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A Novel Approximate Inertial Manifold and Its Related Postprocessing Procedure

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Abstract

In this paper, a novel approximate inertial manifold for the two-dimensional Navier-Stokes equations and its related postprocessing procedure are discussed. This new approximate inertial manifold intends to seek some kind of approximate relationship between the standard Galerkin approximate solution (approximate large eddies) and its residue (approximate small eddies). The result shows that this manifold can approximate the system attractor up to a more accurate level than other approximate inertial manifolds we ever had and its associated postprocessing procedure can get a more accurate approximate solution at any given time with great savings in computing time.

1 Introduction

Despite the considerable increasing in the available computing power during the past few years, numerically solving the partial differential equations, especially the integration of evolution partial differential equations on large time intervals, and under physically realistic situations still remains a difficult problem whose solution is not close at hand. We thereby intend to solve dissipative evolution partial differential equations in dynamically nontrivial situations, i.e., when the long-term behavior is not merely the convergency to a steady state. In this case, the solution to be simulated remains time dependent and, as time goes to infinity, it convergences to a set, the attractor, which can be a complicated set (a fractal). Studying the complicated structure of this set which is reflected by the long term behavior of the solution to some extent is of great importance to understand the nature of turbulent phenomena. That is one of the main reason why people are so interested in the long-term behavior of the solution.

An important attempt in that direction is the theory of inertial manifold, which was introduced by Foias, Sell and Temam [5]; Fabes, Luskin and Sell [3]; Temam[14]; Constantin et al.[2]; and the references therein. The idea is to imbed the attractor in a smooth finite-dimensional manifold and to reduce all the dynamics to this manifold. Unfortunately, the existence of such finite-dimensional manifold requires a very strict condition, the spectral gap condition, which is not valid for some important dissipative systems including the Navier-Stokes equations. A more flexible and less restrictive idea is to look for a so-called approximate inertial manifold, which should be a smooth finite-dimensional manifold approximating the attractor up to a certain accuracy, and to build numerical schemes providing orbits lying on this manifold. This manifold is in fact the graph of some smooth finite-dimensional mapping. From a physical point of view, this kind of mappings can be viewed as approximate interactive laws between large and small eddies.

For a given Hilbert space H in which the solutions of dissipative systems are sought, traditional construction of approximate inertial manifold is to seek some finite-dimensional mappings from H_m (large eddy subspace) to \hat{H} (small eddy subspace), where H_m is a finite-dimensional linear space spanned by the first m functions of the Galerkin basis and \hat{H} is the L^2 orthogonal complement of H_m in H , so that people can use it to approximately describe the interactive relation between large and small eddies. For example, we refer readers to [4], [11], [12], [9] and etc. This leads to solving coupled systems of large and small eddy components at each time step. As has been illustrated in [15], their associated numerical scheme, nonlinear Galerkin method, is so computationally costly that it is generally less efficient than the standard Galerkin method in spite of its higher accuracy. On the other hand, the standard Galerkin method can provide us an approximation of the solution in H_m . According to the idea of approximate inertial manifold, there should be some approximate interactive laws between large and small eddies of the solution. Now that the standard Galerkin approximation in H_m is a reasonable approximation of the large eddies, there is no reason to prevent us from seeking some similar approximate relationship between this approximate large eddy components and the residue of it which can be regarded as a reasonable approximation of small eddies. These consideration leads to construction of a certain novel approximate inertial manifold which is the graph of some finite-dimensional mapping from H_m to H . If this kind of approximate inertial manifold could be constructed successfully, we can use it to get an approximation of the residue corresponding to the standard Galerkin approximation and we expect that the standard Galerkin approximation together with this kind of approximate residue can generate a more accurate approximation of the solution. Because this kind of approximate inertial manifold is used to describe an approximate interactive law between standard Galerkin approximation and its residue, the related numerical scheme is expected to obtain a more suitable approximation of the solution at any given time. That is the reason why we will call this related numerical scheme the postprocessing procedure. Obviously, this procedure will lead to substantial savings of the computing time. Those mentioned above are the goals of our paper. We will proceed our investigation in the case of the two-dimensional Navier-Stokes equations. However, all the results which will be obtained here can be extended to many other dissipative equations.

The goal of this paper is similar to that of [15]. That is we also want to get a more accurate approximate solution at any given time by postprocessing the standard Galerkin solution. The main differences are as following. The construction of the postprocessing procedure in [15] is still based on the traditional construction of approximate inertial manifold, which is to get an approximation of the real small eddy components in \hat{H} ($\hat{H} \cap H_m = \phi$) from the approximate large eddy components (Galerkin solution) in H_m . Of course the final accuracy is restricted by both the accuracy of standard Galerkin solution in H_m and the accuracy of the approximate small eddy components in \hat{H} . The idea in this paper as we said just now is to get an approximation of the residue in the whole space H from standard Galerkin solution. So the final accuracy only depends on the approximation of the residue and may be better than the large eddy accuracy of the standard Galerkin solution (accuracy in H_m). Our result shows that the H^1 error of our postprocessing scheme is of order $O(L_m \lambda_{m+1}^{-\frac{3}{2}})$ while it is of order $O(L_m^2 \lambda_{m+1}^{-1})$ in [15]. Here L_m is of order $O(1 + \sqrt{\ln(1 + \lambda_m)})$. On the other hand, we also construct a novel approximate inertial manifold which is the graph of a smooth mapping from H_m into H and can approximate the system attractor up to order $O(L_m \lambda_{m+1}^{-\frac{3}{2}})$. In fact, our postprocessing scheme is a direct application of this new approximate inertial manifold.

The paper is organized as follows. In section 2, we give some functional settings of the two-dimensional Navier-Stokes equations. In section 3, we investigate an approximate interactive law between the standard Galerkin approximation and its residue for the solution of the steady Navier-Stokes equations. Also a numerical scheme based on this law is presented together with its error analysis and it demonstrates a great improvement of the convergence rate of the

standard Galerkin method. In section 4, we give the definition of our novel approximate inertial manifold based upon the mapping derived in section 3. And some important properties of this manifold are also studied in detail in this section too. It will show this approximate inertial manifold can approximate the attractor up to a higher level than any other approximate inertial manifolds we ever had. In the last section, a related postprocessing procedure is constructed together with its error analysis.

2 Functional Settings

Let P be the classical L^2 orthogonal divergence free projection on $L^2(\Omega)^2$ and H be the Hilbert space of the projection of $L^2(\Omega)^2$ by P with usual L^2 scalar product (\cdot, \cdot) and norm $|\cdot| = (\cdot, \cdot)^{\frac{1}{2}}$. Here $\Omega \subset \mathcal{R}^2$ is a bounded domain. We consider the following functional evolution Navier-Stokes equations in H

$$\begin{cases} \frac{du}{dt} + \nu Au + B(u, u) = f, \\ u(0) = a, \end{cases} \quad (1)$$

where u is the velocity, $a \in H$ the initial velocity, $\nu > 0$ the viscosity parameter and $f \in H$ the time independent external force. $A = -P\Delta$ is the Stokes operator which is known to be a linear self-adjoint unbounded and positive closed operator in H with domain $D(A)$. And A^{-1} is compact. We can then define the powers A^s of A for $s \in \mathcal{R}$; A^s maps $D(A^s)$ into H and $D(A^s)$ is a Hilbert space when equipped with nature scalar product $(A^s \cdot, A^s \cdot)$ and norm $|A^s \cdot|$. We set $V = D(A^{\frac{1}{2}})$ and denote by $((\cdot, \cdot))$ the inner product in V associated with the norm $\|\cdot\| = |A^{\frac{1}{2}} \cdot|$. The nonlinear term $B(u, u) = P[(u \cdot \nabla)u]$ defines a bilinear operator from $V \times V$ into V' and we denote by b the following trilinear form on V given by

$$b(u, v, w) = \langle B(u, v), w \rangle_{V'}, \quad \forall u, v, w \in V.$$

Furthermore, we give some traditional properties of $b(\cdot, \cdot, \cdot)$ [14]:

$$\begin{cases} b(u, v, w) = -b(u, w, v), \quad \forall u, v, w \in V, \\ b(u, v, w) \leq c_1 |u|_{L^\infty} \|v\| |w|, \quad \forall u \in L^\infty(\Omega)^2, v \in V, w \in H, \\ b(u, v, w) \leq c_1 |u| |A^{\frac{1}{2}} v|_{L^\infty} |w|, \quad \forall u, w \in H, A^{\frac{1}{2}} v \in L^\infty(\Omega)^2, \\ b(u, v, w) \leq c_1 |u| \|v\| |w|_{L^\infty}, \quad \forall u \in H, v \in V, w \in L^\infty(\Omega)^2, \\ b(u, v, w) \leq c_1 |A^{\frac{s_1}{2}} u| |A^{\frac{s_2+1}{2}} v| |A^{\frac{s_3}{2}} w|. \end{cases} \quad (2)$$

The last estimation holds for any $u \in D(A^{\frac{s_1}{2}})$, $v \in D(A^{\frac{s_2+1}{2}})$ and $w \in D(A^{\frac{s_3}{2}})$ with $s_1 + s_2 + s_3 \geq 1$ and $(s_1, s_2, s_3) \neq (1, 0, 0), (0, 1, 0)$ and $(0, 0, 1)$.

The above settings apply in particular to the two-dimensional Navier-Stokes equations in a bounded domain Ω associated to the nonslip or space-periodic boundary condition.

To introduce the Galerkin approximation of (1), let us denote by $w_k, k \in \mathcal{N}$ an orthonormal basis of H consisting of the eigenvectors of Stokes operator A ,

$$Aw_k = \lambda_k w_k, \quad 0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k \leq \dots \rightarrow \infty \quad \text{as } k \rightarrow \infty.$$

For any fixed $m \in \mathcal{N}$, we denote by P_m the L^2 orthogonal projection from H onto the space H_m spanned by the first m eigenvectors w_1, \dots, w_m . We also set $Q_m = I - P_m$. It is classical that:

$$|A^\alpha Q_m v| \leq \lambda_{m+1}^{\alpha-\mu} |A^\mu v|, \quad |A^\mu P_m v| \leq \lambda_m^{\mu-\alpha} |A^\alpha v|, \quad \forall \alpha < \mu, v \in D(A^\mu). \quad (3)$$

Furthermore, we give the following Brezis-Gallouet inequality[1] for any $v \in D(A)$:

$$|v|_{L^\infty} \leq c \|v\| \left(1 + \sqrt{\ln\left(1 + \frac{|Av|}{\|v\|}\right)}\right),$$

and especially, for $v \in H_m$, we have

$$\|v\|_{L^\infty} \leq L_m \|v\|, \quad (4)$$

where $L_m \sim 1 + \sqrt{\ln(1 + \lambda_m)}$. Now the standard Galerkin approximate equations of (1) admits: find $u_m(t) \in H_m$ such that

$$\begin{cases} \frac{du_m}{dt} + \nu Au_m + P_m B(u_m, u_m) = P_m f, \\ u_m(0) = P_m a. \end{cases} \quad (5)$$

As is said in the introduction, we are interested in the interaction between u_m and the difference between u and u_m which will be denoted by

$$\eta = u - u_m$$

in the rest for the sake of convenience. So we can get the equations of η by substituting $u = u_m + \eta$ into (1). By noticing (5), we finally get

$$\begin{cases} \frac{d\eta}{dt} + \nu A\eta + B(u_m, \eta) + B(\eta, u_m) + B(\eta, \eta) = Q_m[f - B(u_m, u_m)], \\ \eta(0) = Q_m a. \end{cases} \quad (6)$$

As usual, we define the following bilinear form on $V \times V$:

$$a(u, v) = \nu((u, v)), \quad \forall u, v \in V.$$

It is very easy to verify that $a(\cdot, \cdot)$ is symmetric continuous and positive on $V \times V$. In fact

$$a(u, u) = \nu \|u\|^2.$$

Then we can give the weak formulas of (1), (5) and (6) as

$$\left(\frac{du}{dt}, v\right) + a(u, v) + b(u, u, v) = (f, v), \quad \forall v \in V, \quad (7)$$

$$\left(\frac{du_m}{dt}, v\right) + a(u_m, v) + b(u_m, u_m, v) = (f, v), \quad \forall v \in H_m, \quad (8)$$

$$\begin{aligned} \left(\frac{d\eta}{dt}, v\right) + a(\eta, v) + b(u_m, \eta, v) + b(\eta, u_m, v) + b(\eta, \eta, v) \\ = (Q_m f, v) - b(u_m, u_m, Q_m v), \quad \forall v \in V. \end{aligned} \quad (9)$$

The estimations of η are classical and we will not address them here. By using the methods which are classical for these equations, for example see [7], we can check that the initial boundary (or initial) value problems (1) and (5) have unique solutions $u = u(t)$ and $u_m = u_m(t)$ for $t > 0$. Moreover, if we demand

$$a \in D(A), \quad f \in H, \quad (10)$$

there exist finite constants $M_1, M_2, M_\infty > 0$ such that

$$\begin{aligned} \sup_{t>0} \|u(t)\|, \|u_m(t)\| &\leq M_1, \\ \sup_{t>0} |Au(t)|, |Au_m(t)| &\leq M_2, \\ \sup_{t>0} \|u(t)\|_{L^\infty}, \|u_m(t)\|_{L^\infty} &\leq M_\infty. \end{aligned} \quad (11)$$

It is well known that, for the above problem (1) and (5) (or (7) and (8)), regularity properties hold under the condition (10). In particular, we can state the following results involving \dot{u} and \dot{u}_m in

Lemma 2.1 *Under the condition (10), we have*

$$|\dot{u}(t)|_{H^2(\Omega)^2}, |\dot{u}_m(t)|_{H^2(\Omega)^2} \leq \kappa(t), \quad \forall t > 0,$$

where $\kappa(t) > 0$ depends on (t, a, f) and is bounded on $[t_0, \infty)$ for any $t_0 > 0$.

The above lemma can be proved, for example, by combining the techniques in Johnson et al. [8] and Marion [10].

3 Finite-Dimensional Mapping: Steady State Case

Although we are interested in the long-term behavior of system (1) (or (7)) in dynamically nontrivial situation, we also believe that the studying of the steady case will be very helpful and may enlighten us when we consider the unsteady case.

In this section, we will study the relationship between u_m and η for the steady Navier-Stokes equations. We want to construct a finite-dimensional mapping from H_m into V such that it can reflect the interaction between the approximate large eddies and small eddies, namely, u_m and η . Then in the next section, we will show that this mapping is suitable to be applied to unsteady case to generate a novel approximate inertial manifold which can approximate the attractor with accuracy of order $L_m \lambda_{m+1}^{-\frac{3}{2}}$. Here we will not write down the steady equations associated with (1), (5) and (6) (or (7)~(9)). Readers should keep in mind that we mean their associated steady equations when we talk about these equations within this section.

As is said in the previous section, there is a natural decomposition of u when its standard Galerkin approximation u_m is at hand, namely

$$u = u_m + \eta. \tag{12}$$

If we can get some mapping Φ from H_m into V such that

$$\|\Phi(u_m) - \eta\| = o(\|\eta\|),$$

the following function

$$\tilde{u} := u_m + \Phi(u_m) \tag{13}$$

is a more accurate approximation of u than u_m . Though the residue equations (6) can provide us a manner to derive η exactly from u_m , it is in fact to solve the Navier-Stokes equations and has no practical meanings. On the other hand, when we consider the space derivatives, $B(\eta, u_m) + B(\eta, \eta)$ should be a higher order small quantity compared with $\nu A\eta + B(u_m, \eta)$. According to this observation,

$$\nu A\hat{\eta} + B(u_m, \hat{\eta}) = Q_m[f - B(u_m, u_m)]$$

may be a reasonable approximation of (6). Noticing (2) and Lax-Milgram theorem, the following property is obvious.

Lemma 3.1 *The following problem*

$$\begin{cases} \text{for any } \phi \in H_m, \text{ find } \Phi(\phi) \in V \text{ such that} \\ a(\Phi(\phi), v) + b(\phi, \Phi(\phi), v) = (Q_m[f - B(\phi, \phi)], v), \quad \forall v \in V \end{cases} \tag{14}$$

can determine a single value mapping from H_m into V .

By using this mapping, we will give our main result of this section in

Theorem 3.1 For $f \in H$, we have

$$\|u - \tilde{u}\| \leq c_2(|\eta| \|\eta\| + L_m |A^{-\frac{1}{2}} \eta|),$$

where u is the solution of (7), \tilde{u} defined by (13) and c_2 is a positive constant.

To prove this theorem, we need a new estimation on the trilinear form $b(\cdot, \cdot, \cdot)$.

Lemma 3.2 For any $v \in D(A^{-\frac{1}{2}})$, $w \in V$ and $\phi \in H_m$, there exists a constant $c_3 > 0$ independent of v , w , ϕ and m such that

$$b(v, \phi, w) \leq c_3 L_m |A^{-\frac{1}{2}} v| |A\phi| \|w\|.$$

And if we suppose that $w \in D(A)$, we have

$$b(\phi, w, v) \leq c_3 L_m \|\phi\| |Aw| |A^{-\frac{1}{2}} v|.$$

Proof. Let us recall the definition of b :

$$b(v, \phi, w) = \int_{\Omega} (v \cdot \nabla) \phi \cdot w dx.$$

We denote $w = (w_1, w_2)^T$ and $\phi = (\phi_1, \phi_2)^T$ where $w_i, \phi_i, i = 1, 2$, are scalar functions on Ω . By a simple calculation, it is easy to rewrite the kernel of the integration as

$$(v \cdot \nabla) \phi \cdot w = (w \cdot \nabla \phi) \cdot v$$

where $w \cdot \nabla \phi := w_1 \nabla \phi_1 + w_2 \nabla \phi_2$. Therefore,

$$b(v, \phi, w) = (w \cdot \nabla \phi, v).$$

Meanwhile, we denote by \mathcal{D} any space derivative on \mathcal{R}^2 . Then

$$\mathcal{D}(w \cdot \nabla \phi) = (\mathcal{D}w \cdot \nabla \phi) + (w \cdot \nabla \mathcal{D}\phi).$$

For any $\psi \in H$, it yields

$$\begin{aligned} (\mathcal{D}(w \cdot \nabla \phi), \psi) &= (\mathcal{D}w \cdot \nabla \phi, \psi) + (w \cdot \nabla \mathcal{D}\phi, \psi) = ((\psi \cdot \nabla) \phi, \mathcal{D}w) + ((\psi \cdot \nabla) \mathcal{D}\phi, w) \\ &= b(\psi, \phi, \mathcal{D}w) + b(\psi, \mathcal{D}\phi, w). \end{aligned}$$

Noticing the estimations of b in (2), the Brezis-Gallouet inequality (4) and $\phi \in H_m$, we have

$$\begin{aligned} (\mathcal{D}(w \cdot \nabla \phi), \psi) &= b(\psi, \phi, \mathcal{D}w) - b(\psi, w, \mathcal{D}\phi) \\ &\leq c_1 |\psi| |A^{\frac{1}{2}} \phi|_{L^\infty} \|\mathcal{D}w\| + c_1 |\psi| \|w\| |A^{\frac{1}{2}} \phi|_{L^\infty} \leq 2c_1 L_m |A\phi| \|w\| |\psi|. \end{aligned}$$

Thus we have

$$|\mathcal{D}(w \cdot \nabla \phi)| \leq 2c_1 L_m |A\phi| \|w\|.$$

Now let us return to our estimation of b .

$$\begin{aligned} b(v, \phi, w) &= (w \cdot \nabla \phi, v) = (A^{\frac{1}{2}}(w \cdot \nabla \phi), A^{-\frac{1}{2}} v) \leq |A^{\frac{1}{2}}(w \cdot \nabla \phi)| |A^{-\frac{1}{2}} v| \\ &\leq |\mathcal{D}(w \cdot \nabla \phi)| |A^{-\frac{1}{2}} v| \leq 2c_1 L_m |A\phi| \|w\| |A^{-\frac{1}{2}} v|. \end{aligned}$$

Denote $c_3 = 3c_1$ and we can get the first estimation.

Similarly, we have

$$\begin{aligned} b(\phi, w, v) &= (A^{\frac{1}{2}}B(\phi \cdot \nabla)w, A^{-\frac{1}{2}}v) \leq |A^{\frac{1}{2}}(\phi \cdot \nabla)w| |A^{-\frac{1}{2}}v| \\ &\leq |\mathcal{D}((\phi \cdot \nabla)w)| |A^{-\frac{1}{2}}v| \leq (|\mathcal{D}\phi \cdot \nabla w| + |(\phi \cdot \nabla)\mathcal{D}w|) |A^{-\frac{1}{2}}v|. \end{aligned}$$

For any $\psi \in H$

$$\begin{aligned} ((\mathcal{D}\phi \cdot \nabla)w, \psi) &= b(\mathcal{D}\phi, w, \psi) = b(\mathcal{D}\phi, P_m w, \psi) + b(\mathcal{D}\phi, Q_m w, \psi) \\ &\leq c_1 \|\phi\| |A^{\frac{1}{2}}P_m w|_{L^\infty} |\psi| + c_1 |A^{\frac{3}{4}}\phi| |A^{\frac{3}{4}}Q_m w| |\psi| \\ &\leq c_1 L_m \|\phi\| |Aw| |\psi| + c_1 \|\phi\| |Aw| |\psi| \\ &\leq 2c_1 L_m \|\phi\| |Aw| |\psi|, \\ ((\phi \cdot \nabla)\mathcal{D}w, \psi) &= b(\phi, \mathcal{D}w, \psi) \leq c_1 L_m \|\phi\| |Aw| |\psi|. \end{aligned}$$

Combine the above three inequalities, we can get the second estimation. \square

In the proof of lemma 3.2, we used a well known result that the semi-norm in V is equivalent to the complete norm $|A^{\frac{1}{2}}\cdot| = \|\cdot\|$. To avoid having too many constants, we simply treated the constant related to these two equivalent norms as unit and this will not cause any significant difference.

Proof of Theorem 3.1. Noting (12) and (13), we have

$$u - \tilde{u} = \eta - \Phi(u_m).$$

Here $\Phi(u_m)$ is the solution of

$$\begin{cases} \text{find } \Phi(u_m) \in V \text{ such that} \\ a(\Phi(u_m), v) + b(u_m, \Phi(u_m), v) = (Q_m[f - B(u_m, u_m)], v), \quad \forall v \in V. \end{cases} \quad (15)$$

Denoting $\epsilon = \eta - \Phi(u_m)$ and subtracting the above equations from (9), we have

$$a(\epsilon, v) + b(u_m, \epsilon, v) = -b(\eta, u_m, v) - b(\eta, \eta, v).$$

Taking $v = \epsilon$ and using lemma 3.2 and (2), we can get

$$\begin{aligned} \nu \|e\|^2 &\leq |b(\eta, u_m, \epsilon)| + |b(\eta, \eta, \epsilon)| \leq c_3 L_m |A^{-\frac{1}{2}}\eta| |Au_m| \|e\| + c_1 |\eta|_{\frac{1}{2}}^2 \|e\| \\ &\leq c_3 L_m |Au_m| |A^{-\frac{1}{2}}\eta| \|e\| + c_1 c_0^2 |\eta| \|\eta\| \|e\|. \end{aligned}$$

Here we used the following Sobolev interpolation inequality

$$|\phi|_{\frac{1}{2}} \leq c_0 |\phi| \|\phi\|, \quad \forall \phi \in V. \quad (16)$$

Then by noticing (11),

$$\|e\| \leq \frac{c_3}{\nu} L_m M_2 |A^{-\frac{1}{2}}\eta| + \frac{c_1 c_0^2}{\nu} |\eta| \|\eta\|.$$

Denote

$$c_2 = \max\left\{\frac{c_3 M_2}{\nu}, \frac{c_1 c_0^2}{\nu}\right\},$$

we can get the result. \square

From the result of theorem 3.1, we know that $\|\eta - \Phi(u_m)\|$ is a higher order small quantity compared with $\|\eta\|$. Then we can construct the following postprocessing procedure for (8).

Postprocessing Procedure 1

(Step 1) Solve (8) to get u_m ;

(Step 2) Solve (15) to get $\Phi(u_m)$;

(Step 3) Form the postprocessed solution $\tilde{u} = u_m + \Phi(u_m)$.

The result of theorem 3.1 indicates that this postprocessing procedure can produce a more accuracy approximate solution of (7).

4 Approximate Inertial Manifold

In this section, we will show that the finite dimensional mapping Φ derived in last section is applicable to generate a kind of novel approximate inertial manifold for system (1). First of all, we want to do a little modification on Φ .

We recall that the Navier-Stokes equations (1) has a compact absorbing set $B \subset V$. For any initial value a , there must be some positive time t_a such that $u(t)$ will enter this compact set B and can not escape from it since $t > t_a$. For convenience, we define

$$B = \{v \in V, \|v\| \leq 2M_1\}.$$

Here we used the constant M_1 appeared in (11) to denote the radius of B . Generally speaking, M_1 appeared in (11) is dependent on the initial value a . Concerning the absorbing property, if we give M_1 in advance such that B is still an absorbing set of the system, there must be a finite $t_a > 0$ such that $u(t) \in B$ when $t > t_a$ for any $a \in D(A)$. Without loss of generality, we will always assume $a \in B$.

Now, we introduce a smooth function

$$\theta(s) \in C^\infty(\mathcal{R}^+)$$

with the following properties

$$\theta(s) \in [0, 1], \quad |\theta'(s)| \leq 2, \quad \theta(s) = 1, \forall s \in [0, 1], \quad \theta(s) = 0, \forall s \in [2, +\infty).$$

And we do a little modification of (14) by using $\theta(s)$.

$$\begin{cases} \text{for any } \phi \in H_m, \text{ find } \Phi(\phi) \in V \text{ such that} \\ a(\Phi(\phi), v) + \theta(\frac{\|\phi\|}{M_1})b(\phi, \Phi(\phi), v) = \theta(\frac{\|\phi\|}{M_1})(Q_m[f - B(\phi, \phi)], v), \quad \forall v \in V. \end{cases} \quad (17)$$

Because of the same argument of lemma 3.1, (17) can determine a single value mapping Φ from H_m into V .

Theorem 4.1 *The finite dimensional mapping Φ defined by (17) have the following properties:*

i) $\Phi(\phi) = 0$ for any $\phi \notin B$;

ii) For any $\phi \in H_m$,

$$\|\Phi(\phi)\| \leq c_8 L_m \lambda_{m+1}^{-\frac{1}{2}} \triangleq \rho_m, \quad (18)$$

$$|A\Phi(\phi)| \leq \frac{1}{\nu}(c_1|\phi|_{L^\infty}\rho_m + c_1M_1|\phi|_{L^\infty} + |f|), \quad (19)$$

where $c_8 = \nu^{-1}(|f|/L_m + 4c_1M_1^2)$. Moreover, if we assume $f \in V$, we have

$$|A\Phi(\phi)| \leq c_9 L_m^2 \lambda_{m+1}^{-\frac{1}{2}} \triangleq \hat{\rho}_m, \quad (20)$$

$$|A^{\frac{3}{2}}\Phi(\phi)| \leq \frac{1}{\nu}(c_3M_1L_m\hat{\rho}_m + c_3M_1|A\phi|_{L_m} + \|f\|) \triangleq L_m\tilde{\rho}_m, \quad (21)$$

where $c_9 = \nu^{-1}(c_1 c_8 M_1 + \|f\|/L_m^2 + c_3 M_1 |A\phi|/L_m)$. Obviously, $\hat{\rho}_m$ will tend to zero as m tends to ∞ and $\hat{\rho}_m$ is bounded if we suppose $\phi \in B \cap D(A)$.

iii) Φ is a global Lipschitz smooth mapping. There exists some constant $l_m > 0$ such that

$$\|\Phi(\phi_1) - \Phi(\phi_2)\| \leq l_m \|\phi_1 - \phi_2\|, \quad \forall \phi_1, \phi_2 \in V. \quad (22)$$

And $l_m \rightarrow 0$ when $m \rightarrow +\infty$.

iv) If we denote

$$\mathcal{M} = \text{Graph}\Phi,$$

for any given initial value $a \in D(A)$ and external force $f \in V$ of the system (1), there exists a positive constant t_a such that

$$\text{dist}_{t > t_a}(u(t), \mathcal{M})_V \leq c_4 L_m \lambda_{m+1}^{-\frac{3}{2}}, \quad (23)$$

where $c_4 > 0$ is independent of m and t .

Proof. i) $\phi \notin B$ means $\|\phi\| > 2M_1$. Then, (17) becomes

$$a(\Phi(\phi), v) = 0, \quad \forall v \in V.$$

Obviously, we have $\Phi(\phi) = 0$.

ii) For $\phi \notin B \cap H_m$, the result is obvious because of i).

Now let us consider $\phi \in B \cap H_m$. Take $v = \Phi(\phi)$ in (17) and notice (2), we have

$$\begin{aligned} \nu \|\Phi(\phi)\|^2 &\leq |(f, Q_m \Phi(\phi))| + |b(\phi, \phi, Q_m \Phi(\phi))| \leq |f| |Q_m \Phi(\phi)| + c_1 |\phi|_{L^\infty} \|\phi\| |Q_m \Phi(\phi)| \\ &\leq \lambda_{m+1}^{-\frac{1}{2}} |f| \|\Phi(\phi)\| + c_1 L_m \lambda_{m+1}^{-\frac{1}{2}} \|\phi\|^2 \|\Phi(\phi)\| \leq \lambda_{m+1}^{-\frac{1}{2}} (|f| + 4c_1 L_m M_1^2) \|\Phi(\phi)\|. \end{aligned}$$

Denote $c_8 = \nu^{-1}(|f|/L_m + 4c_1 M_1^2)$ and $\rho_m = c_8 L_m \lambda_{m+1}^{-\frac{1}{2}}$, we can get (18).

Again, by taking $v = A\Phi(\phi)$ in (17), we have

$$\begin{aligned} \nu |A\Phi(\phi)|^2 &\leq |b(\phi, \Phi(\phi), A\Phi(\phi))| + |(Q_m[f - B(\phi, \phi)], A\Phi(\phi))| \\ &\leq c_1 |\phi|_{L^\infty} \|\Phi(\phi)\| |A\Phi(\phi)| + (|f| + c_1 |\phi|_{L^\infty} \|\phi\|) |A\Phi(\phi)|. \end{aligned}$$

By using the result we just got, we can get the estimation (19).

Now we consider the further regularity result of $\Phi(\phi)$ for $f \in V$. First of all, we can get a rigorous estimation on $|A\Phi(\phi)|$. In fact, following the proceeding estimation procedure and noticing lemma 3.2, we have

$$\begin{aligned} \nu |A\Phi(\phi)|^2 &\leq c_1 M_1 L_m \rho_m |A\Phi(\phi)| + \lambda_{m+1}^{-\frac{1}{2}} \|f\| |A\Phi(\phi)| + \lambda_{m+1}^{-\frac{1}{2}} |A^{\frac{1}{2}} B(\phi, \phi)| |A\Phi(\phi)| \\ &\leq c_1 M_1 L_m \rho_m |A\Phi(\phi)| + \lambda_{m+1}^{-\frac{1}{2}} (\|f\| + c_3 L_m M_1 |A\phi|) |A\Phi(\phi)|. \end{aligned}$$

If we denote

$$c_9 = \frac{1}{\nu} (c_1 c_8 M_1 + \|f\|/L_m^2 + c_3 M_1 |A\phi|/L_m) \quad \text{and} \quad \hat{\rho}_m = c_9 L_m^2 \lambda_{m+1}^{-\frac{1}{2}}, \quad (24)$$

we can get (20). Of course, if we suppose $\phi \in B \cap D(A)$, we know that $\hat{\rho}_m \rightarrow 0$ as $m \rightarrow +\infty$.

For the estimation of $|A^{\frac{3}{2}}\Phi(\phi)|$, we have

$$\nu |A^{\frac{3}{2}}\Phi(\phi)|^2 \leq |b(\phi, \Phi(\phi), A^2\Phi(\phi))| + |(Q_m f, A^2\Phi(\phi))| + |(Q_m B(\phi, \phi), A^2\Phi(\phi))|.$$

For each term on the right hand side of the above inequality, we have

$$|b(\phi, \Phi(\phi), A^2\Phi(\phi))| \leq c_3 L_m \|\phi\| |A\Phi(\phi)| |A^{\frac{3}{2}}\Phi(\phi)| \leq c_3 M_1 L_m \hat{\rho}_m |A^{\frac{3}{2}}\Phi(\phi)|,$$

$$|(Q_m f, A^2\Phi(\phi))| \leq \|f\| |A^{\frac{3}{2}}\Phi(\phi)|,$$

$$|(Q_m B(\phi, \phi), A^2\Phi(\phi))| \leq |A^{\frac{1}{2}}B(\phi, \phi)| |A^{\frac{3}{2}}\Phi(\phi)| \leq c_3 L_m M_1 |A\phi| |A^{\frac{3}{2}}\Phi(\phi)|.$$

Combine these estimations and we can get the regularity result (21).

iii) For any $\phi_1, \phi_2 \in H_m$, we denote

$$w_1 = \Phi(\phi_1), \quad w_2 = \Phi(\phi_2), \quad \theta_1 = \theta\left(\frac{\|\phi_1\|}{M_1}\right), \quad \theta_2 = \theta\left(\frac{\|\phi_2\|}{M_1}\right).$$

Moreover, we introduce

$$\phi_\epsilon = \phi_1 - \phi_2, \quad w_\epsilon = w_1 - w_2, \quad \theta_\epsilon = \theta_1 - \theta_2.$$

Then we have

$$\begin{aligned} & a(w_\epsilon, v) + \theta_1 [b(\phi_\epsilon, w_1, v) + b(\phi_2, w_\epsilon, v)] + \theta_\epsilon b(\phi_2, w_2, v) \\ & = \theta_\epsilon (Q_m f, v) + \theta_1 [b(\phi_\epsilon, \phi_1, Q_m v) + b(\phi_2, \phi_\epsilon, Q_m v)] + \theta_\epsilon b(\phi_2, \phi_2, Q_m v) \end{aligned} \quad (25)$$

For $\phi_1, \phi_2 \notin B \cap H_m$, we have $w_1 = w_2 = 0$ by i) and (22) is valid for this case.

For the case of $\phi_1, \phi_2 \in B \cap H_m$, we take $v = w_\epsilon$ in (25).

$$\begin{aligned} & \nu \|w_\epsilon\|^2 + \theta_1 b(\phi_\epsilon, w_1, w_\epsilon) + \theta_\epsilon b(\phi_2, w_2, w_\epsilon) \\ & = \theta_\epsilon (Q_m f, w_\epsilon) + \theta_1 [b(\phi_\epsilon, \phi_1, Q_m w_\epsilon) + b(\phi_2, \phi_\epsilon, Q_m w_\epsilon)] + \theta_\epsilon b(\phi_2, \phi_2, Q_m w_\epsilon). \end{aligned} \quad (26)$$

To estimate each term in (26), we must estimate θ_ϵ in advance. From the definition of θ , it is easy to know that

$$|\theta_\epsilon| \leq |\theta'(\xi)| \left| \frac{\|\phi_1\| - \|\phi_2\|}{M_1} \right| \leq \frac{2}{M_1} \|\phi_\epsilon\|, \quad (27)$$

where ξ is a constant between $\frac{\|\phi_1\|}{M_1}$ and $\frac{\|\phi_2\|}{M_1}$. Noticing ii) and (27), we majorize the estimation of each term in (26) as follows.

$$\begin{aligned} \theta_1 b(\phi_\epsilon, w_1, w_\epsilon) & \leq |b(\phi_\epsilon, w_1, w_\epsilon)| \leq c_1 \|\phi_\epsilon\| \|w_1\| \|w_\epsilon\| \\ & \leq c_1 \rho_m \|\phi_\epsilon\| \|w_\epsilon\| \leq \frac{\nu}{12} \|w_\epsilon\|^2 + \frac{3c_1^2 \rho_m^2}{\nu} \|\phi_\epsilon\|^2, \end{aligned}$$

$$\begin{aligned} \theta_\epsilon b(\phi_2, w_2, w_\epsilon) & \leq \frac{2}{M_1} \|\phi_\epsilon\| |b(\phi_2, w_2, w_\epsilon)| \leq \frac{2c_1}{M_1} \|\phi_\epsilon\| \|\phi_2\| \|w_2\| \|w_\epsilon\| \\ & \leq 2c_1 \rho_m \|\phi_\epsilon\| \|w_\epsilon\| \leq \frac{\nu}{12} \|w_\epsilon\|^2 + \frac{12c_1^2 \rho_m^2}{\nu} \|\phi_\epsilon\|^2, \end{aligned}$$

$$\begin{aligned} \theta_\epsilon (Q_m f, w_\epsilon) & \leq \frac{2}{M_1} \|\phi_\epsilon\| |f| |Q_m w_\epsilon| \leq \frac{2\lambda_{m+1}^{-\frac{1}{2}}}{M_1} |f| \|\phi_\epsilon\| \|w_\epsilon\| \\ & \leq \frac{\nu}{12} \|w_\epsilon\|^2 + \frac{12|f|^2 \lambda_{m+1}^{-1}}{M_1^2 \nu} \|\phi_\epsilon\|^2, \end{aligned}$$

$$\begin{aligned}
\theta_1 b(\phi_\varepsilon, \phi_1, Q_m w_\varepsilon) &\leq |b(\phi_\varepsilon, \phi_1, Q_m w_\varepsilon)| \leq c_1 \|\phi_\varepsilon\|_{L^\infty} \|\phi_1\| \|Q_m w_\varepsilon\| \\
&\leq c_1 M_1 L_m \lambda_{m+1}^{-\frac{1}{2}} \|\phi_\varepsilon\| \|w_\varepsilon\| \leq \frac{\nu}{12} \|w_\varepsilon\|^2 + \frac{3c_1^2 M_1^2 L_m^2 \lambda_{m+1}^{-1}}{\nu} \|\phi_\varepsilon\|^2, \\
\theta_1 b(\phi_2, \phi_\varepsilon, Q_m w_\varepsilon) &\leq c_1 M_1 L_m \lambda_{m+1}^{-\frac{1}{2}} \|\phi_\varepsilon\| \|w_\varepsilon\| \leq \frac{\nu}{12} \|w_\varepsilon\|^2 + \frac{3c_1^2 M_1^2 L_m^2 \lambda_{m+1}^{-1}}{\nu} \|\phi_\varepsilon\|^2, \\
\theta_\varepsilon b(\phi_2, \phi_2, Q_m w_\varepsilon) &\leq \frac{2c_1}{M_1} \|\phi_\varepsilon\| \|\phi_2\|_{L^\infty} \|\phi_2\| \|Q_m w_\varepsilon\| \leq 2c_1 M_1 L_m \lambda_{m+1}^{-\frac{1}{2}} \|\phi_\varepsilon\| \|w_\varepsilon\| \\
&\leq \frac{\nu}{12} \|w_\varepsilon\|^2 + \frac{12c_1^2 M_1^2 L_m^2 \lambda_{m+1}^{-1}}{\nu} \|\phi_\varepsilon\|^2.
\end{aligned}$$

Combining the above estimations with (26), we can finally get (22) with

$$l_m = \frac{1}{\nu} \sqrt{30c_1^2 \rho_m^2 + 24|f|^2 M_1^{-2} \lambda_{m+1}^{-1} + 36c_1^2 M_1^2 L_m^2 \lambda_{m+1}^{-1}}.$$

We know that $l_m \rightarrow 0$ when $m \rightarrow \infty$ from the definition of ρ_m , which is the expected result.

At last, we consider that one of ϕ_1 and ϕ_2 is not in $B \cap H_m$ and the other one is inside this set. Without loss of generality, we assume $\phi_1 \notin B \cap H_m$. Then we have $\theta_1 = 0$. Again, by taking $v = w_\varepsilon$ in (26), we have

$$\nu \|w_\varepsilon\|^2 + \theta_\varepsilon b(\phi_2, w_2, w_\varepsilon) = \theta_\varepsilon(Q_m f, w_\varepsilon) + \theta_\varepsilon b(\phi_2, \phi_2, Q_m w_\varepsilon).$$

Noticing (27) and the estimations we just did, we have

$$\begin{aligned}
\theta_\varepsilon b(\phi_2, w_2, w_\varepsilon) &\leq 2c_1 \rho_m \|\phi_\varepsilon\| \|w_\varepsilon\| \leq \frac{\nu}{6} \|w_\varepsilon\|^2 + \frac{6c_1^2 \rho_m^2}{\nu} \|\phi_\varepsilon\|^2, \\
\theta_\varepsilon(Q_m f, w_\varepsilon) &\leq \frac{2|f| \lambda_{m+1}^{-\frac{1}{2}}}{\nu} \|\phi_\varepsilon\| \|w_\varepsilon\| \leq \frac{\nu}{6} \|w_\varepsilon\|^2 + \frac{6|f|^2 \lambda_{m+1}^{-1}}{M_1^2 \nu} \|\phi_\varepsilon\|^2, \\
\theta_\varepsilon b(\phi_2, \phi_2, Q_m w_\varepsilon) &\leq 2c_1 M_1 L_m \lambda_{m+1}^{-\frac{1}{2}} \|\phi_\varepsilon\| \|w_\varepsilon\| \leq \frac{\nu}{6} \|w_\varepsilon\|^2 + \frac{6c_1^2 M_1^2 L_m^2 \lambda_{m+1}^{-1}}{\nu} \|\phi_\varepsilon\|^2.
\end{aligned}$$

Combining the above estimations, we can get (22) again with

$$l_m = \frac{2\sqrt{3}}{\nu} (c_1 \rho_m + \frac{|f|}{M_1} \lambda_{m+1}^{-\frac{1}{2}} + c_1 M_1 L_m \lambda_{m+1}^{-\frac{1}{2}}),$$

and it is obvious that $l_m \rightarrow 0$ when $m \rightarrow \infty$. Then we can draw our conclusion.

iv) As we said before, for any given initial value $a \in D(A)$, we can choose $t_a > 0$ such that

$$u(t) \in B \cap D(A), \quad \forall t > t_a.$$

In fact, this t_a can be determined easily if one notices the classical energy estimation of $u(t)$.

To simplify our presentation, we denote

$$p(t) = P_m u(t), \quad q(t) = Q_m u(t)$$

and

$$\phi_p(t) = \Phi(p(t)).$$

For any $t > t_a$, we know that

$$\hat{u} := p(t) + \phi_p(t) \in \mathcal{M}.$$

If we denote

$$\epsilon = u(t) - \hat{u}(t) = q(t) - \phi_p(t),$$

we have

$$\text{dist}_{t>t_a}(u(t), \mathcal{M})_V \leq \|\epsilon\|.$$

Because of $u(t) \in B$, we have $p(t) \in B \cap H_m$. Therefore $\theta(\frac{\|p(t)\|}{M_1}) = 1$. Equations (17) now become

$$a(\phi_p, v) + b(p, \phi_p, v) = (Q_m[f - B(p, p)], v), \quad \forall v \in V. \quad (28)$$

And $q(t)$ satisfies

$$\left(\frac{dq}{dt}, v\right) + a(q, v) + b(p, q, v) + b(q, p, v) + b(q, q, v) = (f - B(p, p), v), \quad \forall v \in Q_m V. \quad (29)$$

On the other hand, we decompose ϵ as

$$\epsilon = P_m \epsilon + Q_m \epsilon \triangleq \epsilon_p + \epsilon_q.$$

Now let us estimate ϵ_p and ϵ_q respectively. It is obvious that $\epsilon_p = -P_m \phi_p$. Taking $v = \epsilon_p$ in (28) and noticing the regularity result in ii), we have,

$$\nu \|\epsilon_p\|^2 \leq |b(p, Q_m \phi_p, \epsilon_p)| \leq c_1 |p|_{L^\infty} |Q_m \phi_p| \|\epsilon_p\| \leq c_1 M_\infty L_m \tilde{\rho}_m \lambda_{m+1}^{-\frac{3}{2}} \|\epsilon_p\|.$$

Thus

$$\|\epsilon_p\| \leq \frac{c_1 M_\infty}{\nu} \tilde{\rho}_m L_m \lambda_{m+1}^{-\frac{3}{2}}. \quad (30)$$

Because $|Au| \leq M_2$, it is easy to verify that $\tilde{\rho}_m$ is bounded. And for ϵ_q , by restricting (28) on $Q_m V$ and subtracting it from (29), we have

$$a(\epsilon_q, v) + b(p, \epsilon, v) = -\left(\frac{dq}{dt}, v\right) - b(q, p, v) - b(q, q, v), \quad \forall v \in Q_m V.$$

Since we consider $t > t_a > 0$, we know $\kappa(t)$ in lemma 2.1 is bounded and we will use κ to denote this time independent bound. Taking $v = \epsilon_q$ in the last equation and noticing (3), it yields

$$\begin{aligned} \nu \|\epsilon_q\|^2 &\leq |b(p, \epsilon_p, \epsilon_q)| + \left|\left(\frac{dq}{dt}, \epsilon_q\right)\right| + |b(q, p, \epsilon_q)| + |b(q, q, \epsilon_q)| \\ &\leq c_1 |p|_{L^\infty} \|\epsilon_p\| |\epsilon_q| + |A^{-\frac{1}{2}} \frac{dq}{dt}| \|\epsilon_q\| + c_3 L_m |A^{-\frac{1}{2}} q| |Ap| \|\epsilon_q\| + c_1 |q|_{\frac{1}{2}}^2 \|\epsilon_q\| \\ &\leq (c_1 M_\infty \lambda_{m+1}^{-\frac{1}{2}} \|\epsilon_p\| + \kappa \lambda_{m+1}^{-\frac{3}{2}} + c_3 M_2^2 L_m \lambda_{m+1}^{-\frac{3}{2}} + c_1 M_2^2 \lambda_{m+1}^{-\frac{3}{2}}) \|\epsilon_q\|. \end{aligned}$$

Combining this inequality with (30), we know there must exist some constant $c_4 > 0$ independent of t and m such that

$$\|\epsilon\| \leq c_4 L_m \lambda_{m+1}^{-\frac{3}{2}}.$$

□

This theorem indicates that Φ is a smooth Lipschitz mapping from H_m into V and the associated manifold \mathcal{M} is bounded and the attractor of the system (1) is in the $L_m \lambda_{m+1}^{-\frac{3}{2}}$ neighborhood of it.

5 Postprocessing Algorithm

The finite dimensional mapping Φ and its associated inertial manifold \mathcal{M} provide us some kind of relationship between the large and small eddies of the long-term behavior of the solution of (1). In fact, they reflect the relation between the approximate large eddies (the standard Galerkin approximation) and the approximate small eddies (the associated residue). The theorem 4.1 indicates that all of the solutions of (1) will enter a $L_m \lambda_{m+1}^{-\frac{3}{2}}$ neighborhood of \mathcal{M} after a short time period $t_a > 0$. So, we could use this kind of relation, namely Φ , to improve the accuracy of the standard Galerkin approximation and get a more suitable approximation at any time $t > t_a$. That is the problem we will address in this section.

To avoid the contradiction in regularity, we assume

- (H)** the initial value a of system (1) is the evolution result of another initial value $a' \in D(A)$ given at $t = -t_a$, where t_a is a positive constant such that the constant $\kappa(t)$ in lemma 2.1 is bounded for $t \geq 0$ just like what we demanded in the proof of theorem 4.1. And we still denote this bound by κ which depends on t_a and a .

Under this assumption, we give our postprocessing procedure based on Φ in the following three steps.

postprocessing Procedure 2

(Step 1) Solve (8) to get the standard Galerkin approximate solution $u_m(t)$;

(Step 2) At any time when you want to get a more accurate approximate solution, please solve the following linear equations in V

$$\begin{cases} \text{find } \Phi(u_m(t)) \in V \text{ such that} \\ a(\Phi(u_m(t)), v) + b(u_m(t), \Phi(u_m(t)), v) = (Q_m[f - B(u_m(t), u_m(t))], v), \quad \forall v \in V; \end{cases}$$

(Step 3) Get a more accurate approximation of u at time t

$$\tilde{u}(t) = u_m(t) + \Phi(u_m(t)).$$

Notice that we did not use the function θ in (Step 2) as what has been done in (28). The reason is that $\theta = 1$ because of the estimations in (11). Therefore, we know \tilde{u} in (Step 3) is on the approximate inertial manifold \mathcal{M} .

Before we give the error estimation of this postprocessing procedure, we need some estimation on $\eta(t)$. We denote

$$\hat{e} = Q_m \dot{\eta}, \quad e_m = P_m \dot{\eta}.$$

Then from the residue equations (6) and assumption (H), we have

$$\begin{cases} \frac{de_m}{dt} + \nu A e_m + P_m [B(\dot{u}_m, \eta) + B(u_m, e_m + \hat{e}) + B(e_m + \hat{e}, u_m) \\ \quad + B(\eta, \dot{u}_m) + B(e_m + \hat{e}, \eta) + B(\eta, e_m + \hat{e})] = 0 \\ e_m(0) = 0. \end{cases} \quad (31)$$

Lemma 5.1 *Under the assumption (H), we have*

$$|A^{-\frac{1}{2}} e_m| \leq \kappa_1(t) L_m (\lambda_{m+1}^{-\frac{3}{2}} + \|A^{-\frac{1}{2}} \eta\|_t), \quad \text{for any given } t < \infty,$$

where $\|A^{-\frac{1}{2}} \eta\|_t = \sup_{0 \leq s \leq t} |A^{-\frac{1}{2}} \eta(s)|$, and $\kappa_1(t) > 0$ is independent of m .

Proof. The proof of this lemma is quite simple and we only sketch it in the following.

Multiplying the both sides of (31) with $A^{-1}e_m$ and integrating them on Ω , we have

$$\begin{aligned} \frac{1}{2} \frac{d|A^{-\frac{1}{2}}e_m|^2}{dt} + \nu|e_m|^2 &\leq |b(\dot{u}_m, \eta, A^{-1}e_m)| + |b(u_m, e_m, A^{-1}e_m)| \\ &+ |b(u_m, \hat{e}, A^{-1}e_m)| + |b(e_m, u_m, A^{-1}e_m)| + |b(\hat{e}, u_m, A^{-1}e_m)| \\ &+ |b(\eta, \dot{u}_m, A^{-1}e_m)| + |b(e_m, \eta, A^{-1}e_m)| + |b(\hat{e}, \eta, A^{-1}e_m)| \\ &+ |b(\eta, e_m, A^{-1}e_m)| + |b(\eta, \hat{e}, A^{-1}e_m)|. \end{aligned}$$

Now we need to estimate each term on the right hand side of the last inequality by using (2) and lemma 3.2. The estimation of each term can be obtained without difficulty if one notice the rigorous estimation of b in lemma 3.2. Here we only give the final result.

$$\frac{d|A^{-\frac{1}{2}}e_m|}{dt} \leq \frac{c_5}{\nu}|A^{-\frac{1}{2}}e_m| + \frac{c_6}{\nu}L_m|A^{-\frac{1}{2}}\eta| + \frac{c_7}{\nu}L_m\lambda_{m+1}^{-\frac{3}{2}},$$

where $c_5 = c_5(M_1) > 0$, $c_6 = c_6(M_1, \kappa) > 0$ and $c_7 = c_7(M_1, \kappa) > 0$ are time independent constants. Now by integrating the above differential inequality of $|A^{-\frac{1}{2}}e_m|$, we can get the result with

$$\kappa_1(t) := \frac{c_6 + c_7}{c_5} e^{\frac{c_5}{\nu}t}.$$

□

By using this lemma, we can get an error estimation of our postprocessing procedure 2. Firstly, we denote

$$E(t) = u(t) - \tilde{u}(t) = \eta(t) - \Phi(u_m(t)).$$

Theorem 5.1 *Under the assumption (H) and choosing m large enough such that*

$$\lambda_{m+1} \geq 4c_1^2 M_\infty^2 \nu^{-2},$$

we have

$$\|E(t)\| \leq \kappa_2(t)L_m(\lambda_{m+1}^{-\frac{3}{2}} + |\eta| \|\eta\| + \| |A^{-\frac{1}{2}}\eta \| \|t\|), \quad \text{for any given } t < +\infty,$$

where $\kappa_2(t) > 0$ is independent of m .

Proof. Subtracting the equations of (Step 2) from (9), we have

$$a(E, v) + b(u_m, E, v) = -(\dot{\eta}, v) - b(\eta, u_m, v) - b(\eta, \eta, v), \quad \forall v \in V. \quad (32)$$

We decompose E as

$$E = P_m E + Q_m E \triangleq E_m + \hat{E}.$$

Taking $v = E_m$ in (32) and by using (3) and (16), we have

$$\begin{aligned} \nu \|E_m\|^2 &\leq |(e_m, E_m)| + |b(\eta, u_m, E_m)| + |b(\eta, \eta, E_m)| + |b(u_m, \hat{E}, E_m)| \\ &\leq |A^{-\frac{1}{2}}e_m| \|E_m\| + c_3 L_m |A^{-\frac{1}{2}}\eta| |Au_m| \|E\| + c_1 c_0^2 |\eta| \|\eta\| \|E_m\| + c_1 |u_m|_{L^\infty} \|E_m\| \|\hat{E}\| \\ &\leq (|A^{-\frac{1}{2}}e_m| + c_3 M_2 L_m |A^{-\frac{1}{2}}\eta| + c_1 c_0^2 |\eta| \|\eta\| + c_1 M_\infty \lambda_{m+1}^{-\frac{1}{2}} \|\hat{E}\|) \|E_m\|. \end{aligned}$$

Thus

$$\|E_m\| \leq \nu^{-1} (|A^{-\frac{1}{2}}e_m| + c_3 M_2 L_m |A^{-\frac{1}{2}}\eta| + c_1 c_0^2 |\eta| \|\eta\| + c_1 M_\infty \lambda_{m+1}^{-\frac{1}{2}} \|\hat{E}\|). \quad (33)$$

Taking $v = \hat{E}$ in (32), it admits

$$\begin{aligned}
\nu \|\hat{E}\| &\leq |(\hat{e}, \hat{E})| + |b(\eta, u_m, \hat{E})| + |b(\eta, \eta, \hat{E})| + |b(u_m, E_m, \hat{E})| \\
&\leq |A^{-\frac{1}{2}} \hat{e}| \|\hat{E}\| + c_3 L_m |A^{-\frac{1}{2}} \eta| |A u_m| \|\hat{E}\| + c_1 c_0^2 |\eta| \|\eta\| \|\hat{E}\| + c_1 |u_m|_{L^\infty} \|E_m\| \|\hat{E}\| \\
&\leq (\kappa \lambda_{m+1}^{-\frac{3}{2}} + c_3 M_2 L_m |A^{-\frac{1}{2}} \eta| + c_1 c_0^2 |\eta| \|\eta\| + c_1 M_\infty \lambda_{m+1}^{-\frac{1}{2}} \|E_m\|) \|\hat{E}\|.
\end{aligned}$$

Therefore, we have

$$\|\hat{E}\| \leq \nu^{-1} (\kappa \lambda_{m+1}^{-\frac{3}{2}} + c_3 M_2 L_m |A^{-\frac{1}{2}} \eta| + c_1 c_0^2 |\eta| \|\eta\| + c_1 M_\infty \lambda_{m+1}^{-\frac{1}{2}} \|E_m\|). \quad (34)$$

Combining (33)~(34) and demanding

$$\lambda_{m+1} \geq 4c_1^2 M_\infty^2 \nu^{-2},$$

we can get

$$\begin{aligned}
\|E\| &\leq 2\nu^{-1} (\kappa \lambda_{m+1}^{-\frac{3}{2}} + |A^{-\frac{1}{2}} e_m| + 2c_3 M_2 L_m |A^{-\frac{1}{2}} \eta| + 2c_1 c_0^2 |\eta| \|\eta\|) \\
&\leq 2\nu^{-1} \left(\frac{\kappa}{L_m} + \kappa_1(t) \right) L_m \lambda_{m+1}^{-\frac{3}{2}} \\
&\quad + 2\nu^{-1} (2c_3 M_2 + \kappa_1(t)) L_m \| |A^{-\frac{1}{2}} \eta| \|_t + 4\nu^{-1} c_1 c_0^2 |\eta| \|\eta\|.
\end{aligned}$$

If we denote

$$\kappa_2(t) = 2\nu^{-1} \max \left\{ \frac{\kappa}{L_m} + \kappa_1(t), \kappa_1(t) + 2c_3 M_2, 2c_1 c_0^2 \right\},$$

the result can be achieved. \square

Remark 1. *It is classical that*

$$|\eta| \leq C(t) \lambda_{m+1}^{-1}, \quad \|\eta\| \leq C(t) \lambda_{m+1}^{-\frac{1}{2}}.$$

And from the appendix we know that there also holds

$$|A^{-\frac{1}{2}} \eta| \leq C(t) \lambda_{m+1}^{-\frac{3}{2}}.$$

Then the result of theorem 5.1 tells us that

$$\|E\| \leq C_1(t) L_m \lambda_{m+1}^{-\frac{3}{2}}.$$

Of course, the postprocessing procedure 2 can greatly improve the convergence rate of standard Galerkin approximation from $\lambda_{m+1}^{-\frac{1}{2}}$ to $L_m \lambda_{m+1}^{-\frac{3}{2}}$ at any time t , especially for large m . The substantial saving of the computing time is then obvious because we should solve the standard Galerkin approximate equations in a larger finite dimensional subspace H_M with $M \sim m^3$ to get an approximate solution of order $\lambda_{m+1}^{-\frac{3}{2}}$.

Remark 2. *As is said before, Φ , which is used to generate the novel approximate inertial manifold and to construct the postprocessing procedures in our context, is a finite dimensional mapping from H_m into V . Then for (Step 2) in both postprocessing procedure 1 and 2, we must solve the linear problem in the whole space V . But, a simple investigation of Φ will show that this mapping is in fact a finite dimensional mapping from H_m to some subspace of V .*

First of all, we will construct a new projection \mathcal{P}_m from H onto H_m based upon the solution of (5). For this purpose, we introduce some new symbols:

$$\mathcal{L}(w, v) := a(w, v) + b(u, w, v), \quad \forall w, v \in V,$$

$$\mathcal{L}_m(w, v) := a(w, v) + b(u_m, w, v), \quad \forall w, v \in V,$$

where u and u_m are the solutions of (7) and (8) respectively.

Obviously, both $\mathcal{L}(\cdot, \cdot)$ and $\mathcal{L}_m(\cdot, \cdot)$ are continuous on $V \times V$ and positive if we notice (2) and (11). Then by Lax-Milgram theorem, the following variational problem

$$\begin{cases} \text{find } w \in H_m \text{ such that} \\ \mathcal{L}_m(w, v) = \langle g, v \rangle_{V'}, \quad \forall v \in H_m, \end{cases} \quad (35)$$

has a unique solution for any given $g \in V'$. Now let us give the definition of our new projection.

$$\begin{cases} \text{for any given } w \in H, \text{ find } \mathcal{P}_m w \in H_m \text{ such that} \\ \mathcal{L}_m(w - \mathcal{P}_m w, v) = 0, \quad \forall v \in H_m. \end{cases} \quad (36)$$

Because (35) is well-posed for any given $g \in V'$, (36) can really define a projection from H onto H_m and we denote by $\mathcal{Q}_m = I - \mathcal{P}_m$. Now for any vector in H , we have the following decomposition

$$w = \mathcal{P}_m w + \tilde{w}, \quad \text{with } \tilde{w} = \mathcal{Q}_m w.$$

And (36) indicates that \tilde{w} is orthogonal with H_m in the sense of $\mathcal{L}_m(\cdot, \cdot)$. Similarly, we can decompose space V as

$$V = H_m \oplus \tilde{V}, \quad \text{with } \tilde{V} = \mathcal{Q}_m V.$$

It is obvious that $\Phi(u_m) \in \tilde{V}$ because we have

$$a(\Phi(u_m), v) + b(u_m, \Phi(u_m), v) = 0, \quad \forall v \in H_m.$$

So, to get the solution of (Step 2) in our postprocessing procedures, we only need to solve the linear problem in \tilde{V} , a subspace of V . Unfortunately, the construction of \tilde{V} is also a time consuming procedure. And for the unsteady case, the structure of \tilde{V} is dependent on time t . Concerning about these facts, we still prefer to solve this linear problem in whole space V unless we can find a simple construction of \tilde{V} or at least its approximation.

In practice, the second step of postprocessing procedure 2 must be accomplished in some finite dimensional subspace of V whose dimension should be much larger than m . That is, for some $N \in \mathcal{N}$ with $N \gg m$, we should restrict (Step 2) and (Step 3) on V_N :

$$\text{(Step 2')} \quad \begin{cases} \text{find } \Phi_N(u_m) \in V_N \text{ such that} \\ a(\Phi_N(u_m), v) + b(u_m, \Phi_N(u_m), v) = (Q_m[f - B(u_m, u_m)], v), \quad \forall v \in V_N \end{cases}$$

$$\text{(Step 3')} \quad \tilde{u}_N(t) = u_m(t) + \Phi_N(u_m(t)).$$

Then the error of the practical postprocessing procedure (Step 1), (Step 2') and (Step 3') satisfies

$$\|u(t) - \tilde{u}_N(t)\| \leq \|u(t) - \tilde{u}(t)\| + \|\Phi(u_m(t)) - \Phi_N(u_m(t))\| = \|E(t)\| + \|\tilde{E}(t)\|,$$

where $\tilde{E}(t) = \tilde{u}(t) - \tilde{u}_N(t) = \Phi(u_m(t)) - \Phi_N(u_m(t))$. Now we give the error estimate of this practical postprocessing procedure as following corollary. Because the proof is quite simple, we will omit it and only state the result.

Corollary 5.1 *Under the conditions of theorem 5.1, the error of the scheme (Step 1), (Step 2') and (Step 3') admits*

$$\|u(t) - \tilde{u}_N(t)\| \leq C_1(t) L_m \lambda_{m+1}^{-\frac{3}{2}} + C_2 \lambda_{N+1}^{-\frac{1}{2}},$$

where $C_2 > 0$ is a constant independent of t , m and N .

It is quite clear that, to balance the two error terms in the corollary, we should take N large enough such that $N \sim \frac{m^3}{L_m^2}$. In other words, the result of the postprocessing procedure is equivalent to solve the equation (1) in a very large finite subspace V_N . This obviously leads to significantly saving of the CPU time.

Appendix

In this section, we will present a further property of η , the estimate of $|A^{-\frac{1}{2}}\eta|$.

First of all, let us decompose η as

$$\eta = P_m\eta + Q_m\eta \stackrel{\Delta}{=} p + q. \quad (37)$$

Noticing (6), it is easy to know that p satisfies

$$\begin{cases} \frac{dp}{dt} + \nu Ap + P_mB(u_m, p + q) + P_mB(p + q, u_m) + P_mB(p + q, p + q) = 0, \\ p(0) = 0. \end{cases} \quad (38)$$

On the other hand, it is well known that Stokes operator A can generate an analytic semigroup $\{e^{-\nu At}\}_{t \geq 0}$ on H with (see [6])

$$\|A^\alpha e^{-\nu At}\|_{\mathcal{L}(H, H)} \leq c_{10}(\nu t)^{-\alpha} e^{-\nu \delta t}, \quad t > 0, \alpha > 0, \quad (39)$$

where $\delta > 0$ is only dependent on A and $c_{10} > 0$ is a constant depending only on α .

By using the semi-group presentation, we can rewrite (38) as

$$\begin{aligned} p(t) &= - \int_0^t e^{-\nu A(t-s)} P_m \{B(u_m, p) + B(p, u_m) + B(p, p) + B(p, q) + B(q, p) \\ &\quad + B(u_m, q) + B(q, u_m) + B(q, q)\} ds \\ &= - \int_0^t e^{-\nu A(t-s)} P_m B_1(p) ds - \int_0^t e^{-\nu A(t-s)} P_m B_2(q) ds, \end{aligned}$$

where

$$\begin{aligned} B_1(p) &= B(u_m, p) + B(p, u_m) + B(p, p) + B(p, q) + B(q, p), \\ B_2(q) &= B(u_m, q) + B(q, u_m) + B(q, q). \end{aligned}$$

Then by using (39), we have

$$\begin{aligned} |A^{-\frac{1}{2}}p(t)| &\leq \int_0^t |A^{-\frac{1}{2}}e^{-\nu A(t-s)} P_m B_1(p)| ds + \int_0^t |A^{-\frac{1}{2}}e^{-\nu A(t-s)} P_m B_2(q)| ds \\ &= \int_0^t |A^{\frac{1}{2}}e^{-\nu A(t-s)} A^{-1} P_m B_1(p)| ds + \int_0^t |A^{\frac{1}{2}}e^{-\nu A(t-s)} A^{-1} P_m B_2(q)| ds \\ &\leq c_{10} \nu^{-\frac{1}{2}} \int_0^t (t-s)^{-\frac{1}{2}} e^{-\delta(t-s)} |A^{-1} P_m B_1(p)| ds \\ &\quad + c_{10} \nu^{-\frac{1}{2}} \int_0^t (t-s)^{-\frac{1}{2}} e^{-\delta(t-s)} |A^{-1} P_m B_2(q)| ds \end{aligned}$$

Let us estimate each term of $|A^{-1}B_1(p)|$ and $|A^{-1}B_2(q)|$. First of all, we consider each term of $|A^{-1}B_1(p)|$. For example, consider $|A^{-1}P_mB(u_m, p)|$. To do this, we need following property of trilinear form $b(\cdot, \cdot, \cdot)$ whose proof is almost the same as lemma 3.2.

Proposition For any $v \in D(A)$, $w \in D(A^{\frac{3}{4}})$ and $\phi \in D(A^{-\frac{1}{2}})$, we have

$$|b(\phi, v, w)| \leq c_3 |A^{-\frac{1}{2}}\phi| |Av| |A^{\frac{3}{4}}w|, \quad (40)$$

$$|b(w, v, \phi)| \leq c_3 |A^{\frac{3}{4}}w| |Av| |A^{-\frac{1}{2}}\phi|, \quad (41)$$

where c_3 is a positive constant which has the same meaning as in lemma 3.2.

Now let us continue our estimation. From (41), we have for any $v \in H_m$

$$|b(u_m, p, A^{-1}v)| = |b(u_m, A^{-1}v, p)| \leq c_3 |A^{\frac{3}{4}}u_m| |A^{-\frac{1}{2}}p| |v|.$$

Thus we have

$$|A^{-1}P_m B(u_m, p)| \leq c_3 |A^{\frac{3}{4}}u_m| |A^{-\frac{1}{2}}p|. \quad (42)$$

Similarly, we can derive

$$|A^{-1}P_m B(p, p)| \leq c_3 |A^{\frac{3}{4}}u| |A^{-\frac{1}{2}}p|, \quad (43)$$

$$|A^{-1}P_m B(q, p)| \leq c_3 |A^{\frac{3}{4}}u| |A^{-\frac{1}{2}}p|. \quad (44)$$

For the other two terms, we will use (40) to cope with them. For any $v \in H_m$

$$|b(p, u_m, A^{-1}v)| = |b(p, A^{-1}v, u_m)| \leq c_3 |A^{\frac{3}{4}}u_m| |A^{-\frac{1}{2}}p| |v|.$$

So we get

$$|A^{-1}P_m B(p, u_m)| \leq c_3 |A^{\frac{3}{4}}u_m| |A^{-\frac{1}{2}}p|. \quad (45)$$

By doing this to the last term in the same way, we have

$$|A^{-1}P_m B(p, q)| \leq c_3 |A^{\frac{3}{4}}u| |A^{-\frac{1}{2}}p|. \quad (46)$$

Combining (42)~(46), we derive the first estimation

$$|A^{-1}P_m B_1(p)| \leq 3c_3 (\| \|A^{\frac{3}{4}}u\| \| + \| \|A^{\frac{3}{4}}u_m\| \|) |A^{-\frac{1}{2}}p|, \quad (47)$$

where $\| \cdot \| = \sup_{s \geq 0} | \cdot (s) |$ and we will also denote $\| \cdot \|_t = \sup_{0 \leq s \leq t} | \cdot (s) |$ in the following.

For the estimation of $|A^{-1}P_m B_2(q)|$, the method is completely the same as the above one. We can use (41) to deal with $B(u_m, q) + B(q, q)$ and (40) to deal with $B(q, u_m)$. So we just give the result in the following

$$|A^{-1}P_m B_2(q)| \leq 2c_3 (\| \|A^{\frac{3}{4}}u\| \| + \| \|A^{\frac{3}{4}}u_m\| \|) |A^{-\frac{1}{2}}q|. \quad (48)$$

Obviously,

$$\sup_{t \geq 0} \int_0^t (t-s)^{-\frac{1}{2}} e^{-\delta(t-s)} ds < \delta^{-1/2} \gamma_{\frac{1}{2}} < +\infty$$

where

$$\gamma_\alpha = \int_0^\infty s^{-\alpha} e^{-s} ds = \Gamma(-1 - \alpha).$$

By introducing the following constants

$$c_{11} = 2c_{10}c_3 \delta^{-\frac{1}{2}} \gamma_{\frac{1}{2}} (\| \|A^{\frac{3}{4}}u\| \| + \| \|A^{\frac{3}{4}}u_m\| \|), \quad c_{12} = 3c_{10}c_3 (\| \|A^{\frac{3}{4}}u\| \| + \| \|A^{\frac{3}{4}}u_m\| \|),$$

we can get a new inequality about $|A^{-\frac{1}{2}}p|$. That is

$$|A^{-\frac{1}{2}}p(t)| \leq c_{12} \nu^{-\frac{1}{2}} \int_0^t (t-s)^{-\frac{1}{2}} e^{-\delta(t-s)} |A^{-\frac{1}{2}}p| ds + c_{11} \nu^{-\frac{1}{2}} \| \|A^{-\frac{1}{2}}q\| \|_t.$$

Set

$$g(s) = |A^{-\frac{1}{2}}p(s)| e^{\delta s},$$

and we have

$$g(t) \leq c_{11} \nu^{-\frac{1}{2}} e^{\delta t} \| \|A^{-\frac{1}{2}}q\| \|_t + c_{12} \nu^{-\frac{1}{2}} \int_0^t (t-s)^{-\frac{1}{2}} g(s) ds.$$

To give the estimation of g , we must introduce an inequality. Many inequalities of this type can be found in Henry[6]. The following special version, lemma A.1, was proven in [13].

Lemma A.1 *Let T, α, β and θ be positive constants, $0 < \theta < 1$. Then for any continuous function $f : [0, T] \rightarrow [0, +\infty)$ that satisfies*

$$f(t) \leq \alpha + \beta \int_0^t (t-s)^{-\theta} f(s) ds, \quad 0 \leq t \leq T,$$

we have

$$f(t) \leq c' \alpha \exp\{c' \beta^{1/(1-\theta)} t\}, \quad 0 \leq t \leq T,$$

with a positive constant c' that depends only on θ .

Now by using lemma A.1, we can immediately obtain

$$g(t) \leq c_{11} c' \nu^{-\frac{1}{2}} e^{\delta t} \exp\{c' c_{12}^2 \nu^{-1} t\} \|A^{-\frac{1}{2}} q\|_t.$$

Denoting by $T_1(t) > 0$ the constant $c_{11} c' \exp\{c' c_{12}^2 \nu^{-1} t\} + \nu^{\frac{1}{2}}$, we have

$$|A^{-\frac{1}{2}} p(t)| \leq (\nu^{-\frac{1}{2}} T_1(t) - 1) \|A^{-\frac{1}{2}} q\|_t. \quad (49)$$

Now we summarize the above deducing into the following

Theorem A.1 *For any given data $a \in D(A)$ and $f \in H$, we know the Navier-Stokes equations (1) and its Galerkin approximate equations (5) have unique solutions*

$$u(t) \in L^\infty(\mathcal{R}^+, D(A)), \quad u_m(t) \in L^\infty(\mathcal{R}^+, D(A)).$$

Then for any $t > 0$, we have

$$|A^{-\frac{1}{2}} \eta| \leq \nu^{-\frac{1}{2}} \lambda_{m+1}^{-\frac{1}{2}} T_1(t) \|\eta\|_t.$$

Proof From [7] and (49), the result can be obtained immediately. \square

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