\psi(h)} k_t^{\gamma/2}(h)\,(A_i\psi)(h)\, c_{ji}(h)\,(A_j k_t^{\gamma/2})(h) + R \quad .$$

The first term on the right hand side, which only depends on the hermitian part of the matrix $C = (c_{ij})$ of principal coefficients, can now be bounded below using the subellipticity condition,

$$\sum_{i,j=1}^{d'}\int_G d\hat{h}\, e^{-\psi(h)}(A_i k_t^{\gamma/2})(h)\, c_{ji}(h)\,(A_j k_t^{\gamma/2})(h) \geq \mu_C \sum_{i=1}^{d'}\int_G d\hat{h}\, e^{-\psi(h)}|(A_i k_t^{\gamma/2})(h)|^2 \quad ,$$

and the second term can be bounded by an $\varepsilon, \varepsilon^{-1}$ argument, e.g.,

$$\Big|\sum_{i,j=1}^{d'}\int_G d\hat{h}\, e^{-\psi(h)} k_t^{\gamma/2}(h)\,(A_i\psi)(h)\, c_{ji}(h)\,(A_j k_t^{\gamma/2})(h)\Big|$$
$$\leq \|C\|_\infty \sum_{i=1}^{d'}\int_G d\hat{h}\, e^{-\psi(h)}\Big(\varepsilon\,|(A_i k_t^{\gamma/2})(h)|^2 + (4\varepsilon)^{-1} k_t^{\gamma}(h)\|A_i\chi\|_\infty^2\Big) \quad .$$

The lower order terms contained in the remainder R can be bounded in a similar manner with an $\varepsilon, \varepsilon^{-1}$ argument. (Care has to be taken with the terms with no derivatives on the right hand side. These contain a factor $K_t(g\,;h)$ but this can be replaced by $k_t(h)$ at the cost of introducing an extra factor $K_t(g\,;h)/K_t^{(\delta)}(g\,;h)$. But this factor satisfies $0 \leq K_t(g\,;h)/K_t^{(\delta)}(g\,;h) \leq 1$ and consequently plays no role in the estimate.) Therefore one obtains a differential inequality

$$H'_\gamma(t) \geq 2\gamma^{-1}(1-\gamma)\mu_C \sum_{i=1}^{d'}\int_G d\hat{h}\, e^{-\psi(h)}|(A_i k_t^{\gamma/2})(h)|^2 - \nu_\gamma H_\gamma(t) \tag{15}$$

where ν_γ is a constant which depends only on γ, μ_C, $\|C\|_\infty$, $\|c\|_\infty$, $\|c_0\|_\infty$ and $\|A_i\chi\|_\infty$.

Similarly,

$$H_1'(t) = \sum_{i,j=1}^{d'} \int_G d\hat{h}\, e^{-\psi(h)}(A_i\psi)(h)\, c_{ji}(h)\,(A_jk_t)(h) + R \quad .$$

So there exists a $c > 0$ such that

$$H_1'(t) \geq -c\sum_{i=1}^{d'} \int_G d\hat{h}\, e^{-\psi(h)}|(A_ik_t)(h)| - c\int_G d\hat{h}\, e^{-\psi(h)}k_t(h)$$

where the value of c depends only on μ_C, $\|C\|_\infty$, $\|c\|_\infty$, $\|c_0\|_\infty$ and $\|A_i\chi\|_\infty$. But

$$|(A_ik_t)(h)| \leq \varepsilon\, k_t^{\gamma-2}(h)\,|(A_ik_t)(h)|^2 + (4\varepsilon)^{-1}k_t^{2-\gamma}(h)$$

for any $\varepsilon > 0$. Therefore

$$H_1'(t) \geq -4c\gamma^{-2}\varepsilon\sum_{i=1}^{d'} \int_G d\hat{h}\, e^{-\psi(h)}|(A_ik_t^{\gamma/2})(h)|^2$$
$$- (4\varepsilon)^{-1}c\,d' \int_G d\hat{h}\, e^{-\psi(h)}k_t^{2-\gamma}(h) - c\int_G d\hat{h}\, e^{-\psi(h)}k_t(h) \quad .$$

Next using the upper bounds for K_t of Theorem 1.1, which have already been established, and making the choice $\varepsilon = (2c)^{-1}\gamma\,(1-\gamma)\,\mu_C$ one deduces that there exists $\tau_\gamma > 0$, whose value depends on γ, μ_C, $\|C\|_\infty$, $\|c\|_\infty$, $\|c_0\|_\infty$ and $\|A_i\chi\|_\infty$ such that

$$H_1'(t) \geq -2\gamma^{-1}(1-\gamma)\,\mu_C\sum_{i=1}^{d'} \int_G d\hat{h}\, e^{-\psi(h)}|(A_ik_t^{\gamma/2})(h)|^2 - \tau_\gamma t^{-(1-\gamma)D'/2}H_1(t) \qquad (16)$$

uniformly for all $t \in \langle 0, 1]$. Hence

$$(H_\gamma + H_1)'(t) \geq -(\nu_\gamma + \tau_\gamma t^{-(1-\gamma)D'/2})(H_\gamma + H_1)(t)$$

by (15) and (16). Thus with $\gamma = 1 - (D')^{-1}$ and $\omega_\gamma = \nu_\gamma + \tau_\gamma$ one has

$$(H_\gamma + H_1)'(t) \geq -\omega_\gamma t^{-1/2}(H_\gamma + H_1)(t)$$

and since this inequality is valid for all $t \in \langle 0, 1]$ one may integrate between t_0 and t and conclude that

$$\log(H_\gamma(t) + H_1(t)) \geq -2\omega_\gamma + \log(H_\gamma(t_0) + H_1(t_0)) \geq -2\omega_\gamma + \log H_1(t_0) \quad .$$

But $\lim_{t_0\to 0} H_1(t_0) = e^{-\psi(g)} + \delta \geq e^{-\chi(e)}$. Therefore

$$H_\gamma(t) + H_1(t) \geq c_\gamma$$

for some $c_\gamma > 0$, whose value depends on the coefficients only through μ_C, $\|C\|_\infty$, $\|c\|_\infty$ and $\|c_0\|_\infty$, uniformly for $g \in G$ and $t \in \langle 0, 1]$. Note that c_γ is independent of the parameter δ used to modify the kernel. Since $H_\gamma(t) \leq (H_1(t))^\gamma$, it follows that $H_1(t) \geq c$ for a suitable $c > 0$, which is again independent of δ, uniformly for all $t \in \langle 0, 1]$ and $g \in G$. Therefore

$$\int_G d\hat{h}\, K_t(g\,;h) \geq \int_G d\hat{h}\, e^{-\psi(h)}K_t(g\,;h) = H_1(t) - \delta \geq c - \delta \quad .$$

Finally taking the limit $\delta \to 0$ one concludes that (14) is valid.

Combination of the Gaussian upper bounds of of Theorem 1.1 and the L_1-lower bounds of (14) now give bounds that indicate that K_t is localized near the diagonal $g = h$ uniformly for all small t.

Lemma 3.1 *There exist $c', \lambda > 0$ such that*

$$\int_{B'(g;\lambda t^{1/2})} d\hat{h}\, K_t(g\,;h) \geq c'$$

uniformly for all $t \in \langle 0, 1]$ and $g \in G$.

Proof It follows from the Gaussian upper bounds on the kernel and a quadrature estimate (see, for example, [Rob] pages 223–224) that one has bounds

$$\int_{G \backslash B'(g;\lambda t^{1/2})} d\hat{h}\, K_t(g\,;h) \leq a\, e^{\omega t} \lambda^{-1}$$

for all $\lambda, t > 0$, uniform in $g \in G$. Therefore using (14) one deduces that

$$\int_{B'(g;\lambda t^{1/2})} d\hat{h}\, K_t(g\,;h) \geq c - a\, e^{\omega} \lambda^{-1}$$

for all $t \in \langle 0, 1]$ and $g \in G$. Consequently the statement of the lemma holds, with $c' = 2^{-1}c$, for all $\lambda \geq 2\, a\, c^{-1} e^{\omega}$. □

The next lemma is the key to pointwise local lower bounds. It is a version of an estimate first given by Nash [Nas] in his fundamental analysis of strongly elliptic operators on $\mathbf{R}^d$. Nash's idea was to use the relative entropy as a measure of localization. Nash considered the entropy of the semigroup kernel relative to a Gaussian measure and used its properties to obtain bounds

$$K_1(x\,;y) \geq c > 0 \tag{17}$$

for all x, y such that $|x - y| \leq \kappa$. Then a scaling argument gives the crucial local bounds

$$K_t(x\,;y) \geq c\, t^{-d/2} \tag{18}$$

for all $t \in \langle 0, 1]$ and all x, y such that $|x - y| \leq \kappa\, t^{1/2}$.

One can apply Nash's arguments directly in the current context and deduce the Lie group equivalent of (17) but these bounds are difficult to exploit since one does not have scaling arguments. Therefore there is no obvious way of deducing the analogue of the bounds (18). But Saloff-Coste and Stroock [SaS] realized that this difficulty can be circumvented by considering the relative entropy with respect to a suitably chosen family of measures with compact support. We will closely follow their reasoning.

For $r > 0$ define $\rho_r, \sigma_r : G \to [0, \infty\rangle$ by $\rho_r = (\sigma_r)^2$ and $\sigma_r(g) = 1 - |g|' r^{-1}$ if $|g|' \leq r$ and $\sigma_r(g) = 0$ if $|g|' > r$. Both ρ_r and σ_r are weakly* differentiable and it follows from the triangle inequality that $\sum_{i=1}^{d'} \|A_i \sigma_r\|_\infty^2 \leq a\, r^{-2}$ for a suitable $a > 0$. Moreover, there exists a local weighted Poincaré inequality with density function ρ_r. Define the weighted average of φ by $\langle \varphi \rangle_{r,\rho} = \left(\int_G d\hat{g}\, \rho_r(g) \right)^{-1} \int_G d\hat{g}\, \rho_r(g)\, \varphi(g)$.

Proposition 3.2 *There exist $R_0 \in \langle 0,1]$ and $c > 0$ such that*

$$\int_G d\hat{g}\, \rho_r(g)|\varphi(g) - \langle\varphi\rangle_{r,\rho}|^2 \leq c\,r^2 \int_G d\hat{g}\, \rho_r(g) \sum_{i=1}^{d'} |(A_i\varphi)(g)|^2$$

uniformly for all $r \in \langle 0, R_0]$ and $\varphi \in L'_{2;1}(B'(2))$.

Proof This has been proved essentially in the appendix of [SaS]. In [SaS] the group is polynomial, but for a general group all the estimates are valid locally, just as in Jerison [Jer]. □

This Poincaré inequality will be used in the derivation of the following result.

Lemma 3.3 *There exist $c, \lambda > 0$ and $t_0 \in \langle 0,1]$ such that*

$$\int_G d\hat{h}\, \rho_{\lambda t^{1/2}}(h) \log\left(t^{D'/2}K_t(g\,;h)\right) \geq -c \int_G d\hat{h}\, \rho_{\lambda t^{1/2}}(h)$$

uniformly for all $t \in \langle 0, t_0]$ and $g \in B'(4^{-1}\lambda t^{1/2})$.

Remark 3.4 It is not evident that the relative entropy defined by the integral on the left hand side of the above inequality is finite. But this will be established in the following proof.

Proof Let $\lambda \geq 1$ and $R_0 \in \langle 0,1]$ be as in Lemma 3.1 and Proposition 3.2. Set $t_0 = \lambda^{-2}R_0^2$. Next fix $t \in \langle 0, t_0]$ and $g \in B'(4^{-1}\lambda t^{1/2})$ and for $s, \delta \in \langle 0,1]$ set $l_s^{(\delta)}(h) = t^{D'/2}K_{st}^{(\delta)}(g\,;h)$ where $K_t^{(\delta)} = K_t + \delta$ as above. We simplify notation by setting $\rho(h) = \rho_{\lambda t^{1/2}}(h)$ and $\sigma(h) = \sigma_{\lambda t^{1/2}}(h)$. Then $l_s^{(\delta)} \geq t^{D'/2}\delta > 0$ and we can introduce the functions $\Phi_\delta \colon \langle 0,1] \to \mathbf{R}$ by

$$\Phi_\delta(s) = \left(\int_G d\hat{h}\, \rho(h)\right)^{-1} \int_G d\hat{h}\, \rho(h) \log l_s^{(\delta)}(h) \quad .$$

One must prove that for suitable $c, \lambda > 0$ one has $\Phi_\delta(1) \geq -c$ uniformly for all $\delta \in \langle 0,1]$, $t \in \langle 0, t_0]$ and $g \in B'(4^{-1}\lambda t^{1/2})$. Once this is established it follows from the monotone convergence theorem, by setting $\delta = n^{-1}$ and taking the limit $n \to \infty$, that the relative entropy integral exists. Moreover, in the limit $\delta \to 0$, the required bounds are valid.

Differentiating Φ_δ and then integrating by parts one finds

$$\begin{aligned}\int_G d\hat{h}\, \rho(h)\, \Phi'_\delta(s) &= -t \int_G d\hat{h}\, \rho(h)\, l_s^{(\delta)}(h)^{-1}(\widetilde{H} l_s^{(0)})(h) \\ &= t\Big\{ \int_G d\hat{h}\, \rho(h) \sum_{i,j=1}^{d'} (A_i \log l_s^{(\delta)})(h)\, c_{ji}(h)\, (A_j \log l_s^{(\delta)})(h) \\ &\qquad - 2\int_G d\hat{h}\, \rho(h) \sum_{i,j=1}^{d'} (A_i\sigma)(h)\, c_{ji}(h)\, \sigma(h)(A_j \log l_s^{(\delta)})(h)\Big\} + R\end{aligned}$$

where we have again explicitly exhibited the contribution to the differential equation of the part of $\widetilde{H}$ containing two derivatives and used R to indicate the contribution of the lower order terms. Note that we have also used the identity $A_j l_s^{(0)} = A_j l_s^{(\delta)}$.

Therefore by using subellipticity on the first term on the right and an $\varepsilon, \varepsilon^{-1}$ estimate on the second term one obtains bounds

$$\int_G d\hat{h}\,\rho(h)\,\Phi'_\delta(s) \geq (1-\varepsilon)\mu_C\, t\int_G d\hat{h}\,\rho(h)\sum_{i=1}^{d'}|(A_i\log l_s^{(\delta)})(h)|^2$$
$$-\,\varepsilon^{-1}\mu_C^{-1}\,\|C\|_\infty^2\, t\int_G d\hat{h}\,\rho(h)\sum_{i=1}^{d'}|(A_i\sigma)(h)|^2 + R$$

for all $\varepsilon \in \langle 0,1]$. Since $\sum_{i=1}^{d'}\|A_i\sigma\|_\infty^2 \leq a\lambda^{-2}t^{-1}$ one then finds

$$\int_G d\hat{h}\,\rho(h)\,\Phi'_\delta(s) \geq (1-\varepsilon)\,\mu_C\, t\int_G d\hat{h}\,\rho(h)\sum_{i=1}^{d'}|(A_i\log l_s^{(\delta)})(h)|^2$$
$$-\,\varepsilon^{-1}\mu_C^{-1}\,\|C\|_\infty^2\, a\,\lambda^{-2}|B'_{\lambda t^{1/2}}| + R \quad .$$

The remainder R can, however, be dealt with by similar $\varepsilon, \varepsilon^{-1}$ estimates. (Care has to be taken again with the terms in $\widetilde{H}$ with no derivatives on the right hand side. These contain a factor $l_s^{(0)}$ but this can be replaced by $l_s^{(\delta)}$ at the cost of introducing an extra factor $l_s^{(0)}/l_s^{(\delta)}$. But this factor takes values in $[0,1]$ and plays no essential role in the estimate which is expressed totally in terms of $l^{(\delta)}$ with all constants independent of δ.) Then, choosing ε appropriately, one obtains bounds

$$\int_G d\hat{h}\,\rho(h)\,\Phi'_\delta(s) \geq 2^{-1}\mu_C\, t\int_G d\hat{h}\,\rho(h)\sum_{i=1}^{d'}|(A_i\log l_s^{(\delta)})(h)|^2 - \nu\,|B'_{\lambda t^{1/2}}|$$

with the value of ν independent of δ, t and g. Next we use Proposition 3.2 to deduce that

$$\int_G d\hat{h}\,\rho(h)\sum_{i=1}^{d'}|(A_i\log l_s^{(\delta)})(h)|^2 \geq a\,\lambda^{-2}\,t^{-1}\int_G d\hat{h}\,\rho(h)|\log l_s^{(\delta)}(h) - \Phi_\delta(s)|^2$$

where a is a constant independent of δ, t, g and the coefficients of H. Therefore combining these bounds and using a straightforward estimate on the integral of ρ one concludes that

$$\Phi'_\delta(s) \geq \mu\Big(\int_G d\hat{h}\,\rho(h)\Big)^{-1}\int_G d\hat{h}\,\rho(h)|\log l_s^{(\delta)}(h) - \Phi_\delta(s)|^2 - \nu'$$

with the values of ν' and μ dependent on μ_C, $\|C\|_\infty$, $\|c\|_\infty$ and $\|c_0\|_\infty$ but independent of the choice of δ, t and g. This is the Nash differential inequality, [Nas], Part II, which is 'solved' by repetition of his original arguments (see, for example, [Dav] p. 95). One needs bounds $\int_{B'(g;4^{-1}\lambda t^{1/2})} d\hat{h}\, l_s(h) \geq c\,t^{D'/2}$ uniformly for all $s \in \langle 0,1]$, $t \in \langle 0,t_0]$ and $g \in B'(4^{-1}\lambda t)$, but these have been proved under weaker restrictions in Lemma 3.1. The conclusion is that $\Phi_\delta(1) \geq -c$ uniformly for all $\delta \in \langle 0,1]$, $t \in \langle 0,t_0]$ and $g \in B'(4^{-1}\lambda t)$. Finally taking the limit $\delta \to 0$ gives the statement of the lemma. □

The required local lower bounds on the kernel follow straightforwardly from the entropy estimate.

Lemma 3.5 *There exist $a, \lambda > 0$ and $t_0 \in \langle 0,1]$ such that*

$$K_t(g\,;h) \geq a\,t^{-D'/2}$$

uniformly for all $t \in \langle 0,t_0]$ and $g, h \in G$ with $|gh^{-1}|' \leq \lambda\,t^{1/2}$.

Proof First note that as $K_t(h\,;g) = \widetilde{K}_t(g\,;h)$ where $\widetilde{K}_t$ is the kernel associated with the adjoint semigroup on the $L_{\hat{p}}$-spaces it follows from Lemma 3.3 that one has estimates

$$\int_G d\hat{h}\,\rho_{\lambda t^{1/2}}(h)\log\Big(t^{D'/2}K_t(g\,;h)\Big) \geq -c\int_G d\hat{h}\,\rho_{\lambda t^{1/2}}(h)$$

$$\int_G d\hat{h}\,\rho_{\lambda t^{1/2}}(h)\log\Big(t^{D'/2}K_t(h\,;g)\Big) \geq -c\int_G d\hat{h}\,\rho_{\lambda t^{1/2}}(h)$$

uniformly for all $t \in \langle 0, t_0]$ and $g \in B'(\lambda t^{1/2})$, for suitable $c, \lambda > 0$ and $t_0 \in \langle 0, 1]$. Now using the convolution semigroup property one has

$$\begin{aligned} t^{D'}K_{2t}(g\,;k) &= \int_G d\hat{h}\, t^{D'/2}K_t(g\,;h)t^{D'/2}\,K_t(h\,;k) \\ &\geq \int_G d\hat{h}\,\rho_{\lambda t^{1/2}}(h)\,t^{D'/2}K_t(g\,;h)\,t^{D'/2}K_t(h\,;k) \quad . \end{aligned}$$

Therefore using concavity of the logarithm one deduces that

$$\begin{aligned} &\log\Big(\Big(\int_G d\hat{h}\,\rho_{\lambda t^{1/2}}(h)\Big)^{-1}t^{D'}K_{2t}(g\,;k)\Big) \\ &\qquad \geq \Big(\int_G d\hat{h}\,\rho_{\lambda t^{1/2}}(h)\Big)^{-1}\int_G d\hat{h}\,\rho_{\lambda t^{1/2}}(h)\Big(\log\big(t^{D'/2}K_t(g\,;h)\big) + \log\big(t^{D'/2}K_t(h\,;k)\big)\Big) \\ &\qquad \geq -2\,c \end{aligned}$$

uniformly for all $t \in \langle 0, t_0]$ and $g, k \in B'(\lambda t^{1/2})$. Hence

$$K_{2t}(g\,;k) \geq e^{-2c}t^{-D'}\int_G d\hat{h}\,\rho_{\lambda t^{1/2}}(h) \geq a\,t^{-D'/2}$$

under the same restrictions on g, k and t. Now the restrictions on g and k can be weakened by noting that $(g,k) \mapsto K_t(gh\,;kh)$ is the kernel associated with the subelliptic operator H conjugated with right translations by h. Since the coefficients of the conjugated operator are the right translates of the original coefficients the ellipticity constant μ_C and the parameters $\|C\|_\infty$ etc. are unchanged. Hence one has similar estimates

$$K_{2t}(gh\,;kh) \geq a\,t^{-D'/2}$$

valid under the previous restrictions but for all $h \in G$. The statement of the lemma follows immediately. □

The final step in the proof of the Gaussian lower bounds on the kernel is to convert the local lower bounds into the Gaussian bounds by the use of the convolution semigroup property. This is achieved by a now standard procedure (see, for example, [Rob], Proposition III.5.2).

Finally we note that if one can establish the local lower bounds for all $t > 0$ then the Gaussian lower bounds of Theorem 1.1 are valid with $\omega' = 0$.

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