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Abstract. In this paper we consider water-drive to recover oil from a strongly heterogeneous porous column. The two-phase model uses Corey relative permeabilities and Brooks-Corey capillary pressure. The heterogeneities are perpendicular to flow and have a periodic structure. This results in one-dimensional flow and a space periodic absolute permeability, reflecting alternating coarse and fine layers. Assuming many - or thin - layers, we use homogenization techniques to derive the effective transport equations. The form of these equations depend critically on the capillary number. The analysis is confirmed by numerical experiments. This paper summarises the results obtained in [10]

1 Introduction

Water-drive, i.e. injection of water into reservoirs to drive oil towards production well, is a widely used technique in oil recovery processes. Rock heterogeneities in the reservoir generally have a negative influence on the recovery rate. If the heterogeneities occur perpendicular to flow from injection to production wells, oil may be trapped at interface from high to low permeability. Consequently, part of the oil becomes inaccessible to flow, thus leading to a reduction in recovery rate. This situation was analysed in [14], [11], and studied experimentally in [16]. In the same context, steady state solutions as well as an averaging procedure were considered in [9].

The main purpose of this paper is to derive in a rational way the effective flow equations corresponding to a periodic medium, when the ratio of micro scale (periodicity length) and macro scale (column length) is small.

To this end we consider a one dimensional flow of two immiscible and incompressible phases (water and oil) through a heterogeneous porous medium. The medium is characterised by a constant porosity Φ and a variable absolute permeability $k = k(x)$. The space-time behaviour of the phases is described in terms of the reduced saturations $0 \le S_\alpha \le 1$, with $\alpha = o, w$. Since only two phases are present we have $S_w + S_o = 1$. The underlying equations are mass and momentum balance for phases (see [3]).

Since the flow is one-dimensional and no internal sources are present, the total specific discharge $q := q_o + q_w$, with q_α denoting the specific discharge of phase α, is constant in space. Throughout this paper we consider it constant in time as well. With $q > 0$ given, the underlying equations can be combined into a single transport equation for one saturation only (see [11] or [10] for details).

This equation involves the relative permeabilities of the fluid phases and the capillary pressure as typical nonlinearities of the model. Relative permeabilities, denoted by $k_{r\alpha}$, arise as a reduction of the absolute permeability due to the pressure of the other phase. In this paper we use Corey [8] expressions with exponent 2. In terms of oil saturation $u := S_o$ (i.e. $S_w = 1 - u$) this means

$$k_{rw} = k_{rw}(u) = (1-u)^2, \quad k_{ro} = k_{ro}(u) = u^2. \tag{1}$$

Capillary pressure arises as a result of interfacial forces on the pore scale. Petroleum engineers often use the Leverett model [15] in which

$$p_c(x, S_w) = \sigma\sqrt{\Phi/k(x)}J(S_w). \tag{2}$$

Here σ denotes the interfacial tension between the phases in the pores and J the Leverett function. Following Brooks-Corey [7] we use

$$J(u) = (1-u)^{-1/\lambda} \quad \text{with } \lambda = 2. \tag{3}$$

Note that $J(0) = 1 > 0$. This implies the existence of an oil entry pressure: a pressure $p_c(x, 0)$ has to be exerted on the oil before it can enter a fully water saturated medium.

Note 1. This paper is based on the results obtained in [10]. There we only used (1) and (3) in the numerical experiments. The theory does not require these specific choices. It is based on generalisations of (1) and (3), having a similar qualitative behaviour. In particular $k_{rw}(1) = k_{ro}(0) = 0$, $J(0) > 0$, $J(1-) = \infty$ and the nonlinearities are monotone.

In dimensionless form the oil-transport equation reads

$$\begin{aligned} &\frac{\partial u}{\partial t} + \frac{\partial F}{\partial x} = 0\,, \\ &F = f(u) - N_c k(x)\lambda(u)\frac{\partial}{\partial x}p_c(x,u)\,. \end{aligned} \tag{4}$$

Here

$$f(u) = \frac{k_{ro}(u)}{k_{ro}(u) + Mk_{rw}(u)} \tag{5}$$

denotes the oil fractional flow function, and

$$\lambda(u) = k_{rw}(u)f(u), \; p_c(x,u) = J(u)/\sqrt{k(x)}. \tag{6}$$

The formulation involves two dimensionless numbers: the capillary number N_c and the viscosity ration M. They are given by

$$N_c = \frac{\sigma\sqrt{K\Phi}}{\mu_w q L_x} \text{ and } M = \frac{\mu_o}{\mu_w}, \tag{7}$$

where K denotes a reference absolute permeability, L_x a macroscopic reference length and μ_α the viscosities of the fluids.

Depending on the specific application, the value of the capillary number may vary considerably. For instance, adding surfactants or polymers may substantially alter σ or μ_w. Likewise, the flow rate q can have different values. Therefore we investigate in Section 2 the consequences of having a moderate and a small value for N_c.

Existence, uniqueness and regularity for equation (4) was established for constant or smooth absolute permeabilities (see, e.g., [2], [1], [12] or [4]). Here we are interested in the case when k is piecewise constant. The situation with a single discontinuity ($k = k^+$ if $x < 0$ and $k = k^-$ if $x > 0$) is studied in [11] and [6]. There equation (4) is considered separately in the two sub-domains with constant k and matching conditions are imposed at the interface. In terms of the Leverett model, these conditions are (assuming, without loss of generality, $k^+ > k^-$, i.e. coarse material for $x < 0$ and fine material for $x > 0$):

$$\text{(i)} \qquad [F(t)] = 0; \tag{8}$$

$$\text{(ii)} \begin{cases} u(0-,t) < u^* \text{ implies } u(0+,t) = 0\,, \\ u(0-,t) \geq u^* \text{ implies } \dfrac{J(u(0-,t))}{\sqrt{k^+}} = \dfrac{J(u(0+,t))}{\sqrt{k^-}}\,, \end{cases} \tag{9}$$

for all $t > 0$, where $[F(t)] = F(0+,t) - F(0-,t)$. The first condition expresses oil-flux continuity. The second condition relates the capillary pressure to a threshold saturation u^*, uniquely defined by

$$\frac{J(u^*)}{\sqrt{k^+}} = \frac{J(0)}{\sqrt{k^-}}. \tag{10}$$

The capillary pressure is continuous only if both phases are present on both sides of the k-discontinuity. If oil is absent in the fine medium (i.e. for $x > 0$), the existence of an oil entry pressure leads to a discontinuous capillary pressure. This is in essence the mechanism for oil trapping in the coarse ($k = k^+$) material, see [11] and [6] for details.

In Section 2 we assume a periodic micro structure of coarse ($k = k^+$) and fine ($k = k^-$) material, each of length $L_y << L_x$ (see [10] for the non-periodic case). This leads to a natural choice of the small expansion parameter $\varepsilon = L_y/L_x$. In this case, trapping occurs at transitions from high to low permeability and we expect to find a trapping related threshold saturation below which the oil becomes immobile. We outline the homogenization procedure, study the resulting auxiliary problems and derive the effective equations for the limit $\varepsilon \searrow 0$. The magnitude of the capillary number N_c is important. Two cases are worked out:

Capillary limit, $N_c = 0(1)$. In this situation the auxiliary problem only has constant state solutions, and the effective equation is found explicitly. It turns out to be of convection-diffusion type, where both convection and diffusion

vanish if the averaged oil saturation drops below $\frac{1}{2}u^*$.
Balance, $N_c = O(\varepsilon)$. In this case a first order conservation law of Buckley-Leverett type is obtained. The upscaled oil-fractional flow function vanishes again if the averaged oil saturation drops below a certain value, related to a specific solution of the auxiliary problem.
Some numerical results are given in Section 3.

2 Homogenization procedure for periodic layers

Let us assume a periodic micro structure where, in original length scale,

$$k(x) = \begin{cases} k^+ \text{ on } & -L_y < x < 0, \\ k^- \text{ on } & 0 < x < L_y, \end{cases}$$

k is $2L_y$ periodic and $k^+ > k^-$. After scaling, the k-discontinuities are located at $\{\varepsilon i : i \in \mathbb{Z}\}$. The corresponding permeability $k^\varepsilon(x)$ is defined by $k^\varepsilon(x) = k(x/\varepsilon)$, where

$$k = \begin{cases} k^+ \text{ on } (2i-1, 2i), \\ k^- \text{ on } (2i, 2i+1). \end{cases} \tag{11}$$

Since (9) does not depend on the direction of the flow (coarse $\rightarrow$ fine or fine $\rightarrow$ coarse), we impose at $x = 2i\varepsilon$

$$\begin{aligned} &\text{if } u(2i\varepsilon - 0) < u^*, \text{ then } u(2i\varepsilon + 0) = 0; \\ &\text{if } u(2i\varepsilon - 0) \geq u^*, \text{ then } \frac{J(u(2i\varepsilon - 0))}{\sqrt{k^+}} = \frac{J(u(2i\varepsilon + 0))}{\sqrt{k^-}}, \end{aligned} \tag{12}$$

and at $x = (2i+1)\varepsilon$

$$\begin{aligned} &\text{if } u((2i+1)\varepsilon + 0) \geq u^*, \text{ then} \\ &\qquad \frac{J(u((2i+1)\varepsilon + 0))}{\sqrt{k^+}} = \frac{J(u((2i+1)\varepsilon - 0))}{\sqrt{k^-}}; \\ &\text{if } u((2i+1)\varepsilon + 0) < u^*, \text{ then } u((2i+1)\varepsilon - 0) = 0. \end{aligned} \tag{13}$$

Replace k by k^ε in equation (4) and let u^ε be a solution satisfying the matching conditions (8), (12) and (13). Denoting the fast scale by $y = x/\varepsilon$, the two scale asymptotic expansion

$$u^\varepsilon(x,t) = u^0(x,y,t) + \varepsilon u^1(x,y,t) + \varepsilon^2 u^2(x,y,t) \dots, \tag{14}$$

is substituted into equation (4). Equating powers of ε, this results in equations for $u^0, u^1 \dots$ (see, e.g., [5] or [13]). We are interested in the average of u^0 with respect to $y \in (-1,1)$, which is expected to be the weak limit of u^ε. This convergence will not be demonstrated. Our purpose is to derive upscaled equations and to study the corresponding auxiliary problems. In doing so, capillary limit and balance are considered separately.

2.1 Capillary limit: $N_c = O(1)$

The homogenization procedure yields the following equations:

$$\varepsilon^{-2}: \qquad -N_c\sqrt{k}D(u^0)\frac{\partial u^0}{\partial y} = F^0 = F^0(x,t), \tag{15}$$

$$\varepsilon^{-1}: \qquad 0 = \frac{\partial F^0}{\partial x} + \frac{\partial F^1}{\partial y} = \frac{\partial}{\partial x}\left\{-N_c\sqrt{k}D(u^0)\frac{\partial u^0}{\partial y}\right\} + \frac{\partial}{\partial y}\left\{f(u^0) - N_c\sqrt{k}\left(D(u^0)\left(\frac{\partial u^1}{\partial y} + \frac{\partial u^0}{\partial x}\right) + D'(u^0)u^1\frac{\partial u^0}{\partial y}\right)\right\} \tag{16}$$

$$\varepsilon^{0}: \qquad \frac{\partial u^0}{\partial t} + \frac{\partial F^2}{\partial y} + \frac{\partial F^1}{\partial x} = 0, \tag{17}$$

where F^0, F^1 and F^2 are obtained after applying expansion (14) to the oil flux $F(u^\varepsilon)$.

We look for y-periodic solutions of (15) satisfying (12) and (13), with x and t as given parameters. Such solutions exist only if $F^0 = 0$ (see [10]). In this case we find

$$u^0(y) = \begin{cases} C > u^* & \text{for } -1 < y < 0, \\ \overline{C} := J^{-1}\left(\sqrt{\frac{k-}{k+}}J(C)\right) & \text{for } 0 < y < 1, \end{cases} \tag{18}$$

or

$$u^0(y) = \begin{cases} C \le u^* & \text{for } -1 < y < 0, \\ 0 & \text{for } 0 < y < 1. \end{cases} \tag{19}$$

After inserting u^0 into (16), F^1 can be written explicitly in terms of C and $\overline{C}$. For $C \le u^*$ there is no flow inside the fine micro-structure, so the effective flux vanishes. In the nontrivial case, where $C > u^*$, we have

$$F^1 = \frac{\frac{f(C)}{\sqrt{k^+}D(C)} + \frac{f(\overline{C})}{\sqrt{k^-}D(\overline{C})}}{\frac{1}{\sqrt{k^+}D(C)} + \frac{1}{\sqrt{k^-}D(\overline{C})}} - N_c\frac{\frac{\partial C}{\partial x} + \frac{\partial \overline{C}}{\partial x}}{\frac{1}{\sqrt{k^+}D(C)} + \frac{1}{\sqrt{k^-}D(\overline{C})}}.$$

Integrating (17) over $(-1,1)$ and using the y-periodicity and continuity of F^2, the effective equation for the averaged oil saturation $U = \frac{1}{2}(C + \overline{C})$ reads

$$\frac{\partial U}{\partial t} + \frac{\partial}{\partial x}\left\{\mathcal{F}(U) - N_c\mathcal{D}(U)\frac{\partial U}{\partial x}\right\} = 0. \tag{20}$$

The graph of $\mathcal{F}$ and $\mathcal{D}$ are shown in Figure 1. Note that $\mathcal{F}(U) = \mathcal{D}(U) = 0$ for $0 \le U \le \frac{1}{2}u^*$, $\mathcal{D}(1) = 0$ and $\mathcal{F}$ is strictly increasing on $\left(\frac{1}{2}u^*, 1\right)$ with $\mathcal{F}(1) = 1$.

2.2 Balance: $N_c = O(\varepsilon)$

Setting $N_c := N_c\varepsilon$, the homogenization procedure now gives

$$\varepsilon^{-1}: \qquad f(u^0) - N_c\sqrt{k}D(u^0)\frac{\partial u^0}{\partial y} = F^0 = F^0(x,t) \tag{21}$$

$$\varepsilon^{0}: \qquad \frac{\partial u^0}{\partial t} + \frac{\partial F^0}{\partial x} + \frac{\partial F^1}{\partial y} = 0. \tag{22}$$

Equation (21) leads to the following auxiliary problem.
Problem A_u: Given $F \in \mathbb{R}$, find $u : [-1,0)\cup(0,1] \to [0,1]$ satisfying

$$f(u) - N_c\sqrt{k}k_{rw}(u)f(u)J'(u)\frac{du}{dy} = F \text{ in } (-1,0)\cup(0,1) \tag{23}$$

subject to the matching condition ($y = 0$)

$$\begin{cases} \text{if } u(0-) < u^*, \text{ then } u(0+) = 0; \\ \text{if } u(0-) \geq u^*, \text{ then } \dfrac{J(u(0-))}{\sqrt{k^+}} = \dfrac{J(u(0+))}{\sqrt{k^-}}. \end{cases} \tag{24}$$

and the periodicity condition ($y = \pm 1$)

$$\begin{cases} \text{if } u(-1+0) < u^* \text{ then } u(1-0) = 0; \\ \text{if } u(-1+0) \geq u^* \text{ then } \dfrac{J(u(-1+0))}{\sqrt{k^+}} = \dfrac{J(u(1-0))}{\sqrt{k^-}}. \end{cases} \tag{25}$$

This problem is studied in detail in [10]. The main results are the following

Theorem 1. *Let $F \in \mathbb{R}$ be given. The following cases can be distinguished:*

(i) $F < 0$, *or* $F > 1$*: there are no solutions to Problem* A_u*;*
(ii) $0 < F < 1$*: Problem* A_u *admits a unique solution, which is strictly increasing on* $(-1,0)$ *and strictly decreasing on* $(0,1)$*;*
(iii) $F = 1$*:* $u \equiv 1$ *uniquely solves Problem* A_u*;*
(iv) $F = 0$*: Problem* A_u *admits a family of solutions; each solution is uniquely determined by* $u(0-) = l$ *(with* $0 \leq l \leq u^*$*) and is identically zero on* $(0,1)$.

By this theorem, any $F \in (0,1]$ uniquely determines the solution $u = u(y,F)$ of Problem A_u. Let $\overline{u}$ denote the maximal element (satisfying $\overline{u}(0-) = u^*$) in the family of solutions corresponding to $F = 0$, see Theorem 1 (iv). Then the cell-averaged oil saturation

$$U = U(F) = \frac{1}{2}\int_{-1}^{1} u(y,F)dy \tag{26}$$

satisfies

Lemma 2. *(i)* $U \in C([0,1])$ *and is strictly increasing;*
(ii) $U(1) = 1$ *and* $U(0+) = \overline{U} := \frac{1}{2}\int_{-1}^{1} \overline{u}(y)dy$.

This statement allows us to define the inverse $F = U^{-1}$, i.e. the homogenized flux function, which satisfies $F \in C([0,1])$ such that $F(U) = 0$ for $0 \le U \le \overline{U}$, F is strictly increasing in $(\overline{U}, 1)$ and $F(1) = 1$.

These properties hold for quite general relative permeabilities and Leverett functions. For the specific Brooks-Corey model defined by (1) - (3) we have in addition

$$F(U) = O((U - \overline{U})^2) \quad \text{as} \quad U \searrow \overline{U} \tag{27}$$

and

$$F(U) = 1 - O((1 - \overline{U})^2) \quad \text{as} \quad U \nearrow 1. \tag{28}$$

The graph of F is shown in Figure 4.

The effective equation is obtained by taking the y-average of equation (22). Using the periodicity and continuity of F^1, and setting $F^0 = F(U)$ one finds

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = 0. \tag{29}$$

Because of (27) and (28) we call this equation of Buckley-Leverett type, although true convex-concave behaviour could not be shown analytically. This was only found numerically, see Figure 4.

It is interesting to note here that the homogenized (or effective/upscaled) flux function involves elements of the local capillary forces. They enter through the solution of the auxiliary problem A_u.

3 Numerical results

In this section we present the results of several numerical experiments. We computed the solution of the equation with micro-structure, i.e. equation (4) with matching conditions (8) and (9) at the interface, in the flow domain $(-1, 1)$ with 160 layers. Thus $\varepsilon = L_y/L_x = 1/80$. We used the specific Brooks-Corey model (1) - (3) and fixed $k^+ = 1$ and $k^- = 0.5$.

Originally, at $t = 0$, the porous medium is saturated with oil. Water is being injected from the left to displace the oil. Thus

$$u(x, 0) = 1 \quad \text{for} \quad -1 < x < 1$$

and

$$u(-1, t) = 0 \quad \text{for} \quad t > 0.$$

In addition we impose $\dfrac{\partial u}{\partial x}(1, t) = 0$.

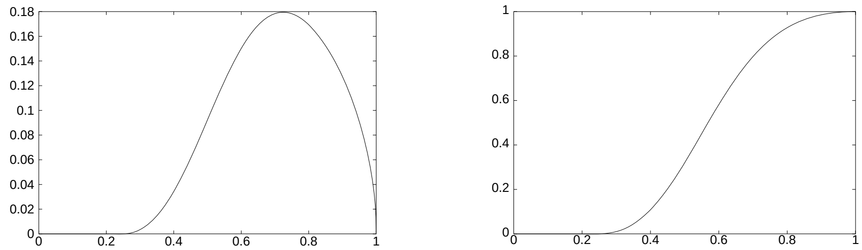

Fig. 1. Capillary limit: effective diffusion (left) and convection (right).

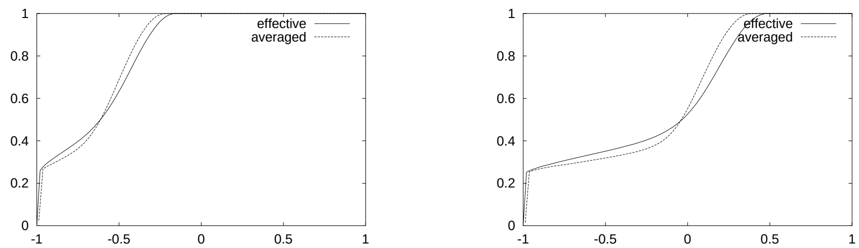

Fig. 2. Averaged and effective solution at $t = 0.3$ (left) and at $t = 0.7$ (right).

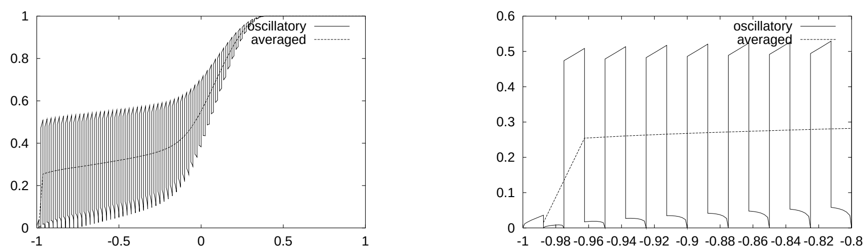

Fig. 3. Full problem: averaged and oscillatory solution at $t = 0.7$.

3.1 Capillary limit with $N_c = 1$

Here the homogenization procedure is quite explicit and thereafter the upscaled convection $\mathcal{F}(U)$ and diffusivity $\mathcal{D}(U)$ are trivially determined. The results are shown in Figure 1. Here $1/2u^* = 0.25$ denotes the macroscopic irreducible oil saturation. Figure 2 shows the averaged oil saturation determined from the full problem and the solution of the effective equation (20). Note that the computed solutions both start at $U = 0.25$. Figure 3 shows the oscillatory solution of the full problem and its average. The figure on the right is an enlargement of the oil saturation in the first 8 cells.

3.2 Balance with $N_c = \varepsilon$

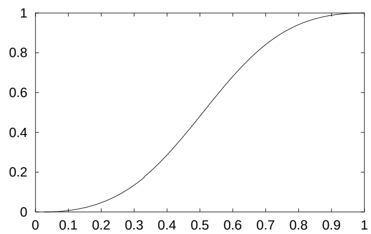

Fig. 4. Balance: effective oil fractional flow function.

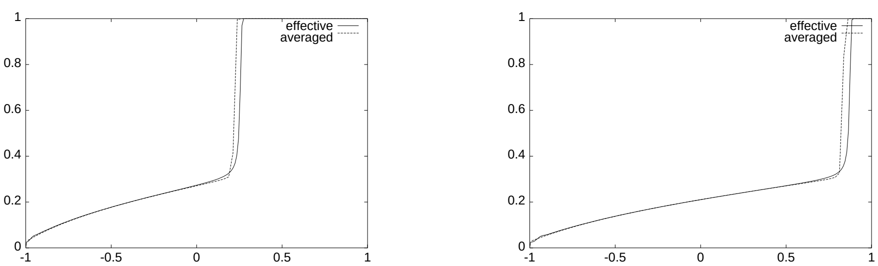

Fig. 5. Averaged and effective solution at $t = 0.3$ (left) and at $t = 0.7$ (right).

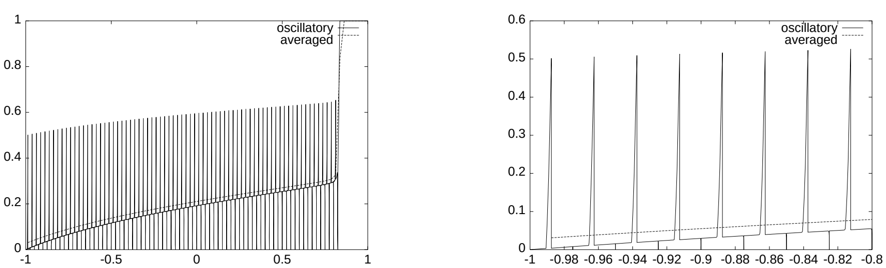

Fig. 6. Full problem: averaged and oscillatory solution at $t = 1.5$.

To find the homogenized oil fractional flow function as it appears in equation (29) is not straightforward, because it involves the auxiliary problem A_u. Details about the construction are given in [10] and the result is shown in Figure 4. Note the small macroscopic irreducible oil saturation $\overline{U} \approx 2.54 \cdot 10^{-2}$.

This is caused by the dominance of convection versus capillary diffusion. Furthermore, note the convex-concave shape, a true Buckley-Leverett behaviour. Figure 5 shows the averaged oil saturation determined from the full problem as the solution of the effective equation (29). Finally, Figure 6 shows the oscillatory solution of the full problem and its average. Again, the figure on the right shows the oil saturation in the first 8 cells. Note the boundary layer behaviour at transitions from coarse to fine material. This causes the significant reduction of the macroscopic irreducible oil saturation.

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