Evolution equations

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EVOLUTION EQUATIONS

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Contents

0. One-parameter semi-groups of operators

Notations:

- X: Banach space with norm $\|\cdot\|$.
- $\mathcal{L}(X)$: Bounded operators on X.
- $A \in \mathcal{L}(X)$ $\|A\| = \sup_{x \in X} \frac{\|Ax\|}{\|x\|}$.

Definition. A semi-group of operators on a Banach space X is a mapping:

$$[0, \infty) \to \mathcal{L}(X),\ t \mapsto P_t,\ \{P_t\}_{t \geq 0}$$

such that

1) $P_0 = I$

2) $\forall_{t \geq 0} \forall_{s \geq 0} \quad P_t P_s = P_{t+s}$

Definition.

- $\{P_t\}_{t \geq 0}$ is called a *uniformly continuous semi-group* on X if $t \mapsto P_t$ is continuous as a mapping: $[0, \infty) \to \mathcal{L}(X)$.
- $\{P_t\}_{t \geq 0}$ is called a *strongly continuous semi-group* on X if $\forall_{x \in X}\, t \mapsto P_t x$ is continuous as a mapping: $[0, \infty) \to X$.

Examples.

- $t \mapsto e^{\alpha t} I$, $\alpha \in \mathbb{R}$ and fixed, is uniformly continuous.
- Take $L_2(\mathbb{R})$ and define $P_t : t \mapsto P_t u$ by $(P_t u)(x) = u(x - t)$. In this case P_t is strongly continuous but not uniformly continuous. Proof?

1. Uniformly continuous semi-groups

Theorem 1.1. Let $A \in \mathcal{L}(X)$, $t \in C$, and define

$$P_t = e^{tA} = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!} \, .$$

Then

1) $\{P_t\}_{t \geq 0}$ is a uniformly continuous semi-group.

2) $A = \lim\limits_{h \downarrow 0} \dfrac{P_h - I}{h}$ in $\mathcal{L}(X)$.

3) $R_\lambda = \int\limits_0^\infty e^{-\lambda t} P_t dt$ for $\lambda \in C$ with $\operatorname{Re} \lambda > \|A\|$.

Note: The resolvent R_λ of A appears as the Laplace transform of $\{P_t\}_{t \geq 0}$.

Remark: A is called the *infinitesimal generator* of the semi-group $\{P_t\}_{t \geq 0}$.

Proof. 1) and 2) $t \mapsto P_t$ is obviously an entire analytic function and hence continuously differentiable. From the expansion:

$$\frac{d}{dt} P_t = A P_t = P_t A \, .$$

Therefore

$$A = A P_0 = \frac{d}{dt} P_t |_{t=0} = \lim_{h \downarrow 0} \frac{P_h - I}{h} \, .$$

3) $$P_t = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!} \Rightarrow \|P_t\| \leq \sum_{n=0}^{\infty} \frac{|t|^n \|A\|^n}{n!} = e^{|t| \cdot \|A\|} \, .$$

So

$$\operatorname{Re} \lambda > \|A\| \quad \Rightarrow \quad \int\limits_0^\infty \|e^{-\lambda t} P_t\| dt < \infty \quad \Rightarrow \quad \int\limits_0^\infty e^{-\lambda t} P_t dt$$

is absolutely convergent.

Now, put $S_\lambda = \int\limits_0^\infty e^{-\lambda t} P_t dt$.

We will show that $S_\lambda = (\lambda - A)^{-1} = R_\lambda$:

$$
\begin{aligned}
(\lambda - A)S_\lambda &= (\lambda - A)\int_0^\infty e^{-\lambda t}P_t dt = \lambda \int_0^\infty e^{-\lambda t}P_t dt - \int_0^\infty e^{-\lambda t}AP_t dt \\
&= \lambda \int_0^\infty e^{-\lambda t}P_t dt - \int_0^\infty e^{-\lambda t}\frac{d}{dt}P_t dt = \\
&= \lambda \int_0^\infty e^{-\lambda t}P_t dt - e^{-\lambda t}P_t|_0^\infty + \int_0^\infty -\lambda e^{-\lambda t}P_t dt = \\
&= \lim_{s\to\infty}(-e^{-\lambda s}P_s) + I\ .
\end{aligned}
$$

Since

$$
\|e^{-\lambda s}P_s\| = e^{-\text{ Re }\lambda s}\|P_s\| \leq e^{-s(\text{ Re }\lambda - \|A\|)}
$$

and Re $\lambda > \|A\|$ it follows that $\lim_{s\to\infty} e^{-\lambda s}P_s = 0$.
So $(\lambda - A)S_\lambda = 1$. Also $AP_t = P_tA$ implies $S_\lambda(\lambda - A) = I$.
Therefore $S_\lambda = (-\lambda - A)^{-1} = R_\lambda$ whenever Re $\lambda > \|A\|$. □

Note: Cf. the highschool formula

$$
\int_0^\infty e^{-\lambda t}e^{at}dt = \frac{1}{\lambda - a} \quad \forall_{a\in\mathbb{C}},\ \text{Re }\lambda > |a|\ .
$$

Definition. $\{P_t\}_{t\geq 0}$ is a called a *contradiction semi-group* if $\forall_{t\geq 0}\, \|P_t\| \leq 1$.

Remark. For a contraction semi-group we have

$$
\text{Re }\lambda > 0 \Rightarrow \int_0^\infty \|e^{-\lambda t}P_t\|dt < \infty\ .
$$

So the resolvent R_λ exists whenever Re $\lambda > 0$.
For the resolvent set $\rho(A)$ of A it now follows $\rho(A) \supseteq \{\lambda \in C \mid \text{ Re }\lambda > 0\}$ and for the spectrum $\sigma(A)$ of A: $\sigma(A) \subseteq \{\lambda \in C \mid \text{ Re }\lambda \leq 0\}$.
Finally

$$
\|R_\lambda\| = \|\int_0^\infty e^{-\lambda t}P_t dt\| \leq \int_0^\infty e^{-t\cdot\text{ Re }\lambda}dt = \frac{1}{\text{Re }\lambda}\ .
$$

Theorem 1.2. (Reverse of theorem 1.1).
Let $\{P_t\}_{t\geq 0}$ be a uniformly continuous semi-group. Then

$$
\exists!\, A \in \mathcal{L}(X)\, \forall_{t\geq 0}\, [P_t = e^{tA}]\ .
$$

(A is called the infinitesimal generator).

Proof.

$$\forall_{a>0}\forall_{s>0} : \frac{1}{a}\int_0^a P_{s+t}dt = \frac{1}{a}\int_0^a P_s P_t dt = P_s \cdot \frac{1}{a}\int_0^a P_t dt \; .$$

P_t is continuous $\Rightarrow \lim_{a\downarrow 0} \frac{1}{a}\int_0^a P_t dt = I$.

For $a > 0$, sufficiently small, $\|I - \frac{1}{a}\int_0^a P_t dt\| < 1$.

But then $I - \left(I - \frac{1}{a}\int_0^a P_t dt\right) = \frac{1}{a}\int_0^a P_t dt$ is invertible and $\left(\frac{1}{a}\int_0^a P_t dt\right)^{-1} \in \mathcal{L}(X)$.

Further, $\frac{1}{a}\int_0^a P_{s+t}dt = \frac{1}{a}\int_s^{s+a} P_t dt$ is differentiable to s. Hence, also

$$P_s = \frac{1}{a}\int_0^a P_{s+t}dt \cdot \left(\frac{1}{a}\int_0^a P_t dt\right)^{-1}$$

is differentiable to s, whereas

$$P_s' = \lim_{\varepsilon\to 0} \frac{P_{s+\varepsilon} - P_s}{\varepsilon} = \lim_{\varepsilon\downarrow 0} \frac{P_\varepsilon - I}{\varepsilon} P_s \; .$$

If we put $\lim_{\varepsilon\downarrow 0} \frac{P_\varepsilon - I}{\varepsilon} = A$ then we have $P_s' = AP_s = P_s A$.

Let $Q_t = e^{-tA}P_t$. Then $Q_t' = e^{-tA}P_t' - e^{-tA}AP_t = 0$. This implies $Q_t =$ constant $= Q_0 = I$.

Finally, $e^{-tA}P_t = I$ or $P_t = e^{tA}$. □

2. Strongly continuous semi-groups

Lemma. Let $\{P_t\}_{t\geq 0}$ be strongly continuous on X.
Let $\varepsilon > 0$, $s > 0$, $x \in X$ and put

$$A_\varepsilon = \frac{P_\varepsilon - I}{\varepsilon} \text{ and } B_s x = \frac{1}{s}\int_0^s P_t x \, dt \, .$$

Then B_s is a bounded operator, $B_s \in \mathcal{L}(X)$, and

$$A_\varepsilon B_s x = A_s B_\varepsilon x = B_s A_\varepsilon x \, .$$

Proof. The mapping $t \mapsto P_t x$ is continuous. Hence

$$\forall_{s>0} \forall_{x\in X} \Big[\sup_{0\leq t\leq s} \|P_t x\| < \infty \Big] \, .$$

Then, with Banach-Steinhaus, $\sup\limits_{0\leq t\leq s} \|P_t\| < \infty$.
This implies the boundedness of B_s.
Concerning the algebraic part of the Lemma:

$$\begin{aligned} A_\varepsilon B_s x &= \frac{1}{\varepsilon s}\int_0^s (P_{t+\varepsilon} - P_t)x \, dt = \frac{1}{\varepsilon s}\Big\{ \int_\varepsilon^{s+\varepsilon} P_t x \, dt - \int_0^s P_t x \, dt \Big\} \\ &= \frac{1}{\varepsilon s}\Big\{ \int_0^{s+\varepsilon} P_t x \, dt - \int_0^\varepsilon P_t x \, dt \Big\} = \frac{1}{\varepsilon s}\int_0^\varepsilon (P_{s+t} - P_t)x \, dt = A_s B_\varepsilon x \, . \end{aligned}$$

$$\begin{aligned} A_\varepsilon B_s x &= \frac{1}{\varepsilon s}\int_0^s (P_{t+\varepsilon} - P_t)x \, dt = \frac{1}{s}\int_0^s P_t \frac{P_\varepsilon - I}{\varepsilon} x \, dt = \\ &= \frac{1}{s}\int_0^s P_t A_\varepsilon x \, dt = B_s A_\varepsilon x \, . \end{aligned}$$

□

Theorem 2.1. Put $D(A) = \{x \in X \mid \lim\limits_{\varepsilon\downarrow 0} A_\varepsilon x \text{ exists}\}$.
Define $A : D(A) \to X$ by $Ax = \lim\limits_{\varepsilon\downarrow 0} A_\varepsilon x$. Then

1) $\overline{D(A)} = X$.

2) A is a closed linear operator.

(A is called the *infinitesimal generator* of $\{P_t\}_{t\geq 0}$).

Proof.

1) Let $x \in X$. We want to show $x \in \overline{D(A)}$. From $B_s x = \frac{1}{s}\int_0^s P_t x\, dt$ and the strong continuity of $\{P_t\}_{t\geq 0}$ it follows that $\lim_{s\downarrow 0} B_s x = x$. So we are ready if we show that $B_s x \in D(A)$. Indeed A_s continuous $\Rightarrow \lim_{\varepsilon\downarrow 0} A_s B_\varepsilon x = A_s x$.
Lemma $\Rightarrow \lim_{\varepsilon\downarrow 0} A_\varepsilon B_s x = A_s x \Rightarrow B_s x \in D(A)$.

2) Observe $x \in D(A) \Rightarrow \lim_{\varepsilon\downarrow 0} B_s A_\varepsilon x = A_s x = B_s A x$. Consider a sequence $\{x_n\}_{n\geq 0} \subset D(A)$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} Ax_n = y$.
If we show that $x \in D(A)$ and $Ax = y$ the operator A is closed. Indeed, for all $s > 0$

$$B_s y = \lim_{n\to\infty} B_s A x_n = \lim_{n\to\infty} A_s x_n = A_s x \ .$$

But then $y = \lim_{s\downarrow 0} B_s y = \lim_{s\downarrow 0} A_s x$, which says $x \in D(A)$ and $y = Ax$.

□

Theorem 2.2. Consider a strongly continuous semi-group $\{P_t\}_{t\geq 0}$. Then $\{P_t\}_{t\geq 0}$ is uniformly continuous iff the infinitesimal generator A is bounded.

Proof. See theorem 1.2. □

Now we discuss the resolvent of infinitesimal generators.

Lemma. Let $f : [0,\infty) \to \mathbb{R}$ be sub-additive and bounded from above on compact sets, then the limiet $\lim_{t\to\infty} \frac{f(t)}{t}$ exists in $[-\infty,\infty)$.

Proof. Sub-additivity means $\forall_{s,t\geq 0} f(s+t) \leq f(s) + f(t)$. Take $t_0 > 0$ and put $\alpha = \sup_{0\leq t\leq t_0} f(t) < \infty$.

$$\forall_{t\geq 0}\, \exists_{n(t)\in \mathbb{N}}\, \exists_{r(t), 0\leq r(t)\leq t_0} \, [t = n(t)t_0 + r(t)]$$

and one has

$$f(t) = f(n(t)t_0 + r(t)) \leq n(t) f(t_0) + \alpha \ .$$

So

$$\frac{f(t)}{t} \leq \frac{n(t)}{t} f(t_0) + \frac{\alpha}{t} = \frac{f(t_0)}{t_0} + \frac{\alpha}{t} - \frac{r(t)}{t t_0} f(t_0) \ , \quad t \geq 0 \ .$$

Hence

$$\forall_{t_0>0} \quad \limsup_{t\to\infty} \frac{f(t)}{t} \leq \frac{f(t_0)}{t_0} .$$

Therefore also

$$\limsup_{t\to\infty} \frac{f(t)}{t} \leq \inf_{t>0} \frac{f(t)}{t} .$$

It follows that

$$\lim_{t\to\infty} \frac{f(t)}{t} = \inf_{t>0} \frac{f(t)}{t} < \infty .$$

□

Corollary. Let $\{P_t\}_{t\geq 0}$ be a strongly continuous semi-group. Then the function $f(t) = \log \|P_t\|$ is sub-additive and bounded on compact sets. (This follows from $\forall_{x\in X} \forall_{T>0} [\|P_t x\|$ is bdd on on $[0,T]]$ and Banach-Steinhaus). So

$$\lim_{t\to\infty} \frac{\log \|P_t\|}{t} = \omega < \infty .$$

Theorem 2.3. Consider $\{P_t\}_{t\geq 0}$ with infinitesimal generator A. Put

$$\omega = \lim_{t\to\infty} \frac{\log \|P_t\|}{t} .$$

Then for $\lambda \in \mathbb{C}$, with $\operatorname{Re} \lambda > \omega$, one has $\lambda \in \rho(A)$ and

$$(\lambda I - A)^{-1} x = R_\lambda x = \int_0^\infty e^{-\lambda t} P_t x \, dt .$$

Proof. Choose $a \in \mathbb{R}$ with $\omega < a < \operatorname{Re} \lambda$. Then

$$\exists_{t_0>0} \forall_{t>t_0} \left[\frac{\log \|P_t\|}{t} \leq a \text{ or } \|P_t\| \leq e^{ta}\right] .$$

Because, via Banach-Steinhaus, $\sup_{0\leq t\leq t_0} \|P_t\| = \alpha < \infty$, we find

$$\exists_{M_a>0} \forall_{t\geq 0} \|P_t\| \leq M_a e^{ta} .$$

Estimate

$$\int_0^\infty \|e^{-\lambda t}P_t x\|dt \le \int_0^\infty e^{-\text{ Re } \lambda t} M_a e^{ta}\|x\|dt = \frac{M_a}{\text{Re }\lambda - a}\|x\| \ .$$

This shows that the integral

$$S_\lambda x = \int_0^\infty e^{-\lambda t} P_t x \ dt$$

converges absolutely and defines a bounded operator S_λ.
We now want to show

$$\lim_{\varepsilon \downarrow 0} (\lambda I - A_\varepsilon) S_\lambda x = \lim_{\varepsilon \downarrow 0} S_\lambda (\lambda I - A_\varepsilon) x = x \ , \quad \text{for all } x \in X \ .$$

Indeed,

$$(\lambda I - A_\varepsilon) S_\lambda x = \lambda S_\lambda x - \int_0^\infty \frac{P_{t+\varepsilon} - P_t}{\varepsilon} e^{-\lambda t} x \ dt$$

$$= \lambda S_\lambda x - \int_\varepsilon^\infty e^{-\lambda t} \frac{e^{\lambda \varepsilon} - 1}{\varepsilon} P_t x \ dt + \frac{1}{\varepsilon} \int_0^\varepsilon e^{-\lambda t} P_t x \ dt$$

$$\to \lambda S_\lambda x - \lambda S_\lambda x + x = x \ , \quad \text{as } \varepsilon \downarrow 0 \ .$$

We conclude $S_\lambda x \in D(A)$ and $(\lambda I - A) S_\lambda x = x$, for all $x \in X$. It also follows that $(\lambda I - A)$ is surjective. Because of $A_\varepsilon P_t = P_t A_\varepsilon$ for all $\varepsilon, t > 0$ we also have

$$\lim_{\varepsilon \downarrow 0} S_\lambda (\lambda I - A_\varepsilon) x = x = S_\lambda (\lambda I - A) x \ ,$$

for all $x \in D(A)$, which implies that $(\lambda I - A)$ is injective.
Conclusion: For all $\lambda \in C$, Re $\lambda > \omega$, $\lambda I - A : D(A) \to X$ is bijective.
This means $\lambda \in \rho(A)$ and $(\lambda I - A)^{-1} x = S_\lambda x = R_\lambda x$. □

Corollary.

$$\|R_\lambda^n\| \le \frac{M_a}{(\text{ Re } \lambda - a)^n} \ , \quad \lambda \in C, \ \text{ Re } \lambda > a > \omega \ .$$

Proof.

$$R_\lambda^n x = \int_0^\infty e^{-\lambda t_n} P_{t_n} dt_n \int_0^\infty e^{-\lambda t_{n-1}} P_{t_{n-1}} dt_{n-1} \cdot \ldots \cdot \int_0^\infty e^{-\lambda t_1} P_{t_1} x \ dt_1$$

$$= \int_0^\infty e^{-\lambda t_n} dt_n \int_0^\infty e^{-\lambda t_{n-1}} dt_{n-1} \cdot \ldots \cdot \int_0^\infty e^{-\lambda t_1} P_{t_n} P_{t_{n-1}} \cdot \ldots \cdot P_{t_1} x \, dt_1 \, .$$

$$\|R_\lambda^n x\| \le \int_0^\infty |e^{-\lambda t_n}| dt_n \int_0^\infty |e^{-\lambda t_{n-1}}| dt_{n-1} \cdot \ldots \cdot \int_0^\infty |e^{-\lambda t_1}| e^{a(t_n + \ldots + t_1)} dt_1$$

$$\le M_a \prod_{j=1}^n \int_0^\infty |e^{-\lambda t_j + a t_j}| dt_j = M_a \frac{1}{(\text{ Re } \lambda - a)^n} \, . \qquad \square$$

Remarks.

1) If $\{P_t\}_{t\ge 0}$ is a strongly continuous contraction semi-group then, in Theorem 2.3, we can take $\omega \le 0$, since $\|P_t\| \le 1$ implies $\frac{1}{t} \log \|P_t\| \le 0$.

2) We have $\|P_t\| \le M_a e^{at}$ with $a > \omega$. Observe that $\{Q_t\}_{t\ge 0}$ with $Q_t = e^{-at} P_t$ is again a strongly continuous semi-group and

$$\frac{d}{dt} Q_t x = -a e^{-at} P_t x + e^{-at} \frac{d}{dt} P_t x \, .$$

At $t = 0$ this leads to $B = A - a$ if A and B denote the infinitesimal generators of $\{P_t\}_{t\ge 0}$ and $\{Q_t\}_{t\ge 0}$, respectively.
From $\rho(A) \supseteq \{\lambda \in \mathbb{C} \mid \text{ Re } \lambda > \omega\}$ it follows that

$$\rho(B) \supseteq \{\lambda \in \mathbb{C} \mid \text{ Re } \lambda > 0 > \omega - a\} \, .$$

So, without losing generality we could restrict to uniformly bounded semi-groups, the spectra of whose infinitesimal generators all lie in the closed left half plane.

Theorem 2.4. If two semi-groups $\{P_t\}_{t\ge 0}$ and $\{Q_t\}_{t\ge 0}$ have the same infinitesimal generator, they are the same.

Proof. For suitable $a, b \in \mathbb{R}$ we have

$$\|P_t\| \le M_a e^{at} \quad \text{and} \quad \|Q_t\| \le M_b e^{bt} \, .$$

Let A and B be the infinitesimal generators of $\{P_t\}$ and $\{Q_t\}$ respectively. Then for Re $\lambda > \max\{a, b\}$

$$\int_0^\infty e^{-\lambda t} P_t x \, dt = R_\lambda(A) x = R_\lambda(B) x = \int_0^\infty e^{-\lambda t} Q_t x \, dt \, .$$

It is sufficient to show that $P_t x = Q_t x$, for all $x \in X$.
For Re $\lambda > \max\{a, b\} = m$ we have

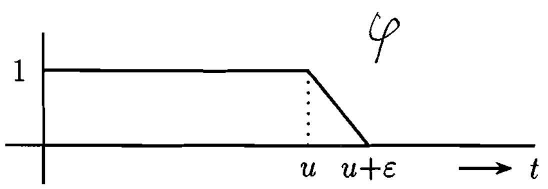
$$\int_0^\infty e^{-(\lambda-m)t}e^{-mt}(P_tx - Q_tx)dt = 0 \ .$$

Since span$\{t \mapsto e^{-(\lambda-m)t}\}$ is dense in $C_0([0,\infty))$ we find

$$\int_0^\infty \varphi(t)e^{-mt}(P_tx - Q_tx)dt = 0 \quad \text{for all } \varphi \in C_0([0,\infty)) \ .$$

Take φ:
and let $\varepsilon \downarrow 0$.
Then

$$\forall_{u\geq 0} \quad \int_0^u e^{-mt}(P_tx - Q_tx)dt = 0 \ .$$



Differentiate to u,

$$e^{-mu}(P_ux - Q_ux) = 0 \ .$$

Hence $P_ux = Q_ux$, $\forall_{u\geq 0}\forall_{x\in X}$.

Theorem 2.5. (Hille-Yosida). Consider a closed operator A in the Banach space X; $\overline{D(A)} = X$. Then A is the infinitesimal generator of a strongly continuous semi-group $\Leftrightarrow \exists_{M\geq 0}\exists_{a\in \mathbb{R}}\forall_{\lambda<a}\forall_{n\in \mathbb{N}}$

$$\left[\lambda \in \rho(A) \text{ and } \|R_\lambda^n\| \leq \frac{M}{(\lambda - a)^n}\right] .$$

Proof. $\Rightarrow$ Theorem 2.3.
$\Leftarrow$ For all $\lambda > a$ we have $R_\lambda = (\lambda - A)^{-1} \in \mathcal{L}(X)$. Form $B_\lambda = -\lambda(I - \lambda R_\lambda) \in \mathcal{L}(X)$ for $\lambda > a$. We will construct the operators P_t of the desired semi-group as the strong limit for $\lambda \to \infty$ from the operator e^{tB_λ}.

1) Let

$$t \geq 0,\ \lambda > a;\ e^{tB_\lambda} = e^{-\lambda t + \lambda^2 tB_\lambda} = e^{-\lambda t}\sum_{n=0}^\infty \frac{(\lambda^2 t)^n R_\lambda^n}{n!} \ .$$

This implies

$$\|e^{tB_\lambda}\| \le Me^{-\lambda t}\sum_{n=0}^{\infty}\frac{(\lambda^2 t)^n}{n!(\lambda-a)^n} = Me^{-\lambda t}\cdot e^{\frac{\lambda^2}{\lambda-a}} = Me^{\frac{\lambda ta}{\lambda-a}} .$$

Pick $a_1 > a$. Since $\lim\limits_{\lambda\to\infty}\dfrac{\lambda ta}{\lambda-a} = ta$, it follows that

$$(*)\qquad \exists_{\lambda_0\in\mathbb{R}}\forall_{\lambda\ge\lambda_0>a}[\|e^{tB_\lambda}\| < Me^{a_1 t}] .$$

2) We show that $\forall_{x\in D(A)}[\lim\limits_{\lambda\to\infty} B_\lambda x = Ax]$. Let $x \in D(A)$ and $\lambda > a$

- $\|\lambda R_\lambda x - x\| = \|\lambda R_\lambda x - R_\lambda(\lambda - A)x\| = \|R_\lambda Ax\| \le \dfrac{M\|Ax\|}{\lambda - a} \to 0$, as $\lambda\to\infty$.
- $\exists_{\lambda_1\in\mathbb{R}}\forall_{\lambda>\lambda_1}\|\lambda R_\lambda\| \le \dfrac{M|\lambda|}{\lambda-a} < 2M.$
- Let $\varepsilon > 0$ and $y \in X$; $\exists_{x_0\in D(A)}\left[\|x_0 - y\| < \min\left(\frac{\varepsilon}{6M},\frac{\varepsilon}{3}\right)\right]$.

We derive, for λ sufficiently large,

$$\begin{aligned}\|\lambda R_\lambda y - y\| &\le \|\lambda R_\lambda y - \lambda R_\lambda x_0\| + \|\lambda R_\lambda x_0 - x_0\| + \|x_0 - y\| \\ &\le 2M\|y - x_0\| + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon .\end{aligned}$$

So

$$\forall_{y\in X}\left[\lim_{\lambda\to\infty}\lambda R_\lambda y = y\right]$$

and from this, $\forall_{x\in D(A)}$

$$\begin{aligned}B_\lambda x &= -\lambda(I - \lambda R_\lambda)x = -\lambda(R_\lambda(\lambda - A) - \lambda R_\lambda)x = \\ &= \lambda R_\lambda Ax \to Ax,\ \text{as } \lambda\to\infty .\end{aligned}$$

3) Put $S_\lambda(t) = e^{tB_\lambda}$ for $\lambda > a$ and $t \in \mathbb{R}^+$. Then $\{S_\lambda(t)\}_{t\ge 0}$ is a uniformly continuous semi-group. We will show that - for $x \in D(A)$ - $S_\lambda(t)x$ converges to a limit $P_t x$ as $\lambda\to\infty$, uniformly on bounded intervals in $[0,\infty)$.
Let $\lambda,\mu > a$. Since $R_\lambda R_\mu = R_\mu R_\lambda$, it follows that $B_\lambda B_\mu = B_\mu B_\lambda$ and $S_\lambda(t)B_\mu = B_\mu S_\lambda(t)$. Let $x \in D(A)$, on $[0,t]$ the function $S_\mu(t-s)S_\lambda(s)x = e^{(t-s)B_\mu + sB_\lambda}$ is continuously differentiable to S, the derivative is

$$e^{(t-s)B_\mu+sB_\lambda}(B_\lambda - B_\mu)x .$$

We estimate

$$\|S_\lambda(t)x - S_\mu(t)x\| = \|\int_0^t e^{(t-s)B_\mu + sB_\lambda}(B_\lambda - B_\mu)x\, ds\|$$

$$\leq \int_0^t \|e^{(t-s)B_\mu}\|\,\|e^{sB_\lambda}\|\,\|B_\lambda x - B_\mu x\| ds \overset{(*)}{\leq}$$

$$\leq Me^{a_1(t-s)}Me^{a_1 s} \cdot t \cdot \|B_\lambda x - B_\mu x\| =$$

$$= M^2 t e^{a_1 t}\|B_\lambda x - B_\mu x\|, \text{ for all } \lambda, \mu \geq \lambda_0 .$$

Because of thos the limit $P_t x = \lim_{\lambda \to \infty} S_\lambda(t)x$, $x \in D(A)$, exists and the convergence is uniformly on bounded intervals.

4) Let $x \in D(A)$ and $\varepsilon > 0$. Then

$$\exists_{\lambda_2 > a} \forall_{\lambda \geq \lambda_2} \|P_t x\| \leq \|S_\lambda(t)x\| + \varepsilon .$$

Therefore

$$\|P_t x\| \leq \|S_\lambda(t)\|\,\|x\| + \varepsilon \overset{(*)}{<} Me^{ta_1}\|x\| + \varepsilon .$$

So

$$\forall_{x \in D(A)} \|P_t x\| \leq Me^{ta_1}\|x\| .$$

Together with $\overline{D(A)} = X$ this implies that P_t extends to a bounded linear operator on X. The extension is again denoted by P_t.
Note that $\|P_t\| \leq Me^{ta_1}$.

5) Let $x \in X$ and $y \in D(A)$. Estimate

$$\|P_t x - S_\lambda(t)x\| \leq \|P_t x - P_t y\| + \|P_t y - S_\lambda(t)y\| + \|S_\lambda(t)y - S_\lambda(t)x\|$$

$$\leq \|P_t\|\,\|x - y\| + \|P_t y - S_\lambda(t)y\| + \|S_\lambda(t)\|\,\|y - x\| \overset{(*)}{\leq}$$

$$\leq Me^{ta_1}\|x - y\| + \|P_t y - S_\lambda(t)y\| + Me^{ta_1}\|x - y\| .$$

Then, with 3), for all $x \in X$ $P_t x = \lim_{\lambda \to \infty} S_\lambda(t)x$, uniformly in t on bounded intervals. From this the continuity of $t \mapsto P_t x$ is immediate. In order to get the semi-group property for P_t we estimate

$$\|P_{s+t}x - P_s P_t x\| \leq \|P_{s+t}x - S_\lambda(s+t)x\| + \|S_\lambda(s)\|\,\|S_\lambda(t)x - P_t x\| +$$

$$+\|(S_\lambda(s) - P_s)P_t x\| \overset{(*)}{\leq} \|P_{s+t}x - S_\lambda(s+t)x\| +$$

$$Me^{a_1 s}\|S_\lambda(t)x - P_t x\| + \|(S_\lambda(s) - P_s)P_t x\| .$$

Let $\lambda \to \infty$ and it follows that $P_{s+t} = P_s P_t$. We know that $S_\lambda(0) = I = P_0$. Thus we have shown that $\|P_t\|_{t\geq 0}$ is a strongly continuous semi-group.

6) Finally, we want to show that A is the infinitesimal generator of $\|P_t\}_{t\geq 0}$. Analoguous to 3), $\forall_{x\in D(A)}$

$$(**) \qquad S_\lambda(t)x - x = \int_0^t S_\lambda(s) B_\lambda x \; ds \; .$$

Because of

$$\|S_\lambda(s)B_\lambda x - P_s Ax\| \leq \|S_\lambda(s)(B_\lambda - A)x\| +$$

$$+\|(S_\lambda(s) - P_s)Ax\| \overset{(*)}{\leq} Me^{sa_1}\|B_\lambda x - Ax\| + \|(S_\lambda(s) - P_s)Ax\|$$

and $P_s x = \lim\limits_{\lambda\to\infty} S_\lambda(s)x$m for all $x \in X$, uniformly in s on $[0,t]$, it follows that for all $x \in D(A)$ $P_s Ax = \lim\limits_{\lambda\to\infty} S_\lambda(s)B_\lambda x$, uniformly in s on $[0,t]$. That is why we may interchange integral and limit in $(**)$, so that

$$P_t x - x = \int_0^t P_s Ax \; ds \; .$$

Hence

$$\forall_{x\in D(A)} \lim_{t\downarrow} \frac{P_t x - x}{t} = \lim_{t\downarrow 0} \frac{1}{t} \int_0^t P_s Ax \; ds = Ax \; .$$

This means that the infinitesimal generator of $\{P_t\}_{t\geq 0}$, call it B, is an extension of A:

$$D(B) \supseteq D(A) \text{ and } \forall_{x\in D(A)} [Bx = Ax] \; .$$

However for $\lambda \in I\!R$ sufficiently large

$$\lambda \in \rho(A) \cap \rho(B) \text{ and } X = (\lambda - A)D(A) = (\lambda - B)D(A) \; ,$$

and also $X = (\lambda - B)D(B)$.
Since $R_\lambda(A)$ and $R_\lambda(B)$ exist both we finally have $D(A) = D(B)$ and therefore $A = B$. □

Corollary. (Hille-Yosidas for contraction semi-groups). Let A be a closed operator in a Banach space X. $\overline{D(A)} = X$. Then A is infinitesimal generator of a strongly continuous contraction semi-group $\Leftrightarrow$

$$\forall_{\lambda>0}[\lambda \in \rho(A) \text{ and } \|R_\lambda\| \leq \frac{1}{\lambda}] \,.$$

Proof.
$\Rightarrow$) $[\forall_{t\geq 0}\|P_t\| \leq 1] \Rightarrow \omega = \lim_{t\to\infty} \dfrac{\log\|P_t\|}{t} \leq 0.$

From Theorem 2.3, in this case, $\lambda > 0 \Rightarrow \lambda \in \rho(A)$, and $\forall_{x\in X}[R_\lambda x = \int_0^\infty e^{-\lambda t} P_t x \, dt]$.

So

$$\|R_\lambda x\| \leq \int_0^\infty e^{-\lambda t}\|x\| dt = \frac{1}{\lambda}\|x\| \text{ and } \|R_\lambda\| \leq \frac{1}{\lambda} \,.$$

$\Leftarrow$) $\|R_\lambda^n\| \leq \|R_\lambda\|^n \leq \dfrac{1}{\lambda^n}$.
Apply Theorem 2.5 with $M = 1$, $a = 0$.
Further, $\|e^{tB_\lambda}\| \leq M e^{\frac{\lambda t a}{\lambda - a}}$, so that $\|e^{tB_\lambda}\| \leq 1$. Let $\lambda \to \infty$, then still $\|P_t\| \leq 1$. □

Theorem 2.6. (Perturbations). Let A be the infinitesimal generator of a strongly continuous semi-group and $B \in \mathcal{L}(X)$ then $A + B$ is again the infinitesimal generator of a strongly continuous semi-group.

Proof. See [BM].

Theorem 2.7. (Hille's Inversion Formula).

$$P_t x = \lim_{n\to\infty} (I - \frac{t}{n} A)^{-n} x$$

uniformly on bounded sets in $[0, \infty)$.

Theorem 2.8. (A regularity result). Let $\{P_t\}_{t\geq 0}$ be a strongly continuous semi-group in a Banach space X. Let A be its infinitesimal generator. Let $n \in \mathbb{N}$ and $x \in D(A^n)$. Then

i) $\forall_{t\geq 0} P_t x \in D(A^n)$.

ii) $t \mapsto P_t x$ is n times continuously differentiable.

iii) $\dfrac{d^n}{dt^n}(P_t x) = A^n P_t x = P_t A^n x$.

Proof. Note that $D(A^n) = R_\lambda^n(X)$ for λ sufficiently large. Take $n = 1$, $x \in D(A)$ and put $x = R_\lambda y = (\lambda - A)^{-1} y$. We have

$$u(t) = P_t x = P_t R_\lambda y = R_\lambda P_t y = \int_0^\infty e^{-\lambda s} P_{t+s} y \, ds =$$

$$= e^{\lambda t} \int_t^\infty e^{-\lambda\tau} P_\tau y \, d\tau \in D(A) \ ,$$

which is obviously differentiable. Calculate

$$\frac{du}{dt} = \lambda P_t x - P_t y = \lambda P_t x - (\lambda - A) R_\lambda P_t y$$

$$= \lambda P_t x = (\lambda - A) P_t R_\lambda y = A P_t x = A u(t) \ .$$

If $n > 1$ this procedure can be repeated, replacing x by $A^k x$ with $1 \leq k \leq n-1$, successively. □

Corollary. Obviously $u(t) = P_t x$ solves the Cauchy-problem

$$\begin{cases} \dfrac{du(t)}{dt} = Au(t) \\ u(0) = x \in D(A) \ . \end{cases}$$

A sort of motivation for writing $P_t = e^{tA}$ is the following:

Theorem 2.9. Let $\{P_t\}_{t\geq 0}$ be a strongly continuous semi-group in a Banach space X. Let A be its infinitesimal generator. Then $\forall_{t\geq 0} \forall_{x\in X}$

$$P_t x = \lim_{n\to\infty} (I - \frac{t}{n} A)^{-n} x \ .$$

Proof. (Sketch). By induction it easily follows that

$$(\lambda - A)^{-n} x = \frac{1}{(n-1)!} \int_0^\infty s^{n-1} e^{-\lambda s} P_s x \, ds \ , \quad n \in \mathbb{N} \ .$$

$$(I - \frac{t}{n} A)^{-n} x = \frac{n^n}{t^n} \Big(\frac{n}{t} - A\Big)^{-n} x = \frac{n^n}{(n-1)! t^n} \int_0^\infty s^{n-1} e^{-\frac{n}{t} s} P_s x \, ds \ .$$

Notice that $s^{n-1} e^{-\frac{n}{t}s}$ has its maximum at $s = (1 - \frac{1}{n})t$ and also that

$$\forall_{n\in \mathbb{N}} \quad \frac{1}{(n-1)!} \int_0^\infty \frac{n^n}{t^n} s^{n-1} e^{-\frac{n}{t}s} ds = 1 \ .$$

Now, show that the norm of

$$P_t x - (I - \tfrac{t}{n} A)^{-n} x = \int_0^\infty \frac{n^n}{(n-1)! t^n} s^{n-1} e^{-\frac{n}{t}s} (P_s x - P_t x) ds$$

tends to zero as $n \to \infty$.

3. Homomorphic semi-groups

In this chapter we suppose that $\{P_t\}_{t\geq 0}$ is a strongly continuous semi-group of bounded operators on a Banach space X. Its infinitesimal generator is denoted by A.

Lemma. Suppose

$$\forall_{t>0}\forall_{x\in X}\,[P_t x \in D(A)]\ .$$

Then

i) $\quad \forall_{t>0}\forall_{x\in X}\,[P_t x \in D(A^\infty)]$

ii) $\quad \forall_{x\in X}\,\forall_{t>0}\,[P_t x \text{ is } \infty - \text{differentiable at } t]$

ii) $\quad \forall_{n\in \mathbb{N}}\,\forall_{t>0}\forall_{x\in X}\,[P_t^{(n)}x = A^n P_t x = (P'_{\frac{t}{n}})^n x = (AP_{\frac{t}{n}})^n x]\ .$

Proof. By induction we show

- $\forall_{t>0}\forall_{n\in \mathbb{N}}\forall_{x\in X}\,[P_t x \in D(A^\infty)]$.
- $t \mapsto P_t x$ is n-times continuously differentiable at $t > 0$.
- $P_t^{(n)}x = A^n P_t x$.

$\boxed{\text{n=1}}$ Let $0 < t_0 < t$, $P_t x = P_{t-t_0}P_{t_0}x$. Apply Theorem 2.8.

$$P_{t_0}x \in D(A) \Rightarrow \begin{cases} P_{t-t_0}P_{t_0}x \in D(A) \\ P_{t-t_0}P_{t_0}x \text{ is continuously differentiable at } t \\ P'_t x = P'_{t-t_0}P_{t_0}x = P_{t-t_0}AP_{t_0}x = AP_t x \end{cases}$$

$\boxed{\text{n=k}} \Rightarrow \boxed{\text{n=k+1}}$ Again by Theorem 2.8,

$$P_t^{(k)}x = A^k P_t x = P_{t-t_0}P_{\frac{t_0}{2}}A^k P_{\frac{t_0}{2}}x \in D(A) \text{ and continuously differentiable}$$

$$\frac{d}{dt}P_t^{(k)}x = P'_{t-t_0}P_{\frac{t_0}{2}}A^k P_{\frac{t_0}{2}}x = AP_{t-t_0}P_{\frac{t_0}{2}}A^k P_{\frac{t_0}{2}}x = A^{k+1}P_t x\ .$$

Finally, since P_t and A commute

$$P_t^{(n)}x = A^n P_t x = (AP_{\frac{t}{n}})^n x = (P'_{\frac{t}{n}})^n x\ .$$

Theorem 3.1. (Yosida) Assume $\exists_{M,1\le M<\infty} \forall_{t>0} \|P_t\| \le M$.
Then the following 3 conditions are equivalent

I. $\forall_{x\in X} \forall_{t>0} [P_t x \in D(A)]$, and

$$\exists_{\alpha>0} \forall_{t,0<t\le 1} [\|tP_t'\| = \|tAP_t\| \le \alpha] .$$

II. a) $\{P_t\}_{t>0}$ has a holomorphic extension, locally given by

$$P_\lambda x = \sum_{n=0}^{\infty} \frac{(\lambda - t)^n}{n!} P_t^{(n)} x, \quad x \in X, \, |\arg\lambda| < \arctan\frac{1}{\alpha e} .$$

b) $\exists_{\delta,0<\delta<1} \exists_{K>0} \forall_{\lambda,|\arg\lambda|<\arctan(\delta\frac{1}{\alpha e})} [\|e^{-\lambda} P_\lambda\| \le K]$

III. $\exists_{\beta>0} \exists_{\varepsilon>0} \forall_{\lambda,\ \mathrm{Re}\ \lambda\ge 1+\varepsilon} \|\lambda(\lambda I - A)^{-1}\| \le \beta$.

Proof. $\boxed{I \Rightarrow II}$

a) For $\lambda > 0$, $t > 0$ we write Taylor's formula

$$P_\lambda x = \sum_{h=0}^{N-1} \frac{(\lambda - t)^n}{n!} P_t^{(n)} x + R_N(\lambda - t) ,$$

with

$$R_N(\lambda - t) = \frac{1}{(N-1)!} \int_0^{\lambda - t} \tau^{N-1} P_{\lambda-\tau}^{(N)} x \, d\tau = \text{(Lemma)}$$

$$\frac{1}{(N-1)!} \int_0^{\lambda-t} \tau^{N-1} (P'_{\frac{\lambda-\tau}{N}})^N x \, d\tau = \frac{N^N}{(N-1)!} \int_0^{\lambda-t} \frac{\tau^{N-1}}{(\lambda-\tau)^N} \left\{ (\frac{\lambda-\tau}{N})(P'_{\frac{\lambda-\tau}{N}}) \right\}^N x \, d\tau .$$

<u>Case</u> $\lambda - t \ge 0$. Choose N so large that $\frac{1}{N}(1 + \frac{1}{\alpha e}) < 1$, then $\lambda < (1 + \frac{1}{\alpha e})t \Rightarrow \frac{\lambda - \tau}{N} < 1$, and we estimate

$$\|R_N(\lambda - \tau)\| \le \frac{N^N}{(N-1)!} \int_0^{\lambda-t} \frac{\tau^{N-1}}{(\lambda-\tau)^N} \alpha^N d\tau \le \frac{N^N}{(N-1)!} \alpha^N \int_0^{\lambda-t} \frac{\tau^{N-1}}{t^N} dt \le \frac{e^N \alpha^N}{t^N} (\lambda - t)^N .$$

Therefore the power series converges to P_λ if $0 \le \frac{\alpha e}{t}(\lambda - t) < 1$. <u>Case</u> $\lambda - t \le 0$.
Choose N so large that $\frac{t}{N} < 1$.
Then $\frac{\lambda - \tau}{N} < \frac{t}{N} < 1$ and

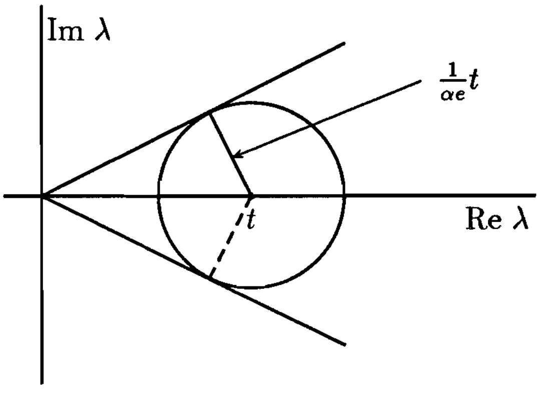
$$\|R_N(\lambda - t)\| \le \frac{N^N}{(N-1)!}\alpha^N \int_{\lambda-t}^{0} \frac{|\tau|^{N-1}}{(\lambda-\tau)^N}\,d\tau \le \frac{e^N\alpha^N}{\lambda^N}(t-\lambda)^N \,.$$

Therefore the power series converges to P_λ if $0 \le \frac{\alpha e}{\lambda}(t-\lambda) < 1$.

Conclusion: On every compact interval in $\left(\frac{t}{1+\frac{1}{\alpha e}}, \left(1+\frac{1}{\alpha e}\right)t\right)$ the power series for P_λ converges uniformly to P_λ.

The radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(\lambda-t)^n}{n!} P_t^{(n)}x \quad \text{at } t>0$$



is at least $\frac{1}{\alpha e}t$ and in the sector $|\arg\lambda| < \arcsin\frac{1}{\alpha e}$, the function P_t can be continued analytically to P_λ. If it so happens that $\frac{1}{\alpha e} > 1$ this implies that P_λ is analytic at $\lambda = 0$ and hence $D(A) = X$.

Note that analytic extension is guaranteed in the sector $|\arg\lambda| < \arctan\frac{1}{\alpha e}$.

b) $S_t = e^{-t}P_t$ is a semi-group with infinitsimal generator $A - I$ and has the property

$$\forall_{x\in X}\,\forall_{t>0}\,[S_t x \in D(A-I) = D(A)] \,.$$

We have

$$0 < t \le 1 : \|tS_t'\| = \|te^{-t}P_t' - te^{-t}P_t\| \le \alpha + M \le M(1+\alpha) \,.$$

$$t > 1 : \|tS_t'\| = \|te^{-t}AP_1P_{t-1} - te^{-t}P_t\| \le M\alpha + M \,.$$

Therefore

$$\forall_{t>0}\,\|(tS_t')^n\| \le M^n(1+\alpha)^n = \frac{1}{\delta_1^n}, \quad \text{with } \delta_1 = \frac{1}{M(1+\alpha)} < 1 \,.$$

According to a) we have the representation

$$e^{-\lambda}P_\lambda x = S_\lambda x = \sum_{n=0}^{\infty} \frac{(\lambda - \operatorname{Re}\lambda)^n}{n!} S^{(n)}_{\operatorname{Re}\lambda}x, \; |\arg\lambda| < \arctan\left(\frac{\delta_1}{e}\right) .$$

Restrict to $|\arg\lambda| \le \arctan(\delta_2\delta_1 e^{-1})$, with $0 < \delta_2 < 1$. Then

$$\|S_\lambda x\| \le \sum_{n=0}^{\infty} \frac{|\lambda - \text{ Re } \lambda|^n}{|\text{ Re } \lambda|^n} \frac{n^n}{n!} \frac{1}{(\delta_1)^n} \|\Big(\frac{\text{Re } \lambda}{n} \delta_1 S'_{\frac{\text{Re } \lambda}{n}}\Big)^n x\|$$

$$\le \sum_{n=0}^{\infty} (\delta_2 \delta_1 e^{-1})^n \frac{n^n}{n!} \frac{1}{(\delta_1)^n} \|x\| \le \frac{1}{1-\delta_2} \|x\| \ .$$

Take $\delta = \delta_1 \delta_2$. This proves b).

Corollary. (Hille).

$$\limsup_{t \downarrow 0} \|t P_t'\| < e^{-1} \Rightarrow D(A) = X \ .$$

Proof. According to d'Alembert the series

$$\sum_{n=0}^{\infty} \frac{(\lambda - t)^n}{n!} P_t^{(n)} x = \sum_{n=0}^{\infty} \frac{(\lambda - t)^n}{t^n} \frac{n^n}{t^n} \Big(\frac{t}{n} P'_{\frac{t}{n}}\Big)^n x$$

converges in the sector $\{\lambda \mid \frac{|\lambda - t|}{t} < 1 + \delta, \delta > 0\}$. This sector contains $\lambda = 0$ and $P_\lambda x$ analytic at $\lambda = 0 \Rightarrow D(A) = X$.

II→ III

For $\lambda \in \mathbb{R}$ the assertion follows from Theorem 2.3 and its Corollary. Indeed we have here $\omega = 0$, take $a = 1$ and $\varepsilon > 0$, then

$$\|R_\lambda\| \le \frac{M_1}{(\lambda - 1)} \quad \text{and} \quad \|\lambda R_\lambda\| \le \frac{M_1}{\left(\frac{\lambda-1}{\lambda}\right)} \le \frac{M_1}{\frac{\varepsilon}{1-\varepsilon}} ,$$

since $\lambda \ge 1 + \varepsilon \Rightarrow 1 - \frac{1}{\lambda} \ge \frac{\varepsilon}{1+\varepsilon}$. Now for λ complex

$$(\lambda R_\lambda) x = \lambda \int_0^\infty e^{-\lambda} P_t x \ dt \ , \qquad \text{Re } \lambda > 0, \ x \in X \ .$$

With $\lambda = 1 + \sigma + i\tau$, $\sigma \ge \varepsilon > 0$, $\tau \in \mathbb{R}$ and $S_t = e^{-t} P_t$ this becomes

$$(\sigma + 1 + i\tau) R_{\sigma+1+i\tau} x = (\sigma + 1 + i\tau) \int_0^\infty e^{-(\sigma + i\tau)t} S_t x \ dt \ .$$

Let $\tau > 0$. Deform the path of integration to a radius $re^{i\theta}$ in the sector $0 < \arg \lambda < \arctan(\frac{\delta}{\alpha e})$

$$((\sigma+1+i\tau)R_{\sigma+1+i\tau})x = (\sigma+1+i\tau)\int_0^\infty e^{-(\sigma+i\tau)re^{i\theta}} S_{re^{i\theta}}x\; e^{i\theta}dr$$

with estimate

$$\|((\sigma+1+i\tau)R_{\sigma+1+i\tau})x\| \le \|x\|\; |\sigma+1+i\tau|\cdot K\int_0^\infty e^{(-\sigma\cos\theta+\tau\sin\theta)r}dr$$

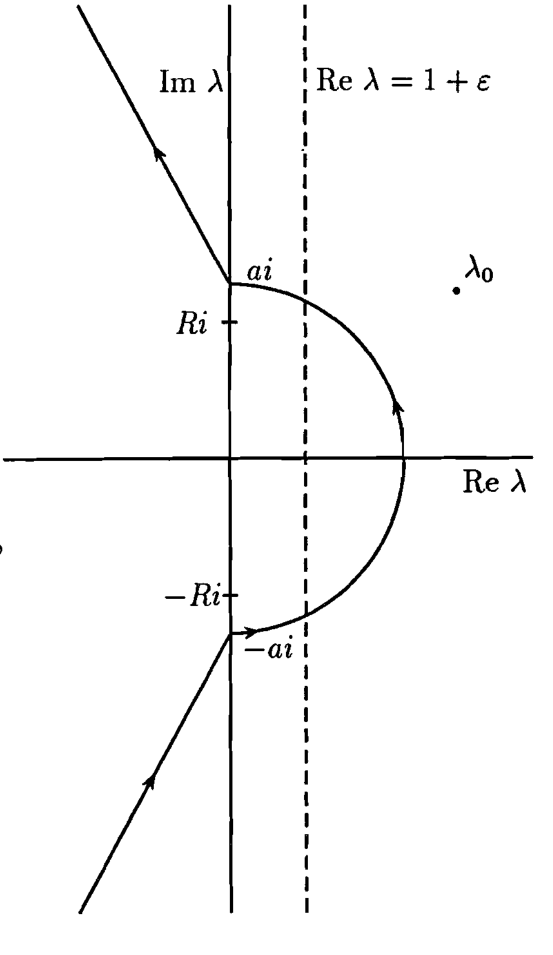
$$\le K\frac{|\sigma+1+i\tau|\;\|x\|}{|-\tau\sin\theta+\sigma\cos\theta|} \le K\left(\frac{1+\frac{1}{\varepsilon}}{\cos\theta}+\frac{1}{\sin\theta}\right)\|x\|\,,$$

since

$$\frac{|\sigma+1+i\tau|}{(-\tau\sin\theta+\sigma\cos\theta)} < \frac{|\sigma+1|}{\sigma\cos\theta}+\frac{-\tau}{-\tau\sin\theta} \le \frac{1+\frac{1}{\varepsilon}}{\cos\theta}+\frac{1}{\sin\theta}\,.$$

An analogous estimate can be given for $\tau > 0$. Choose for β the worst constant in the estimates above. In fact we showed that for all $\varepsilon > 0$ an estimate III can be given. Finally,

III⇒ I



Take Re $\lambda_0 \ge 1+\varepsilon$, then

$$\|(\lambda-\lambda_0)^n R^n_{\lambda_0}x\| \le \frac{|\lambda-\lambda_0|^n}{|\lambda_0|^n}\beta^n\|x\|\,.$$

The resolvent series

$$R_\lambda = \sum_{n=0}^\infty (\lambda_0-\lambda)^n R^{n+1}_{\lambda_0}$$

is therefore convergent if $\dfrac{|\lambda-\lambda_0|}{|\lambda_0|}\beta < 1$.

That's why the sectors

$$\frac{\pi}{2} \le \arg\lambda \le \theta_0 \text{ and } -\theta_0 \le \arg\lambda \le -\frac{\pi}{2}\,,$$

$|\lambda| \ge R$, with θ_0 sufficiently small and FR sufficiently large certainly belong to the resolvent set $\rho(A)$ of A.
In *these sectors and in the right half plane* we now want the estimate

$$\|R_\lambda\| \le \frac{K_1}{|\lambda|} \quad \text{if} \quad |\lambda| \ge R\,. \;(*)$$

For $\operatorname{Re}\lambda \geq 1+\varepsilon$ this already follows from assumption III. From the above resolvent series

$$\|R_\lambda\| \leq \Big(1-\beta\frac{|\lambda-\lambda_0|}{|\lambda_0|}\Big)^{-1}\|R_{\lambda_0}\| \leq \Big(\frac{1}{\beta}-\frac{|\lambda-\lambda_0|}{|\lambda_0|}\Big)^{-1}\frac{1}{|\lambda_0|} \qquad (**)$$

where we imposed $|\lambda-\lambda_0| < \frac{1}{\beta}|\operatorname{Im}\lambda_0|$.

Consider R_λ in the domain $\{1+\varepsilon \geq \operatorname{Re}\lambda \geq -\frac{1}{2\beta}|\operatorname{Im}\lambda|, |\lambda| \geq R\}$. Take in (**) $\lambda_0 = 1+\varepsilon+i\operatorname{Im}\lambda$

$$\begin{aligned}\|R_\lambda\| &\leq \frac{1}{\frac{1}{\beta}|\lambda_0|-|\lambda-\lambda_0|} \leq \frac{1}{\frac{1}{\beta}|\operatorname{Im}\lambda|+\operatorname{Re}\lambda-(1+\varepsilon)}\\ &\leq \frac{1}{\min(1,\frac{1}{\beta})\{|\operatorname{Im}\lambda|+|\operatorname{Re}\lambda|\}-(1+\varepsilon)} \quad \text{if } \operatorname{Re}\lambda \geq 0\ . \end{aligned} \qquad (\dagger)$$

For $-\frac{1}{2\beta}|\operatorname{Im}\lambda| \leq \operatorname{Re}\lambda \leq 0$ holds $|\operatorname{Re}\lambda| = -\operatorname{Re}\lambda \leq \frac{1}{2\beta}|\operatorname{Im}\lambda|$ and hence also $\frac{3}{4}\frac{1}{\beta}|\operatorname{Im}\lambda| \geq \frac{3}{2}|\operatorname{Re}\lambda|$.
Starting from $\dagger$ we then find

$$\begin{aligned}\|R_\lambda\| &\leq \frac{1}{\frac{1}{4}\frac{1}{\beta}|\operatorname{Im}\lambda|+\frac{3}{4}\frac{1}{\beta}|\operatorname{Im}\lambda|+\operatorname{Re}\lambda-(1+\varepsilon)} \leq\\ &\leq \frac{1}{\min(\frac{1}{4\beta},\frac{1}{2})\{|\operatorname{Im}\lambda|+|\operatorname{Re}\lambda|\}-(1+\varepsilon)} \quad \text{if} \quad -\tfrac{1}{2\beta}|\operatorname{Im}\lambda| \leq \operatorname{Re}\lambda \leq 0\ .\end{aligned}$$

This proves (*).
Now consider the path of integration Γ as drawn. Parametrization in 2nd quadrant: $\lambda = ia+\bar{b}s$,

$$a > R,\ s \geq 0,\ |b| = 1,\ \tfrac{\pi}{2} < \arg b < \tfrac{\pi}{2}+\arctan\tfrac{1}{2}\beta\ .$$

Parametrization in 3rd quadrant: $\lambda = -ia+\bar{b}s$. Define

$$\hat{P}_t x = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}R_\lambda x\, d\lambda, \quad t>0,\ x\in X\ .$$

We want to show that $\hat{P}_t = P_t$. Successively we show

i) $\forall_{x\in D(A)} \lim_{t\downarrow 0} \hat{P}_t x = x$.

ii) $\forall_{t>0}\forall_{x\in X}\ \hat{P}'_t x = A\hat{P}_t x$.

iii) $\|\hat{P}_t x\|$ grows at most exponentially as $t \to \infty$.

Proof of i). Let $x_0 \in D(A)$, λ_0 right of Γ.
Let $(\lambda_0 I - A)x_0 = y_0$

$$\hat{P}_t x_0 = \hat{P}_t R_{\lambda_0} y_0 = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} R_\lambda R_{\lambda_0} y_0 d\lambda =$$

$$= \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} (\lambda_0 - \lambda)^{-1} R_\lambda y_0 d\lambda - \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} (\lambda_0 - \lambda)^{-1} R_{\lambda_0} y_0 d\lambda \ .$$

The second integral is zero, close by a big circle! Because of the estimate for R_λ we may take the limit for $t \downarrow 0$ under the integral sign. So

$$\lim_{t \downarrow 0} \hat{P}_t x_0 = \frac{1}{2\pi i} \int_\Gamma (\lambda_0 - \lambda)^{-1} R_\lambda y_0 d\lambda, \quad y_0 = (\lambda_0 - A)x_0 \ .$$

Close Γ by a big circle on the right: Residu $R_{\lambda_0} y_0 = x_0$.

Proof of ii). Using the closedness of A we find $\hat{P}_t x \in D(A)$ and

$$A\hat{P}_t x = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} A R_\lambda x \, d\lambda = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} \lambda R_\lambda x \, d\lambda - \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} x \, d\lambda \ .$$

The second integral is zero again.
By differentiating the defining integral for $\hat{P}_t x$ the wanted identity ii) follows.

Proof of iii). Straightforward because the part of Γ where $\operatorname{Re} \lambda \geq 0$ has finite length. The unique solvability (cf. Corollary of Theorem 2.8) of the Cauchy-problem leads to the conclusion

$$\hat{P}_t x = P_t x = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} R_\lambda x \, d\lambda \in D(A) \quad \text{if } t > 0 \text{ and}$$

$$(tP_t')x = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} (\lambda t) R_\lambda x \, d\lambda$$

$$\|(tP_t')x\| \leq \frac{K_1 \|x\|}{2\pi} \int_\Gamma |e^{\lambda t}| t |d\lambda| \ .$$

Take $0 < t \leq 1$

$$\int_{\Gamma,\ \operatorname{Re} \lambda \geq 0} |e^{\lambda t}| t |d\lambda| \leq L e^{|\lambda|_{\max}} \ .$$

$$\int_0^\infty |e^{(ia+bs)t} t |b| ds = \int_0^\infty e^{st \operatorname{Re} b} t \, ds = \frac{1}{|\operatorname{Re} b|} \ .$$

Take

$$\alpha = Le^{|\lambda|_{\max}} + \frac{2}{|\operatorname{Re} b|}\,.$$

□

4. Some applications

Definition. A semi-inner product on a Banach space X is a mapping $[\cdot,\cdot] : X \times X \to C$ such that

- $[\alpha x + \beta y, z] = \alpha[x,z] + \beta[y,z], \quad \begin{matrix}\alpha, \beta \in C \\ x, y, z \in X\end{matrix}$
- $|[x,y]| \leq \|x\| \; \|y\|$.
- $[x,x] = \|x\|^2$.

Note that linearity in the second argument is not required. Note that the inner product $(\cdot,\cdot)$ on a Hilbert space is also a semi-inner product.

Example. Consider the Banach space $X = C_0(I\!R)$

$$C_0(I\!R) = \{f \mid f : I\!R \to C,\ f \text{ continuous}, \lim_{|\tau|\to\infty} f(\tau) = 0\} .$$

Choose a mapping $\mu : X \to I\!R$ such that $\mu(f) = a \in I\!R \Rightarrow |f|$ attains its maximum at a. It will be clear that

$$[f,g] = f(b)\overline{g(b)} \text{ with } b = \mu(g) \in I\!R$$

defines a semi-inner product.

Definition. The numerical range $\Sigma(A)$ of an operator A in a Banach space X is defined by

$$\Sigma(A) = \{[Ax, x] \mid x \in D(A),\ \|x\| = 1\} ,$$

Theorem. Let $\lambda \in C \backslash \Sigma(A)$. Let d be the distance between $\Sigma(A)$ and A then $A - \lambda I$ is injective and for all $y \in R(A - \lambda)$ one has $\|(A-\lambda)^{-1}y\| \leq \frac{1}{d}\|y\|$. If in addition A is closed and $\lambda \in \rho(A)$ then this inequality holds for all $y \in X$.

Proof.

$$\forall x \in D(A), \|x\| = 1 \; |[(A-\lambda)x, x]| = |[Ax,x] - \lambda| \geq d\|x\|^2 .$$

Therefore

$$\|x\| \; \|(A-\lambda)x\| \geq d\|x\|^2 .$$

Put $(A-\lambda)x = y$. □

Example 1. Consider $\mathcal{A} = a(x)\frac{d}{dx}$ in $C_0(\mathbb{R})$, $a(\cdot)$ continuous and $0 < \varepsilon < a(x) < M < \infty$. Take $D(\mathcal{A}) = \{u \mid u' \in C_0(\mathbb{R})\}$. Show that $\mathcal{A}$ is closed. Check that at a maximum point of $f = \varphi + i\psi$, φ and ψ real valued, one has $\varphi\varphi' + \psi\psi' = 0$ and $\varphi'^2 + \psi'^2 + \varphi\varphi'' + \psi\psi'' \le 0$. (For the latter the second derivative has to exist of course). We now find

$$[\mathcal{A}f, f] = a(\alpha)f'(\alpha)\bar(f)(\alpha) \text{ with } \alpha = \mu(f)$$
$$= a(\alpha)(\varphi'(\alpha) + i\psi'(\alpha))(\varphi(\alpha) - i\psi(\alpha)) \in i\mathbb{R}\ .$$

Further $\omega > 0$ belongs to the resolvent set since the equation

$$[\omega - a(x)\frac{d}{dx}]x = f$$

is solved by

$$w = -\int_x^\infty \frac{e^{-\omega \int_x^\xi \frac{1}{a(\tau)}d\tau}}{a(\xi)} f(\xi)d\xi\ .$$

Hence we find, applying the above theorem that $\mathcal{A}$ generates a strongly continuous dissipative semi-group in $C_0(\mathbb{R})$.
Exercise. Investigate the operator

$$\mathcal{A} = a(x)\frac{d}{dx} + b(x),\ b(\cdot) \text{ is } \mathbb{C}\text{-valued}$$

in $C_0(\mathbb{R})$ for being an infinitesimal generator.

Example 2. First note that the equation

$$Q_\omega u = \omega u - u_{xx} = f\ , \quad \omega > 0$$

is solved by

$$u(x) = \frac{1}{2\sqrt{\omega}} \int_{-\infty}^\infty e^{-\sqrt{\omega}|x-\xi|} f(\xi)d\xi\ .$$

Now consider the operator $\mathcal{B}$ and the resolvent equation

$$(\mathcal{B}-\omega)u = u_{xx} - \omega u + a(x)u_{xx} + b(u)u_x + c(x)u = f$$

$$[I + (a\partial_{xx} + b\partial_x + c)Q_\omega^{-1}]Q_\omega u = f \ . \qquad (*)$$

Note that in $C_0(I\!R)$ we have $\|Q_\omega^{-1}\| = \dfrac{1}{2\omega}$ and $\|\partial_x Q_x^{-1}\| = \dfrac{1}{2\sqrt{\omega}}$.
If the coefficients a, b, c are complex valued, continuous, bounded and moreover $|(x)| < \frac{2}{3}$ then by taking ω sufficiently large, we can achieve $\|(a\partial_{xx} + b\partial_x + c)Q_\omega^{-1}\| < 1$. The resolvent equation can be solved then

$$u = Q_\omega^{-1}[I + (a\partial_{xx} + b\partial_x + c)Q_\omega^{-1}]^{-1} f \ .$$

With our semi-inner product we now calculate the numerical range $\Sigma(B)$. The operator B is a closed operator on the domain

$$D(B) = \{u \mid u' \in C_0(I\!R),\ u'' \in C_0 I\!R)\} = Q_\omega^{-1}(C_0(I\!R)) \ .$$

Put $f = \varphi + i\psi$ again

$$[\varphi'' + i\psi'', \varphi + i\psi] = \varphi''\varphi + \psi''\psi + i(\psi''\varphi - \varphi''\psi) \quad \text{at} \quad a = \mu(f)$$

$$\leq -\varphi'^2 - \psi'^2 + i(\psi''\varphi - \varphi''\psi) \ .$$

So if we take the coefficient a reëel then $[Bf, f]$ will be in some left half plane $\mathrm{Re}\,\lambda < A$, say.
Gathering our results we find that $B - AI$ is a generator of a dissipative semi-group. Find conditions such that the semi-group is holomorphic.

Example 3. The same evolution equation

$$\frac{\partial u}{\partial t} = u_{xx} + a(x)u_{xx} + b(x)u_x + c(x)u$$

as in Example 2. But now in$L_2(I\!R)$. That case is easier. However 'some' differentiability of a is needed.

Appendix A

Elementary Spectral Properties of Operators in a Banach Space

Definition. Let X denote a Banach space over $\mathbb{C}$.

- A linear operator A, with domain $D(A)$, is
  - i) A linear subspace $D(A) \subset X$.
  - ii) A linear map $A : D(A) \to X$.
- The image of A is $\text{Im } A = \{Ax \mid x \in D((A)\}$.
- 'A densely defined' means $\overline{D(A)} = X$.
- A_1 is called a restriction of A_2 and A_2 is called a prolongation (an extension) of A_1 if $D(A_1) \subset D(A_2)$ and $A_1 x = A_2 x$ if $x \in D(A_1)$. Notation $A_1 \subseteq A_2$ or $A_2 \supseteq A_1$.
  We say $A_1 = A_2$ iff $D(A_1) = D(A_2)$ and $A_1 \subseteq A_2$.

Definition. Sums and Products of operators A and B.

- $D(A+B) = D(A) \cap D(B)$, $(A+B)x = Ax + Bx$
  $D(AB) = \{x \in D(B) \mid Bx \in D(A)\}$, $(AB)x = A(Bx)$.
- If A is injective the inverse A^{-1} is $D(A^{-1}) = \text{ Im } A$ and $A^{-1}y = x$ if $Ax = y$. One has $(A^{-1})^{-1} = A$. Note that in general $0A \subseteq 0$, $A^{-1}A \neq AA^{-1}$ and $A^{-1}A \subseteq I$ (0 is null operator, I identity operator with $D(0) = X$, $D(I) = X$).

Note the simple properties: $(A+B)+C = A+(B+C)$, $A+B = B+A$, $(AB)C = A(BC)$, $(A+B)C = AC + BC$, $C(A+B) \supseteq CA + CB$. $A^{-1}A = I|_{D(A)}$.

Definition.

- An operator A is called continuous iff

$$\exists_{M \geq 0} \forall_{x \in D(A)} \|Ax\| \leq M\|x\| \; .$$

- $\mathcal{L}(X)$ denotes the set of all continuous operators A with domain $D(A) = X$. Note that $\mathcal{L}(X)$, supplied with the norm

$$\|A\| = \sup\{\|Ax\| \mid \|x\| \leq\}$$

  is again a Banach space and even a Banach algebra. One has e.g. $\|AB\| \leq \|A\| \, \|B\|$.

Theorem (Neumann Series). Let $A \in \mathcal{L}(A)$, $\|A\| < 1$, then $I - A : X \to X$ is bijective, $(I-A)^{-1} \in \mathcal{L}(X)$ and

$$(I-A)^{-1} + i + A = A^2 + \ldots + A^n + \ldots = \sum_{n=0}^{\infty} A^n \, .$$

Definition. An operator A is called closed if its graph G_A,

$$G_A = \{(x,y) \mid (x,y) \in D(A) \times X, \; Ax = y\} \subset X \times X$$

is a closed linear subset of $X \times X$.
This is equivalent to

$$[x_n \in D(A), \; x_n \to x, \; Ax_n \to y] \Rightarrow [x \in D(A) \text{ and } y = Ax] \, .$$

Theorem.

- If A is injective and closed then also A^{-1} is closed.
- $[A$ is closed, $B \in \mathcal{L}(X)] \Rightarrow A + B$ is closed.
- $[A$ closed, $\lambda \in C] \Rightarrow \lambda - A = \lambda I - A$ is closed.

Theorem. Let A be closed and $\overline{D(A)} = X$. Then A is continuous iff $D(A) = X$.

Theorem.

- The resolvent set $\rho(A) \subset C$ of A is the set of all $\lambda \in C$, such that
  - i) $\lambda - A$ is injective.
  - ii) Im $(\lambda - A)$ dense in X.
  - iii) $(\lambda - A)^{-1}$ is continuous.
- The spectrum $\sigma(A) \subset C$ of A is the complement $\sigma(A) = C \backslash \rho(A)$.
- For $\lambda \in \rho(A)$ the resolvent (operator) of A is $R_\lambda = R(\lambda, A) = (\lambda - A)^{-1}$.

Theorem.

- Let A be closed then $\lambda \in \rho(A)$ iff $\lambda - A : D(A) \to X$ is bijective.
  In this case one has $R_\lambda \in \mathcal{L}(X)$.
- Conversely, if A is an operator and if there exists $\lambda \in \rho(A)$ with $R_\lambda = (\lambda - A)^{-1} \in \mathcal{L}(X)$ then A is closed.

Theorem. Let A be a closed operator. Then:

i) $\rho(A)$ is an open subset of $\mathbb{C}$.

ii) If $\rho(A) \neq \emptyset$ then $\lambda \mapsto R_\lambda \in \mathcal{L}(X)$ is an analytic (bounded) operator valued function on $\rho(A)$.

iii) $R_\lambda - R_\mu = (\mu - \lambda)R_\lambda R_\mu$ and hence $R_\lambda R_\mu = R_\mu R_\lambda$ for all $\lambda, \mu \in \rho(A)$.

iv) For $\mu \in \rho(A)$ and $|\lambda - \mu|\|R_\mu\| < 1$ one has $\lambda \in \rho(A)$ and

$$R_\lambda = \sum_{n=0}^{\infty} (\mu - \lambda)^n R_\mu^{n+1} .$$

Theorem. If $A \in \mathcal{L}(X)$ then $\sigma(A)$ is non-empty and compact. For an arbitrary closed operator the spectrum may be empty. Also $\rho(A)$ can be empty. Any compact set in $\mathbb{C}$ can be the spectrum of an operator in $\mathcal{L}(X)$.

Examples.

a) Operator with empty spectrum.

$$X = C_0([0,1]) = \{u \mid u : [0,1] \to \mathbb{C},\ u \text{ continuous},\ u(0) = 0\} ,$$

$$\|u\| = \max_{0 \le t \le 1} |u(t)| .$$

$$D(A) = \{v \mid v \text{ continuously differentiable},\ v'(0) = 0\}$$

$$(Av)(t) = \frac{dv}{dt}(t) .$$

Check all the details!

b) Any closed set S in $\mathbb{C}$ is the spectrum of an operator in ℓ_2.

$$X = \ell_2 = \{(x_1, x_2, \ldots) \mid \sum_{j=1}^{\infty} |x_j|^2 < \infty\} .$$

Let $S \subset \mathbb{C}$. $S \neq \emptyset$. Choose a sequence $\{\lambda_n\} \subset S$ which is dense in S.

$$D(A) = \{(x_n) \mid (x_n) \in \ell_2,\ (\lambda_n x_n) \in \ell_2\} .$$

Put $Ax = y$ with $y = (y_n) = (\lambda_n x_n)$.
Note that the λ_n are eigenvalues of A, this means that $\lambda_n - A$ is not injective. The set of eigenvalues is called the discrete spectrum of the operator.

c) Spectrum equal to $\mathbb{C}$ but no eigenvalues.

$$X = L_2(\mathbb{R}^2) = \{u \mid \int\int |u(x,y)|^2 dxdy < \infty\}$$

$$D(A) = \{v \mid \int\int (x^2+y^2)|u(x,y)|^2 dxdy < \infty\}$$

$Au = v$ with $v(x,y) = (x+iy)u(x,y)$. Here A is closed, $\sigma(A) = \mathbb{C}$.

Theorem. Let A be a closed operator with $\rho(A) \neq \emptyset$. Let P be a polynomial of degree $n \geq 1$

$$P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n \ , \quad a_n \neq 0 \ .$$

Then the operator $P(A) = a_0 + a_1 A + \ldots + a_n A^n$ with domain $D(P(A)) = D(A^n)$ is closed and $\sigma(P(A)) = P(\sigma(A))$.

Some hints for the proof.

- If $P(z) = k(\lambda_1 - z)(\lambda_2 - z) \cdot \ldots \cdot (\lambda_n - z)$ then also

$$P(A) = k(\lambda_1 - A)(\lambda_2 - A) \cdot \ldots \cdot (\lambda_n - A) \ .$$

- Proceed by induction and write $P(A) = (\lambda - A)Q(A) + rI$ with r a number.
- Put $\mu - P(z) = k(\mu_1 - z)(\mu_2 - z) \cdot \ldots \cdot (\mu_n - z)$ and observe that $\mu \notin P(\sigma(A)) \Rightarrow \mu_i \in \rho(A)$, $1 \leq i \leq n$.

Appendix B

Integration of functions with values in a Banach space

Let X be a Banach space. Let $I = [a, b]$ denote a compact interval. Denote the length of I by $|I|$.

Definition. A step function $f : I \to X$ is a function which can be written $f = \sum_{i=1}^{n} 1_{I_i} x_i$, where $I = \bigcup_{i=1}^{n} I_i$ a partition of I in sub-intervals and $x_i \in X$.

Definition. The integral of a step function is

$$\int_a^b f(t)dt = \int_I f \, dt = \sum_{i=1}^{n} |I_i| x_i \, .$$

(Verify that the definition does not depend don the choice of the decomposition $\{I_i\}$.

Properties.

1) $$\int_I (f+g)dt = \int_I f \, dt + \int_I g \, dt, \quad \int_I \lambda \, f \, dt = \lambda \int_I f \, dt, \quad \lambda \in C \, .$$

2) $$\| \int_I f \, dt \| \le \int_I \|f\| dt \le |I| \sup_{t \in I} \|f(t)\| \, .$$

3) $$U \in \mathcal{L}(X) \quad \int_I U f(t) dt = U \int_I f(t) dt \, .$$

Definition. A ruled function (F: reglée) is a function $f : I \to X$ which is a uniform limit of step functions.

Remarks.

- Continuous functions $f : I \to X$ are ruled.
- A function $f : I \to X$ is ruled iff at each point $a \in I$ both the limits $\lim_{t \uparrow a} f(t)$ and $\lim_{t \downarrow a} f(t)$ exist.

Definition. Let $f : I \to X$ be ruled. One defines

$$\int_a^b f(t)dt = \int_I f \, dt = \lim_{n \to \infty} \int_I f_n dt$$

where (f_n) is a uniformly approximating sequence of f.

Theorem. The definition is OK and the limit does not depend on the approximating sequence.

Note. In the proof of this, the estimate

$$\|\int_I f_n dt - \int_I f_m dt\| \leq |I| \, \|f_n - f_m\|_\infty$$

plays the key role.

Theorem. Let $f : [a,b] \to X$ be continuous. Let $F(t) = \int_a^t f(s)ds$. Then F is differentiable on $[a,b]$ and $F' = f$. $(F_r'(a) = f(a),\ F_e'(b) = f(b))$.

Lemma. Let $f : [a,b] \to X$ be continuous and assume thet $f'(t) = 0$ for $a < t < b$. Then f is constant.

Theorem.

- Let $f : [a,b] \to X$ be continuous and suppose $F : [a,b] \to X$ be differentiable with $F'(t) = f(t)$. Then $\int_a^b f(t)dt = F(b) - F(a)$.
- Let $t \mapsto s(t)$ from $[\alpha, \beta]$ onto $[a,b]$ then the classical formula holds:

$$\int_a^b f(s)ds = \int_\alpha^\beta f(s(t))s'(t)dt \ .$$

Theorem. Let X, Y, Z be Banach spaces and let $(x,y) \mapsto x \cdot y$ be a continuous bilinear mapping from $X \times Y$ to Z.
If $u : I \to X$ and $v : I \to Y$ are differentiable then $t \mapsto u(t) \cdot v(t)$ is also differentiable and

$$\frac{d}{dt} u(t) \cdot v(t) = u'(t) \cdot v(t) + u(t) \cdot v'(t) \ .$$

Definition. (Absolutely convergent integrals). Suppose

- $f : [a, \infty) \to X$ is continuous.
- $\int_a^b \|f(t)\|dt < \infty$.

Then $\lim_{b \to \infty} \int_a^b f(t)dt$ exists and is written $\int_a^\infty f(t)dt$.

Theorem. Let $f : [a,b] \times (\alpha, \beta) \to X$ be continuous and put $F(\lambda) = \int_a^b f(t,\lambda)dt$.

- If f is continuous then also F is continuous.
- If in addition $\frac{\partial f}{\partial \lambda} : [a,b] \times (\alpha,\beta) \to X$ exists and is continuous, then F is continuously differentiable and

$$F'(\lambda) = \int_a^b \frac{\partial f}{\partial \lambda}(t,\lambda)dt \ .$$

The proof of all these results are standard and similar to the "scalar valued" case. The results for integrals on infinite intervals are also similar to the classical case.

Theorem. Suppose

- $u : (\alpha, \beta) \to X$, $-\infty \leq \alpha < \beta \leq \infty$.
- Let A be a closed operator and suppose
  - $\forall_{t \in (\alpha,\beta)} [u(t) \in D(A)]$
  - $Au : (\alpha,\beta) \to X$ is continuous.
- $\int_\alpha^\beta \|u(t)\|dt < \infty \quad \int_\alpha^\beta \|Au(t)\|dt < \infty$.

Then

$$\int_\alpha^\beta u(t)dt \in D(A) \text{ and } A \int_\alpha^\beta u(t)dt = \int_\alpha^\beta Au(t)dt \ .$$

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