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by

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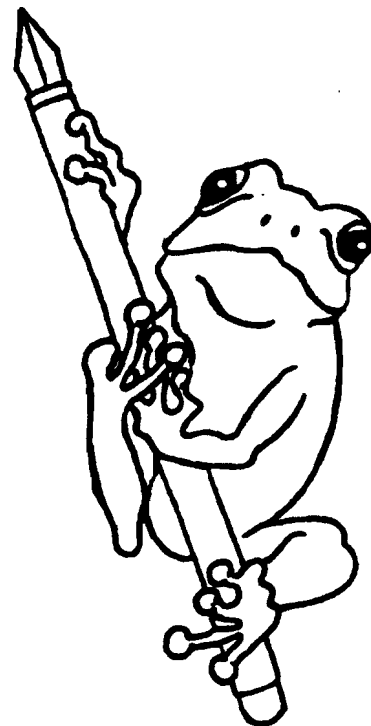
A Least-Squares Method for Linear Elasticity

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Abstract

When the equations of linear elasticity are solved by the standard Galerkin method the equations become stiff for nearly incompressible materials. This results in a perturbed numerical approximation. To avoid this problem a well known technique employs a saddle-point formulation of the equations. In this paper a new technique will be presented which is based on a least-squares (re)formulation of the problem. The original second order elasticity equations are rewritten as a first order system and the least-squares functional arises from this system. Numerical examples are presented to show the difference between this method and the Galerkin method. A drawback of the least-squares method is that the condition number of the corresponding system of equations is depending on the incompressibility of the material, which will also be shown.

KEY WORDS   elasticity equations, least-squares, finite element method

1 Review of linear elasticty

Many structures in everyday life deform when this structue becomes subject to some pressure and this deformation disappears as soon as this pressure is removed. In this case we speak of an elastic deformation of an elastic body. One can think of a pier of a bridge as an example of such a structure. When a train is riding over this bridge, a force works on this pier such that this pier starts to deform. As soon as this force is removed, the pier will return into its original state. Two types of forces working on this pier can be distinguished: the body forces that work on the entire pier and the surface tractions working on a part of boundary, e.g. the gravity is a form of a body force whereas the pressure of the train is to be considered as a surface traction.

In practical situations, when such structures are made of concrete or steel, the deformations are very small. Therefore the problem may be considered linear. Another assumption is that the material is isotropic, i.e. the materialproperties are the same for all directions, and this reduces the number of elasticity constants from 21 to 2.

For the deformation of an isotropic material, these two materialproperties or elasiticy constants are important. The first constant is the Poisson ratio ν, $0 \leq \nu \leq \frac{1}{2}$, and measures the degree of (volumewise) compressibility of the material. $\nu = \frac{1}{2}$ corresponds to a volumewise incompressible material; rubber-like materials have a Poisson ratio close to $\frac{1}{2}$. For steel and concrete the Poisson ratio is 0.29 and 0.2 respectively. The second constant is the elasticity, or Young's, modulus E which is the fraction between stress and strain. For steel and concrete the modulus is $2.15 \, 10^{11} Pa$ and $0.315 \, 10^{11} Pa$ respectively. An alternative for this pair of elasticity constants are the Lamé-coefficients μ and λ which are also depending on the material. The relation between the two pairs is:

$$
\begin{aligned}
\mu &= \frac{E}{2 + 2\nu}, \\
\lambda &= \frac{E\nu}{(1+\nu)(1-2\nu)}.
\end{aligned}
\tag{1}
$$

So, for $\nu \to \frac{1}{2}$, $\lambda \to \infty$.

The problem to be solved can now be formulated as follows and for simplicity the two-dimensional case is considered: find the displacement $u = [u_1, u_2]^T$ of an isotropic, homogeneous body $\Omega \subset \mathbb{R}^2$ subject to a body force F and a surface traction t. Moreover, on $\Gamma_0 \subset \Gamma = \partial\Omega$ the displacement u is prescribed, i.e. $u = f$.
Furthermore the stress σ and strain ε are of importance. The strain is defined by the partial derivatives of u as follows:

$$
\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i, j = 1, 2.
\tag{2}
$$

Although ε is in fact a matrix, we will write it as an array for notational reasons:

$$
\varepsilon = [\varepsilon_{11}, \varepsilon_{12}, \varepsilon_{21}, \varepsilon_{22}]^T.
\tag{3}
$$

The stress is related to the strain by means of constitutive relations, Hooke's law, and for isotropic materials this relation takes the form

$$
\sigma_{ij} = 2\mu\varepsilon_{ij} + \delta_{ij}\lambda \sum_{s=1}^{2} \varepsilon_{ss}.
\tag{4}
$$

Since the strain, and therefore the stress, are functions of the displacement u, $\varepsilon(u)$ will denote the strain with respect to the displacement u.
Subsequently the Lamé-Navier equations of linear elasticity can be formulated. This are

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conditions which must hold if the body Ω is in its equilibrium:

$$\begin{cases} -\sum_{j=1}^{2} \dfrac{\partial \sigma_{ij}(\boldsymbol{u})}{\partial x_j} & = & F_i, \quad i=1,2 \quad \text{in} \quad \Omega, \\[2mm] \sum_{j=1}^{2} \sigma_{ij}(\boldsymbol{u})n_j & = & t_i \quad i=1,2, \quad \text{on} \quad \Gamma_1, \\[2mm] \boldsymbol{u} & = & \boldsymbol{f}, \quad \text{on} \quad \Gamma_0. \end{cases} \tag{5}$$

The equilibrium conditions stated in (5) can also be formulated in an alternative and perhaps more familiar way in terms of $\boldsymbol{u}$:

$$\begin{cases} -\mu \Delta \boldsymbol{u} - (\lambda + \mu) \nabla \nabla \cdot \boldsymbol{u} & = & \boldsymbol{F}, \quad \text{in} \quad \Omega, \\[1mm] \boldsymbol{\sigma} \cdot \boldsymbol{n} & = & \boldsymbol{t}, \quad \text{on} \quad \Gamma_1, \\[1mm] \boldsymbol{u} & = & \boldsymbol{f}, \quad \text{on} \quad \Gamma_0. \end{cases} \tag{6}$$

For later use, (6) is rewritten as

$$\begin{cases} -\nabla \cdot A\nabla \boldsymbol{u} & = & \boldsymbol{F}, \quad \text{in} \quad \Omega, \\[1mm] \boldsymbol{n} \cdot (A\nabla \boldsymbol{u}) & = & \boldsymbol{t}, \quad \text{on} \quad \Gamma_1, \\[1mm] \boldsymbol{u} & = & \boldsymbol{f}, \quad \text{on} \quad \Gamma_0, \end{cases} \tag{7}$$

where

$$A = \begin{pmatrix} \lambda + 2\mu & 0 & 0 & \lambda \\ 0 & \mu & \mu & 0 \\ 0 & \mu & \mu & 0 \\ \lambda & 0 & 0 & \lambda + 2\mu \end{pmatrix}.$$

For this second order boundary value problem, (6) and (7), the solution space $\mathcal{U}$ for $\boldsymbol{u}$ is defined as

$$\mathcal{U} := \{ \boldsymbol{u} \in H^1(\Omega)^2 : -\nabla \cdot A\nabla \boldsymbol{u} \in L^2(\Omega)^2; \ \boldsymbol{u} = \boldsymbol{f} \text{ on } \Gamma_0; \ \boldsymbol{n} \cdot (A\nabla \boldsymbol{u}) = \boldsymbol{t} \text{ on } \Gamma_1 \}.$$

The problem of finding the equilibrium state, i.e. the solution, of (6) is equivalent to the following variational problem (see [10]):
Find $\boldsymbol{u} \in \{ H^1(\Omega)^2 : \boldsymbol{u} = \boldsymbol{f} \text{ on } \Gamma_0 \}$ such that

$$\int_{\Omega} 2\mu \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}) + \lambda (\nabla \cdot \boldsymbol{u})(\nabla \cdot \boldsymbol{v}) d\Omega = \int_{\Omega} \boldsymbol{F} \cdot \boldsymbol{v} d\Omega + \int_{\Gamma_1} \boldsymbol{t} \cdot \boldsymbol{v} d\Gamma, \quad \forall \boldsymbol{v} \in V, \tag{8}$$

where the testfunctions $\boldsymbol{v}$ are taken from the space

$$\mathcal{V} := \{ \boldsymbol{v} \in H^1(\Omega)^2 : \boldsymbol{v}|_{\Gamma_0} = 0 \}.$$

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The numerical solution computed with the Galerkin method is based on this variational formulation.

Note that this variational form arises also from the problem of finding a minimizer u for the energy functional

$$\Phi(u) = \int_\Omega \mu \varepsilon(u)^2 + \frac{\lambda}{2}(\nabla \cdot u)^2 - F \cdot u d\Omega - \int_{\Gamma_1} t \cdot u d\Gamma. \tag{9}$$

In the preceding pages a variational problem (8) was stated and a solution for this problem is an equilibrium state of the boundary value problem (6), since both formulations are equivalent. By Korn's inequality (e.g. [5]) coercivety of the bilinear part of the variational formulation is established. Also boundedness of the bilinear part is readily proven. Hence, the Lax-Milgram Lemma ensures existence and uniqueness of an equilibrium, so we can speak of the equilibrium state of the problem.

In order to compute the solution of the variational problem (8), the Galerkin method can be used (see [2]). This results in a positive definite system of equations, and hence a solution is readily found by use of an iterative solution method. However, in practice it turns out that this direct approach has several disadvantages:

- As the material becomes more incompressible, i.e. $\nu \to \frac{1}{2}$, the differential equation (6) will become stiff, and therefore the variational problem will become more sensitive for small perturbations of the displacement variable. This results in so-called locking of the numerical approximation. For the effects of locking on the finite element approximations see [4] for more details.

- Besides the displacement, other entities like stress and strain can also be of interest in practical situations. Since they are a combination of partial derivatives of u they can be derived directly. However, this will cause loss of accuracy, i.e. the order of approximation weakens from order 2 to 1.

A general technique to avoid these problems is to rewrite the variational problem into a saddle-point problem by introducing the pressure $p = \frac{\lambda}{\mu}\nabla \cdot u$ as auxiliary variable (see [3]). However, this saddle-point formulation shows also some difficulties, when solved numerically:

- The number of unknowns in the resulting linear system of equations is larger than for the Galerkin system, even for a relatively large meshsize.

- The two finite element spaces used in the discretization of the problem have to satisfy a inf-sup condition (in literature known as the Ladyzhenskaya-Babuška-Brezzi (LBB) condition).

- The resulting system of equations will be indefinite in general.

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The last two problems can be solved by defining a least-squares functional based on a first-order system for the original problem and the mixed variable problem. The resulting system of equations is always positive (semi-)definite which facilitates the use of standard iterative solution methods. Moreover, the finite element spaces do not have to satisfy the inf-sup condition and therefore all unknowns can be approximated by the same type of basis functions, if so desired.

Despite these nice properties, there are also a number of disadvantages: ellipticity (i.e. coercivity) of the least-squares functional has to be checked and the system of equations can be ill conditioned.

2 Least-squares method for linear elasticity

For this least-squares approach, first introduce the displacement flux variable $U := \nabla u$, that is

$$U = [\tfrac{\partial}{\partial x}u_1, \tfrac{\partial}{\partial y}u_1, \tfrac{\partial}{\partial x}u_2, \tfrac{\partial}{\partial y}u_2]^T = [(\nabla u_1)^T, (\nabla u_2)^T]^T. \tag{10}$$

This is the same notation as used for the stress and strain in the first seciton.

Operations on this vector function $U = [U_1, U_2]^T$ are defined as

$$\mathcal{D}U = \begin{pmatrix} \mathcal{D}U_1 \\ \mathcal{D}U_2 \end{pmatrix}, \tag{11}$$

for some differential operator $\mathcal{D}$.

This variable U will be the second variable in the mixed variable formulation. Remark that the choice of U is more general than the choice of stress or strain as second variable since the latter can be derived easily from the first.

By the definition of U and the definition of operations on this function (11) it is seen that

$$\nabla \times U = curl(U) = 0$$

because for a two dimensional vector function v, $\nabla\times$ is defined as

$$\nabla \times v = \tfrac{\partial}{\partial x}u_2 - \tfrac{\partial}{\partial y}u_1. \tag{12}$$

Combining all the above, the following first-order system is obtained:

$$\begin{cases} U - \nabla u &= 0, & \text{in} & \Omega \\ -\nabla \cdot AU &= F, & \text{in} & \Omega \\ \nabla \times U &= 0, & \text{in} & \Omega \\ n \cdot AU &= t, & \text{on} & \Gamma_1 \\ u &= f, & \text{on} & \Gamma_0 \end{cases} \tag{13}$$

For the primitive variable u, the solution space is defined by

$$\mathcal{W} := \{u \in H^1(\Omega)^2 : u = f \text{ on } \Gamma_0\}, \tag{14}$$

while for the displacement flux variable U the solution space is defined as

$$\mathcal{V} := \{U \in H^1(\Omega)^4 : n \cdot AU = t \text{ on } \Gamma_1\}.$$

A least-squares functional, based on the system of equations (13), is then defined as

$$I(U, u; F, t) := \|U - \nabla u\|^2 + \|F + \nabla \cdot AU\|^2 + \|\nabla \times U\|^2 + \|n \cdot AU - t\|_{\Gamma_1}^2$$

for $(U, u) \in \mathcal{V} \times \mathcal{W}$, where $\| \cdot \|^2$ denotes the standard $L^2(\Omega)^6$-innerproduct. In order to find an equilibrium, consider the minimization of this functional. This results in the variational formulation:
Find $(U, u) \in \mathcal{V} \times \mathcal{W}$ such that

$$\begin{cases} A(U, V) - b(V, u) &= -(F, \nabla \cdot AV) + (t, n \cdot AV)_{\Gamma_1}, & \forall V \in \mathcal{V}, \\ -b(U, v) + M(u, v) &= 0, & \forall v \in \mathcal{W}, \end{cases} \tag{15}$$

where

$$A(U, V) = \int_\Omega U \cdot V + (\nabla \cdot AU) \cdot (\nabla \cdot AV) + \nabla \times U \cdot \nabla \times V d\Omega +$$
$$\int_{\Gamma_1} (n \cdot AU) \cdot (n \cdot AV) d\Gamma,$$
$$M(u, v) = \int_\Omega \nabla u \cdot \nabla v d\Omega,$$
$$b(U, u) = \int_\Omega U \cdot \nabla u d\Omega,$$
$$(F, \nabla \cdot AV) = \int_\Omega F \cdot (\nabla \cdot AV) d\Omega,$$
$$(t, n \cdot AV)_{\Gamma_1} = \int_{\Gamma_1} t \cdot (n \cdot AV) d\Gamma.$$

In order to show existence and uniqueness of a solution (U, u), ellipticity and continuity, i.e. coercivity and boundedness, of the least-squares functional have to be satisfied. Like what has been done in [9] it is possible to show that the least-squares functional $I(U, u; 0)$ is uniform equivalent with a modified $H^1(\Omega)$-type norm defined as

$$M(U, u) = \|U\|_1^2 + \lambda^2 \|\nabla \text{tr} U\|^2 + \lambda \|\text{tr} U\|^2 + \|\nabla u\|^2. \tag{16}$$

$\text{tr} U$ is defined as $U_1 + U_4$. Uniform equivalence means independent of the Poisson ratio ν. This norm $M(\cdot)$ in turn is equivalent with the $H^1(\Omega)^4 \times H^1(\Omega)^2$-norm on $\mathcal{V} \times \mathcal{W}$. Thus, the results can be summarized as follows:

Lemma 1. *There exists a positive constant C independent of ν such that*

$$\frac{1}{C} M(U, u) \le I(U, u; 0) \le CM(U, u).$$

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Proof. In [9] the relation stated in this Lemma is proven for the least-squares functional

$$I'(\boldsymbol{U}, \boldsymbol{u}; \boldsymbol{F}) = \|\boldsymbol{U} - \nabla \boldsymbol{u}\|^2 + \|\boldsymbol{F} + \nabla \cdot A\boldsymbol{U}\|^2 + \|\nabla \times \boldsymbol{U}\|^2.$$

By adding the boundary integral $\|\boldsymbol{n} \cdot A\boldsymbol{U}\|_{\Gamma_1}^2$ to $I'(\boldsymbol{U}, \boldsymbol{u}; 0)$ and applying a trace inequality, i.e.

$$\|\boldsymbol{n} \cdot A\boldsymbol{U}\|_{\Gamma_1}^2 \le \gamma \|\boldsymbol{U}\|_1^2,$$

the Lemma is proven. $\qquad\square$

Remark that the there are in fact two different C's, a C_l and a C_u for the lower and upper-bound respectively. By taking $C = \max\{C_l, C_u\}$ both are taken into account by one constant.

Lemma 2. *There exist positive constants γ_1 and γ_2 such that*

$$\gamma_1(\|\boldsymbol{U}\|_1^2 + \|\boldsymbol{u}\|_1^2) \le M(\boldsymbol{U}, \boldsymbol{u}) \le \gamma_2(\|\boldsymbol{U}\|_1^2 + \|\boldsymbol{u}\|_1^2).$$

Proof. The lower bound is readily established by applying Friedrich's inequality on $\|\nabla \boldsymbol{u}\|^2$ such that

$$\frac{\alpha}{2}\|\boldsymbol{u}\|_1^2 \le \|\nabla \boldsymbol{u}\|^2,$$

where α is a positive constant.
The upper bound follows from the fact that

$$\lambda^2 \|\nabla \text{tr} \boldsymbol{U}\|_0^2 + \lambda \|\text{tr} \boldsymbol{U}\|_0^2 \le \hat{\gamma}_2 \|\boldsymbol{U}\|_1^2, \tag{17}$$

since

$$
\begin{aligned}
\|\text{tr} \boldsymbol{U}\|_0^2 &= \int_\Omega U_1^2 + U_4^2 + 2U_1 U_4 d\Omega \\
&\le \int_\Omega 2(U_1^2 + U_4^2) d\Omega \\
&\le 2\|\boldsymbol{U}\|_0^2
\end{aligned}
$$

and

$$
\begin{aligned}
\|\nabla \text{tr} \boldsymbol{U}\|_0^2 &= \int_\Omega (\tfrac{\partial}{\partial x}(U_1 + U_4))^2 + (\tfrac{\partial}{\partial y}(U_1 + U_4))^2 d\Omega \\
&\le \int_\Omega 2((\nabla U_1)^2 + (\nabla U_4)^2) d\Omega \\
&\le 2\|\nabla \boldsymbol{U}\|_0^2.
\end{aligned}
$$

Hence, inequality (17) holds for $\hat{\gamma}_2 = 2(\lambda^2 + \lambda)$ and it follows that

$$\min\{1, \frac{\alpha}{2}\}(\|\boldsymbol{U}\|_1^2 + \|\boldsymbol{u}\|_1^2) \le M(\boldsymbol{U}, \boldsymbol{u}) \le (2(\lambda^2 + \lambda) + 1)(\|\boldsymbol{U}\|_1^2 + \|\boldsymbol{u}\|_1^2). \tag{18}$$

$\qquad\square$

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Corollary 1. *The functional $I(U, u; 0)$ and the $H^1(\Omega)^4 \times H^1(\Omega)^2$-norm on $\mathcal{V} \times \mathcal{W}$ are equivalent:*

$$\frac{\min\{1, \frac{\alpha}{2}\}}{C}(\|U\|_1^2 + \|u\|_1^2) \leq I(U, u; 0) \leq C(2(\lambda^2 + \lambda) + 1)(\|U\|_1^2 + \|u\|_1^2). \tag{19}$$

The constants C and α are independent of the Poisson ratio ν.

Proof. Follows by combining the two previous Lemma's. $\qquad\qquad\square$

Hence, it has been proven that the least-squares functional $I(U, u; 0)$ is elliptic and continuous. The Lax-Milgram Lemma ensures the existence of an unique solution $(U, u) \in \mathcal{V} \times \mathcal{W}$ for the variational problem (15).

3 Numerical issues

For the discretization of the variational problem (15) we construct finite element spaces $\mathcal{V}_h \subset \mathcal{V}$ and $\mathcal{W}_h \subset \mathcal{W}$ and the finite element variational formulation then reads:
Find $(U_h, u_h) \in \mathcal{V}_h \times \mathcal{W}_h$ such that

$$\begin{cases} A(U_h, V_h) - b(V_h, u_h) & = & -(F, \nabla \cdot AV_h) + (t, n \cdot AV_h)_{\Gamma_1}, & \forall V_h \in \mathcal{V}_h \\ -b(U_h, v_h) + M(u_h, v_h) & = & 0, & \forall v_h \in \mathcal{W}_h \end{cases} \tag{20}$$

The solutions U_h and u_h of the above variational problem can be expressed in terms of the basis functions as

$$\begin{aligned} U_h = \sum_i \gamma_i^{(1)}(\psi_i, 0, 0, 0) + \sum_i \gamma_i^{(2)}(0, \psi_i, 0, 0) + \\ \sum_i \gamma_i^{(3)}(0, 0, \psi_i, 0) + \sum_i \gamma_i^{(4)}(0, 0, 0, \psi_i) \end{aligned} \tag{21}$$

and

$$u_h = \sum_i \beta_i^{(1)}(\phi_i, 0) + \sum_i \beta_i^{(2)}(0, \phi_i). \tag{22}$$

The finite element basis functions ψ_i and ϕ_i are of piecewise linear conforming type.

By substituting this definition of U_h and u_h into (20) and by taking for V_h and v_h the basis functions ψ_i and ϕ_i respectively, the following linear system of equations in matrix form is obtained:

$$\mathcal{A} \equiv \begin{pmatrix} A & B \\ B^T & M \end{pmatrix} \begin{pmatrix} \gamma \\ \beta \end{pmatrix} = \begin{pmatrix} G \\ g \end{pmatrix}. \tag{23}$$

The contributions for G arise from the body forces F and the surface tractions t; g occurs because of the boundary conditions imposed on Γ_0.

The following observations are obvious: since the original variational formulation (15) is elliptic, the matrix in (23) is positive definite. Furthermore the matrix is symmetric.

Before the actual examples are presented there are two other issues to be discussed: the condition number of the system ($\kappa(\mathcal{A})$) and the discretization error of the finite element approximation.

With regards to $\kappa(\mathcal{A})$ we shall assume that the triangulation $\mathcal{T}$ of Ω satisfies the inverse assumption, i.e. there exists a positive constant θ such that for all $t \in \mathcal{T}$

$$h \leq \theta \mathrm{diam}(t).$$

Under this assumption, many standard finite element spaces satisfy the inverse inequality, i.e.

$$
\begin{aligned}
|U_h|_1 &\leq kh^{-1}\|U_h\|_0 \\
|u_h|_1 &\leq kh^{-1}\|u_h\|_0
\end{aligned}
$$

Furthermore, let $|\gamma|$ and $|\beta|$ denote the l_2-norms of γ and β as used in the notation of U_h and u_h respectively. Then, under the inverse assumption, there exist positive constants a_1, a_2, b_1, b_2 such that

$$a_1 h^2 |\gamma|^2 \leq \|V_h\|_0^2 \leq a_2 h^2 |\gamma|^2$$

and

$$b_1 h^2 |\beta|^2 \leq \|v_h\|_0^2 \leq b_2 h^2 |\beta|^2.$$

Lemma 3. *Under the assumption that the above inequalities hold, we have*

$$\kappa(\mathcal{A}) \leq \frac{(1+k^2)C^2}{\min\{1, \frac{\alpha}{2}\}} \frac{\max\{a_2, b_2\}}{\min\{a_1, b_1\}} (2(\lambda^2 + \lambda) + 1)h^{-2}. \tag{24}$$

Proof. The norm-equivalence stated in Corollary 1 also holds on $\mathcal{V}_h \times \mathcal{W}_h$. Combining this result with the inequalities yields the statement of the Lemma. $\quad\square$

Since U_h is written in the form (21), it follows that

$$\|U_h\|_0^2 = [\gamma^{(1)}, \gamma^{(2)}, \gamma^{(3)}, \gamma^{(4)}]
\begin{pmatrix}
\tilde{A} & & & \\
& \tilde{A} & & \\
& & \tilde{A} & \\
& & & \tilde{A}
\end{pmatrix}
\begin{pmatrix}
\gamma^{(1)} \\
\gamma^{(2)} \\
\gamma^{(3)} \\
\gamma^{(4)}
\end{pmatrix}$$

where

$$\tilde{A}_{kl} = \int_\Omega \psi_k \psi_l d\Omega$$

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Then it is easily found that a_1 and a_2 are the smallest and largest eigenvalues of the matrix $h^{-2}\tilde{A}$ respectively. Similarly the same values are found for b_1 and b_2. In [2] a derivation for estimating these extreme eigenvalues is found and according to that derivation

$$a_1, b_1 \geq 1,$$
$$a_2, b_2 \leq 24.$$

To obtain an estimate for k, note that

$$|U_h|_1^2 = [\gamma^{(1)}, \gamma^{(2)}, \gamma^{(3)}, \gamma^{(4)}] \begin{pmatrix} \tilde{L} & & & \\ & \tilde{L} & & \\ & & \tilde{L} & \\ & & & \tilde{L} \end{pmatrix} \begin{pmatrix} \gamma^{(1)} \\ \gamma^{(2)} \\ \gamma^{(3)} \\ \gamma^{(4)} \end{pmatrix},$$

where

$$\tilde{L}_{kl} = \int_\Omega \nabla \psi_l \cdot \nabla \psi_k d\Omega.$$

It is seen that

$$|U_h|_1^2 \leq c_{\max} |\gamma|^2.$$

where $c_{\max}$ is the largest eigenvalue of $\tilde{L}$.

For the discretization error, we assume that $(U, u) \in \mathcal{V} \times \mathcal{W}$ is the solution of (15) and that $(U_h, u_h) \in \mathcal{V}_h \times \mathcal{W}_h$ is the solution of (20). By use of Céa's Lemma (see [5]) it follows that

$$\|U - U_h\|_1 \leq C \inf_{V_h \in \mathcal{V}_h} \|U - V_h\|_1$$

and

$$\|u - u_h\|_1 \leq C \inf_{v_h \in \mathcal{W}_h} \|u - v_h\|_1.$$

By taking for V_h and v_h the interpolants U_I and u_I respectively and by using interpolation properties of piecewise linear functions it can be shown that

$$\|U - U_h\|_1 \leq Ch|U|_2$$

and

$$\|u - u_h\|_1 \leq Ch|u|_2.$$

We thus find:

Corollary 2. *For the finite element solution (U_h, u_h) of (20) the following a priori error estimate holds:*

$$\|U - U_h\|_1 + \|u - u_h\|_1 \leq Ch(|U|_2 + |u|_2). \tag{25}$$

4 Numerical example

In this example the deformation of a pier of a bridge is computed using the two methods, i.e. the Galerkin and the least-squares methods. Since such a pier can have a very complicated geometry and material properties, the pier will assumed to be square and isotropic to simplify the computations. Thus $\Omega = [0,1]^2$ and the Lamé-coefficients μ and λ are constant over the domain. In order to show the difference between the two methods, the Poisson ratio ν is chosen as 0.4999, a choice which is perhaps a bit unrealistic for structures like a pier.

For this pier Ω, the displacement is assumed to be 0 on

$$\Gamma_0 = \{(x,y) : 0 \le x \le 1, \ y = 0\},$$

while the remainder Γ_1 of the boundary is divided as follows:

$$
\begin{aligned}
\Gamma_1 \ &= \ \Gamma_{1,1} \cup \Gamma_{1,2} \cup \Gamma_{1,3} \\
&= \ \{(x,y) : x = 0, \ 0 \le y \le 1\} \\
&\cup \ \{(x,y) : 0 \le x \le 1, \ y = 1\} \\
&\cup \ \{(x,y) : x = 1, \ 0 \le y \le 1\}
\end{aligned}
$$

The surface tractions t are chosen as $(0, -0.1)^T$ and are working on $\Gamma_{1,2}$. Thus a vertical load is placed on the pier, as is illustrated in Figure 1.
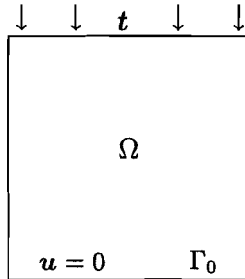

Figure 1: The domain and surface tractions

Figures 2 and 3 illustrate the displacement computed with the Galerkin and least-squares method respectively; the mesh-size has been taken $\frac{1}{16}$. In these graphics only the displacement u_2 is given and note that the solutions are multiplied by -1 in order to get a better view at the graphics and the behavior of the solution.

Figure 2 clearly shows the locking of the Galerkin solution, whereas Figure 3 shows linear deformation. The little irregularity of the solution in Figure 3 is due to the coarseness of the grid and direction of the triangulation.
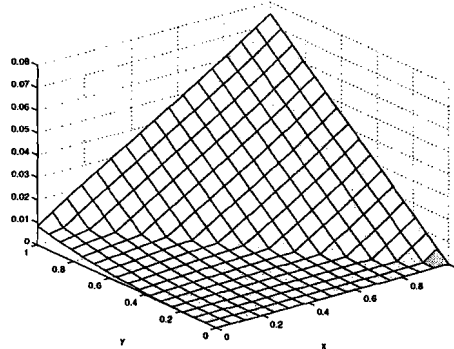
11

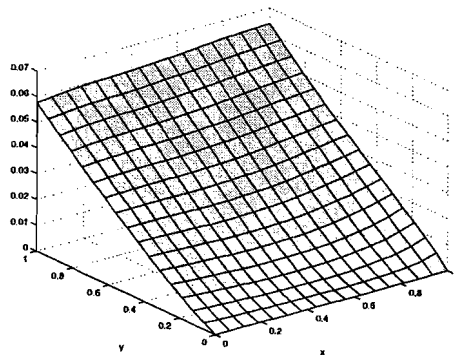. Figure 2: $-u_2$ computed with the Galerkin method

Figure 3: $-u_2$ computed with the least-squares method

We will conclude this example with an analysis for the condition number in this particular case. For the type of triangulation used in this examples, it follows that $c_{\max} \leq 8$, thus $k \leq \sqrt{8}$ (the definitions of $c_{\max}$ and k are stated in section 3).

Hence the bound for the condition number in (24) can be replaced by

$$\frac{24 \cdot 9C^2}{\min\{1, \frac{\alpha}{2}\}}(2(\lambda^2 + \lambda) + 1)h^{-2}. \tag{26}$$

One should recall that the remaining constant C and α are independent of the Poisson ratio ν. When comparing these estimates for the condition number with the actual condition number, the approximation in (26) is relatively accurate since the remaining constant term $C_0 \equiv \frac{(C')^2}{\min\{1, \frac{\alpha}{2}\}}$ is of order $O(1)$ and this is illustrated in Table 1.

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ν	λ	$\kappa(\mathcal{A})$	C_0
0	0	$5.7594 \cdot 10^5$	10.4
0.2	0.2778	$7.2804 \cdot 10^5$	7.70
0.4	1.4286	$2.0462 \cdot 10^6$	4.66
0.45	3.1034	$6.1181 \cdot 10^6$	4.18
0.49	$1.6443 \cdot 10$	$1.1915 \cdot 10^8$	3.75
0.4999	$1.6664 \cdot 10^3$	$1.1176 \cdot 10^{12}$	3.64

Table 1: Condition numbers

5 Conclusions

It has been shown that the least-squares approach for solving the equations of elasticity has certain advantages compared with the standard Galerkin method. For nearly incompressible materials the solution was not subject to 'locking' and besides the displacement the displacement flux was also computed. However, since the number of unknowns is multiplied by 3, the storage of the discretized system of equations can become a problem. Another serious drawback is that the system of equations is ill-conditioned since it depends quadratically on the Lamé-coefficient λ. This is a serious drawback when solving the system of equations iteratively.

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