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Document status and date:
Published: 01/01/1998

Document Version:
Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

A control perspective on synchronization and the
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Abstract

A method, based on ideas from control theory, is described for the synchronization of discrete time transmitter/receiver dynamics. Conceptually, the methodology consists of constructing observer-receiver dynamics that exploit at each time instant the drive signal and past values of the drive signal. In this way, the method can be viewed as a dynamic reconstruction mechanism.

PACS numbers: 02.10.Jf 02.90.+p 05.45.+b 47.52.+j 89.90.+n

1   Introduction

Following PECORA and CAROLL [14] a huge interest in the synchronization of two coupled systems has arisen. This research is partly motivated by its possible use in secure communications, cf. [6]. Often, like in [14] a drive/response, or transmitter/receiver, viewpoint is assumed. In a discrete-time context, this typically allows for a description of the transmitter as a n-dimensional dynamical system

$$x_1(k+1) \quad = \quad f_1(x_1(k), x_2(k)) \qquad (1)$$
$$x_2(k+1) \quad = \quad f_2(x_1(k), x_2(k)) \qquad (2)$$

where $x_1(\cdot)$ and $x_2(\cdot)$ are vectors of dimension m and l, with $m+l = n$ and $x(k) = (x_1(k), x_2(k))$. Given $x_1(\cdot)$

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as the drive signal, the receiver dynamics are taken as a copy of (2)

$$\tilde{x}_2(k+1) \quad = \quad f_2(x_1(k), \tilde{x}_2(k)). \qquad (3)$$

Synchronization of transmitter and receiver now corresponds to the asymptotic matching of (2) and (3), that is

$$\lim_{k \to \infty} \|x_2(k) - \tilde{x}_2(k)\| \quad = \quad 0. \qquad (4)$$

Clearly (4) will not be satisfied in general and, in fact, conditions on f_1 and f_2 that guarantee this condition are only partially known, cf. [13]. For that reason several attempts for achieving synchronization of signals like $x_2(\cdot)$ and $\tilde{x}_2(\cdot)$ have been proposed. In particular we like to recall the (reduced) observer viewpoint advocated in [12] which basically admits the construction of dynamics

$$\tilde{x}_2(k+1) \quad = \quad \tilde{f}_2(x_1(k), \tilde{x}_2(k)). \qquad (5)$$

such that (4) holds, whatever initial conditions (1), (2) and (5) have. Although (5) enlarges the idea of using the copy (3) for (2), there are many systems for which (4) will not be met, no matter how $\tilde{f}_2$ in (5) is chosen.

There is, however, a natural generalization of (5) that consists in exploiting at each time instant k the drive signal $x_1(k)$ *and* $x_1(k-1), ..., x_1(k-N)$. Thus, as receiver dynamics we use the following system

$$\tilde{x}(k+1) \quad = \quad \tilde{f}(\tilde{x}(k), x_1(k), ..., x_1(k-N)). \qquad (6)$$

Here, $\tilde{x}(\cdot)$ is n-dimensional, and $\tilde{f}(\cdot, \cdot)$ and N are such that

$$\lim_{k \to \infty} \|x(k) - \tilde{x}(k)\| = 0. \qquad (7)$$

The receiver (6) acts as an 'extended' observer for the system (1,2), in that also past values of the drive signal $x_1(\cdot)$ are used. It turns out that under fairly weak conditions receiver dynamics (6) exist such that the transmitter (1,2) and (6) synchronize, see Section 2.

Actually, the necessary conditions involved are closely related with *global observability*, cf. [11], or, the Takens-Aeyels-Sauer Reconstruction Theorem, see [16],[1],[2],[15],[13]. However, a crucial difference in our work with the Reconstruction Theorem is that (6) forms a *dynamic* 'inversion' for the state $x(\cdot)$, whereas in the Reconstruction Theorem one computes the state at some time instant by inverting the observability map, which determines $x_2(k)$ from $x_1(k), ..., x_1(k{-}N)$. It is interesting to note that an alternative using look-up tables for this procedure was proposed in [10].

The proposed transmitter/receiver synchronization using a receiver of the form (6) can be demonstrated numerically on several examples from the literature, see e.g. [3, 9]. In this paper, we will, amongst others, consider the example from [3]. The organization of this paper is as follows. In the next section we present a design procedure for observer dynamics (6) where $N = n - 1$. Section 3 presents numerical simulations of some synchronization problems where an observer presented in Section 2 is used. The paper ends with some concluding remarks.

2   Observer design

In this section, we focus on an observer design for non-linear, discrete-time, autonomous, single output systems of the form

$$x(k{+}1) = f(x(k)), \quad y(k) = h(x(k)), \qquad (8)$$

for $k = 0, 1, 2, \cdots$, where $x(\cdot)$ is a vector of dimension n and $y(\cdot)$ is a scalar. Assuming that the Jacobian of h is nonzero - which implies that a nontrivial signal from the dynamics is transmitted - we can, at least locally, rewrite (8) in a form like (1,2) with $y(k) = x_1(k)$ being one-dimensional. Within the context of synchronization, it is desired to reconstruct (asymptotically) the $(n - 1)$-dimensional $x_2(\cdot)$ on the basis of the sequence $x_1(k)$ ($k = 1, 2, \cdots$). We will do this using a suitably selected dynamics of the form (6) which basically means that we treat the synchronization problem as a sort of observer problem, cf. [12]. Without loss of generality we can assume that $f(0) = 0$ and $h(0) = 0$.

For (8) we define the so-called *observability map* ψ by

$$\psi(x) := \begin{bmatrix} h(x) \\ h \circ f(x) \\ \vdots \\ h \circ f^{n-1}(x) \end{bmatrix} \qquad (9)$$

where $h \circ f(x) := h(f(x))$, $f^1 := f$, $f^j := f \circ f^{j-1}$. The system (8) is called *strongly locally observable* around $x = 0$ if the Jacobian $(\partial \psi / \partial x)(0)$ is invertible.

We now sketch a procedure to derive two different types of observers for a strongly locally observable system (8). This procedure was proposed in [7, 8] and represents an extension of the works [4] and [5]. For clarity of presentation, we will restrict to the case that $n = 3$. Extensions to other cases are straightforward.

So, consider a strongly locally observable system (8) with $n = 3$, and define $s_i(x) := h \circ f^{i-1}(x)$ ($i = 1, 2, 3$). Since (8) is strongly locally accessible, $s = \mathrm{col}(s_1, s_2, s_3)$ forms a new set of coordinates for (8) around $x = 0$. In what follows, we will assume throughout that s forms a new set of coordinates *globally*, i.e., ψ in (9) is a global diffeomorphism on $\mathbb{R}^n$. It is straightforwardly checked that in these new coordinates the system (8) takes the form

$$s(k{+}1) = \begin{bmatrix} s_2(k) \\ s_3(k) \\ f_s(s(k)) \end{bmatrix}, \quad y(k) = s_1(k) \qquad (10)$$

where $f_s(s) := h \circ f^3(\psi^{-1}(s))$. Next, define

$$\begin{cases} z_3(k) & := & s_1(k) \\ z_2(k) & := & s_2(k) \\ & & -f_s(y(k-2), y(k-1), s_1(k)) \\ z_3(k) & := & s_3(k) \\ & & -f_s(y(k-1), s_1(k), s_2(k)) \end{cases} \qquad (11)$$

It then follows from (10),(11) that $z = \mathrm{col}(z_1, z_2, z_3)$ satisfies

$$z(k{+}1) = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{E} z(k)$$

$$+ \underbrace{\begin{bmatrix} 0 \\ 0 \\ f_s(y(k-2), y(k-1), y(k)) \end{bmatrix}}_{\Phi(y(k-2), y(k-1), y(k))}, \qquad (12)$$

$$y(k) = z_3(k).$$

An *observer of type 1* now has the form

$$\hat{z}(k+1) = E\hat{z}(k) + \Phi(y(k-2), y(k-1), y(k))$$

$$+ \begin{bmatrix} q_0 \\ q_1 \\ q_2 \end{bmatrix} \big(y(k) - \hat{y}(k)\big), \qquad (13)$$

$$\hat{y}(k) = \hat{z}_3(k)), \quad k \geq 2,$$

where q_0, q_1, q_2 are still to be determined. Defining the error signal $e := \hat{z} - z$, we obtain the error dynamics

$$e(k+1) = \underbrace{\begin{bmatrix} 0 & 0 & -q_0 \\ 1 & 0 & -q_1 \\ 0 & 1 & -q_2 \end{bmatrix}}_{A} e(k) \qquad (14)$$

The characteristic polynomial $p_A(\lambda)$ of A is given by $p_A(\lambda) = \lambda^3 + q_2\lambda^2 + q_1\lambda + q_0$. Choosing q_0, q_1, q_2 in such a way that all eigenvalues of A are located within the unit circle, the observer error $e(k)$ vanishes for $k \to \infty$ and condition (7) is met. With this it follows that the dynamics (13) initialized at an arbitrary point $\hat{z}(0)$ will asymptotically (even exponentially) match the transmitter dynamics (12). Therefore, the receiver dynamics (13) which is fed with the buffered transmitted signal $(y(k-2), y(k-1), y(k))$ synchronizes with (12).

The derivation of an *observer of type 2* starts from the observation that the solutions of (12) satisfy $z_1(k) = z_2(k) = 0$ for $k \geq 2$. This suggests to consider an observer of the form

$$\hat{z}(k+1) = \Phi(y(k-2), y(k-1), y(k))$$

$$+ \begin{bmatrix} \lambda_1 \hat{z}_1(k) \\ \lambda_2 \hat{z}_2(k) \\ \lambda_3(\hat{y}(k) - y(k)) \end{bmatrix}, \qquad (15)$$

$$\hat{y}(k) = \hat{z}_3(k), \quad k \geq 2.$$

Again defining the error signal $e := \hat{z} - z$, we now obtain the error dynamics

$$e(k+1) = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} e(k) \qquad (16)$$

for $k \geq 2$. The convergence rate of the i-th component can now be assigned by λ_i, without affecting the other components. As was the case with observer 1, here we again have that the receiver dynamics (15) which is fed with the buffered transmitted signal $(y(k-2), y(k-1), y(k))$ synchronizes with (12).

Comparing both observer types, we see that the convergence rate of each of the components of observer type 2 can be assigned independently, while this is not the case for observer type 1. Thus, observer type 2 will give a better transient behavior than observer type 1. On the other hand, however, observer type 1 with properly chosen q_0, q_1, q_2 is in general more robust to (measurement) noise than observer type 2 (cf. [7],[8]).

3 Examples

As an example, consider the transmitter system

$$\begin{aligned} x_1(k+1) &= \mu(1-\epsilon)x_1(k)(1-x_1(k)) + \epsilon\, x_2(k) \\ x_2(k+1) &= \mu(1-\epsilon)x_2(k)(1-x_2(k)) + \epsilon\, x_1(k) \end{aligned} \qquad (17)$$

presented by Badola et al. in [3]. Taking $x_1(k)$ as the drive signal* ($m=l=1$), Badola et al. investigated the synchronization of $x_2(k)$ and the receiver signal $x_3(k)$ of which the dynamics were taken as

$$x_3(k+1) = \mu(1-\epsilon)x_3(k)(1-x_3(k)) + \epsilon\, x_1(k). \qquad (18)$$

Our aim is to apply an observer presented in the previous section as receiver dynamics for transmitter (17). With $y(k) = x_1(k)$, it is possible to design observers as in the previous section in order to get the estimates $\hat{x}_1(k)$, $\hat{x}_2(k)$ for the signals $x_1(k)$, $x_2(k)$. The resulting observer equations are omitted for reasons of space. For the subsequent simulations, the initial conditions $x_1(0)=0.2$, $x_2(0)=0.4$, $\hat{x}_1(0)=\hat{x}_2(0)=0.7$ and parameters $\mu = 3.7$, $\epsilon = 0.09$ were used. Following [3], $x_2(k)$ and $x_3(k)$ do not synchronize for these parameters and $x_3(0) = \hat{x}_2(0) = 0.7$ while the observers obtained here show satisfactory behavior. Exemplary simulations of the observer errors applying observer type 1 and 2 can be seen in figures 1 and 2 for $\lambda_1 = \lambda_2 = 0.5$ (for the observer type 1, this corresponds to the choice $q_0 = 0.25, q_1 = -1$). Both observers provide very good estimations after 20 iterations with a maximum absolute observer error less than 0.002. As already mentioned in the previous section, observer type 2 shows smaller observer errors during transient time than observer type 1.

As a second example, we want to extend system (17) to the third order transmitter system

$$\begin{aligned} x_1(k+1) &= \mu(1-\epsilon)x_1(k)(1-x_1(k)) + \epsilon\, x_2(k) \\ x_2(k+1) &= \mu(1-\epsilon)x_2(k)(1-x_2(k)) + \epsilon\, x_3(k) \\ x_3(k+1) &= \mu(1-\epsilon)x_3(k)(1-x_3(k)) + \epsilon\, x_1(k) \end{aligned} \qquad (19)$$

with the drive signal $y(k) = x_1(k)$ ($m = 1$, $l = 2$). In this case, observing the unknown signals $x_2(k)$ and

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*In [3], $x_2(k)$ is considered as the drive signal. Since the coupled system given by (17) is symmetric, we can exchange $x_1(k)$ and $x_2(k)$.
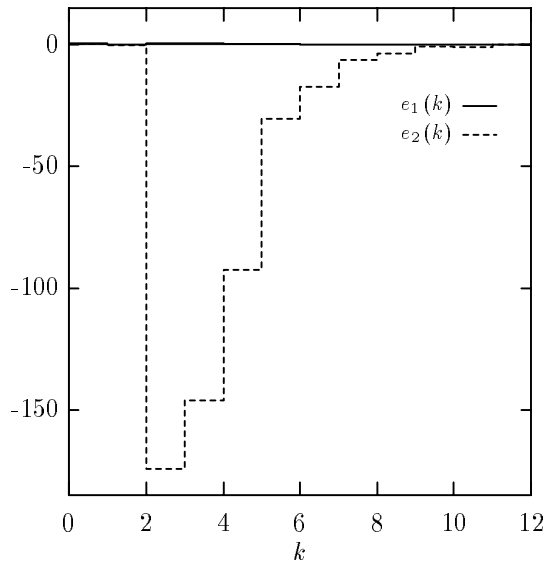
Fig. 1: Observer errors $e_i(k) = \hat{x}_i(k) - x_i(k)$ $(i = 1, 2)$ for system (17) and observer type 1 (13)
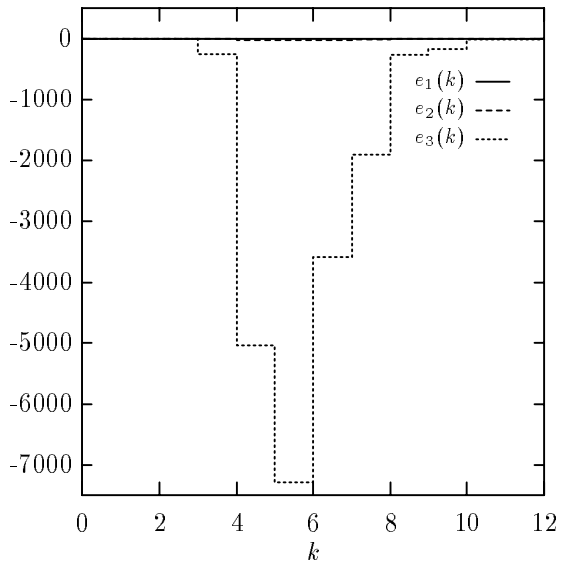

Fig. 3: Observer errors $e_i(k) = \hat{x}_i(k) - x_i(k)$ $(i = 1, 2, 3)$ for system (19) and observer type 1 (13)
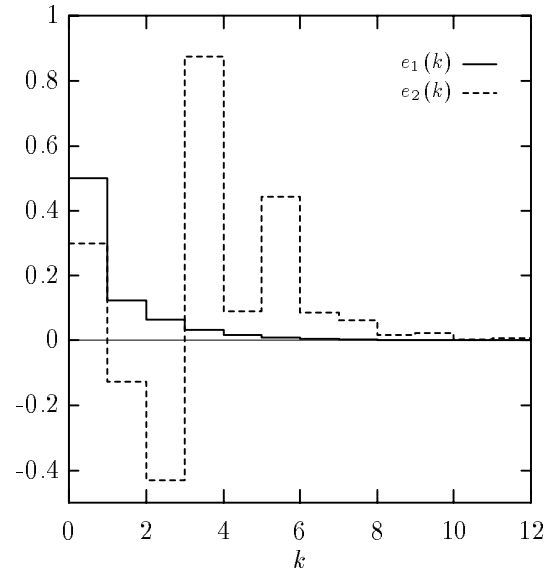

Fig. 2: Observer errors $e_i(k) = \hat{x}_i(k) - x_i(k)$ $(i = 1, 2)$ for system (17) and observer type 2 (15)
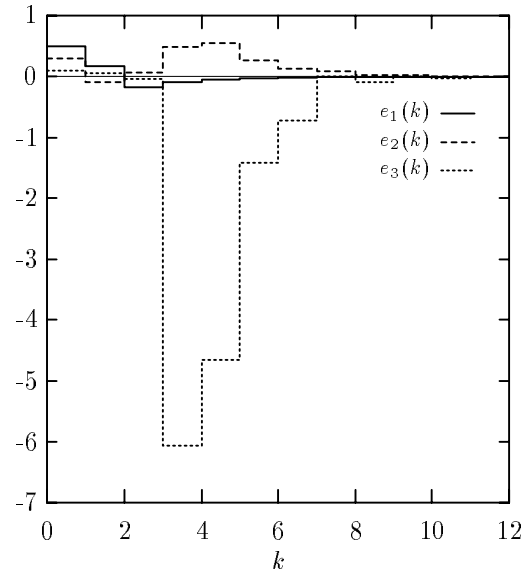

Fig. 4: Observer errors $e_i(k) = \hat{x}_i(k) - x_i(k)$ $(i = 1, 2, 3)$ for system (19) and observer type 2 (15)

$x_3(k)$ is more difficult because $x_3(k)$ does not directly influence the measured drive signal $x_1(k)$ but only via $x_2(k)$. For this reason, the coupling parameter ϵ was increased up to 0.35 while the second parameter $\mu = 3.7$ was not changed. For $x_1(0) = 0.2$, $x_2(0) = 0.4$, $x_3(0) = 0.6$, $\hat{x}_i(0) = 0.7$, $i = 1, 2, 3$, and eigenvalues of the observer error dynamics $\lambda_i = 0.5$, $i = 1, 2, 3$ (for the observer type 1, this corresponds to the choice $q_0 = -0.125$, $q_1 = 0.75$, $q_2 = -1.5$), the observer errors applying observer types 1 and 2 are shown in figures 3 and 4.

It can be seen that $|e_3(k)|$ reaches very high values (up to 7500 with observer type 1) during transient time. Nevertheless, after 20 iterations the maximum absolute observer error is less than 0.007.

The examples show the efficiency of observers taken as

receiver dynamics in synchronization problems, especially when taking into consideration that synchronization of transmitter system and observer is guaranteed if the system is globally observable. Moreover, the eigenvalues of the observer error dynamics and consequently the convergence rate are selectable. For synchronization as presented in [3], one is neither able to guarantee synchronization nor able to influence the number of steps until synchronizations occurs.

4 Concluding remarks

We have presented a control perspective on synchronization of discrete time transmitter systems. The methodology of designing an observer as the receiver system enables the exponential synchronization of

transmitter and receiver, and does not require any condition on conditional Lyapunov exponents as is often the case when identical transmitter and receiver systems are used. Essentially, the observer scheme that is used in this paper exploits at each time instant k the last $n-1$ measurements of the drive signal $y(k), y(k-1), \ldots, y(k-n+1)$, with n being the dimension of the transmitter dynamics, and can be viewed as a dynamic mechanism for the (Takens-Aeyels-Sauer) Reconstruction Theorem, provided the system satisfies a global observability condition. Contrary to [3], our results are valid no matter how the initial conditions are chosen.

The observer viewpoint on the synchronization problem has also been advocated for continuous time systems, see [12], but the scheme as we used here in discrete time has no direct analogue in continuous time. An obvious way to proceed in continuous time therefore could exist in (fast) sampling of the continuous time transmitter and then design a discrete time observer as receiver. In that case the synchronization error becomes small -depending on the sampling time - but not identically zero. However, in many applications this will not be a big problem.

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