

Tiling the surface of a cube by 12 identical pentominoes

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**TILING THE SURFACE OF A CUBE BY 12 IDENTICAL
PENTOMINOES**

by

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TILING THE SURFACE OF A CUBE BY 12 IDENTICAL PENTOMINOES

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Abstract

This report deals with tiling the surface of a cube by twelve congruent copies of a planar pentomino, folded over corners, edges and faces of the cube. Only so-called nice tilings are considered, such that the pentominoes are undeformed and easy to recognize. There are no such tilings for pentominoes U and W. The other pentominoes lead to 1054 tilings distinct modulo rotation and reflection. Most of them are asymmetric and of no particular interest. Only 164 of them have some degree of symmetry, as specified in

X(1), T(1), Z(11), V(2), I(2), F(23), N(3), Y(10), L(30), P(81),

with, in parentheses, the number of tilings for the corresponding pentomino. Every symmetric tiling is shown in terms of an unfold of the cube surface onto the plane. Numerical code and details of symmetry are added.

1. Introduction

The problem of tiling the surface of a cube, by a complete set of the 12 different pentominoes, was solved in a previous report [1]. Let me recall that the total number of solutions was found to be 26,358,584 of which 284,402 are nice. Compared to these large numbers, the 1054 and 164 of the new problem is a mere trifle, and so is the corresponding computing time.

In the old problem, pentomino X played a special role to obtain all solutions different modulo rotation of the cube. In the new problem X can be left alone, because, as is easy to see, X leads to one solution only, and this solution is rotational-invariant.

The cube to be covered is shown in Figure 1 (reflections are eliminated). The 60 squares (cells) are numbered as shown in Figure 2.

For the computer a pentomino placed on the cube is an integer array of dimension five, the elements of which are the five ordinal numbers of cells covered by the pentomino and ordered to increasing value.

Apart from X, the pentominoes are ordered pentomino-wise in the order

U, T, Z, V, W, I, F, N, Y, L, P.

The integer arrays for each pentomino are ordered to increasing lexicographical value. The corresponding matrix of 5 columns and 3576 rows is available from the old problem. In the new problem, backtrack is over small subsets of these rows as correspond to the pentomino under consideration.

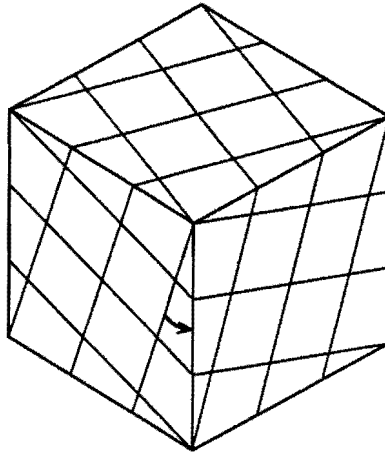


Figure 1. The angle indicated equals $\arctan(1/3)$.

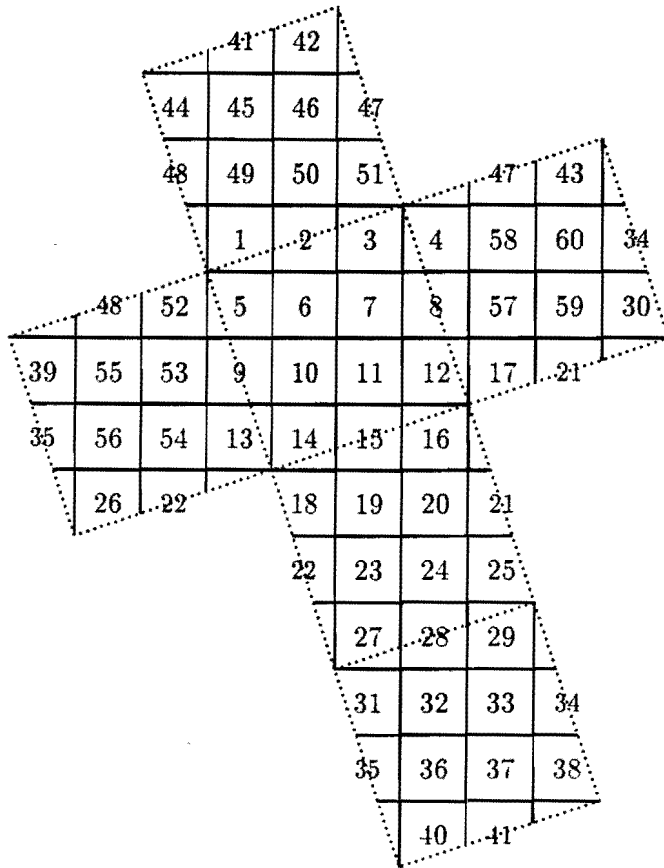


Figure 2. Cell numbers on the cube's layout.

2. Cubic symmetry

The solid cube has many axes of symmetry. First, the three axes from face to face X, Y, Z. Second, the four diagonals from vertex to vertex D1, D2, D3, D4. Third, the six axes from edge to edge T1, T2, T3, T4, T5, T6. See Figure 3, where the corresponding symbols are encircled, at one of two places where the axes cut the cube surface.

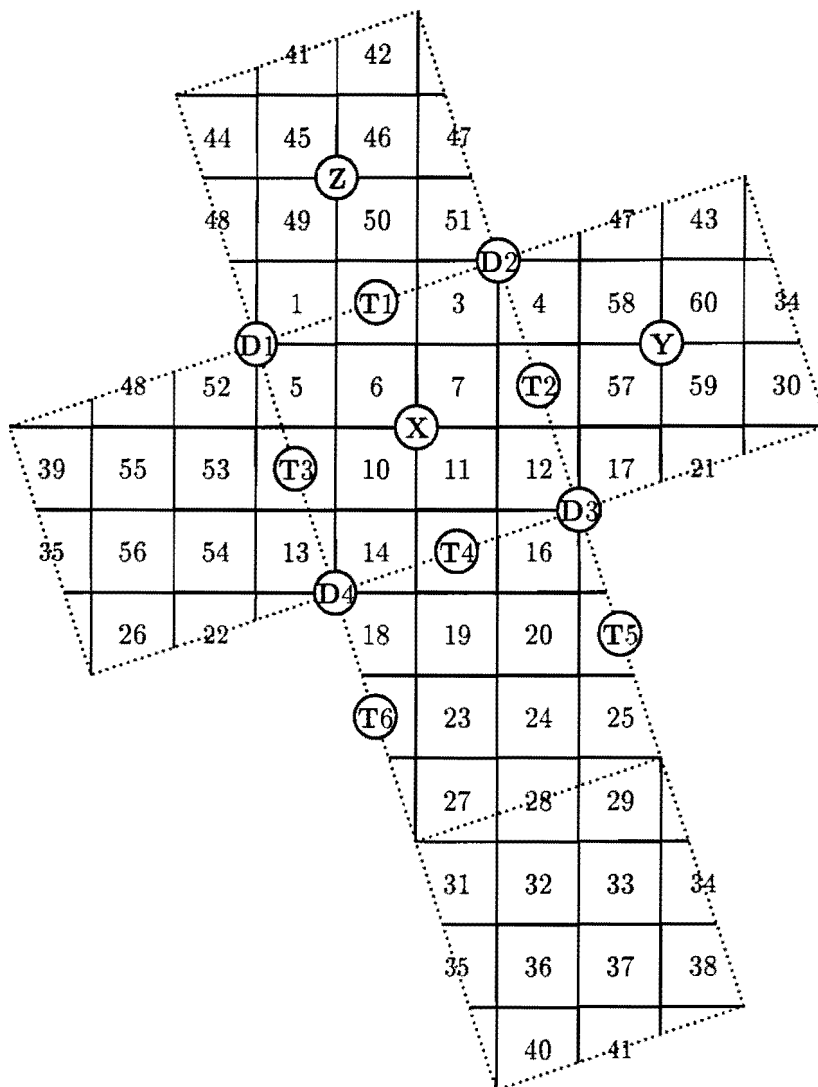


Figure 3. Axes of symmetry within circles.

Now assume that the computer has found a tiling, coded by an ordered set of twelve numbers from 1 to 3576, for a cube fixed in space. If you rotate the cube with its tiling attached, you can describe the tiling with new code numbers, based on the fixed cube. These new numbers can be found by permutation of cell numbers, in combination with sorting procedures, both for numbers and strings.

In the following table these permutations are given explicitly. They were obtained manually, by use of two identical cubes with their cells numbered, one cube held fixed, the other rotated about the axis considered, then rotating the two cubes identically and noting, for 1 to 60 of the first cube, the corresponding cell numbers of the second cube. The lines of IDENTITY are added. With them you can follow permutations more easily.

X stands for rotation over 90 degrees about the x-axis; ROT-1 is the file where the 60 numbers will be stored on disk. X*X stands for rotation over 180 degrees about the x-axis; ROT-2 is the corresponding file of storage. And so forth and so on.

TABLE OF 24 PERMUTATIONS

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
IDENTITY	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
X	4	8	12	16	3	7	11	15	2	6	10	14	1	5	9	13	18	52	53	54
ROT-1	22	48	55	56	26	44	39	35	31	27	40	36	32	28	41	37	33	29	42	38
	34	30	25	43	60	59	21	47	58	57	17	51	50	49	46	45	19	20	23	24
X*X	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	52	51	50	49
ROT-2	48	47	46	45	44	43	42	41	40	39	38	37	36	35	34	33	32	31	30	29
	28	27	26	25	24	23	22	21	20	19	18	17	57	58	59	60	53	54	55	56
X*X*X	13	9	5	1	14	10	6	2	15	11	7	3	16	12	8	4	51	17	57	58
ROT-3	47	21	59	60	43	25	30	34	38	42	29	33	37	41	28	32	36	40	27	31
	35	39	44	26	56	55	48	22	54	53	52	18	19	20	23	24	50	49	46	45
Y	40	41	42	43	44	45	46	47	48	49	50	51	52	1	2	3	4	5	6	7
ROT-4	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
	28	29	30	31	32	33	34	35	36	37	38	39	55	53	56	54	58	60	57	59
Y*Y	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46
ROT-5	47	48	49	50	51	52	1	2	3	4	5	6	7	8	9	10	11	12	13	14
	15	16	17	18	19	20	21	22	23	24	25	26	56	55	54	53	60	59	58	57
Y*Y*Y	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33
ROT-6	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	1
	2	3	4	5	6	7	8	9	10	11	12	13	54	56	53	55	59	57	60	58
Z	44	48	52	5	39	55	53	9	35	56	54	13	31	26	22	18	14	27	23	19
ROT-7	15	28	24	20	16	29	25	21	17	12	30	59	57	8	34	60	58	4	38	43
	47	51	3	42	46	50	2	41	45	49	1	40	36	32	37	33	10	6	11	7
Z*Z	42	41	40	39	38	37	36	35	34	33	32	31	30	29	28	27	26	25	24	23
ROT-8	22	21	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3
	2	1	52	51	50	49	48	47	46	45	44	43	60	59	58	57	56	55	54	53

Z*Z*Z	51 47 43 38 4 58 60 34 8 57 59 30 12 17 21 25 29 16 20 24
ROT-9	28 15 19 23 27 14 18 22 26 31 13 54 56 35 9 53 55 39 5 52
	48 44 40 1 49 45 41 2 50 46 42 3 7 11 6 10 33 37 32 36
D1	5 9 13 18 52 53 54 22 48 55 56 26 44 39 35 31 27 40 36 32
ROT-10	28 41 37 33 29 42 38 34 30 25 43 60 59 21 47 58 57 17 51 4
	8 12 16 3 7 11 15 2 6 10 14 1 49 45 50 46 23 19 24 20
D1*D1	52 48 44 40 1 49 45 41 2 50 46 42 3 51 47 43 38 4 58 60
ROT-11	34 8 57 59 30 12 17 21 25 29 16 20 24 28 15 19 23 27 14 18
	22 26 31 13 54 56 35 9 53 55 39 5 6 7 10 11 37 36 33 32
D2	12 8 4 51 17 57 58 47 21 59 60 43 25 30 34 38 42 29 33 37
ROT-12	41 28 32 36 40 27 31 35 39 44 26 56 55 48 22 54 53 52 18 13
	9 5 1 14 10 6 2 15 11 7 3 16 20 24 19 23 46 50 45 49
D2*D2	43 47 51 3 42 46 50 2 41 45 49 1 40 44 48 52 5 39 55 53
ROT-13	9 35 56 54 13 31 26 22 18 14 27 23 19 15 28 24 20 16 29 25
	21 17 12 30 59 57 8 34 60 58 4 38 37 36 33 32 6 7 10 11
D3	26 22 18 14 27 23 19 15 28 24 20 16 29 25 21 17 12 30 59 57
ROT-14	8 34 60 58 4 38 43 47 51 3 42 46 50 2 41 45 49 1 40 44
	48 52 5 39 55 53 9 35 56 54 13 31 32 33 36 37 11 10 7 6
D3*D3	38 34 30 25 43 60 59 21 47 58 57 17 51 4 8 12 16 3 7 11
ROT-15	15 2 6 10 14 1 5 9 13 18 52 53 54 22 48 55 56 26 44 39
	35 31 27 40 36 32 28 41 37 33 29 42 46 50 45 49 20 24 19 23
D4	17 21 25 29 16 20 24 28 15 19 23 27 14 18 22 26 31 13 54 56
ROT-16	35 9 53 55 39 5 52 48 44 40 1 49 45 41 2 50 46 42 3 51
	47 43 38 4 58 60 34 8 57 59 30 12 11 10 7 6 32 33 36 37
D4*D4	31 35 39 44 26 56 55 48 22 54 53 52 18 13 9 5 1 14 10 6
ROT-17	2 15 11 7 3 16 12 8 4 51 17 57 58 47 21 59 60 43 25 30
	34 38 42 29 33 37 41 28 32 36 40 27 23 19 24 20 49 45 50 46
T1	3 2 1 52 51 50 49 48 47 46 45 44 43 42 41 40 39 38 37 36
ROT-18	35 34 33 32 31 30 29 28 27 26 25 24 23 22 21 20 19 18 17 16
	15 14 13 12 11 10 9 8 7 6 5 4 58 60 57 59 55 53 56 54
T2	25 21 17 12 30 59 57 8 34 60 58 4 38 43 47 51 3 42 46 50
ROT-19	2 41 45 49 1 40 44 48 52 5 39 55 53 9 35 56 54 13 31 26
	22 18 14 27 23 19 15 28 24 20 16 29 33 37 32 36 7 11 6 10
T3	18 22 26 31 13 54 56 35 9 53 55 39 5 52 48 44 40 1 49 45
ROT-20	41 2 50 46 42 3 51 47 43 38 4 58 60 34 8 57 59 30 12 17
	21 25 29 16 20 24 28 15 19 23 27 14 10 6 11 7 36 32 37 33

T4	29 28 27 26 25 24 23 22 21 20 19 18 17 16 15 14 13 12 11 10
ROT-21	9 8 7 6 5 4 3 2 1 52 51 50 49 48 47 46 45 44 43 42
	41 40 39 38 37 36 35 34 33 32 31 30 59 57 60 58 54 56 53 55
T5	39 35 31 27 40 36 32 28 41 37 33 29 42 38 34 30 25 43 60 59
ROT-22	21 47 58 57 17 51 4 8 12 16 3 7 11 15 2 6 10 14 1 5
	9 13 18 52 53 54 22 48 55 56 26 44 45 46 49 50 24 23 20 19
T6	30 34 38 42 29 33 37 41 28 32 36 40 27 31 35 39 44 26 56 55
ROT-23	48 22 54 53 52 18 13 9 5 1 14 10 6 2 15 11 7 3 16 12
	8 4 51 17 57 58 47 21 59 60 43 25 24 23 20 19 45 46 49 50

3. Tilings different modulo rotation

Coping with the symmetry of the tilings is more difficult than in the old problem. First, a tiling will be identified by a string of length 48 derived from a concatenation of its twelve code numbers as computed by the backtrack program. Second, it should be clear that successive strings increase in lexicographical value in the course of backtrack.

If a tiling is not symmetric, it has 24 different codes by rotation. The first of them, that is the smaller, could be stored, and the 23 others ignored. However, by choice we ignore all asymmetric tilings.

If a tiling is symmetric, the first string is not greater than the 23 others, and at least one of the latter is equal to the first.

Assume that the computer finds a tiling, TILING say, with string A\$. Then we compute 23 strings by permutations corresponding to rotating the cube, and order them to increasing or non-decreasing lexicographical value. Let then B\$ be the very first of them. Only if A\$=B\$ is TILING symmetric, and A\$ is stored. For details, the reader should consider, and study, the actual program in GWbasic.

4. Computer program

```

0  CLS:  REM *** This is program    12SELECT.bas/exe    *****
1  REM *** It is about covering a cube with 12 identical pentominoes *****
2  REM *** It computes all codes that are distinct modulo rotation *****
3  REM *** and STORES them in file "NICE"+P$ *****
100 DEFINT A-Z
105 DIM PIS1(3576),PIS2(3576),PIS3(3576),PIS4(3576),PIS5(3576),TAL(12,5)
110 DIM PNUM(3576),B(12),COD(12),SOL(24,12),A$(23),AR(60),BB$(24),A(24),B$(24)
120 DIM BODY(60),COPIS(12),COHOL(12),HOEK(576),COP(12),GETAL(12,5),X(12)
130 DATA "1", "2","3","4","5","6","7","8","9","10","11","12","13","14","15",
        "16","17","18","19","20","21","22","23"
140 FOR I=1 TO 23:READ A$(I):NEXT
200 FOR I= 1 TO 216:PNUM(I)= 2:NEXT
220 FOR I= 457 TO 696:PNUM(I)= 4:NEXT

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230 FOR I= 697 TO 936:PNUM(I)= 5:NEXT
240 FOR I= 937 TO 1176:PNUM(I)= 6:NEXT
250 FOR I=1177 TO 1296:PNUM(I)= 7:NEXT
260 FOR I=1297 TO 1752:PNUM(I)= 8:NEXT
270 FOR I=1753 TO 2232:PNUM(I)= 9:NEXT
280 FOR I=2233 TO 2688:PNUM(I)=10:NEXT
290 FOR I=2689 TO 3168:PNUM(I)=11:NEXT
300 FOR I=3169 TO 3576:PNUM(I)=12:NEXT
400 OPEN "code1" FOR INPUT AS #1:FOR I=1 TO 3576:INPUT #1, PIS1(I):NEXT:CLOSE
410 OPEN "code2" FOR INPUT AS #1:FOR I=1 TO 3576:INPUT #1, PIS2(I):NEXT:CLOSE
420 OPEN "code3" FOR INPUT AS #1:FOR I=1 TO 3576:INPUT #1, PIS3(I):NEXT:CLOSE
430 OPEN "code4" FOR INPUT AS #1:FOR I=1 TO 3576:INPUT #1, PIS4(I):NEXT:CLOSE
440 OPEN "code5" FOR INPUT AS #1:FOR I=1 TO 3576:INPUT #1, PIS5(I):NEXT:CLOSE
460 OPEN "hoek" FOR INPUT AS #1:FOR I=1 TO 576:INPUT #1, HOEK(I):NEXT:CLOSE
470 FOR I=1 TO 576:PIS1(HOEK(I))=0:NEXT: REM *** nice tilings only *****
480 GOSUB 4000
490 OPEN "NICE"+P$ FOR OUTPUT AS #2
500 CLS:FOR I=1 TO 60:BODY(I)=0:NEXT: REM *** Tiling computed in lines ***
600 J=1:FREHOL=1: REM *** 500 through 830 ***
608 BEGIN$=DATE$+" "+TIME$
610 TRYPIS=LOWER
620 IF FREHOL>60 THEN 900
630 IF BODY(FREHOL)<>0 THEN FREHOL=FREHOL+1:GOTO 620
640 IF TRYPIS>UPPER THEN 790
660 IF PIS1(TRYPIS)<>FREHOL THEN TRYPIS=TRYPIS+1:GOTO 640
670 IF BODY(PIS2(TRYPIS))=1 THEN TRYPIS=TRYPIS+1:GOTO 640
680 IF BODY(PIS3(TRYPIS))=1 THEN TRYPIS=TRYPIS+1:GOTO 640
690 IF BODY(PIS4(TRYPIS))=1 THEN TRYPIS=TRYPIS+1:GOTO 640
700 IF BODY(PIS5(TRYPIS))=1 THEN TRYPIS=TRYPIS+1:GOTO 640
710 COHOL(J)=FREHOL:COPIJ(J)=TRYPIS: REM *** begin of filling *****
725 IF CM$="x" THEN 1300: REM *** interupt possible *****
730 BODY(FREHOL)=1
740 BODY(PIS2(TRYPIS))=1
750 BODY(PIS3(TRYPIS))=1
760 BODY(PIS4(TRYPIS))=1
770 BODY(PIS5(TRYPIS))=1
775 CM$=INKEY$: REM *** interupt possible *****
780 J=J+1:FREHOL=FREHOL+1:GOTO 610: REM *** end of filling *****
790 J=J-1:IF J=0 THEN 1300: REM *** begin of erasing *****
800 K1=COHOL(J):K2=COPIJ(J)
810 COHOL(J)=0:COPIJ(J)=0
820 BODY(K1)=0:BODY(PIS2(K2))=0:BODY(PIS3(K2))=0:BODY(PIS4(K2))=0:BODY(PIS5(K2))=0
830 FREHOL=K1:TRYPIS=K2+1:GOTO 640: REM *** end of erasing *****
900 BBB$="": FOR I=1 TO 12:BBB$=BBB$+STR$(COPIJ(I)):NEXT
910 GOTO 2000
1000 REM
1010 IF B$(A(1))<BBB$ THEN 1020 ELSE FOR I=1 TO 12:WRITE #2, COPIJ(I):NEXT

```

```
1020 GOTO 790
1300 REM
1310 PRINT"   Start at "+ BEGIN$
1320 PRINT"   Finish at "+ DATE$+" "+TIME$
1330 REM
1340 END
2000 REM
2010 FOR I=1 TO 12:COD(I)=COPI(I):NEXT
2020 SS=0
2030 FOR JJ=1 TO 23
2035 OPEN "ROT-"+A$(JJ) FOR INPUT AS #1
2040 FOR I=1 TO 60:INPUT #1,AR(I):NEXT:CLOSE #1
2045 FOR I=1 TO 12:COP(I)=COPI(I):NEXT
2047 FOR I=1 TO 12
2050 X(1)=AR(PIS1(COP(I))):X(2)=AR(PIS2(COP(I))):X(3)=AR(PIS3(COP(I)))
2060 X(4)=AR(PIS4(COP(I))):X(5)=AR(PIS5(COP(I)))
2070 N=5: GOSUB 10000
2080 FOR K=1 TO 5:GETAL(I,K)=X(B(K)):NEXT
2090 NEXT
2100 N=12
2110 FOR I=1 TO 12:X(I)=GETAL(I,1):NEXT
2120 GOSUB 10000
2130 FOR I=1 TO 12:FOR K=1 TO 5
2140 TAL(I,K)=GETAL(B(I),K):NEXT:NEXT
2150 FOR I=1 TO 12
2160 FOR M=LOWER TO UPPER
2170 IF (PIS1(M)=TAL(I,1) AND PIS2(M)=TAL(I,2) AND PIS3(M)=TAL(I,3) AND PIS4(M)=
    TAL(I,4) AND PIS5(M)=TAL(I,5) ) THEN COD(I)=M
2180 NEXT M
2190 NEXT I
2200 REM
2220 REM
2230 SS=SS+1:FOR I=1 TO 12: SOL(SS,I)=COD(I):NEXT
2240 REM
2250 NEXT JJ
2260 GOTO 20000
4000 INPUT" Pentomino letter (IN CAPITAL!) ";P$
4010 IF P$="U" THEN LOWER= 1:UPPER= 216
4020 IF P$="T" THEN LOWER= 217:UPPER= 456
4030 IF P$="Z" THEN LOWER= 457:UPPER= 696
4040 IF P$="V" THEN LOWER= 697:UPPER= 936
4050 IF P$="W" THEN LOWER= 937:UPPER=1176
4060 IF P$="I" THEN LOWER=1177:UPPER=1296
4070 IF P$="F" THEN LOWER=1297:UPPER=1752
4080 IF P$="N" THEN LOWER=1753:UPPER=2232
4090 IF P$="Y" THEN LOWER=2233:UPPER=2688
4100 IF P$="L" THEN LOWER=2689:UPPER=3168
```

```

4110 IF P$="P" THEN LOWER=3169:UPPER=3576
4120 RETURN
10000 REM ***** Sorting numbers X(1),X(2),...,X(N) *****
10010 V=N:FOR II=1 TO N:B(II)=II:NEXT
10020 V=INT(V/2):IF V=0 THEN 10080
10030 N1=N-V:FOR KK=1 TO N1:E=KK+V
10040 IF X(B(KK))<=X(B(E)) THEN 10070 ELSE T=B(E):F=KK
10050 B(E)=B(F):E=F:F=F-V:IF F>0 THEN IF X(B(F))>X(T) THEN 10050
10060 B(E)=T
10070 NEXT:GOTO 10020
10080 RETURN
20000 REM
20010 REM
20020 REM
20030 REM
20040 REM
20050 FOR K=1 TO 23:BB$=""
20060 FOR I=1 TO 12
20070 BB$=BB$+STR$(SOL(K,I)):NEXT
20080 B$(K)=BB$
20090 NEXT
20100 N=23
20110 GOSUB 30000
20120 REM
20130 GOTO 1000
30000 REM ***** Sorting strings *****
30010 V=N:FOR I=1 TO N:A(I)=I:NEXT
30020 V=INT(V/2):IF V=0 THEN RETURN
30030 N1=N-V:FOR KK=1 TO N1:E=KK+V
30040 IF B$(A(KK))<=B$(A(E)) THEN 30070 ELSE T=A(E):T$=B$(T):F=KK
30050 A(E)=A(F):E=F:F=F-V:IF F>0 THEN IF B$(A(F))>T$ THEN 30050
30060 A(E)=T
30070 NEXT:GOTO 30020

```

5. Conclusion

Two other programs need be mentioned. The first deals with providing details about the character of symmetry, in terms of axes, with input file "NICE"+P\$. The second program draws the layout on my matrix printer, by inputting the 12 code numbers. The corresponding information is given at bottom of page, but for pentomino X, where code is absent.

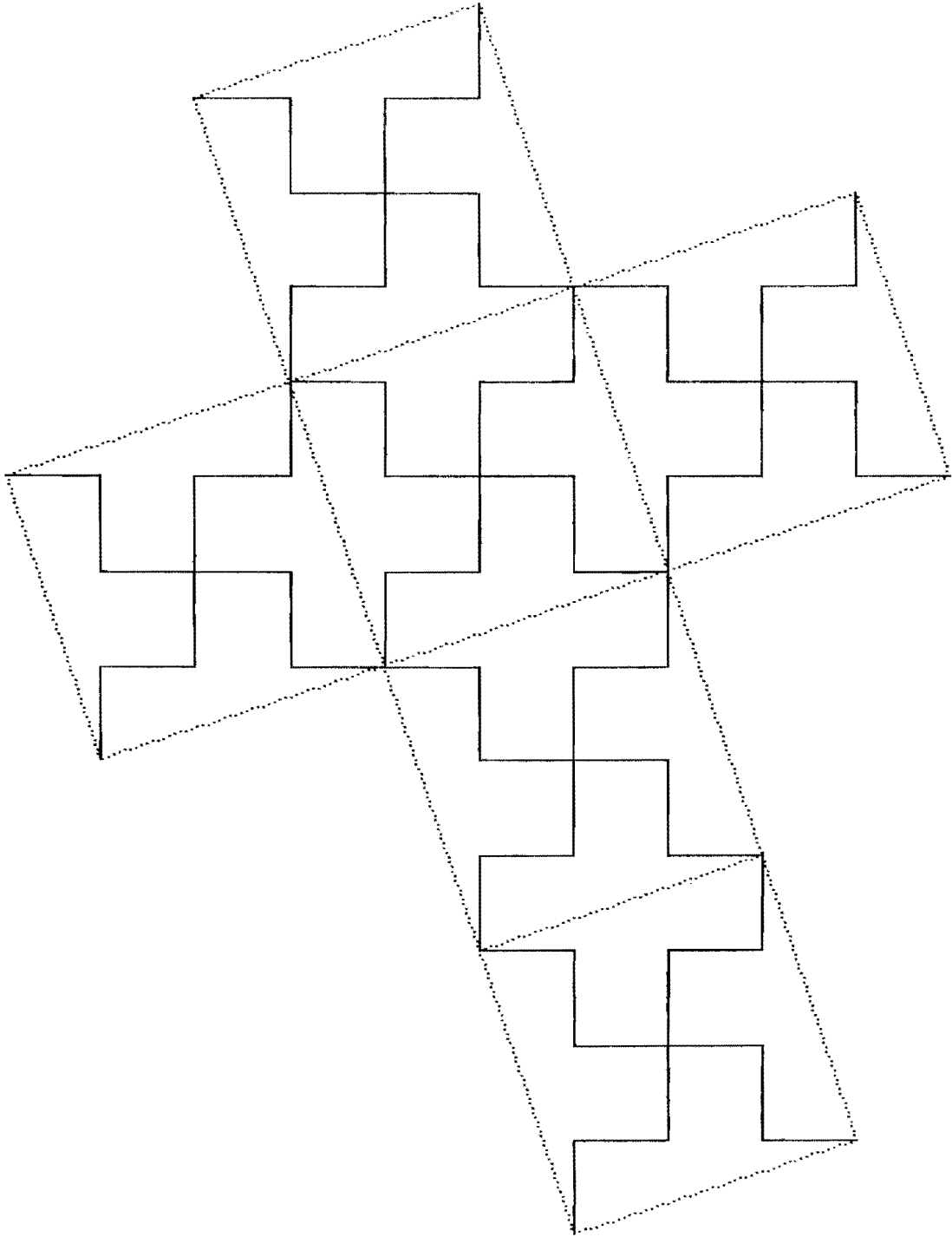
As an ardent puzzler, I made a thin-card-board copy of each layout that has more symmetry than one axis of order 2, after adding a few triangular flaps around the border. Scoring the dashed lines with a blunt knife along a metal straight edge. I can fold the layout into a cube in 3-space and fix it with the flaps inserted between the card-board and 3 squares pasted at the inside. I can unfold the cube at will. Many beautiful geometric objects result!

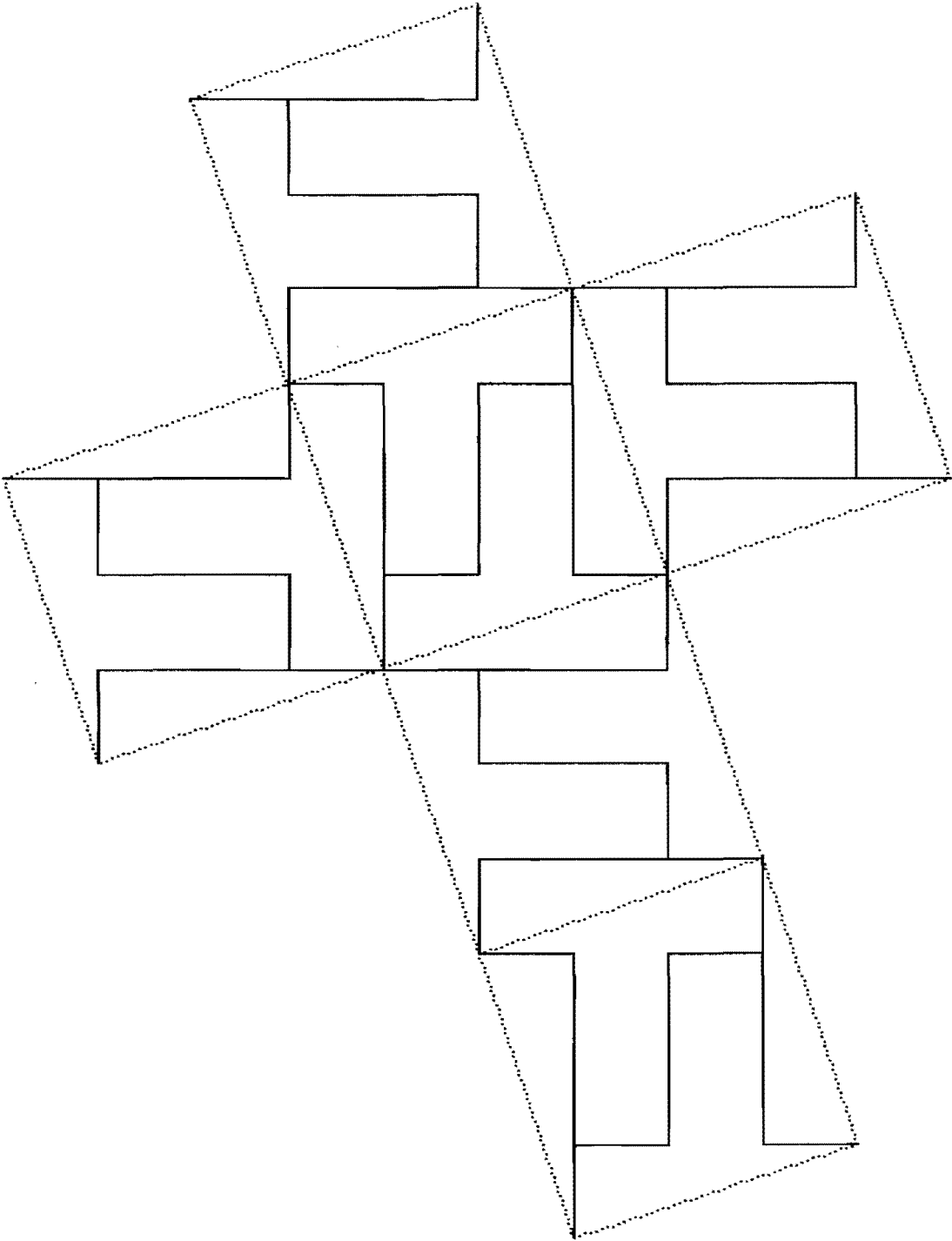
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References

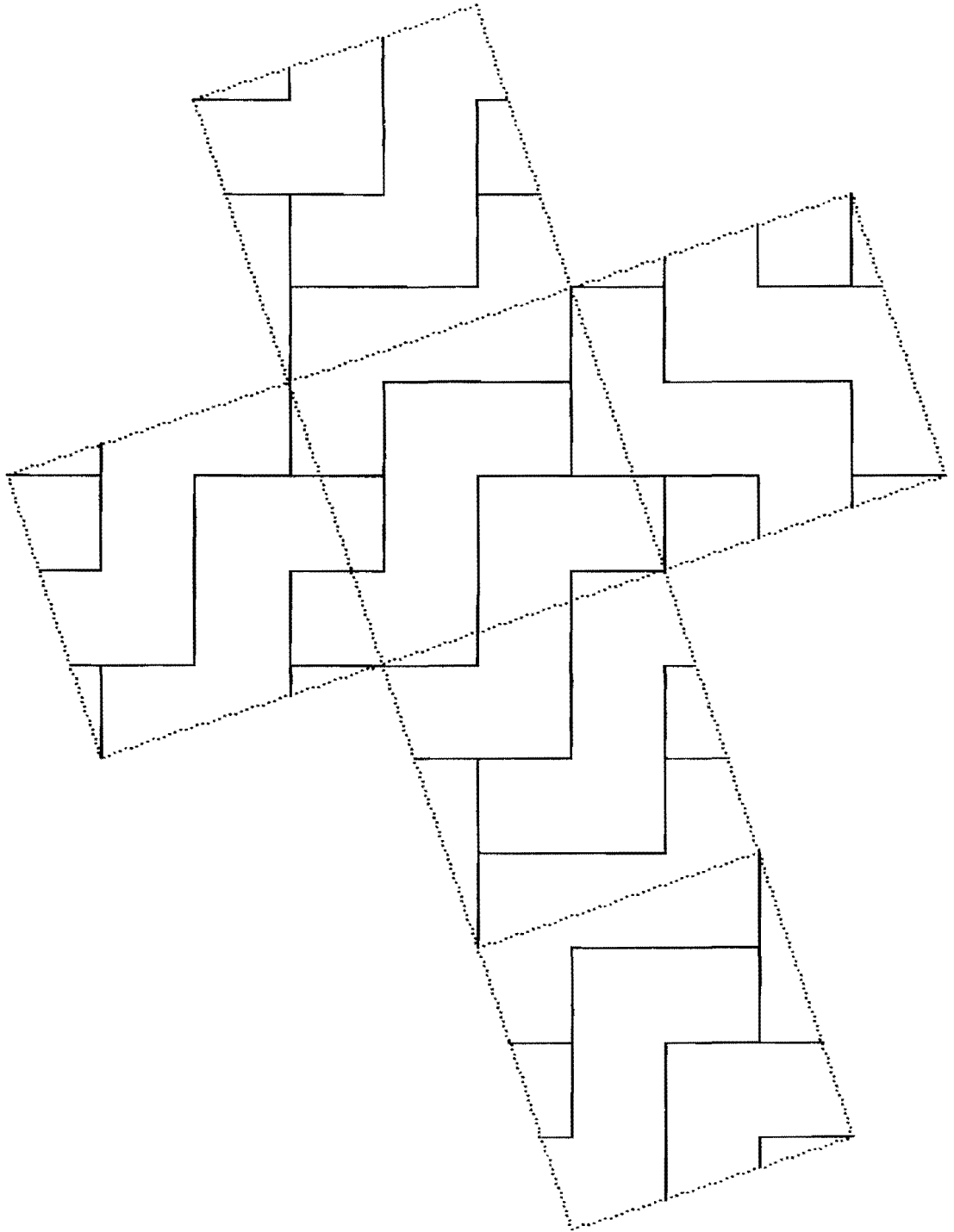
- [1] C.J.Bouwkamp, On Benjamin's Pentomino Cube, EUT Report 97-WSK-01, Eindhoven, December 1997.
- [2] _____, An old pentomino problem revisited, Simplex Sigillum Veri, Eindhoven, 1995, ISBN 90-386-0197-2, pp. 87-96.





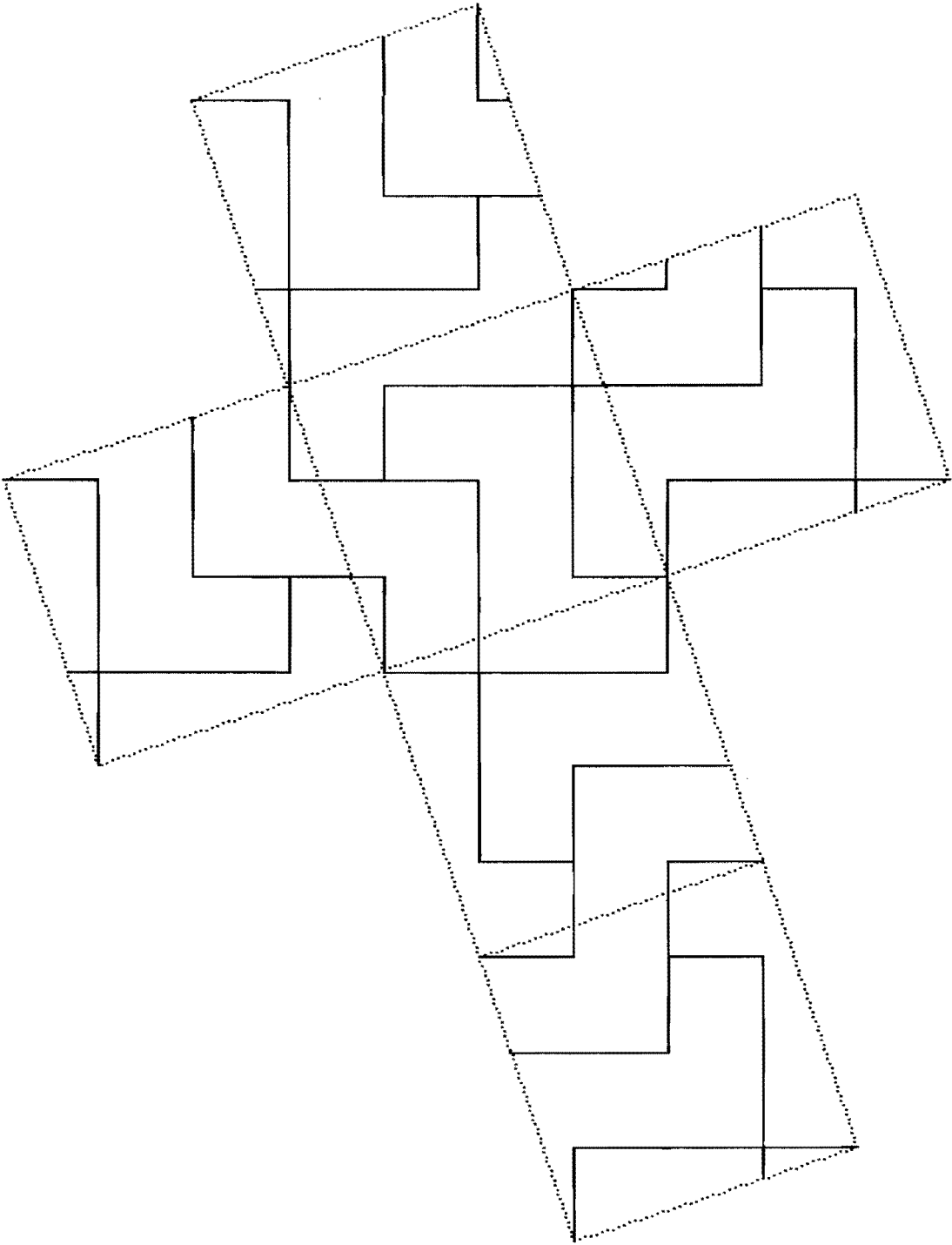
218 264 274 285 333 338 383 395 400 409 453 455

Four axes order 3 (D1 D2 D3 D4) and three axes order 2 (X Y Z)



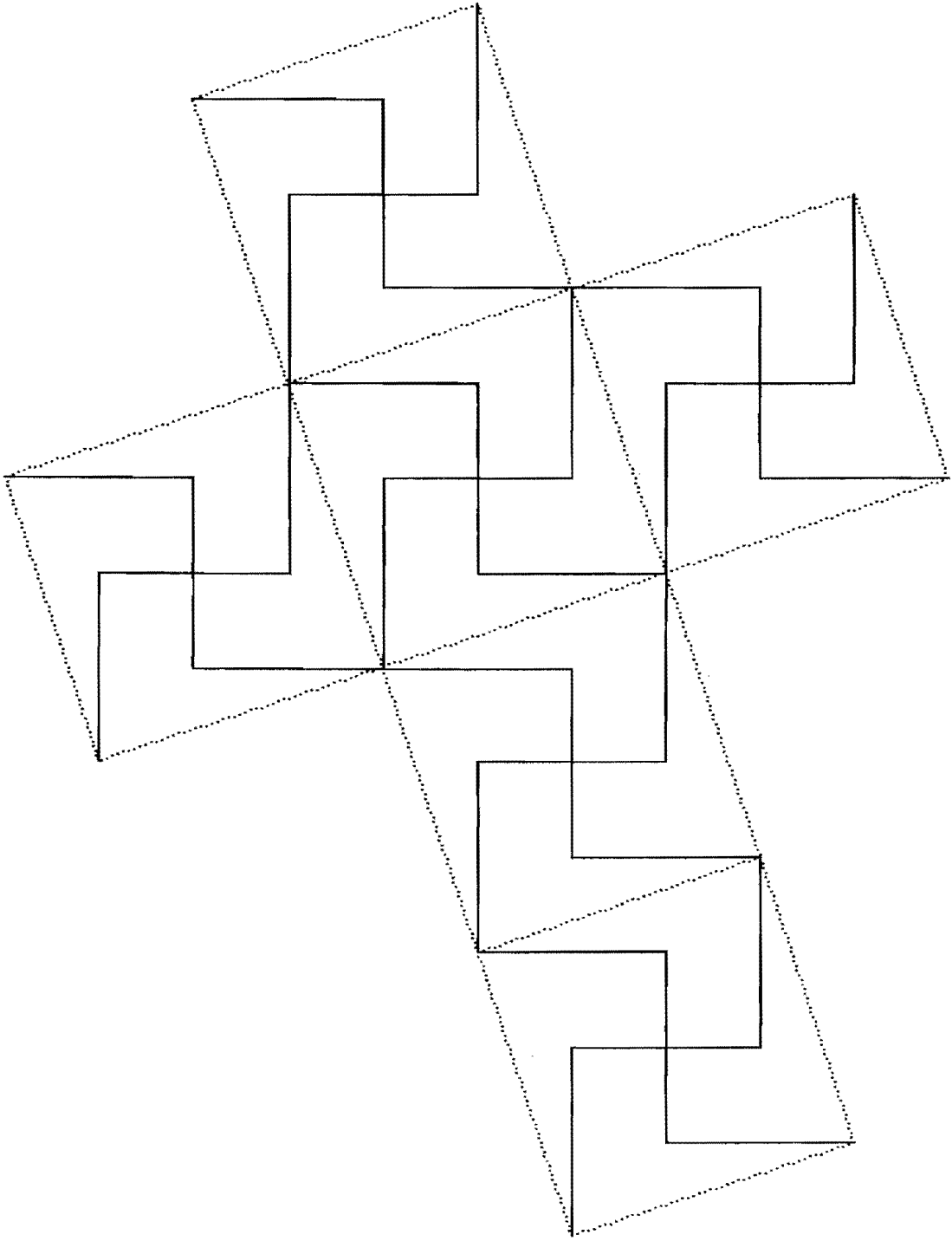
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Three axes order 2 (Y T1 T4)



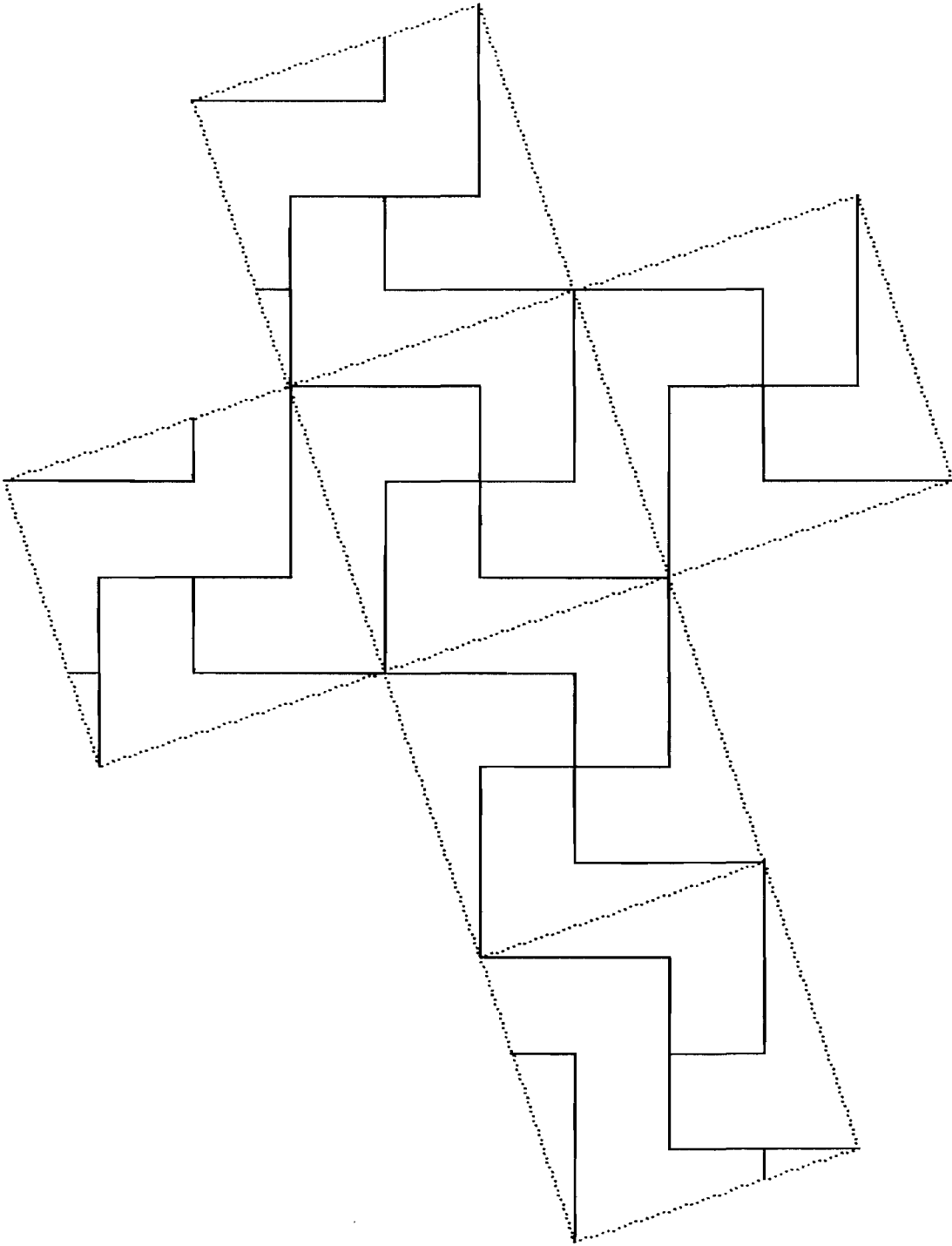
457 508 518 529 534 559 573 606 630 648 682 692

One axis order 3 (D3) and three axes order 2 (T1 T3 T6)

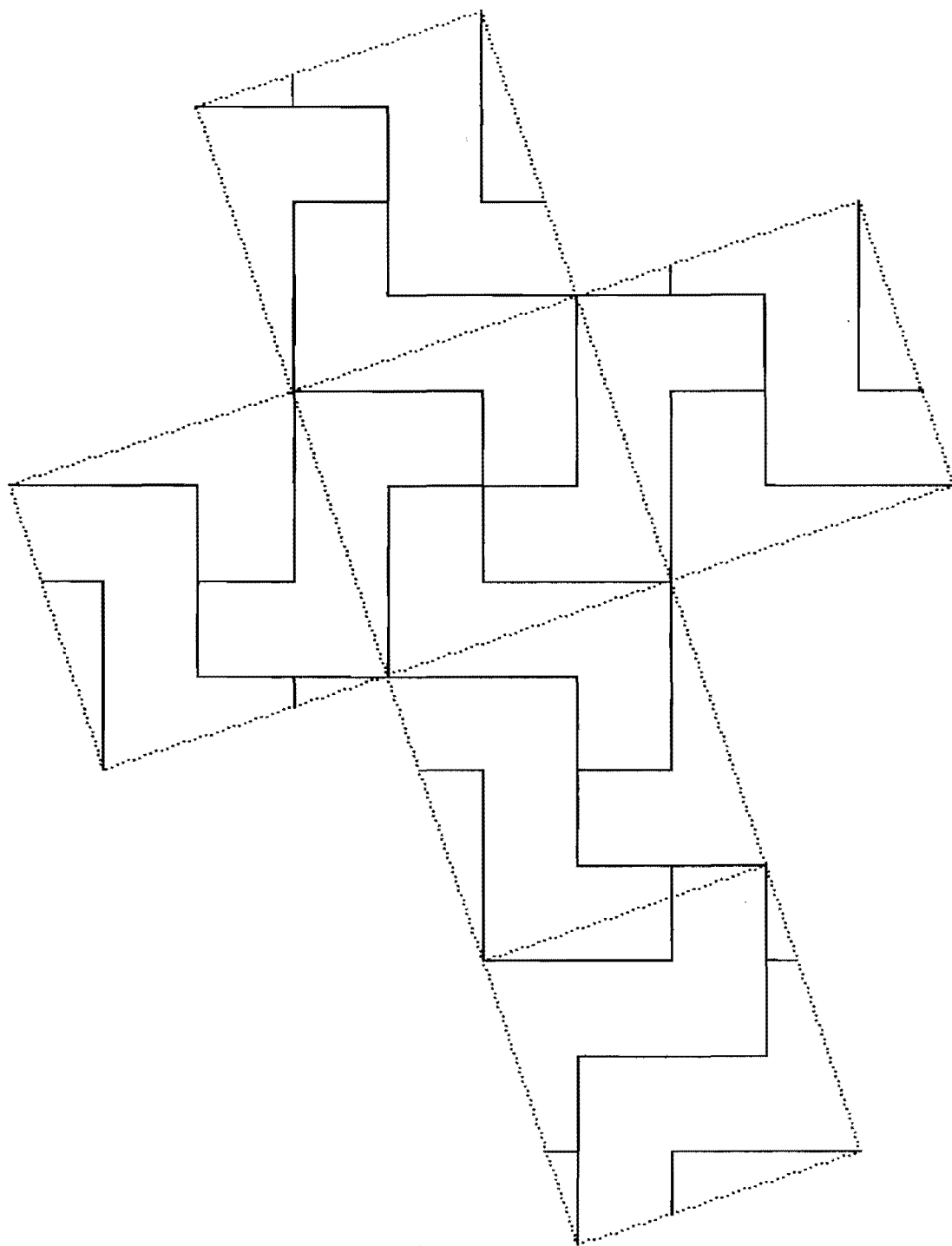


458 503 510 543 574 577 604 634 637 664 690 691

ROTATIONAL-INVARIANT

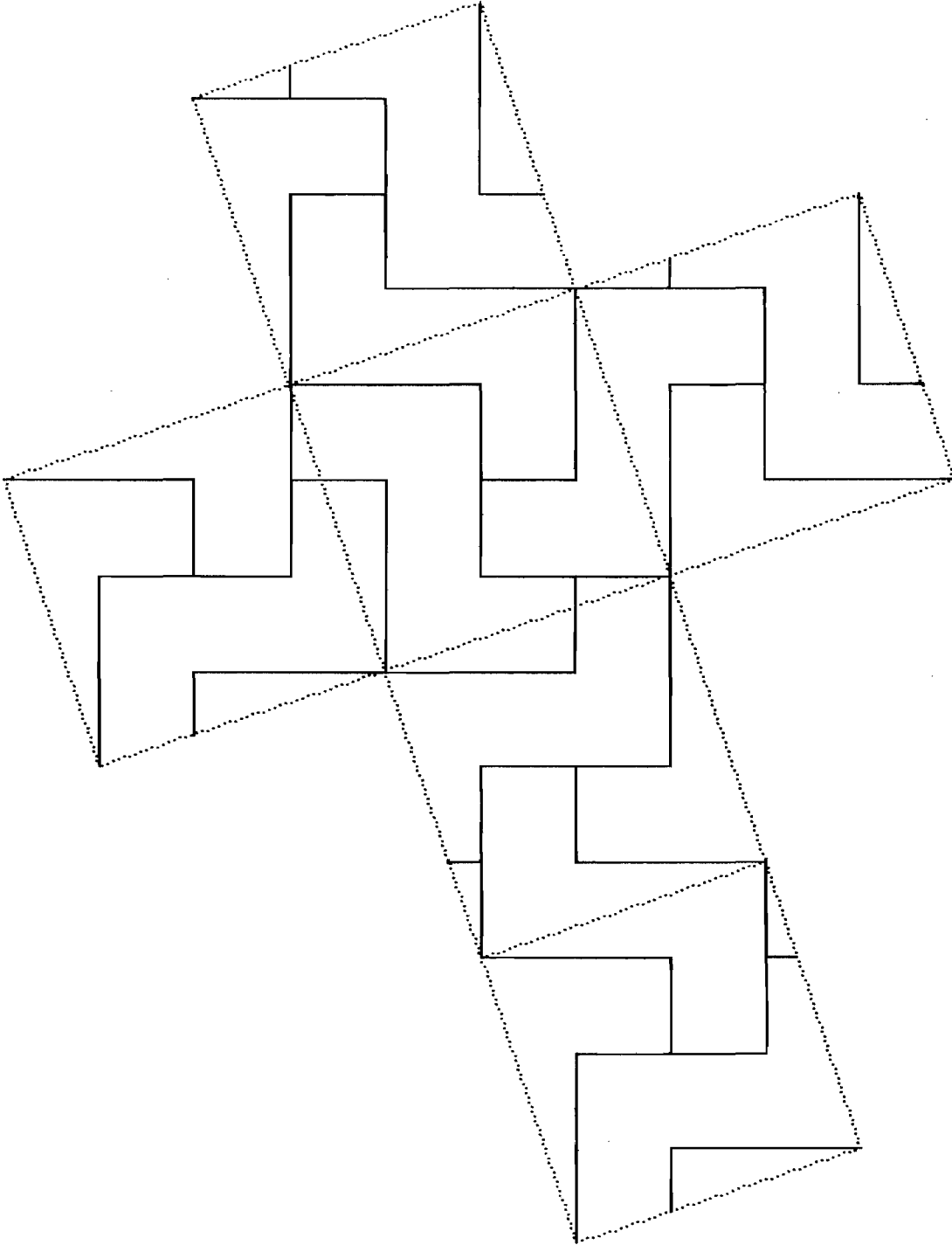


458 503 510 543 574 577 604 634 638 659 688 690



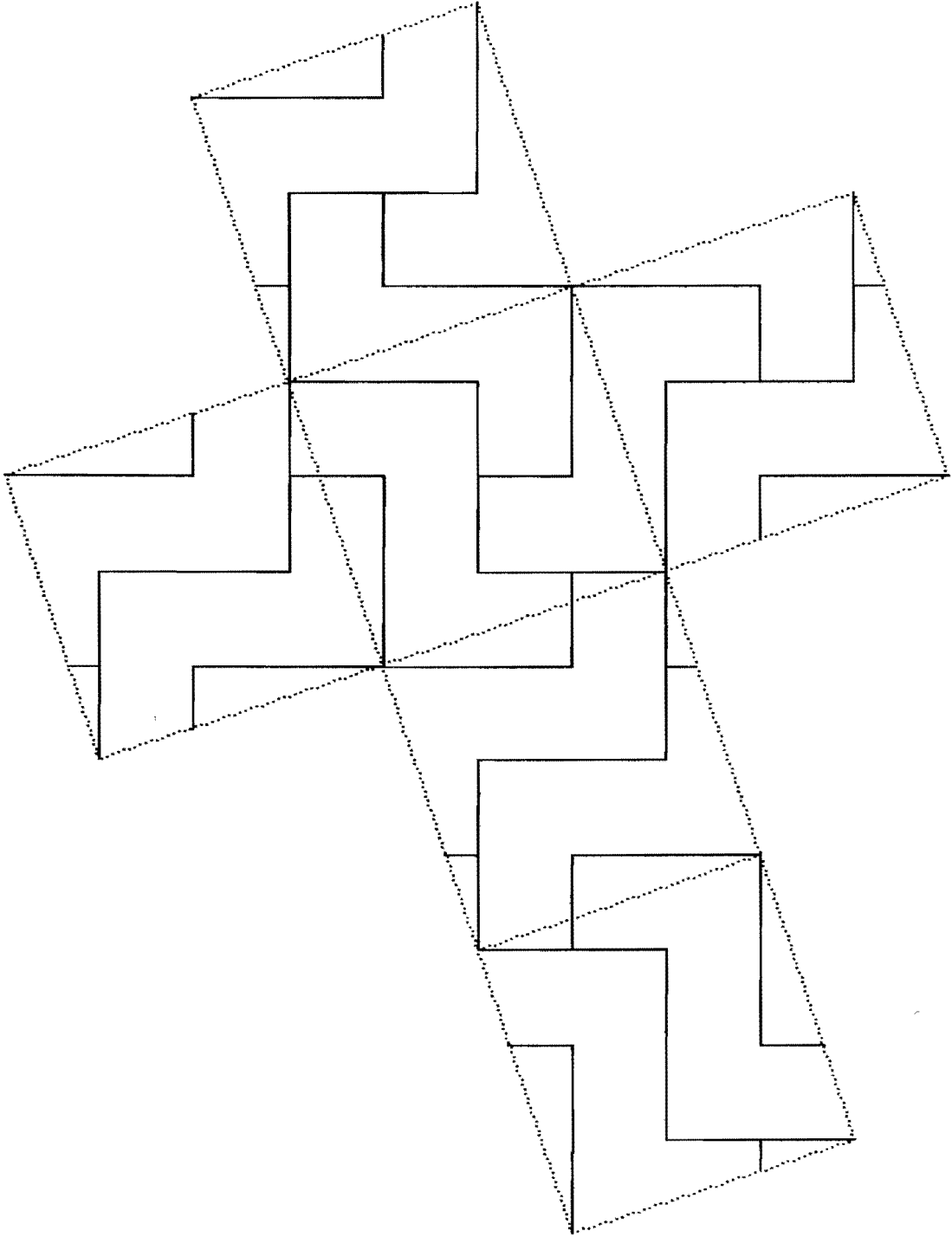
458 503 510 543 574 578 599 631 636 650 685 691

One axis order 2 (X)



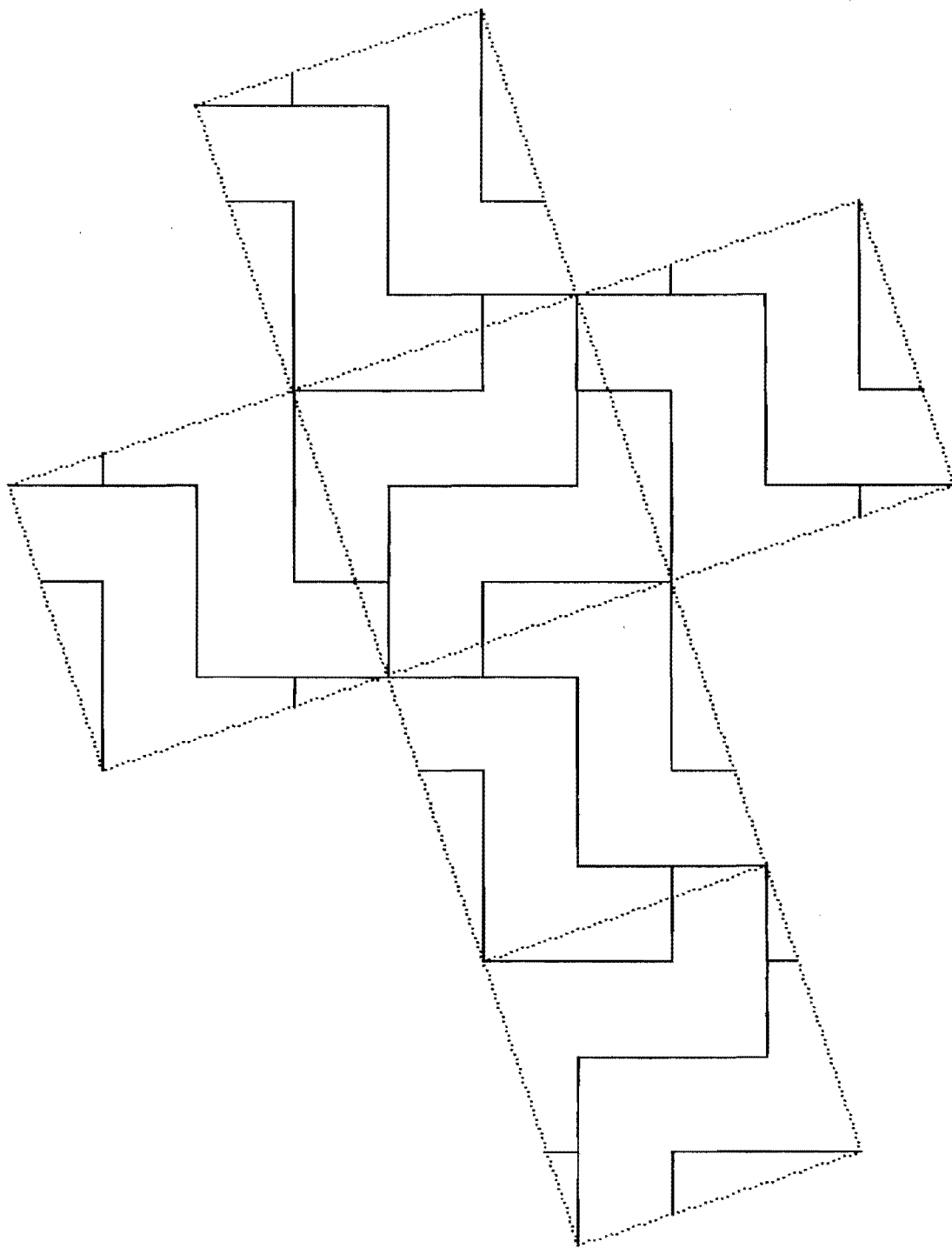
458 503 511 537 571 574 604 636 637 650 685 691

One axis order 3 (D4) and three axes order 2 (T1 T2 T5)



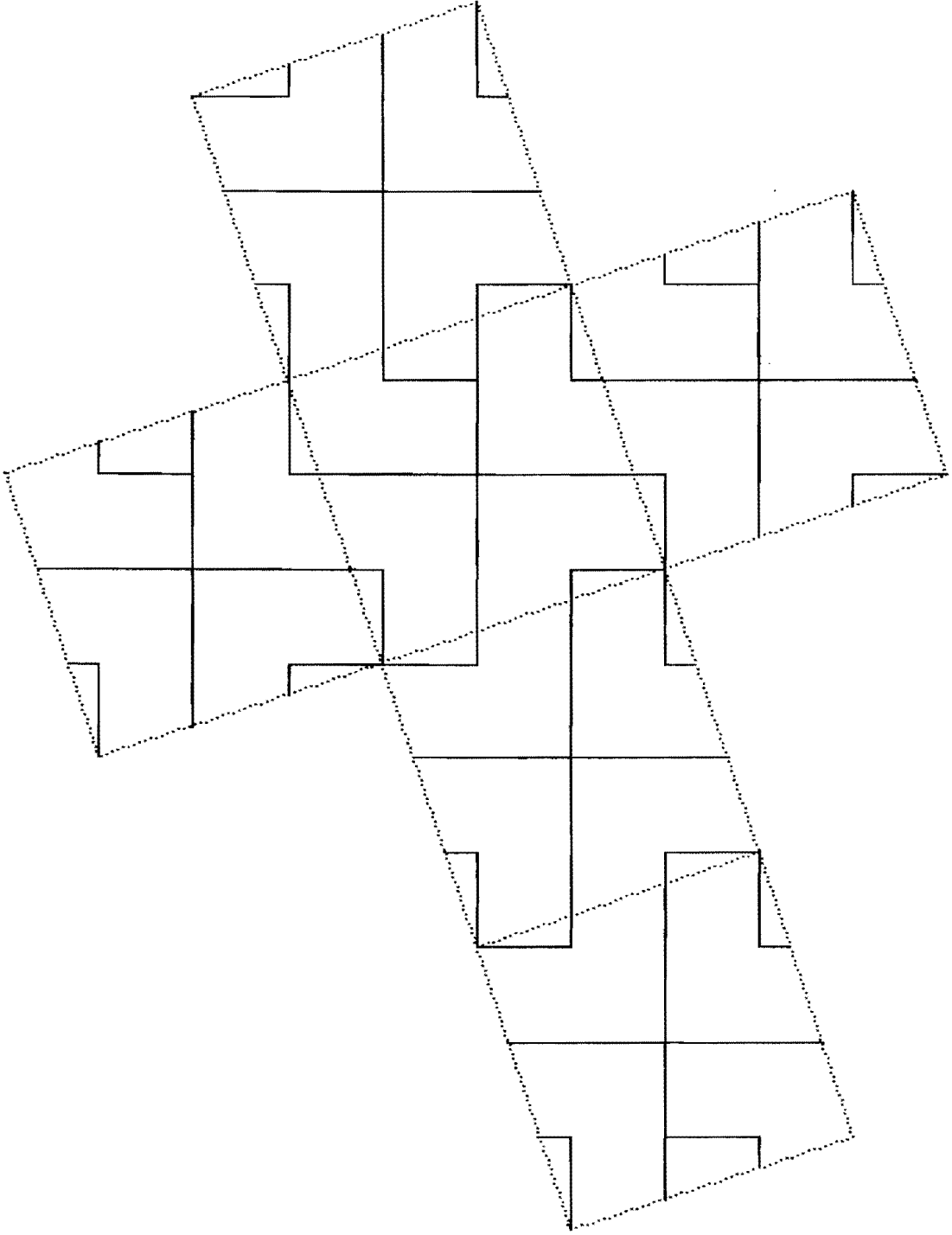
458 503 511 537 571 576 590 627 638 659 688 690

One axis order 3 (D2)



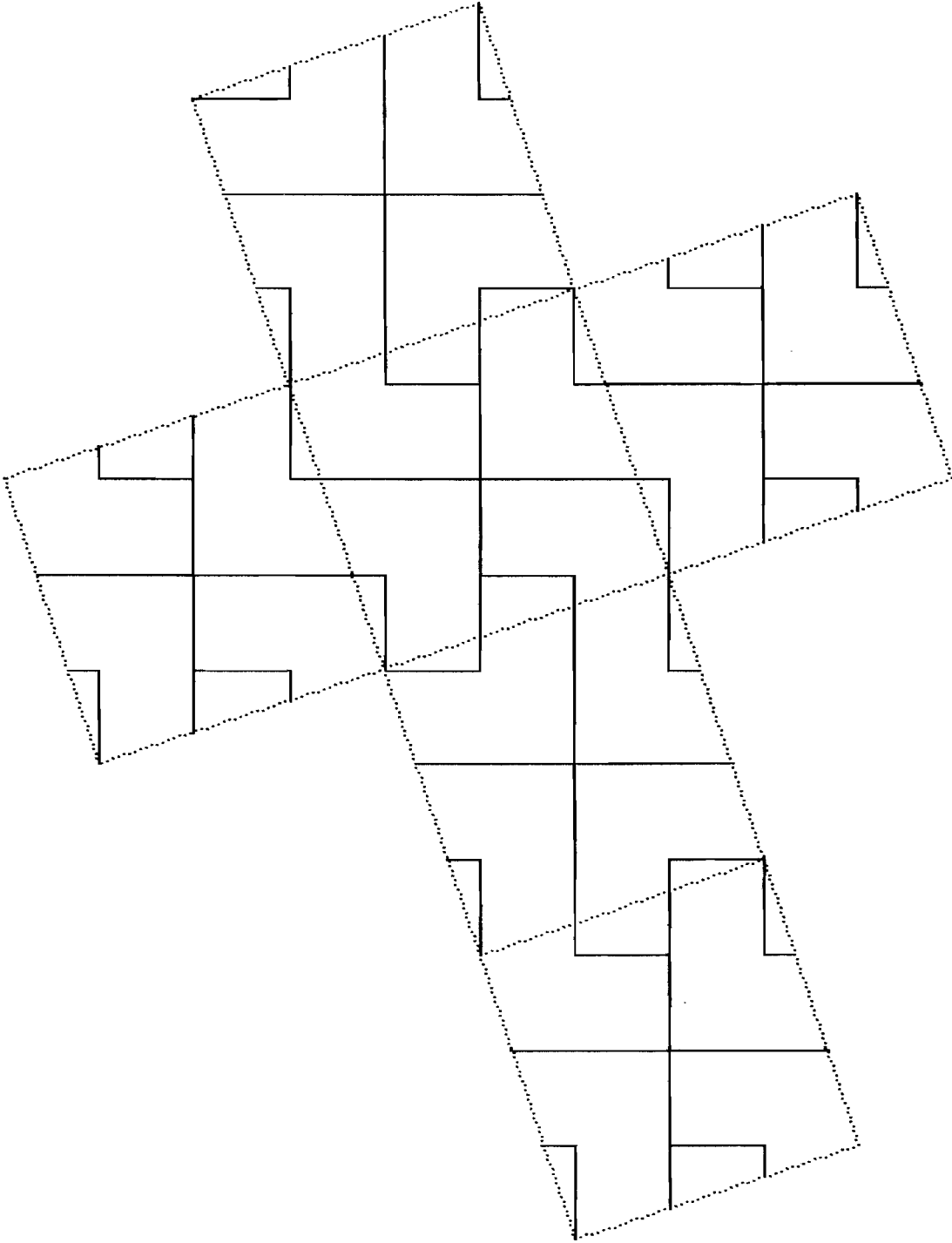
463 497 506 526 562 567 578 599 631 636 650 685

Four axes order 3 (D1 D2 D3 D4) and three axes order 2 (X Y Z)



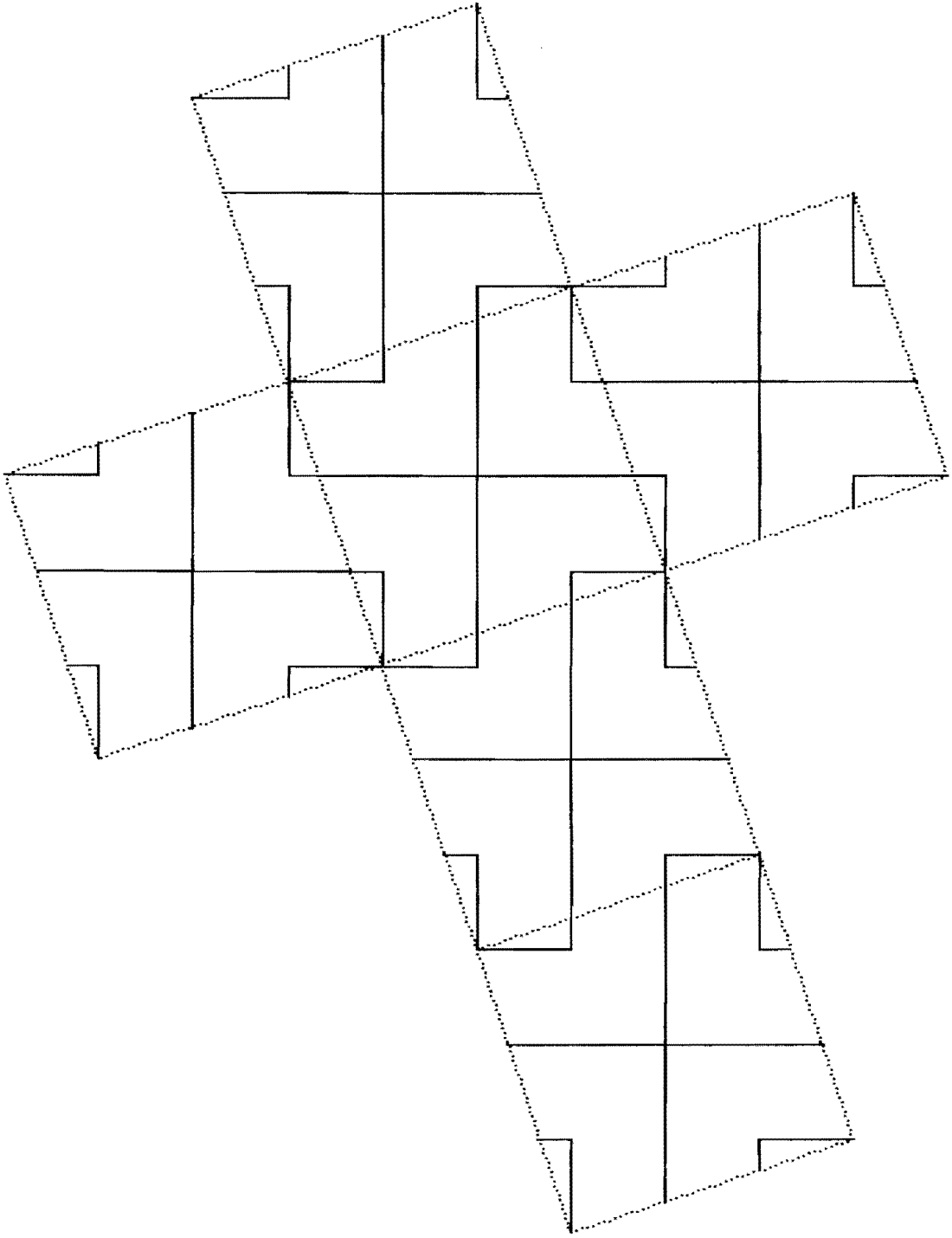
467 483 498 534 545 561 572 606 622 632 667 679

One axis order 4 (Z)



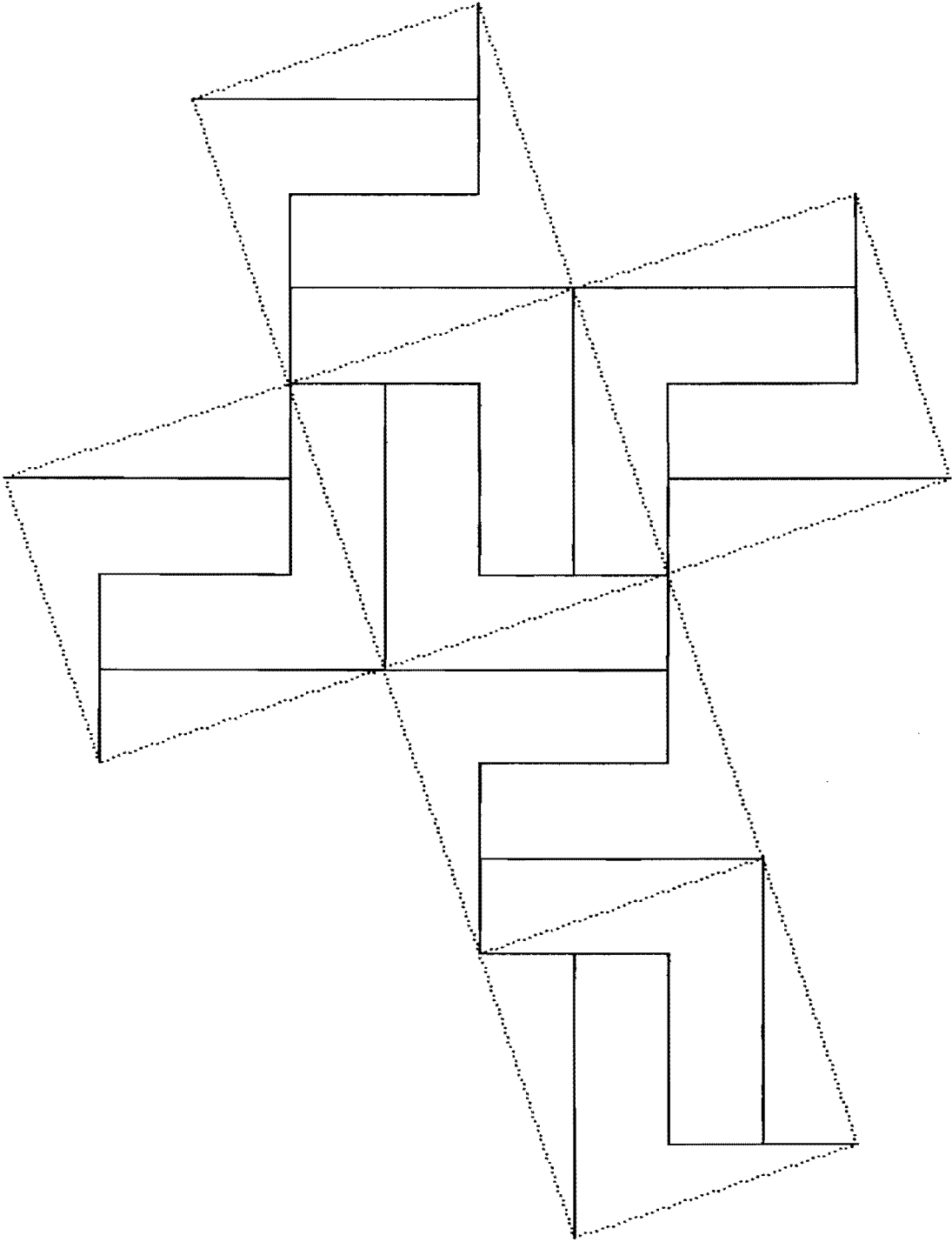
467 483 498 534 546 558 596 605 622 632 667 679

One axis order 4 (Z) and four axes order 2 (X Y T2 T3)



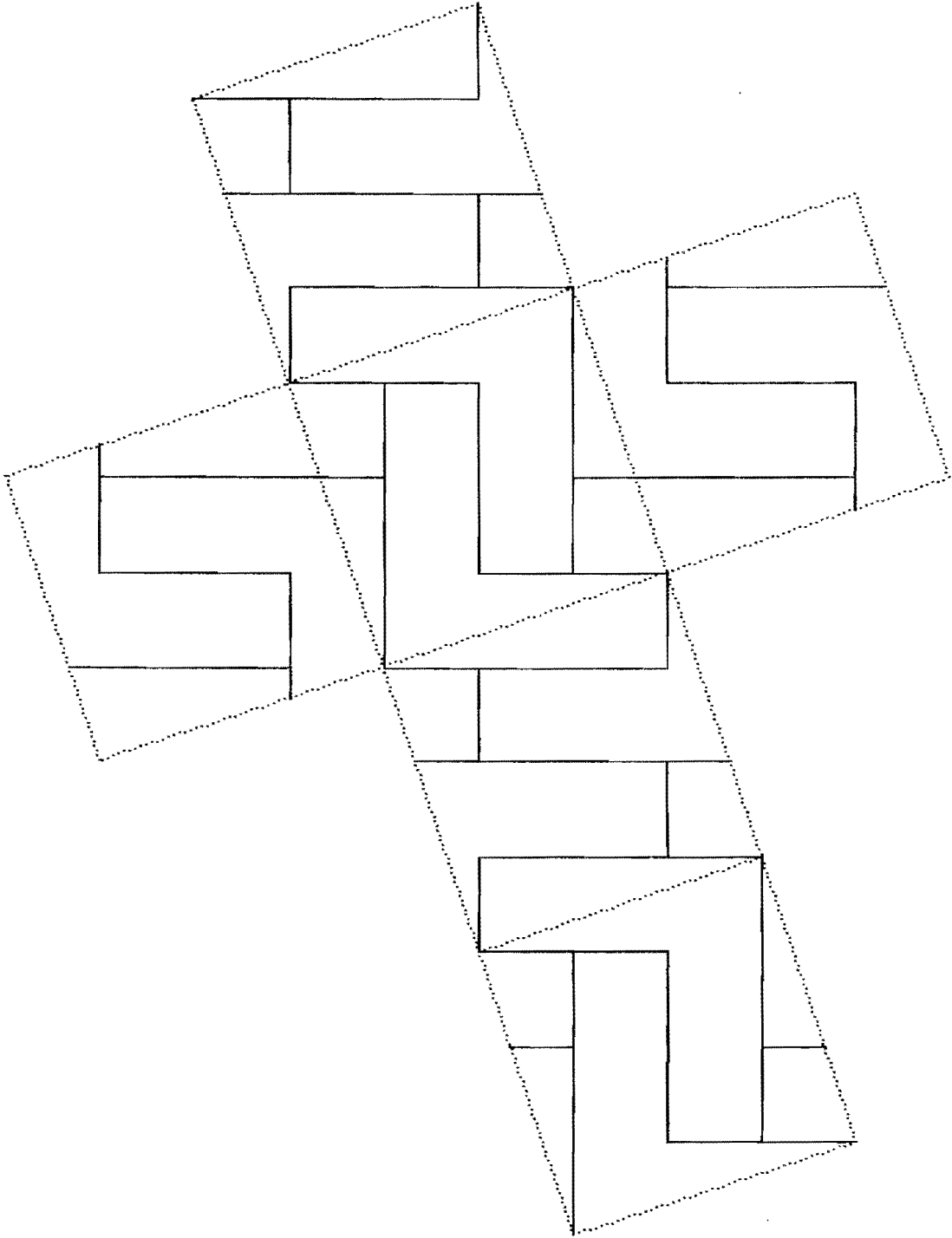
474 484 498 508 534 545 561 572 606 622 632 666

ROTATIONAL-INVARIANT



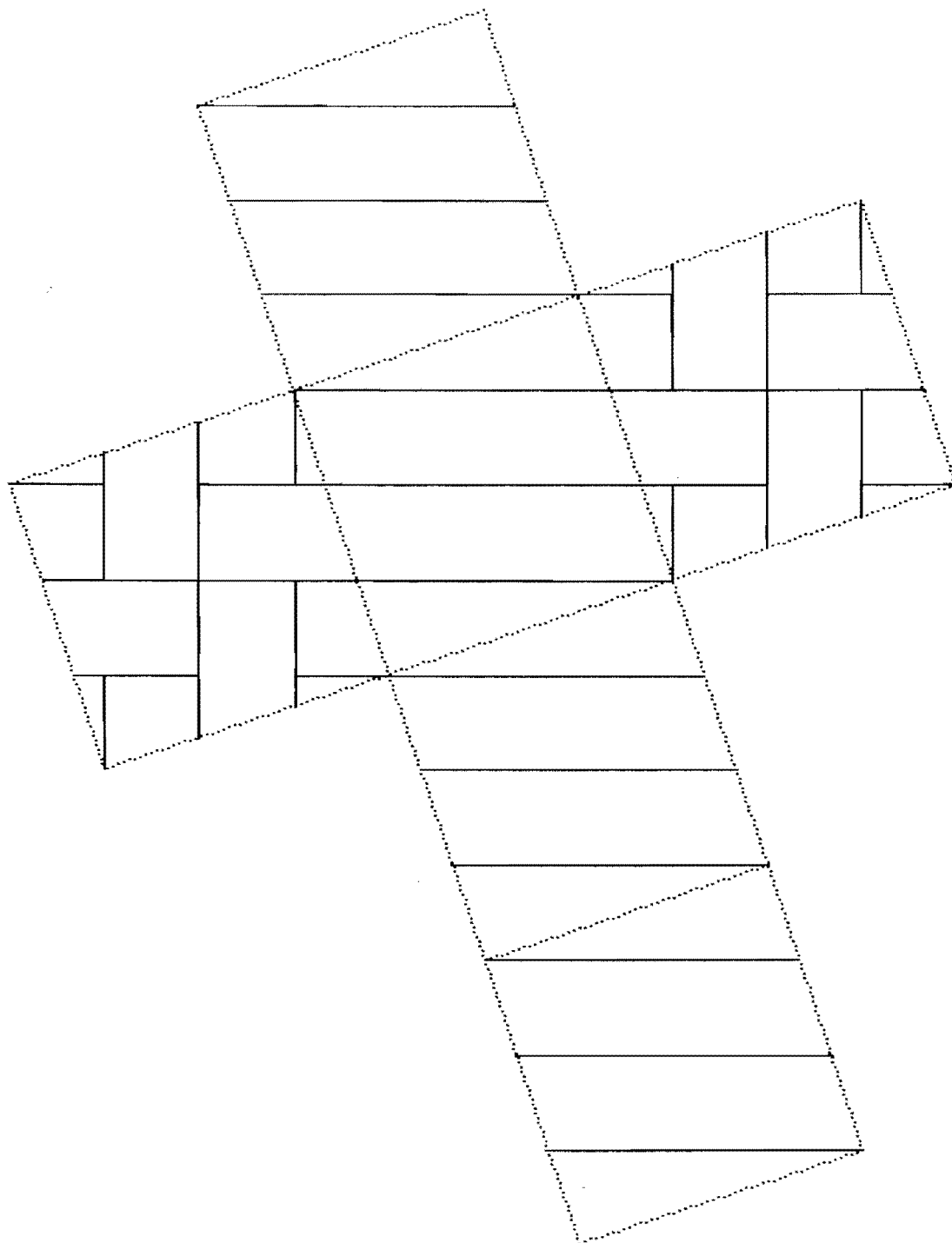
698 744 757 763 817 821 866 879 884 888 933 935

Four axes order 3 (D1 D2 D3 D4) and three axes order 2 (X Y Z)



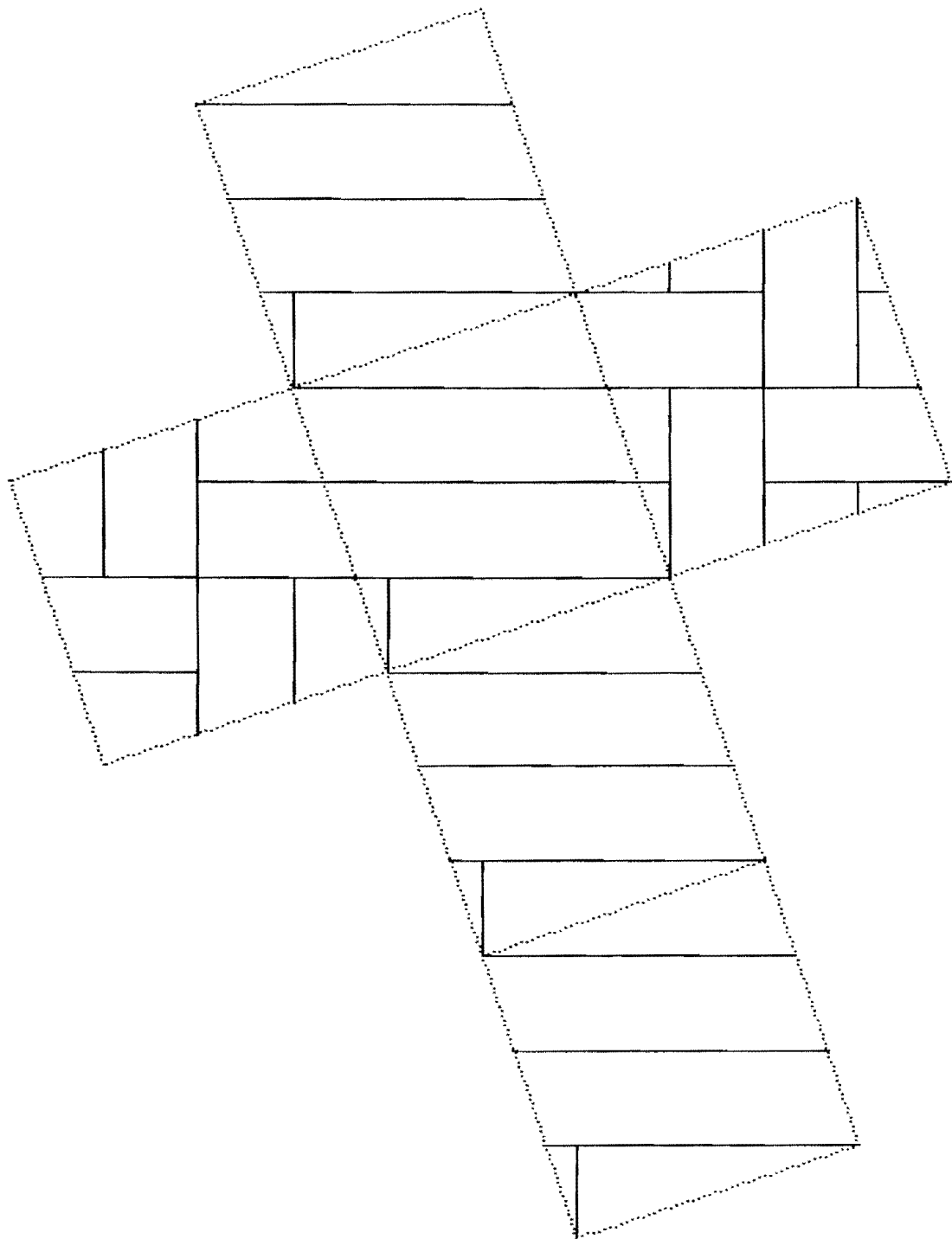
698 746 758 763 780 794 837 857 866 888 903 916

Three axes order 2 (X Y Z)



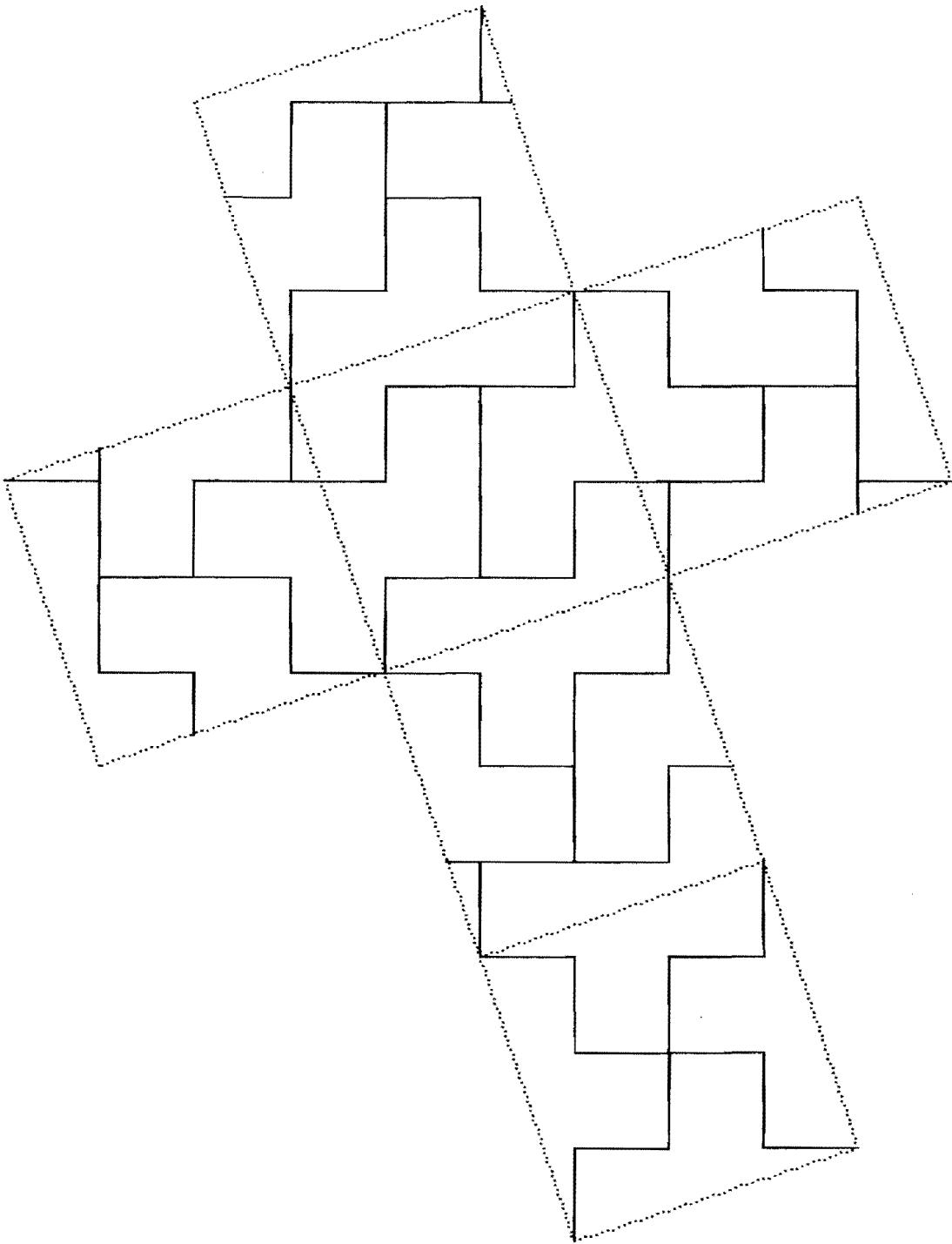
1177 1206 1218 1229 1243 1252 1262 1274 1281 1287 1293 1295

One axis order 4 (Y) and four axes order 2 (X Z T1 T4)



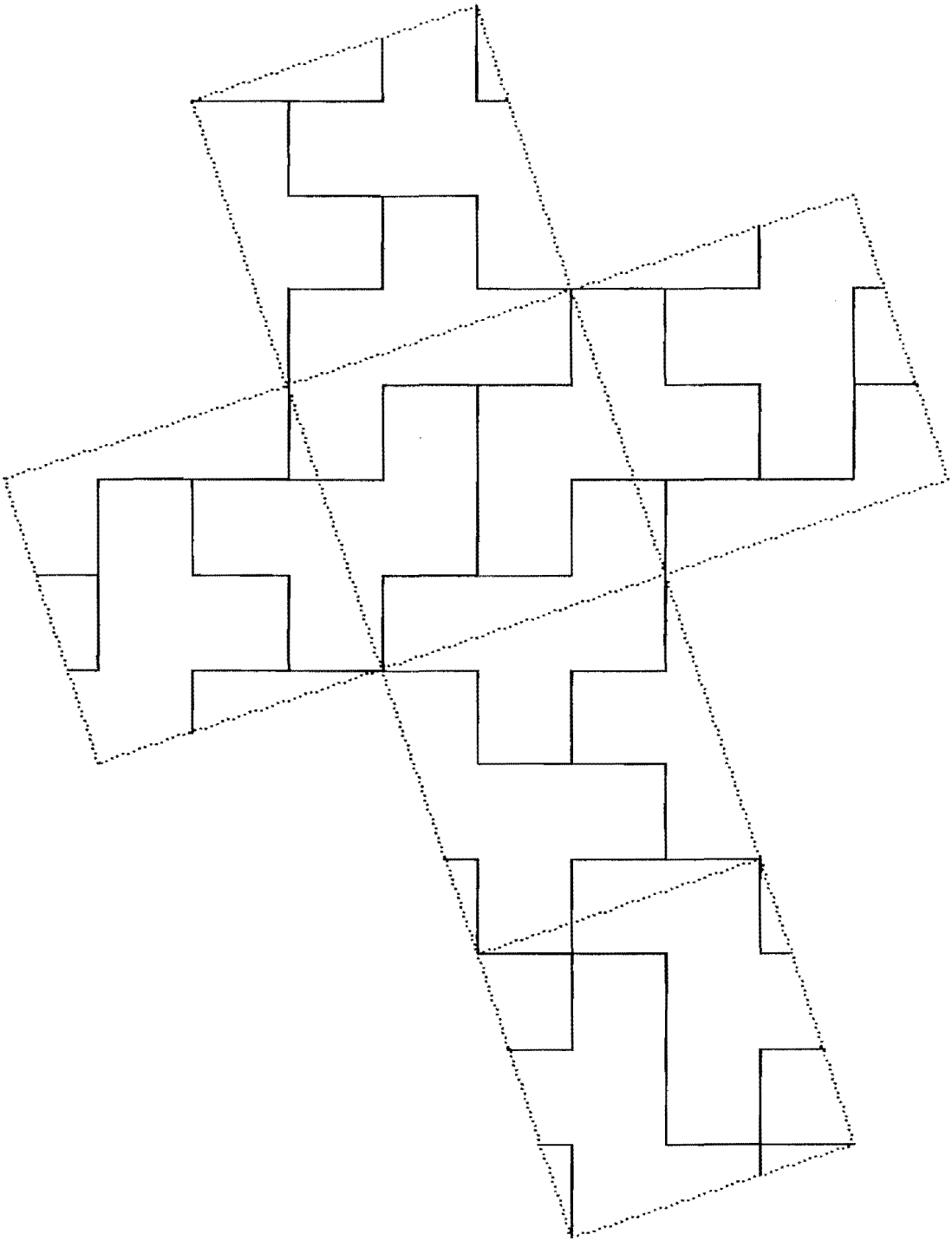
1178 1205 1218 1233 1235 1252 1266 1268 1281 1290 1291 1295

One axis order 4 (Y)



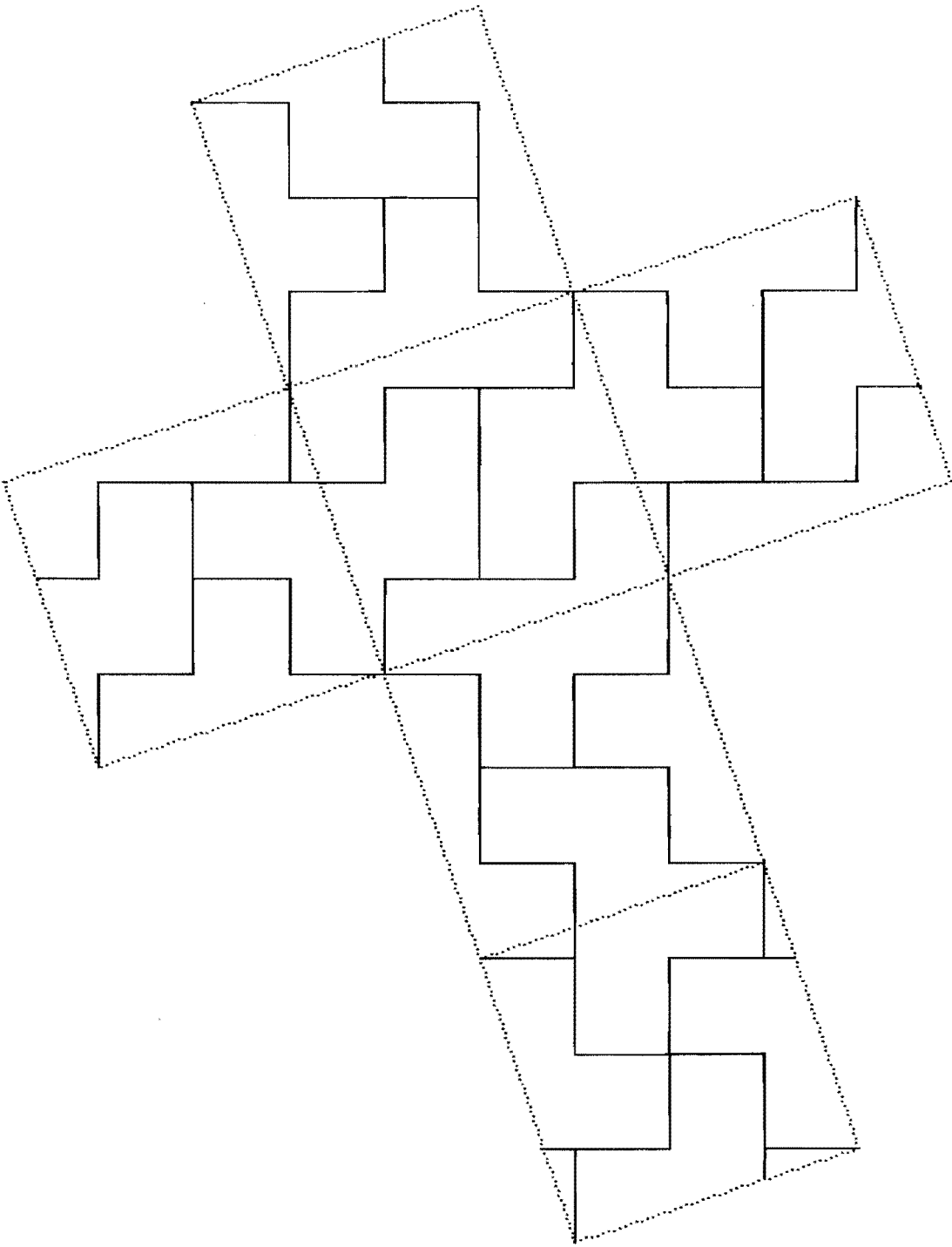
1297 1380 1411 1468 1514 1526 1583 1602 1629 1692 1750 1751

One axis order 2 (X)

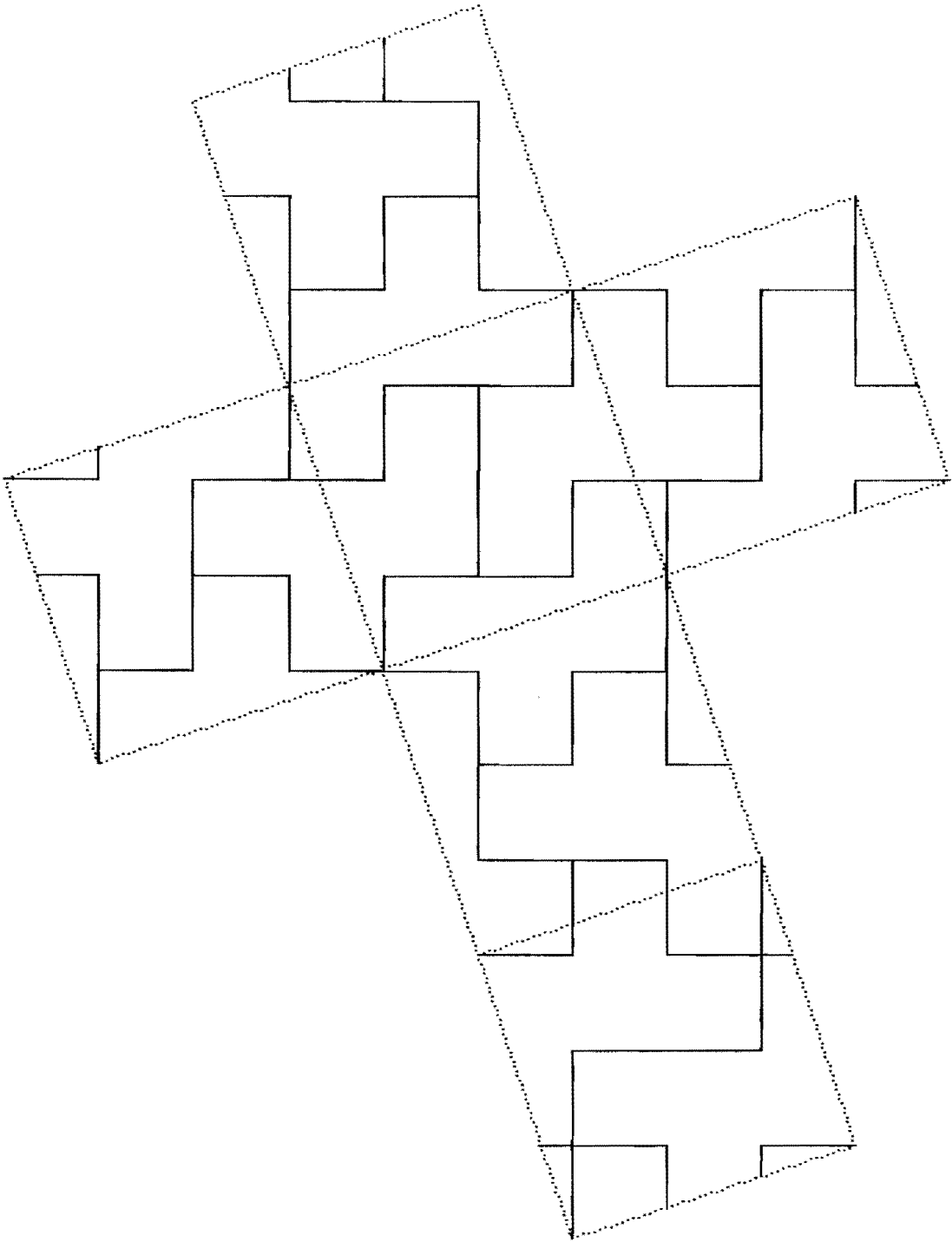


1297 1380 1411 1468 1515 1523 1603 1613 1648 1708 1716 1737

One axis order 2 (X)

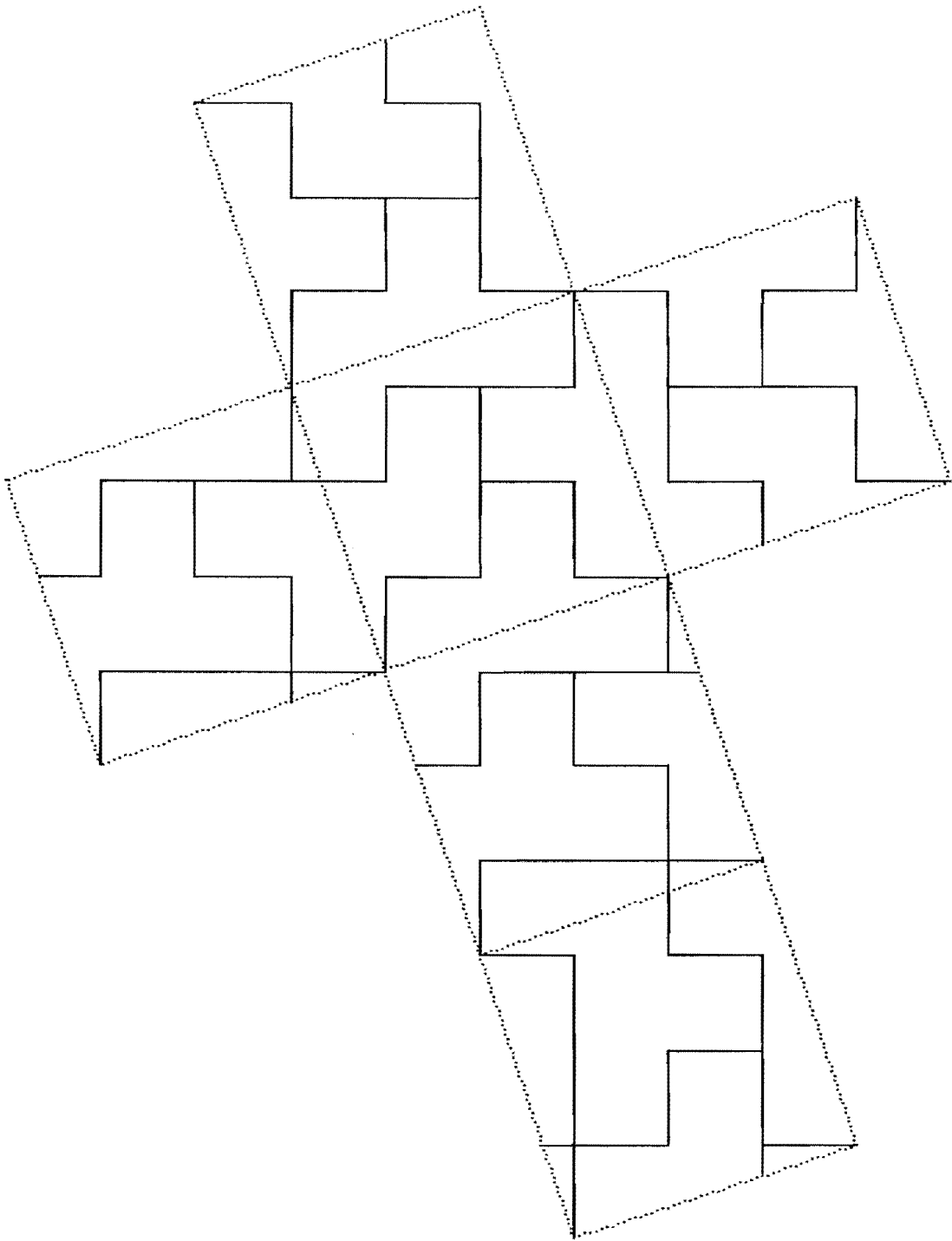


1297 1380 1411 1468 1515 1527 1568 1640 1653 1694 1716 1734

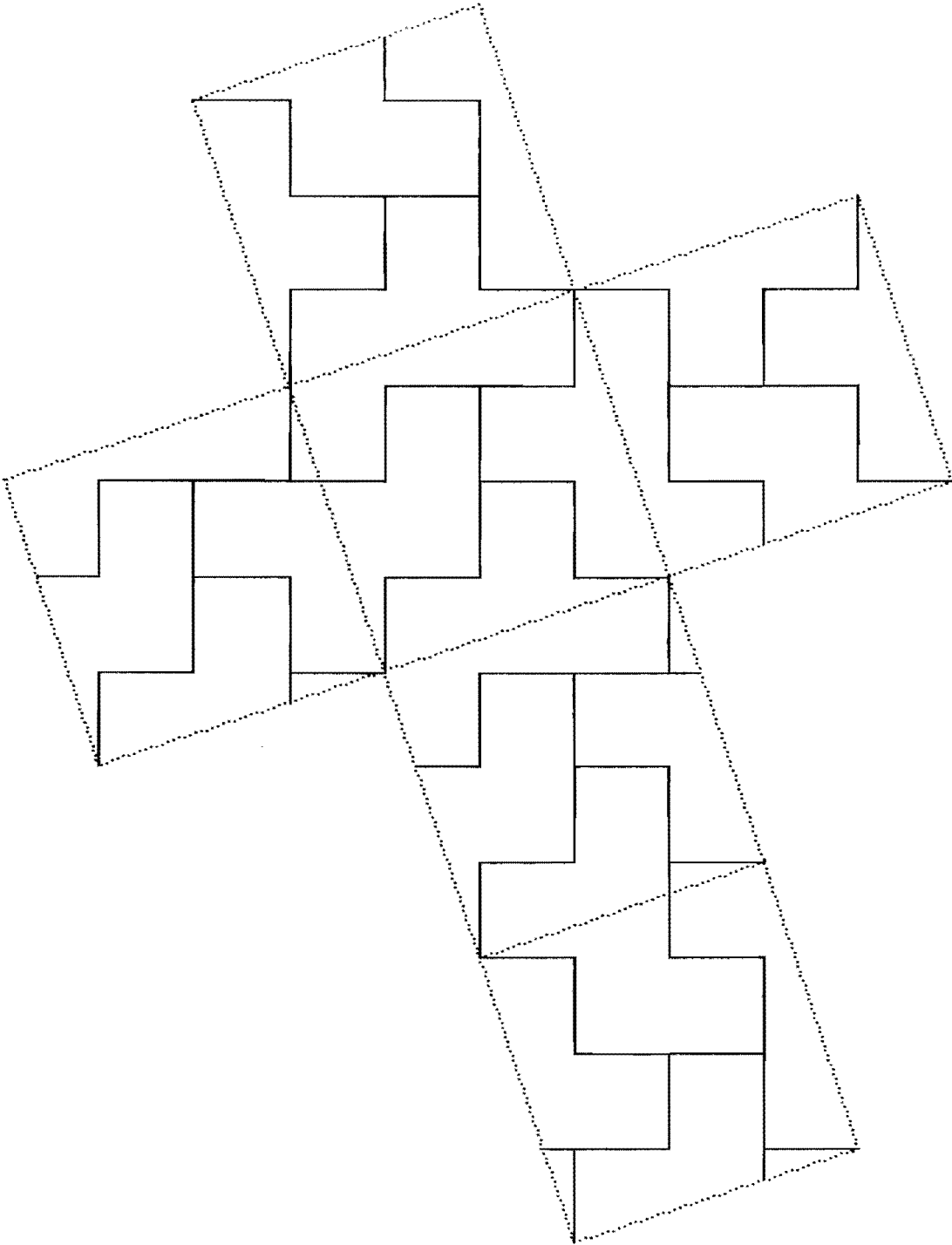


1297 1380 1411 1468 1519 1527 1542 1614 1660 1719 1724 1734

One axis order 2 (X)

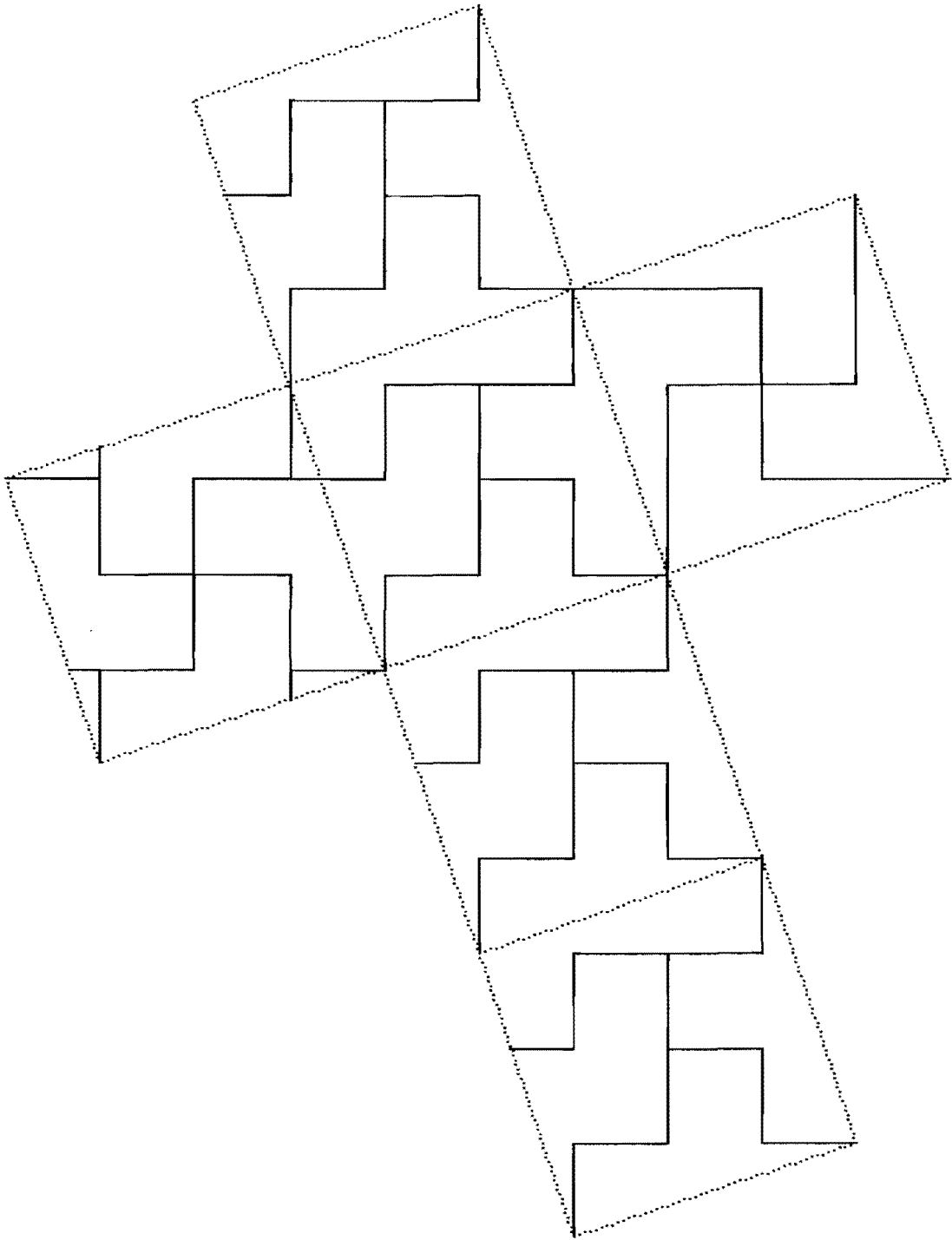


1297 1381 1411 1463 1531 1539 1609 1620 1642 1694 1716 1734

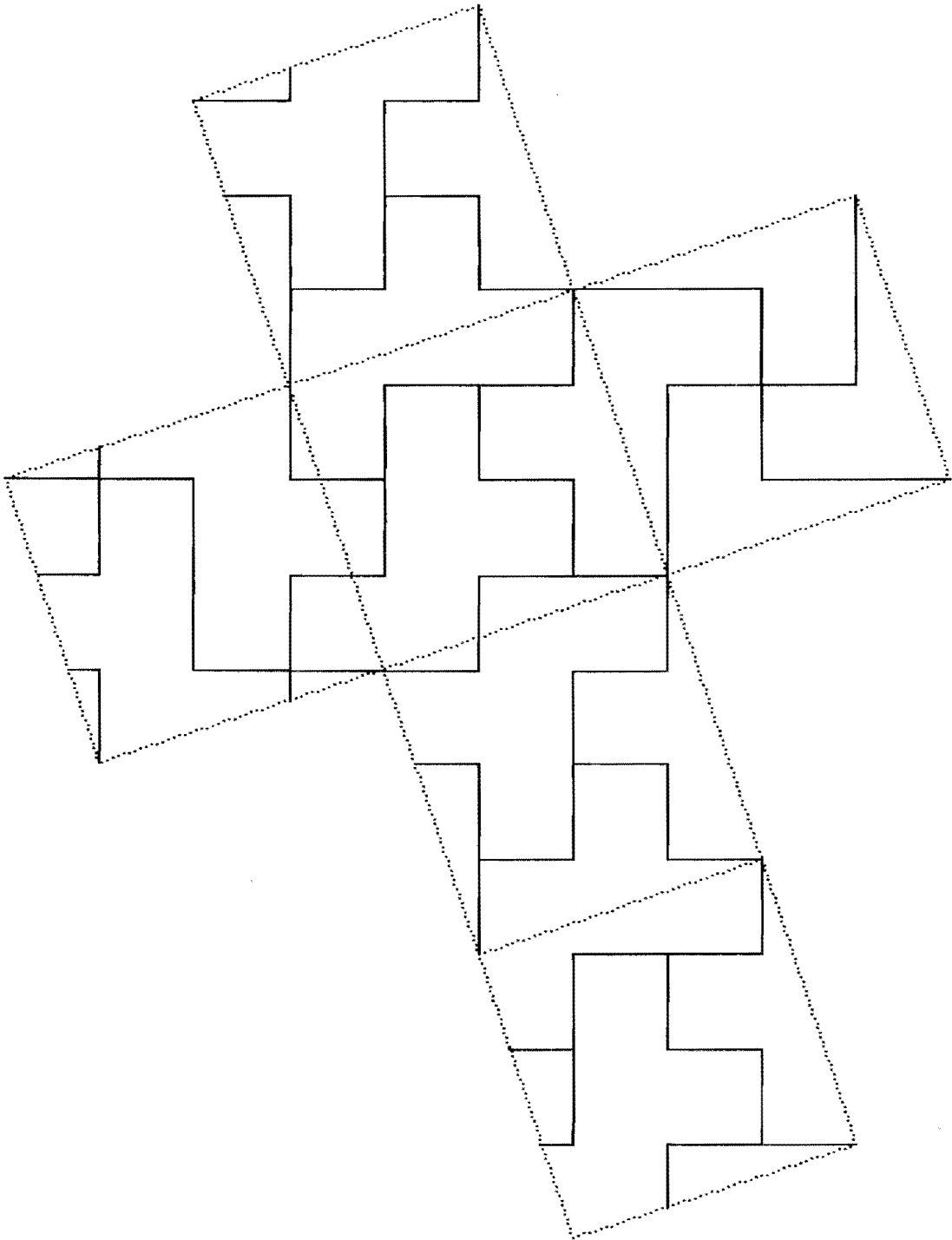


1297 1381 1411 1463 1533 1539 1580 1620 1640 1694 1716 1734

One axis order 3 (D2)

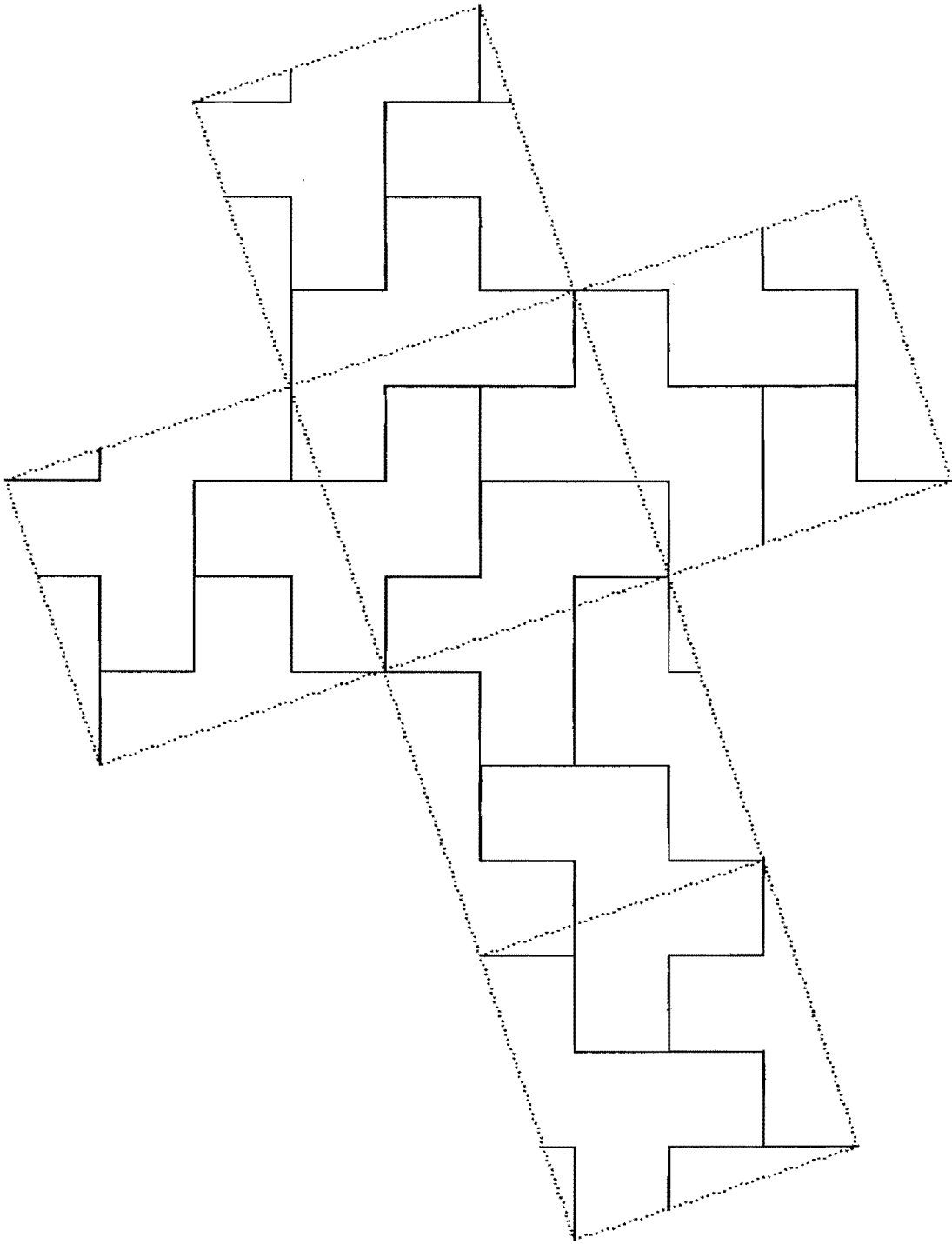


1297 1382 1411 1463 1516 1533 1578 1630 1647 1692 1743 1750

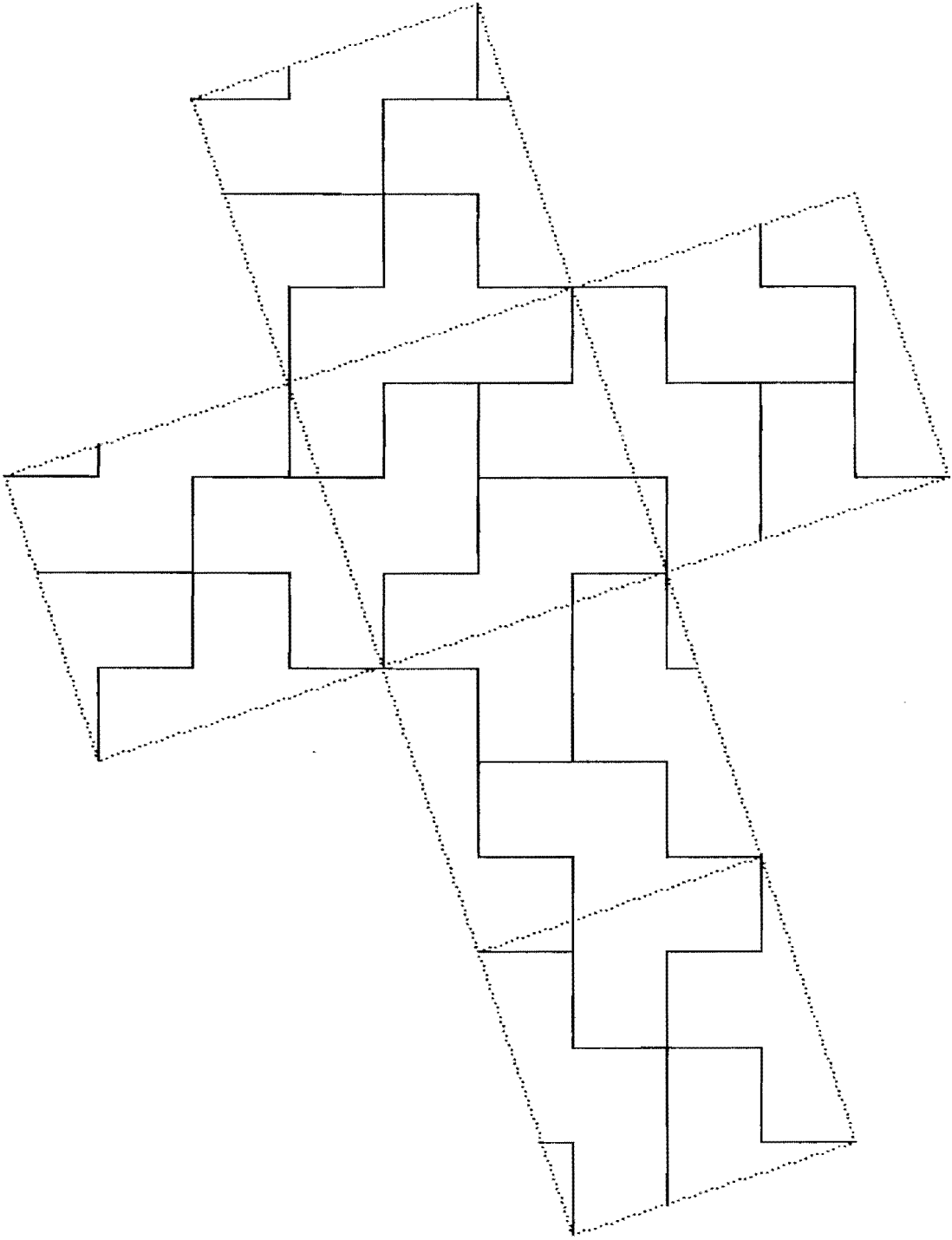


1297 1382 1414 1448 1498 1516 1565 1578 1630 1650 1726 1743

One axis order 2 (Y)

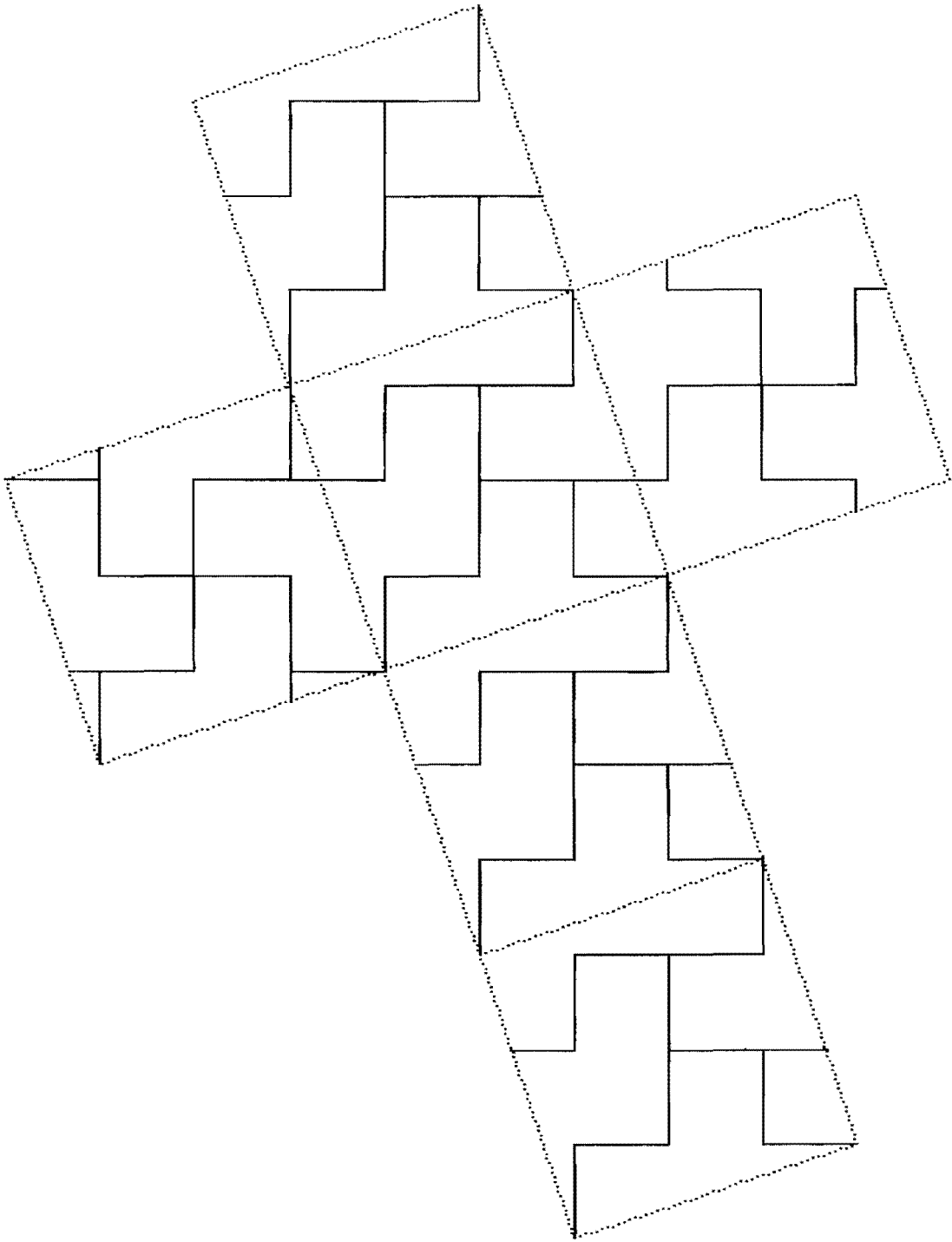


1297 1383 1411 1459 1512 1527 1568 1629 1637 1719 1726 1751



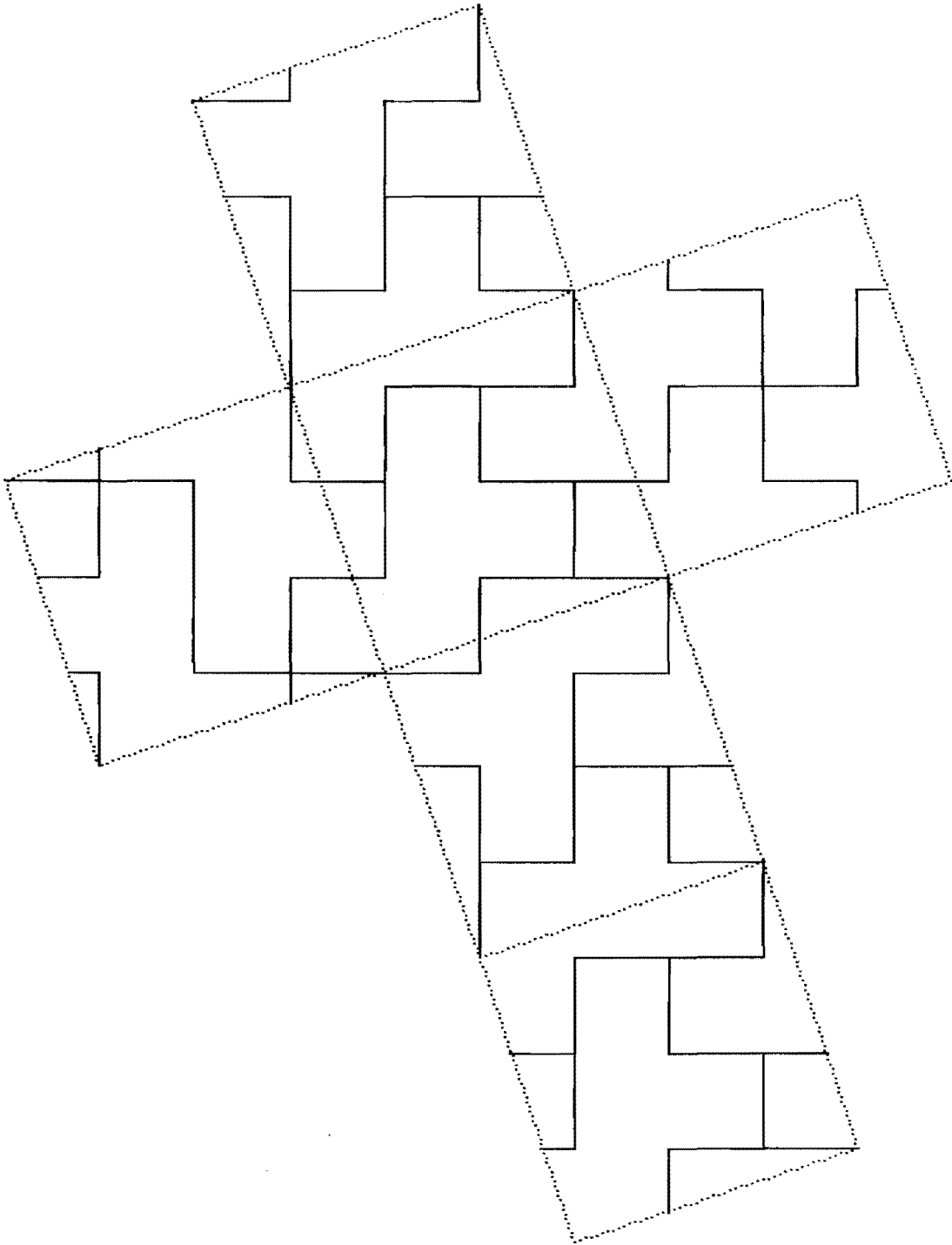
1297 1383 1411 1459 1512 1527 1568 1629 1639 1696 1718 1751

One axis order 3 (D3)



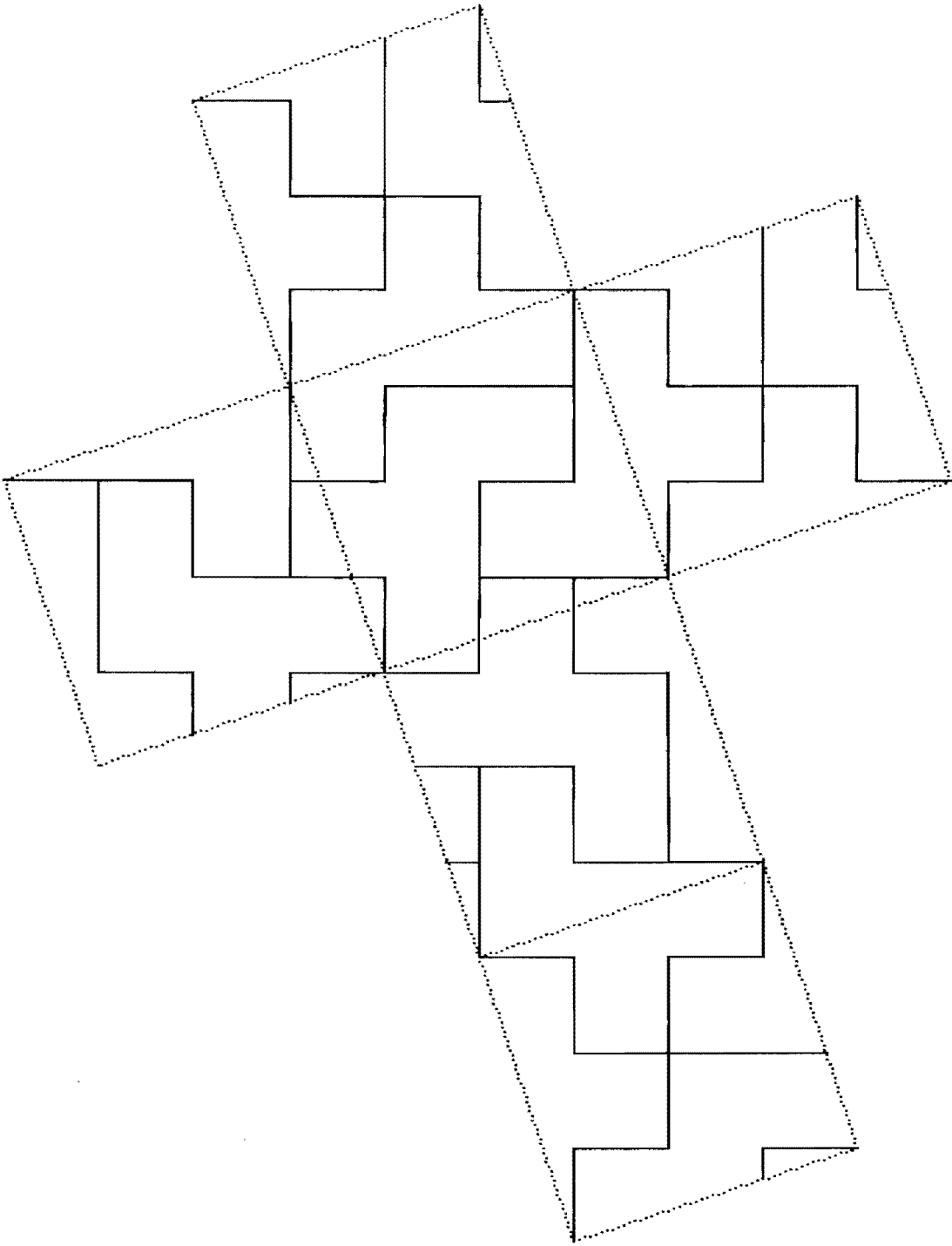
1297 1384 1411 1463 1478 1533 1578 1593 1647 1692 1707 1750

One axis order 4 (Y)



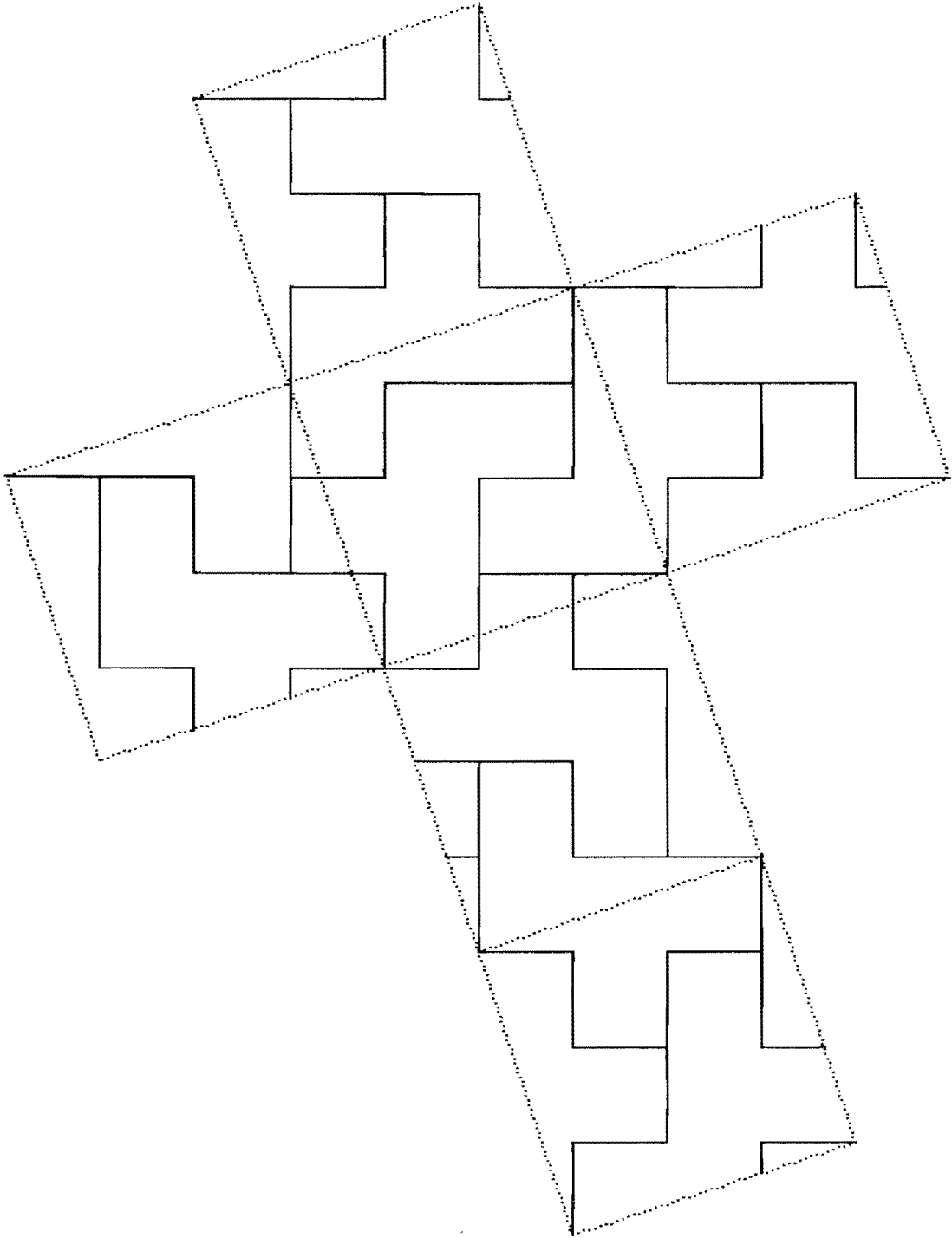
1297 1384 1414 1448 1478 1498 1565 1578 1593 1650 1707 1726

One axis order 2 (Y)



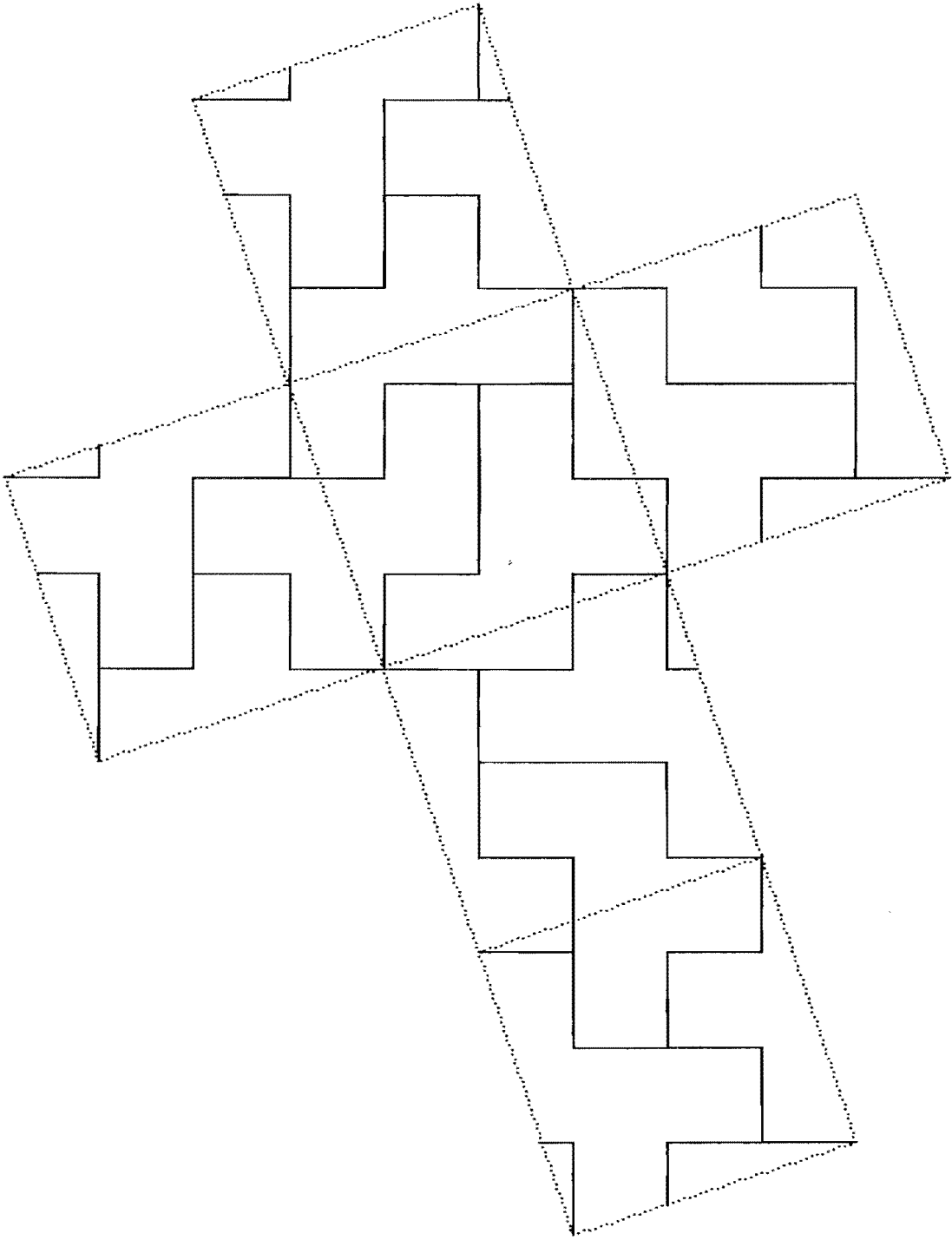
1297 1385 1407 1492 1501 1506 1573 1602 1631 1688 1740 1746

One axis order 3 (D4)



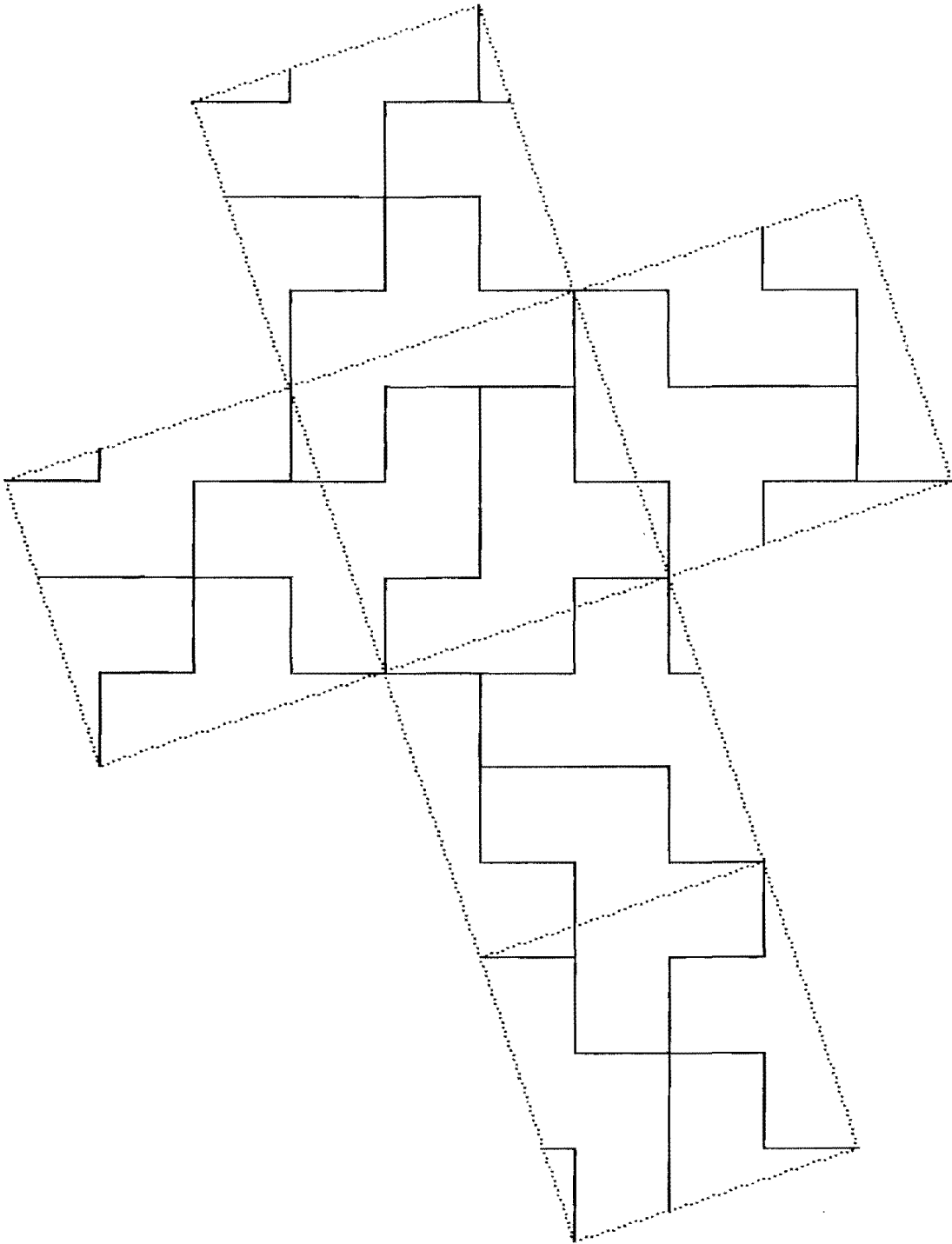
1297 1385 1407 1492 1501 1506 1573 1602 1633 1658 1737 1746

One axis order 3 (D4)



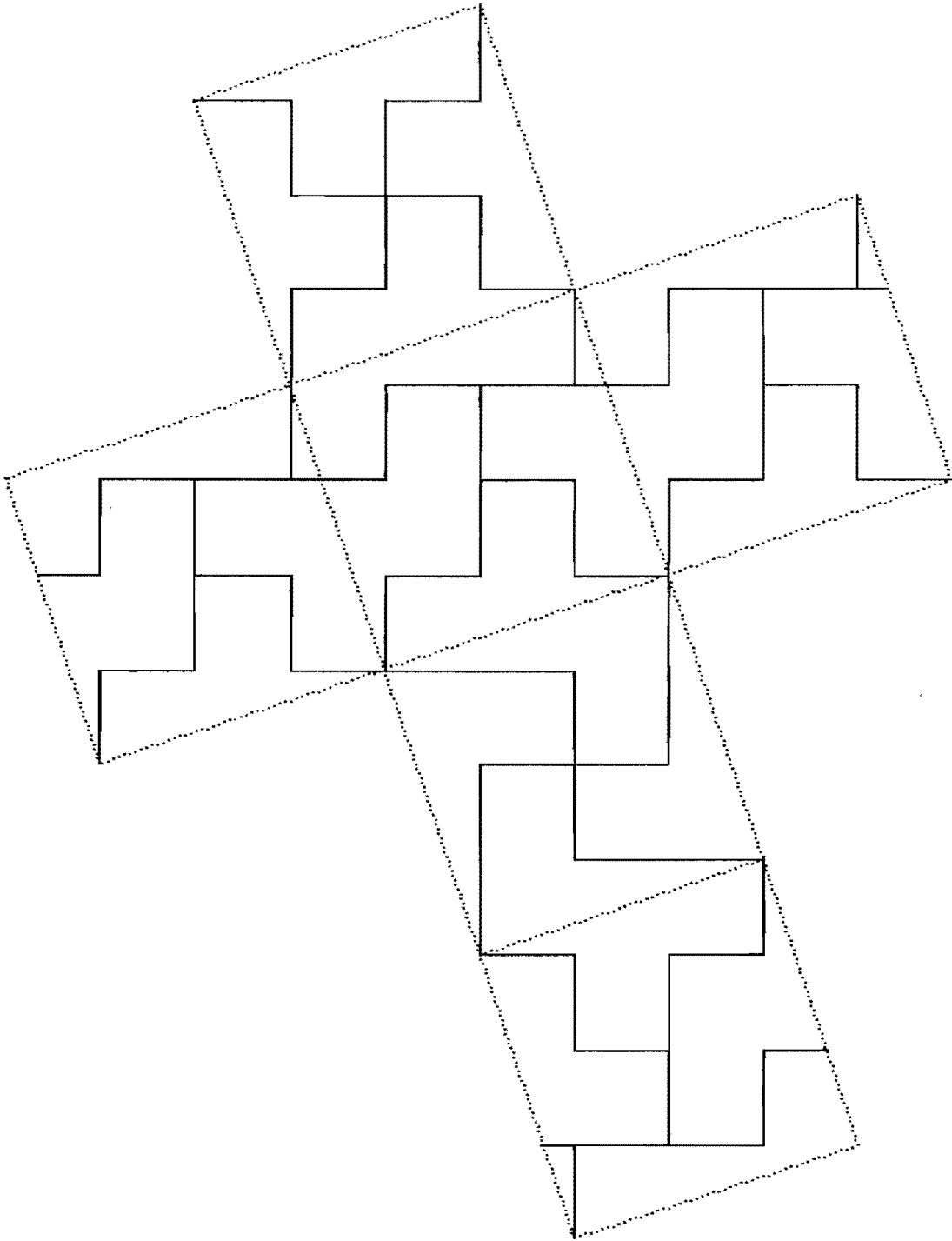
1297 1386 1411 1422 1509 1527 1568 1629 1637 1719 1726 1751

One axis order 3 (D3)

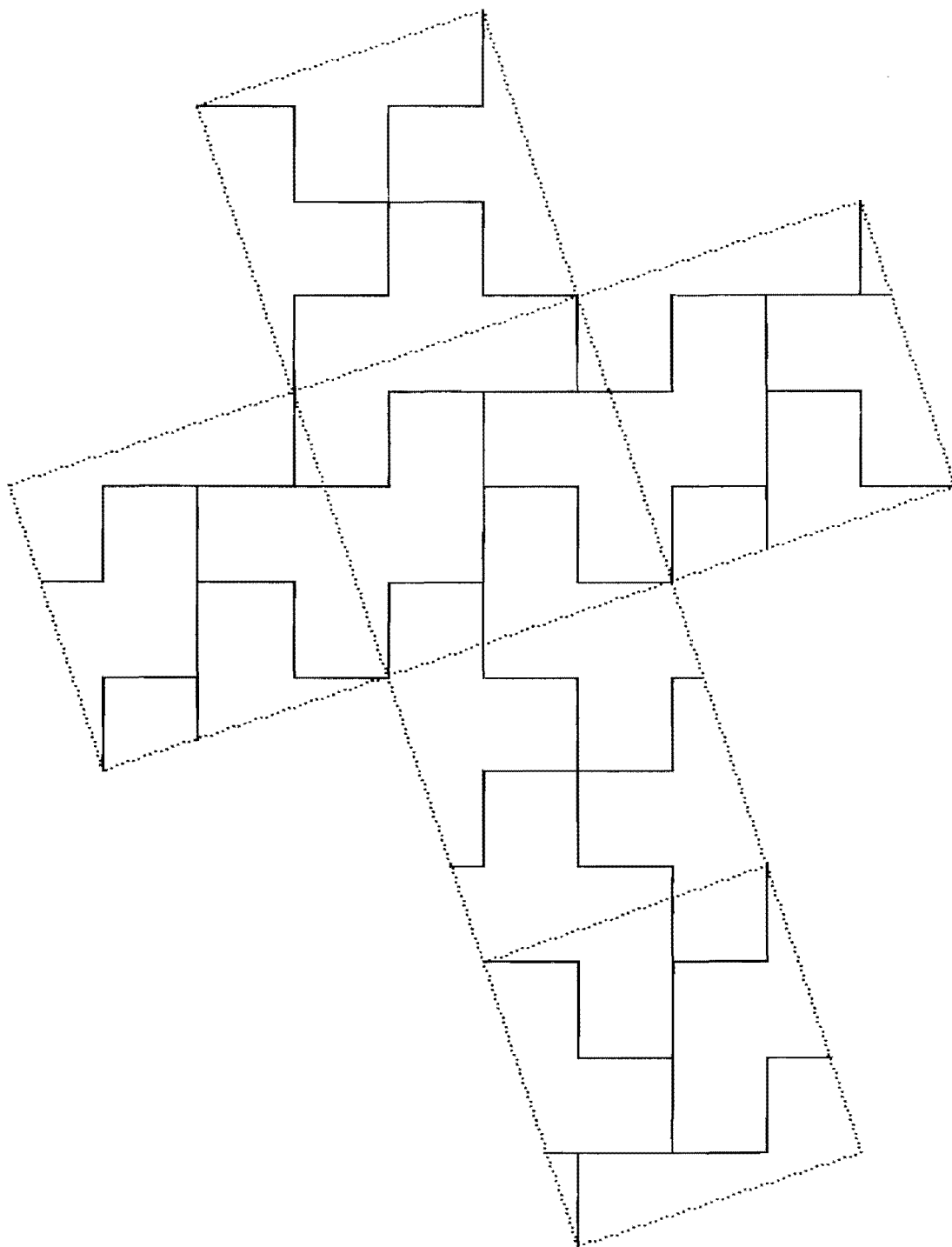


1297 1386 1411 1422 1509 1527 1568 1629 1639 1696 1718 1751

One axis order 3 (D3)

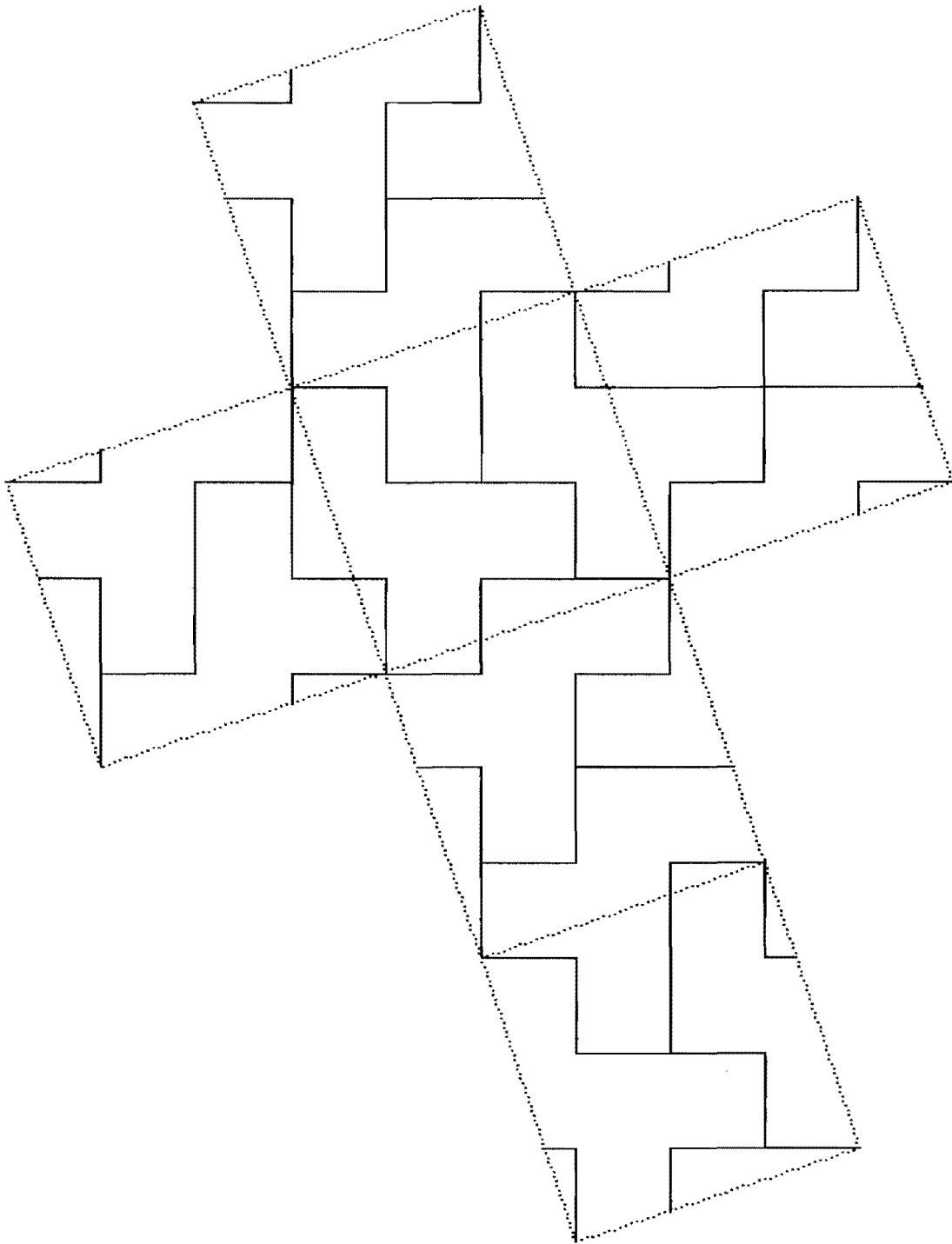


1297 1389 1411 1417 1464 1518 1521 1573 1628 1640 1697 1716



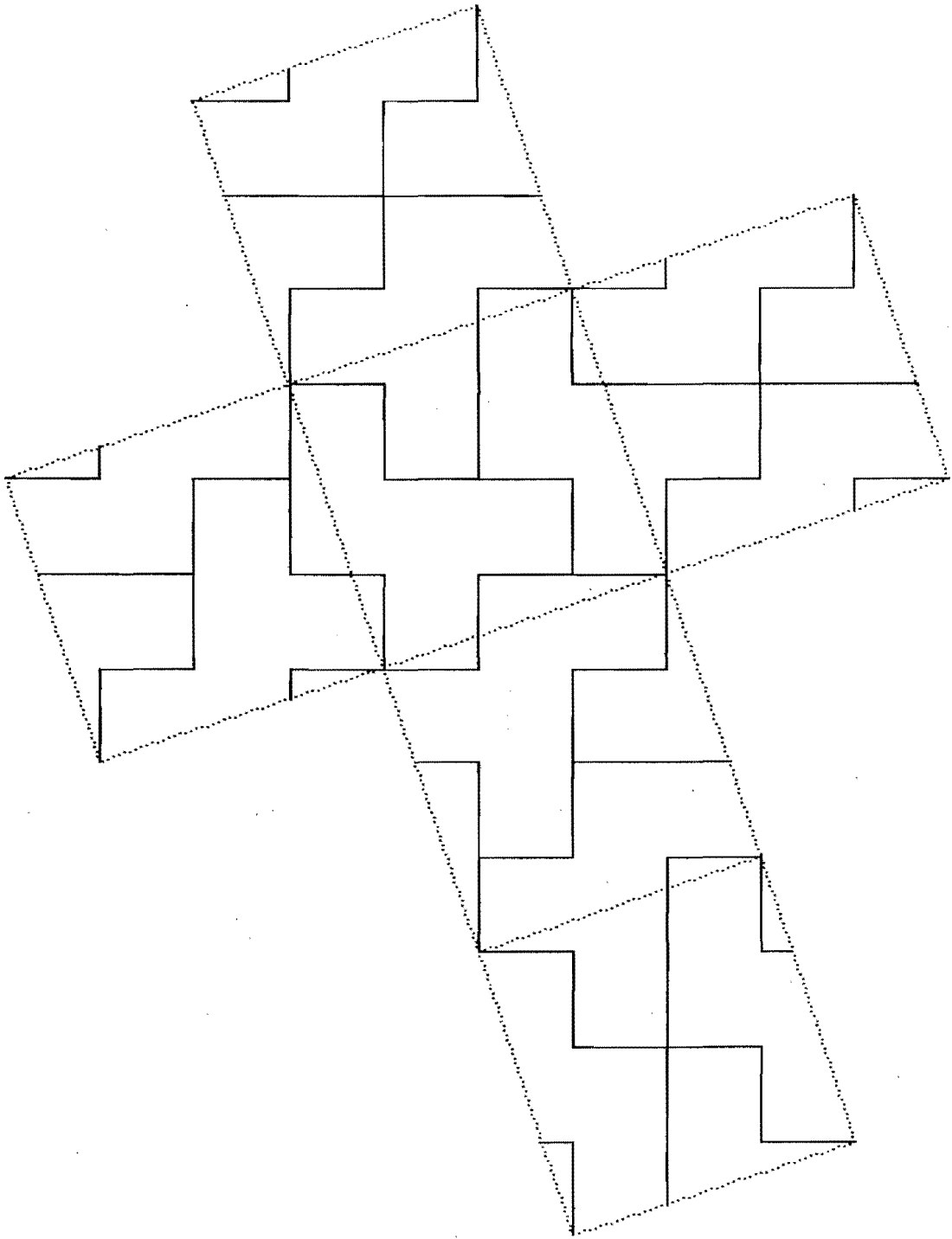
1297 1389 1411 1417 1466 1497 1549 1570 1628 1640 1697 1716

One axis order 4 (Z)



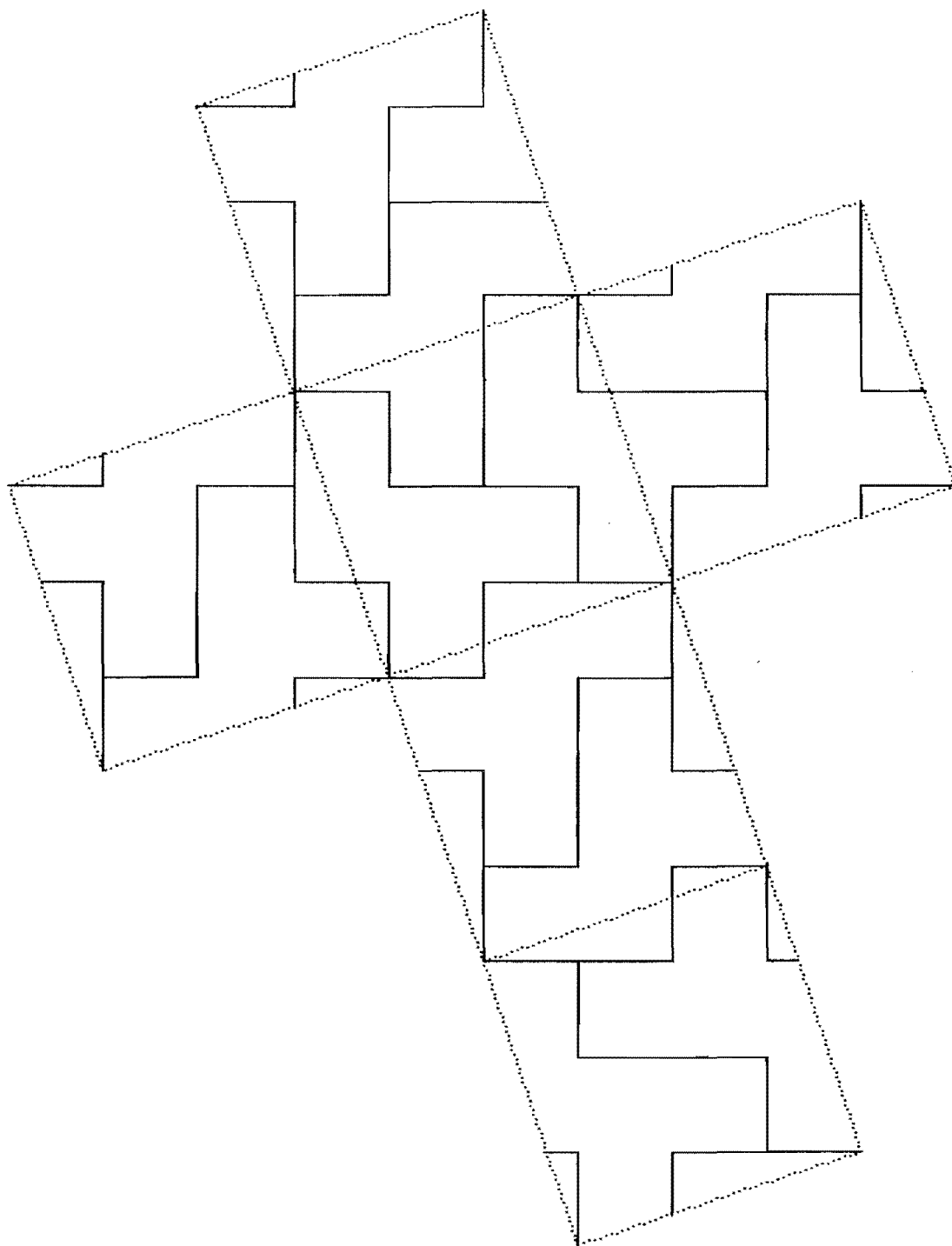
1309 1376 1390 1398 1490 1498 1517 1574 1626 1637 1719 1726

One axis order 2 (Y)



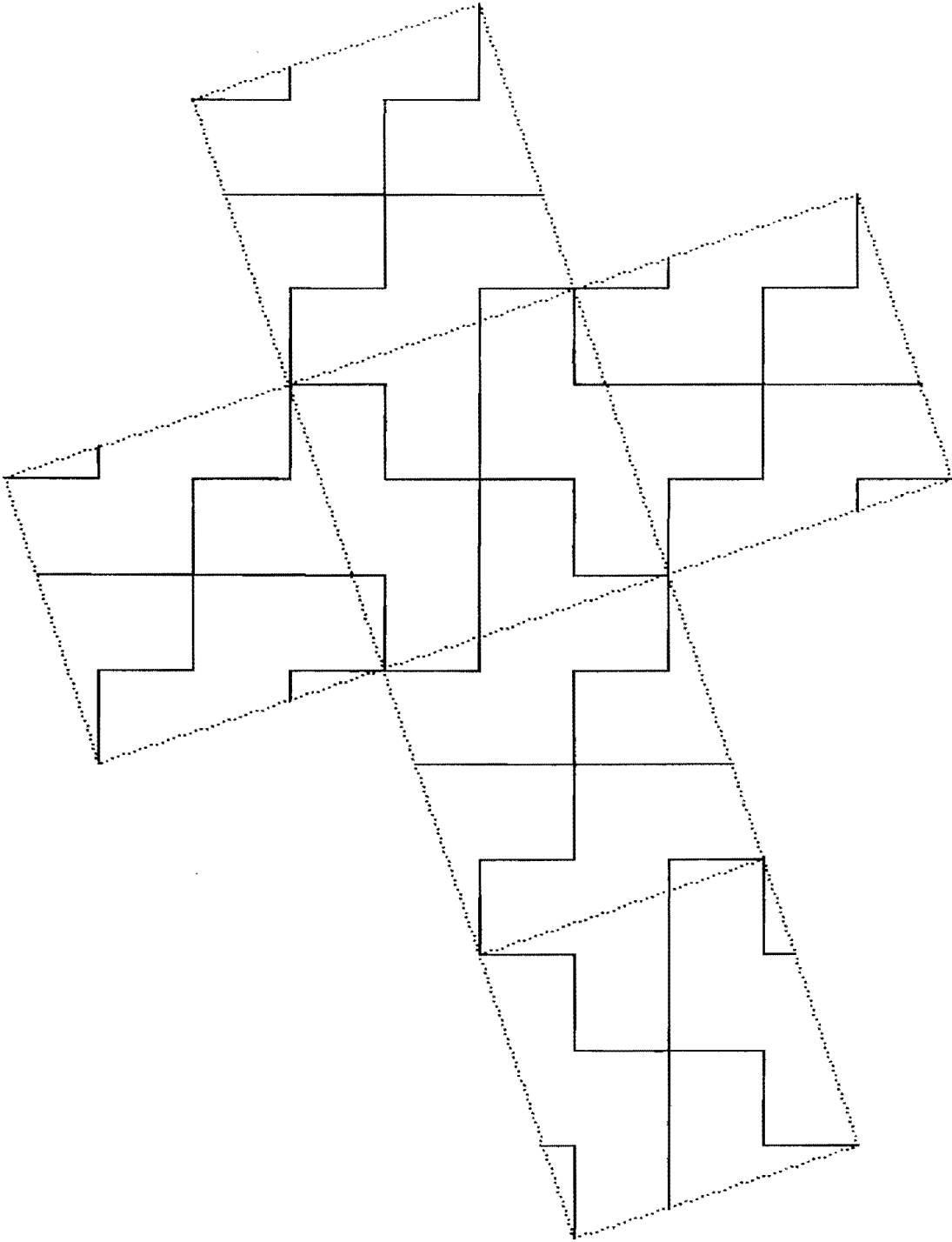
1309 1376 1390 1398 1490 1498 1517 1574 1626 1639 1696 1718

One axis order 3 (D4)



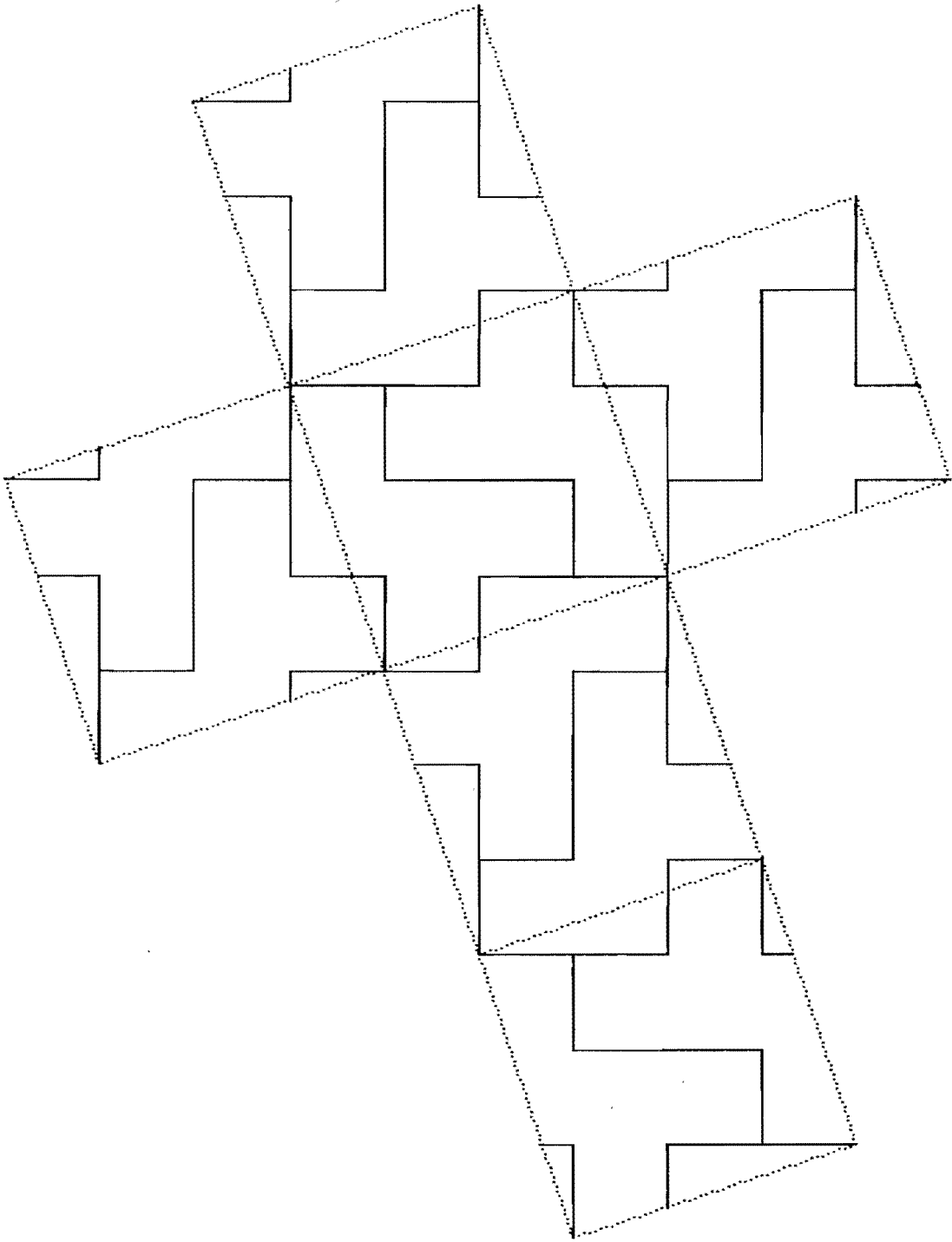
1309 1376 1390 1398 1490 1498 1519 1544 1623 1637 1719 1726

One axis order 3 (D2)



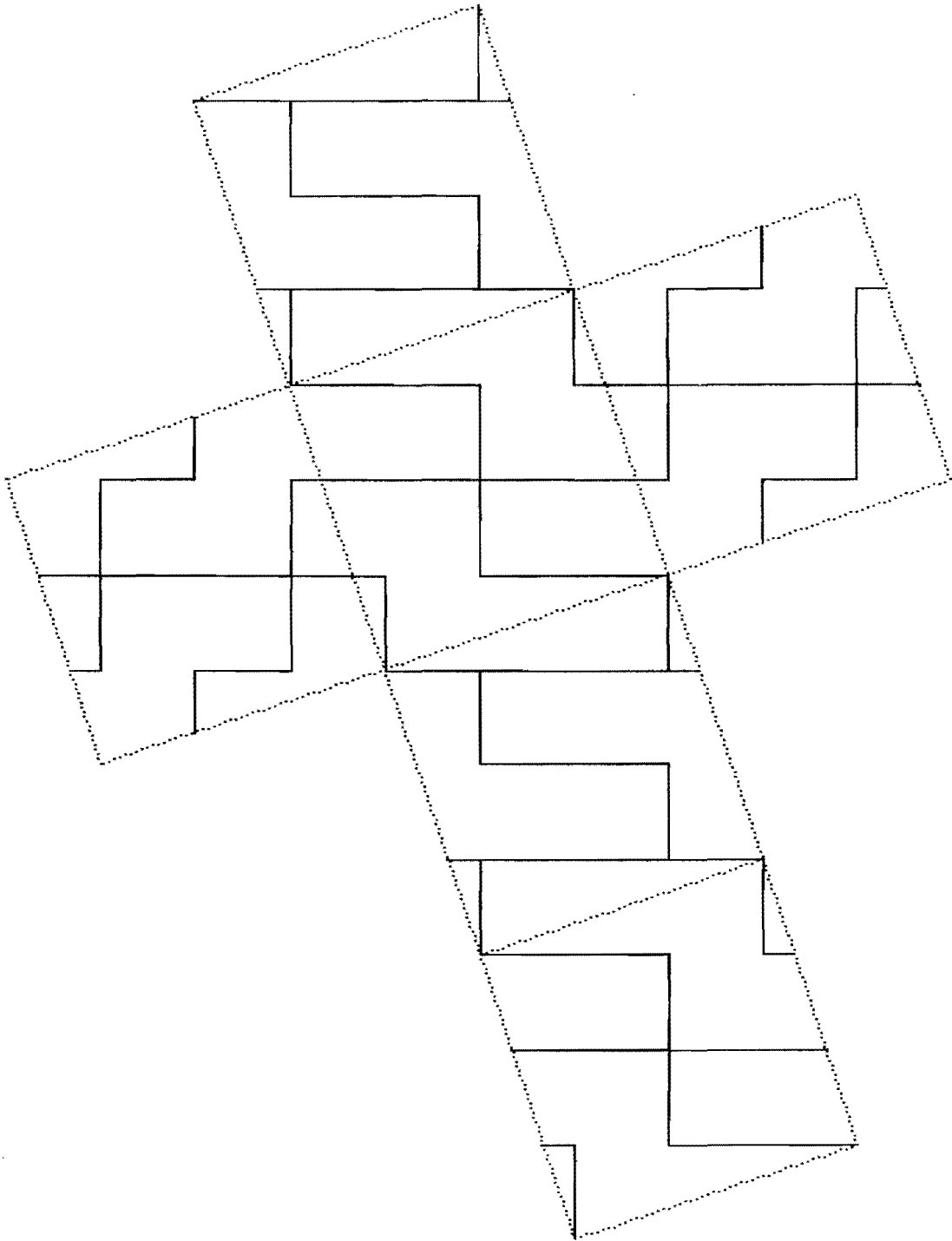
1309 1376 1390 1401 1467 1489 1517 1574 1626 1639 1696 1718

Four axes order 3 (D1 D2 D3 D4) and three axes order 2 (X Y Z)



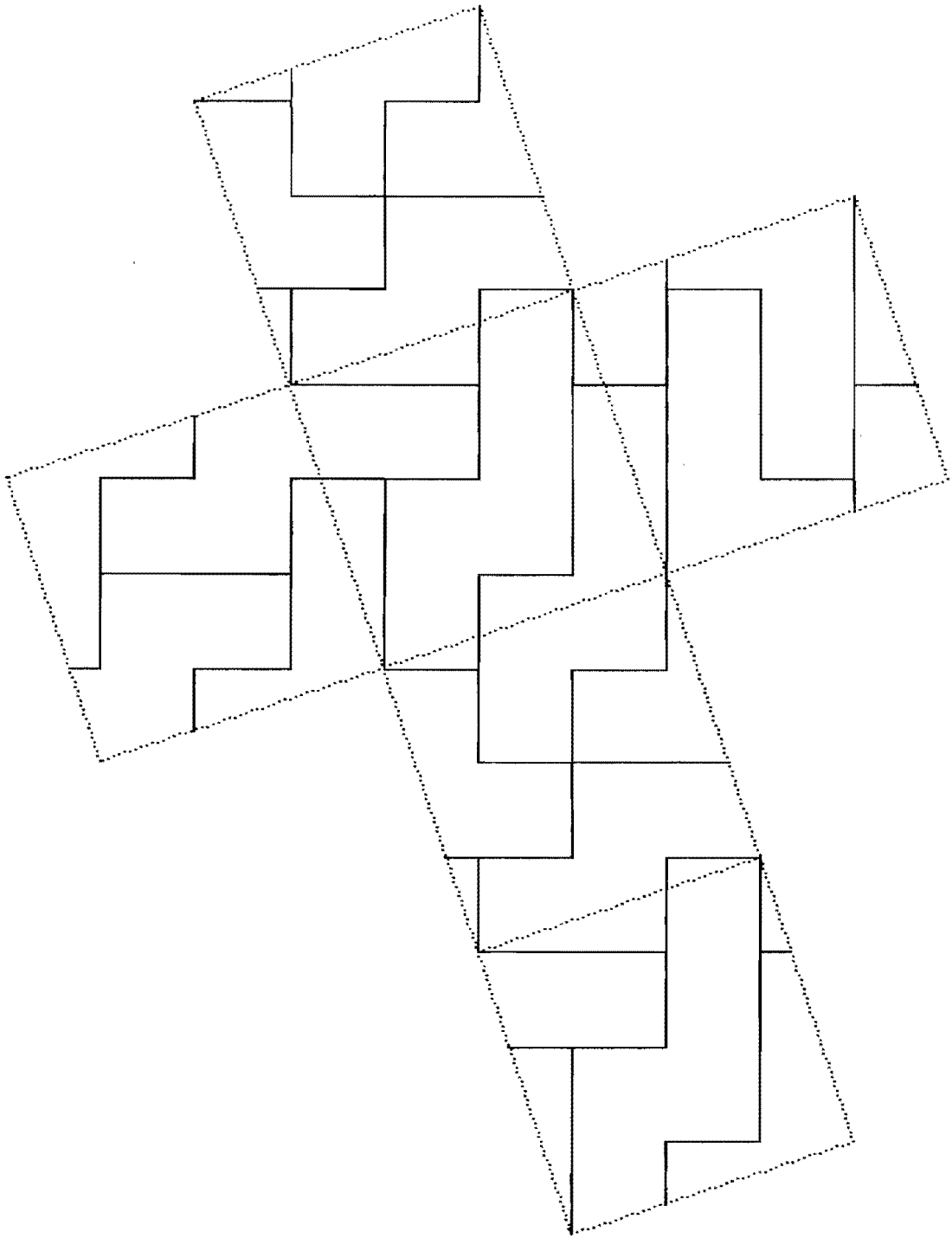
1311 1373 1391 1398 1490 1498 1519 1544 1623 1637 1719 1726

Four axes order 3 (D1 D2 D3 D4) and three axes order 2 (X Y Z)

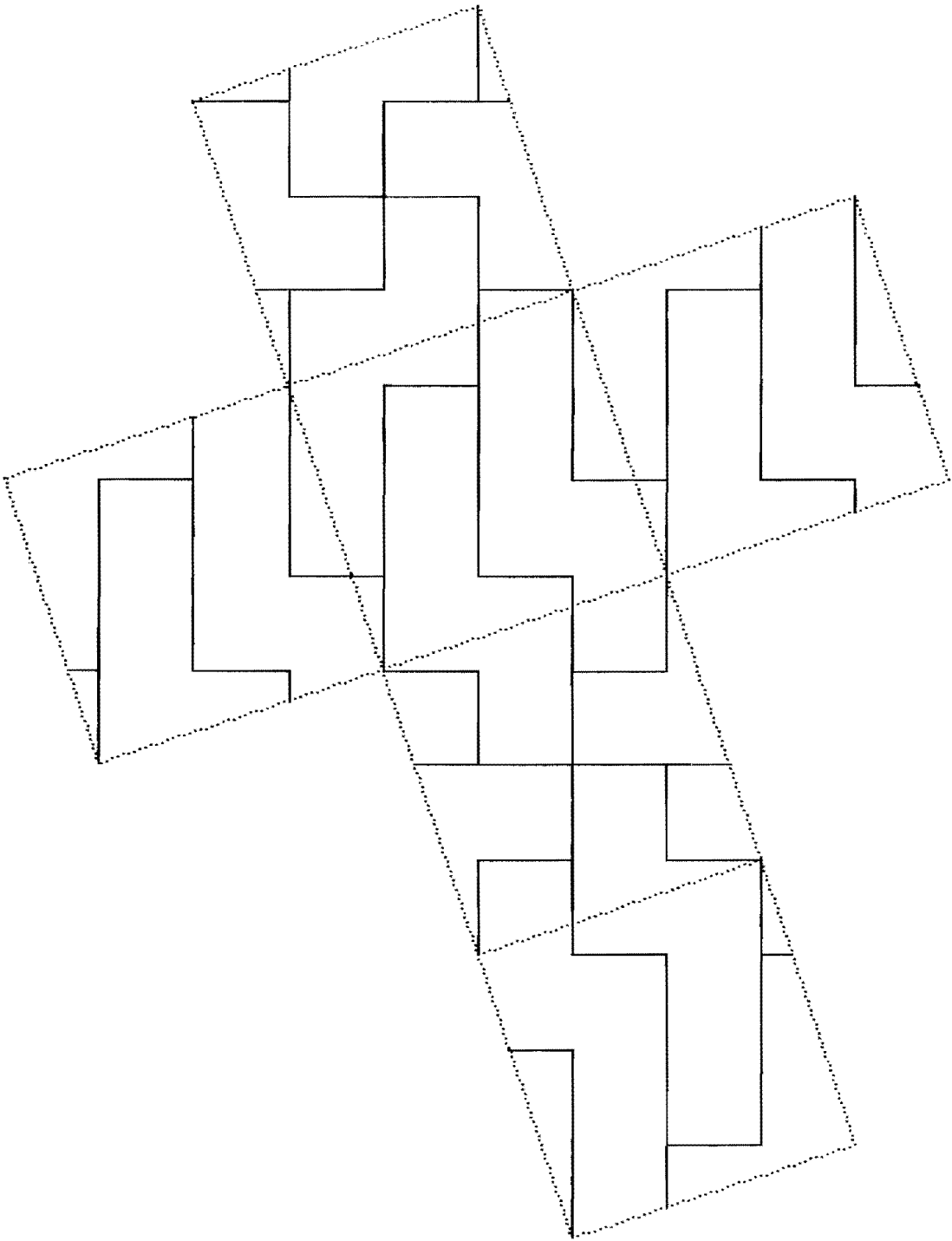


1755 1857 1865 1907 1934 1963 2006 2087 2093 2156 2177 2204

Three axes order 2 (X Y Z)

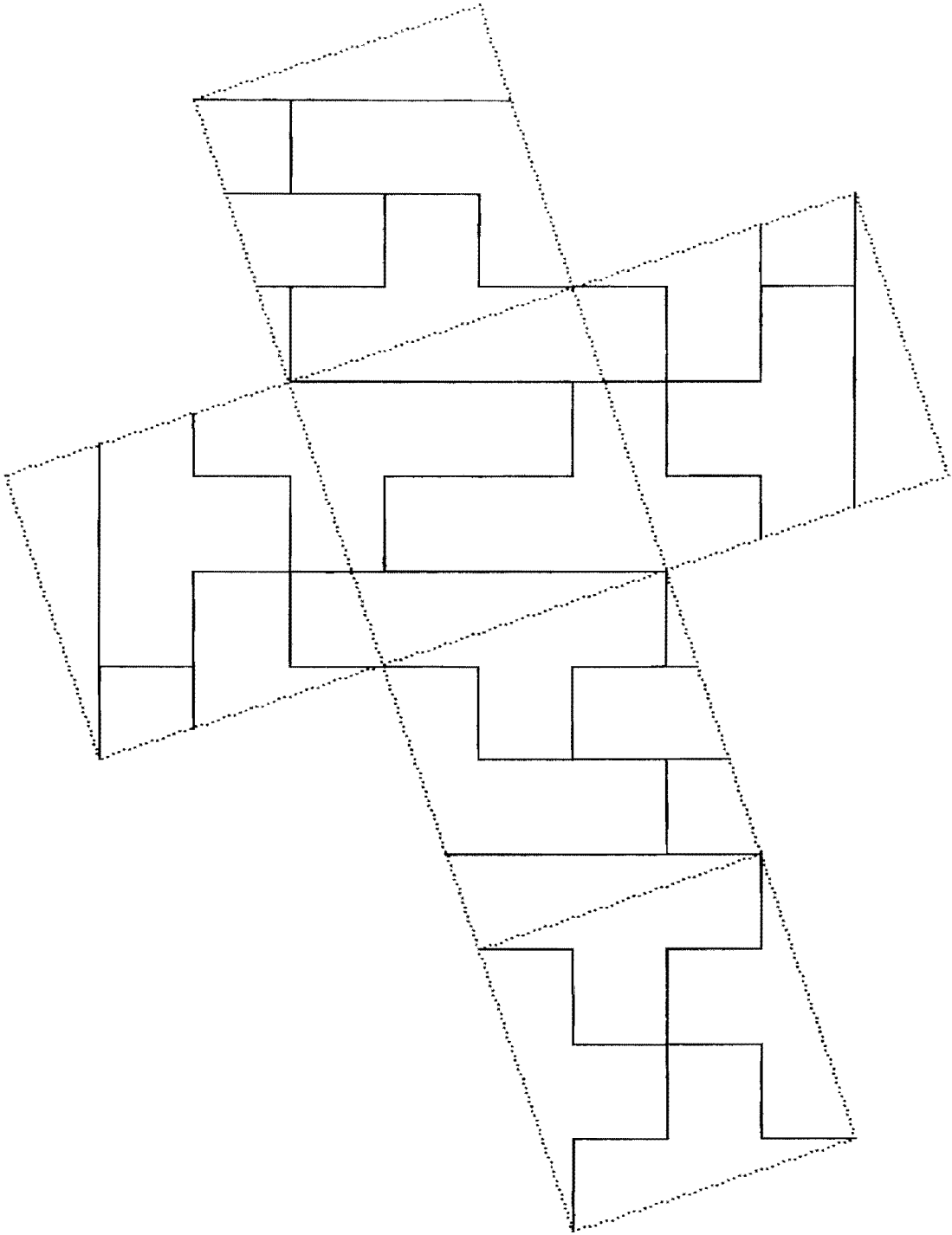


1757 1835 1865 1896 1915 1991 2056 2087 2111 2148 2163 2218



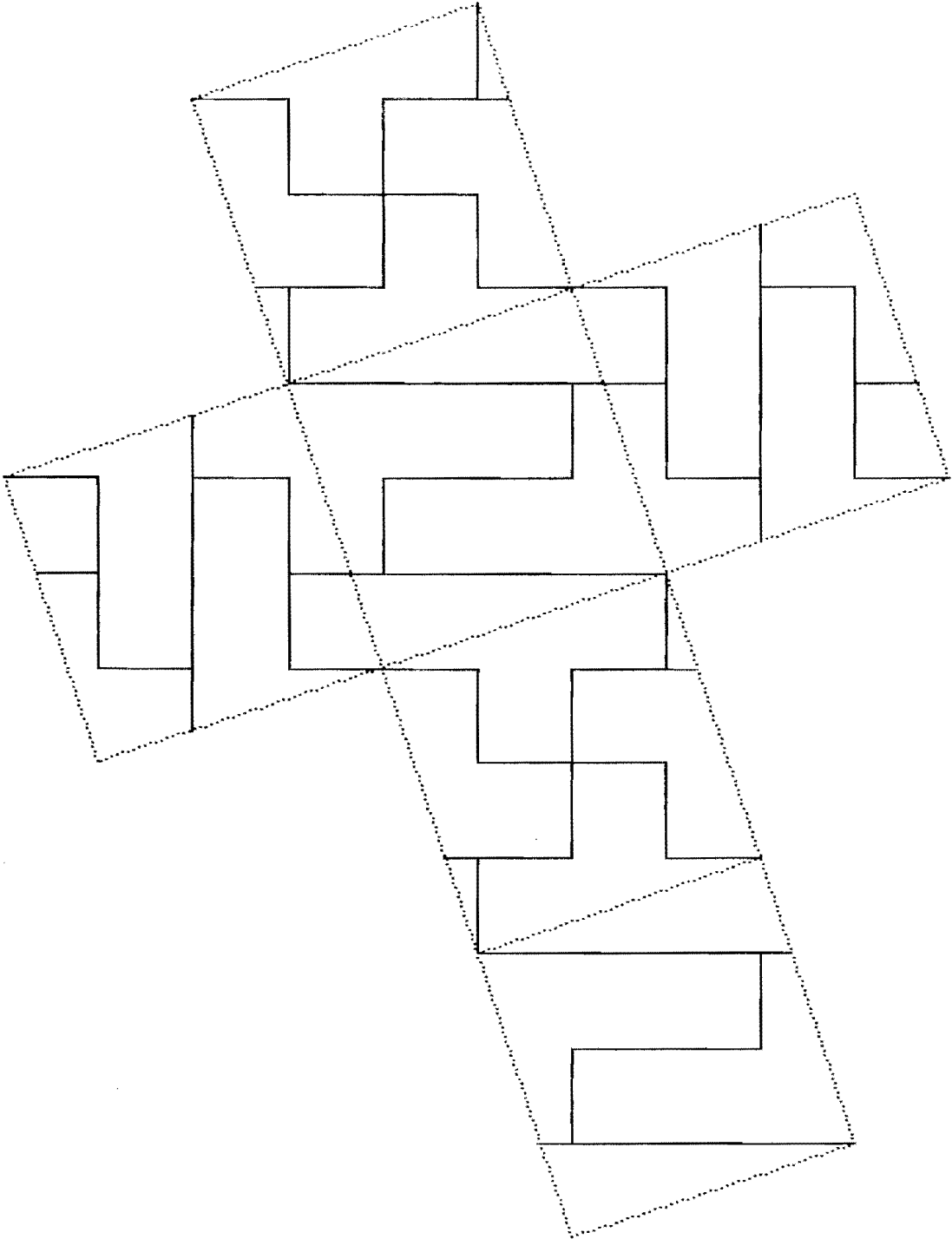
1758 1836 1850 1882 1964 1991 2035 2061 2075 2095 2148 2163

One axis order 4 (Z)



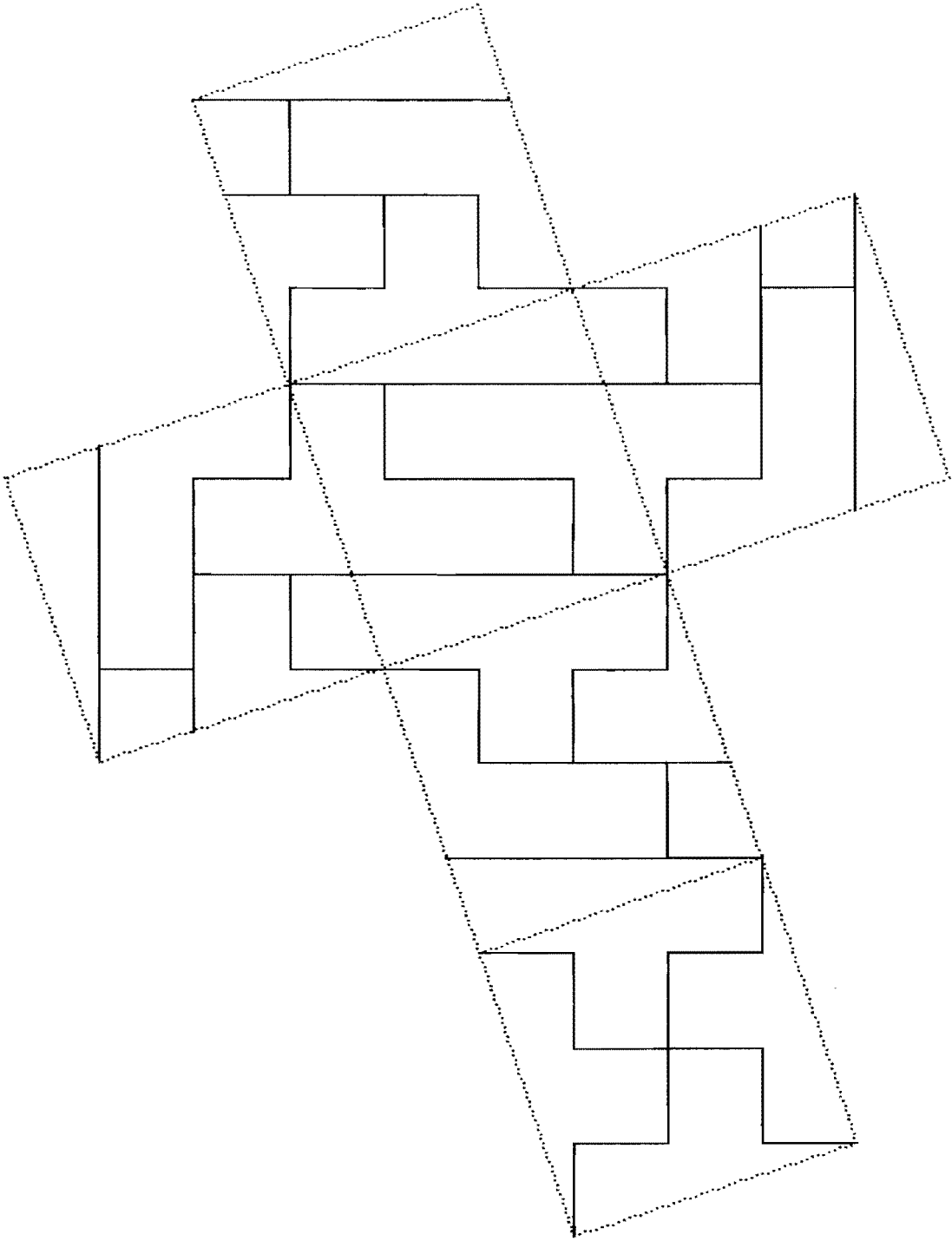
2235 2336 2366 2421 2467 2483 2536 2540 2584 2633 2680 2688

One axis order 2 (X)



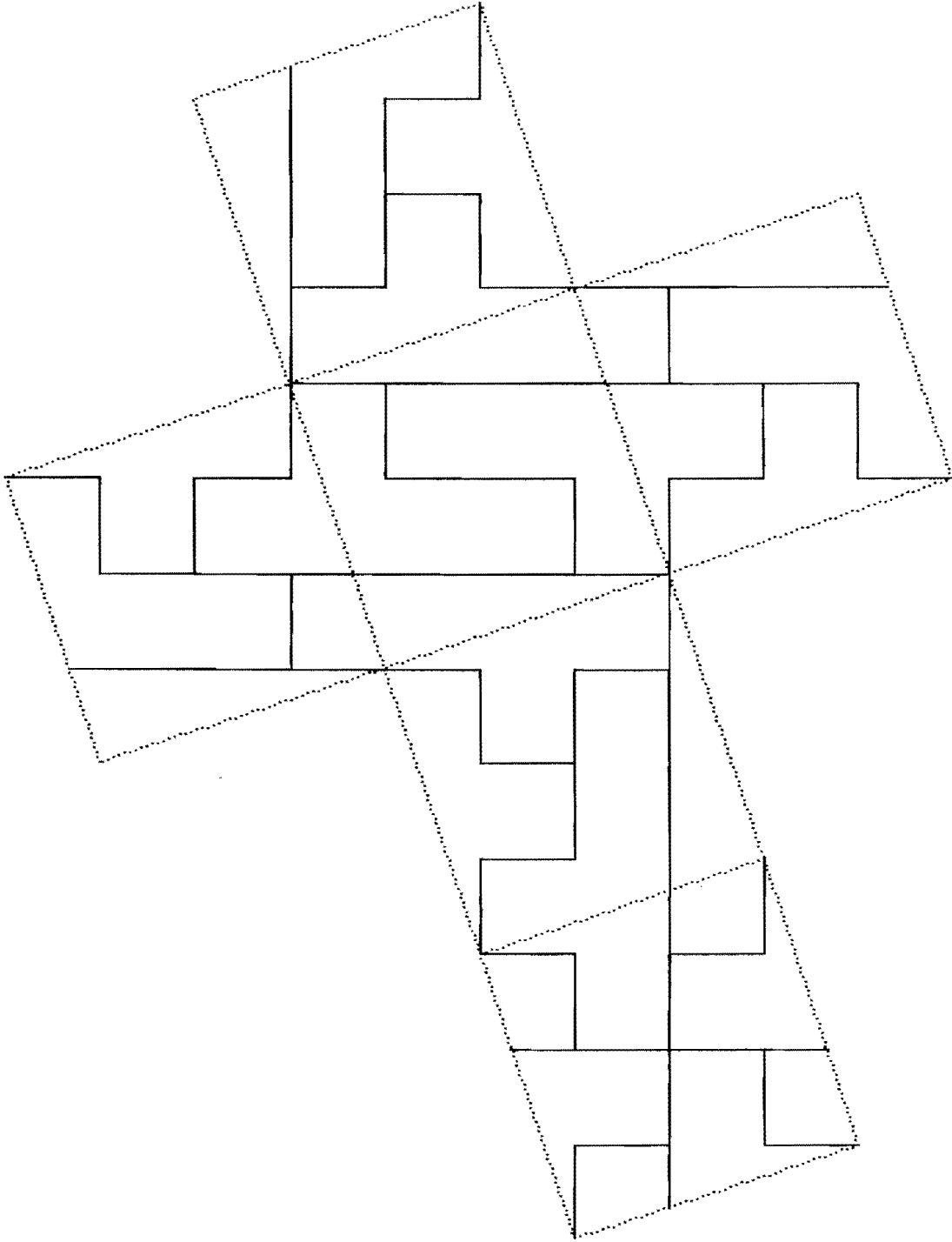
2235 2336 2366 2421 2469 2481 2523 2548 2606 2650 2678 2683

Three axes order 2 (X Y Z)



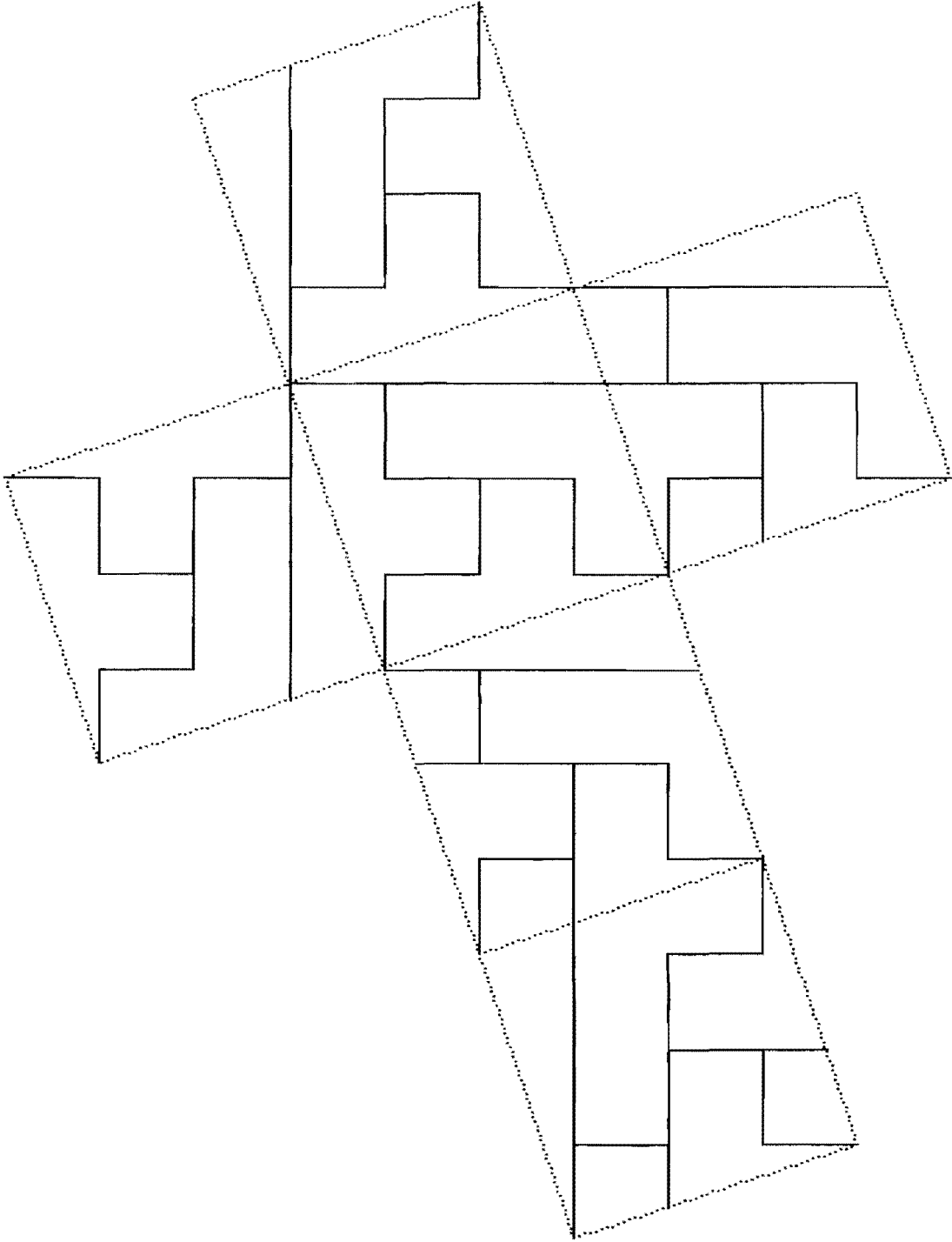
2235 2340 2350 2421 2459 2467 2536 2540 2584 2633 2680 2687

One axis order 2 (X)



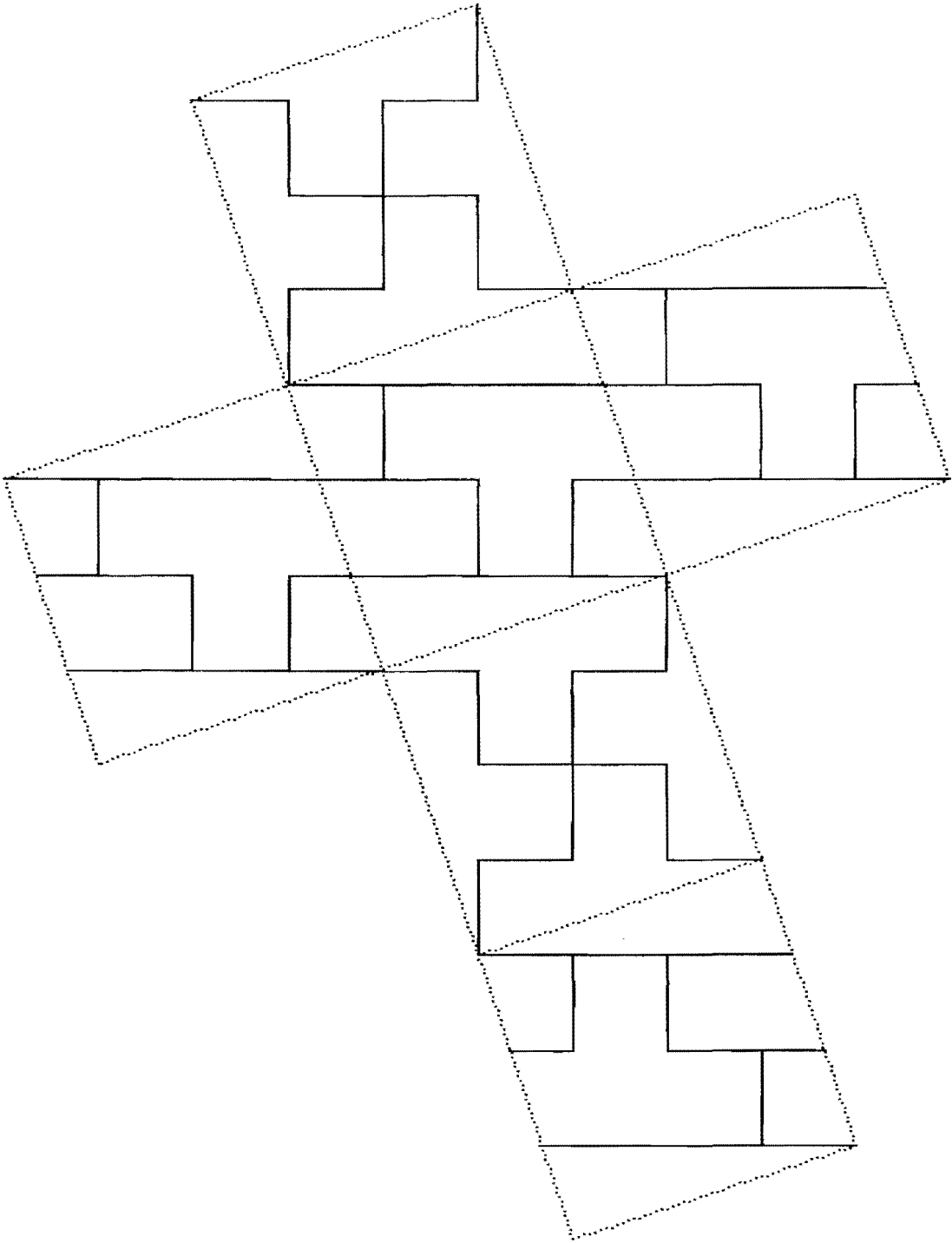
2235 2340 2350 2421 2462 2468 2488 2577 2616 2635 2646 2665

One axis order 2 (X)



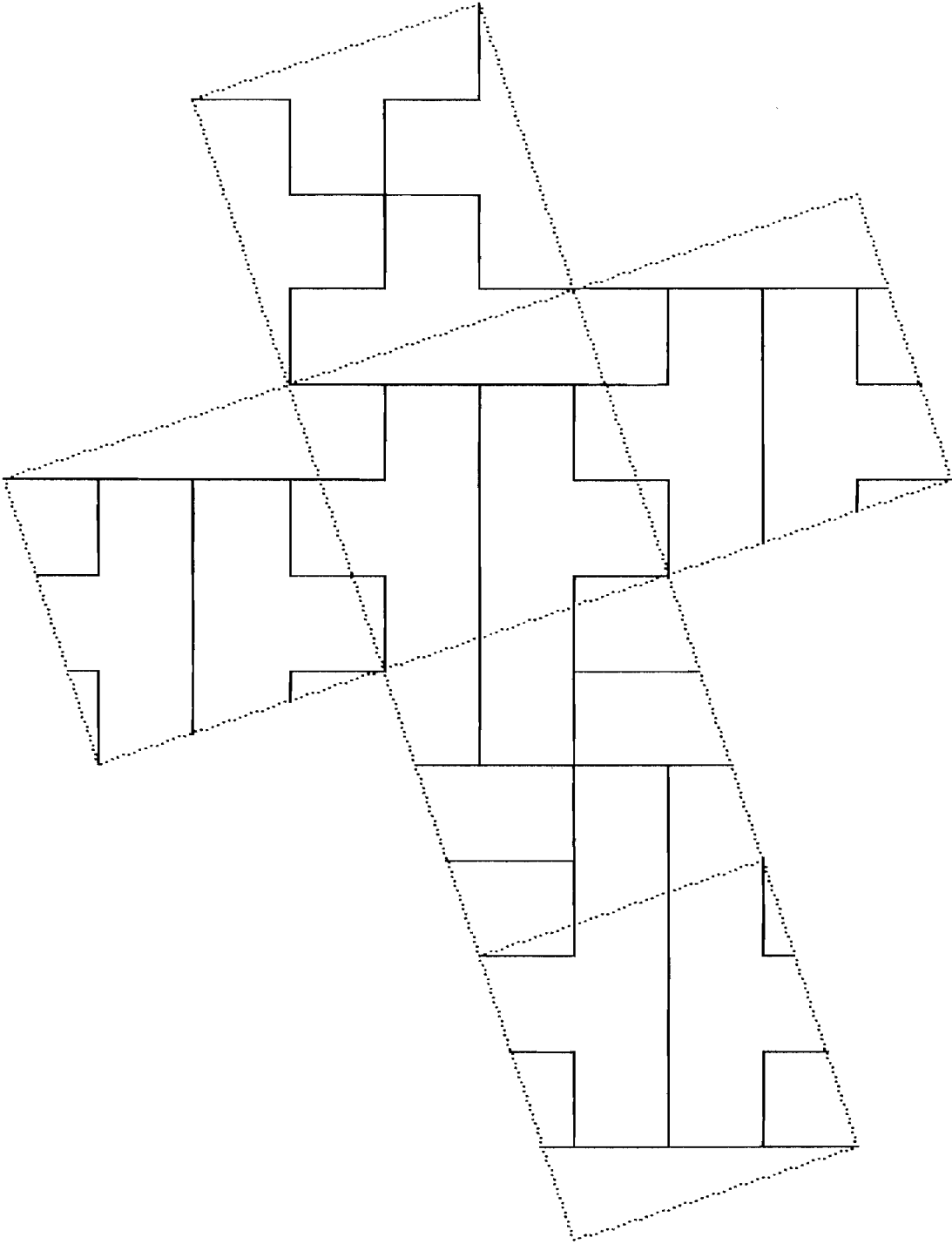
2235 2341 2350 2404 2474 2504 2525 2559 2577 2635 2646 2665

One axis order 3 (D1)



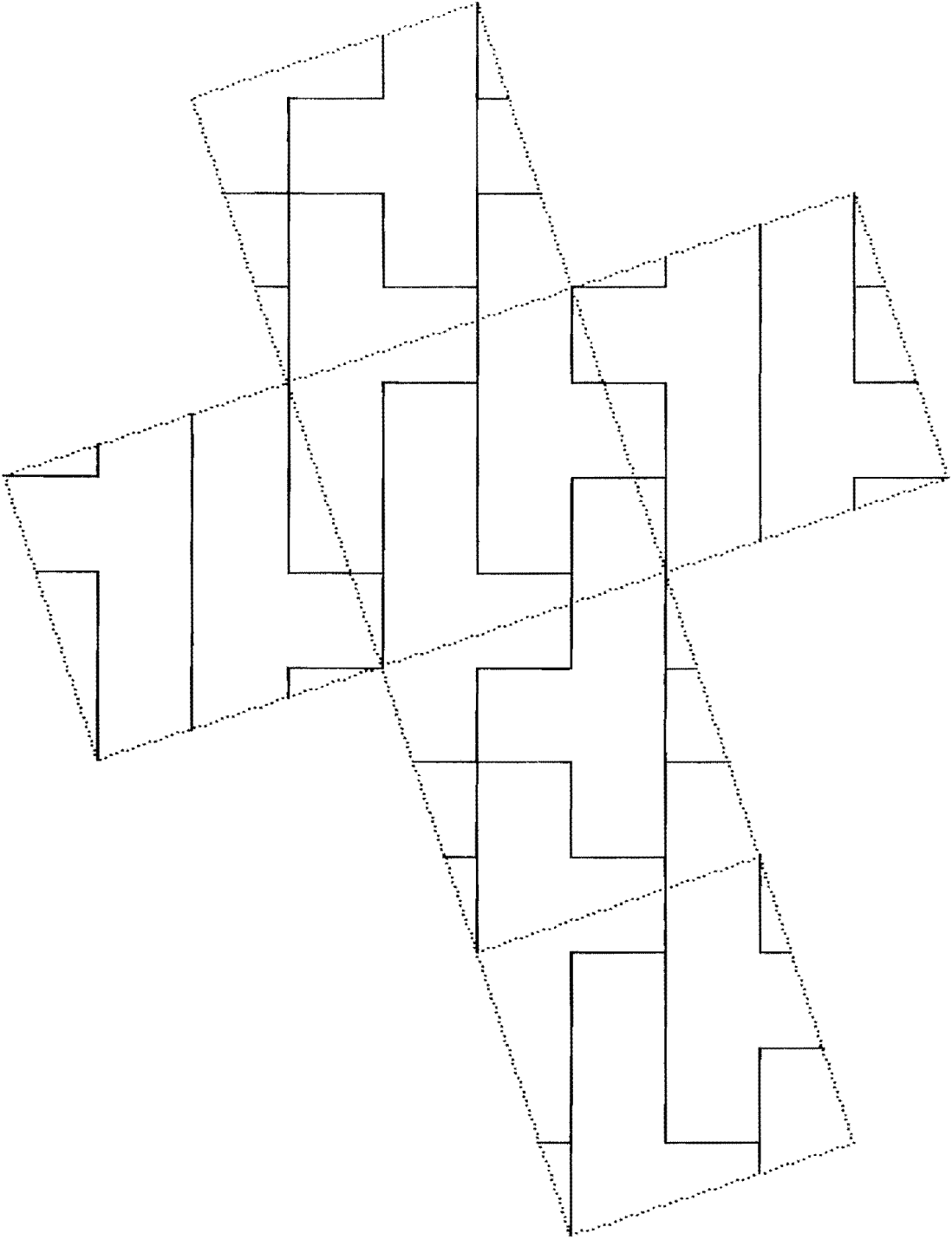
2235 2346 2349 2383 2417 2421 2468 2523 2592 2599 2646 2650

One axis order 4 (Z)



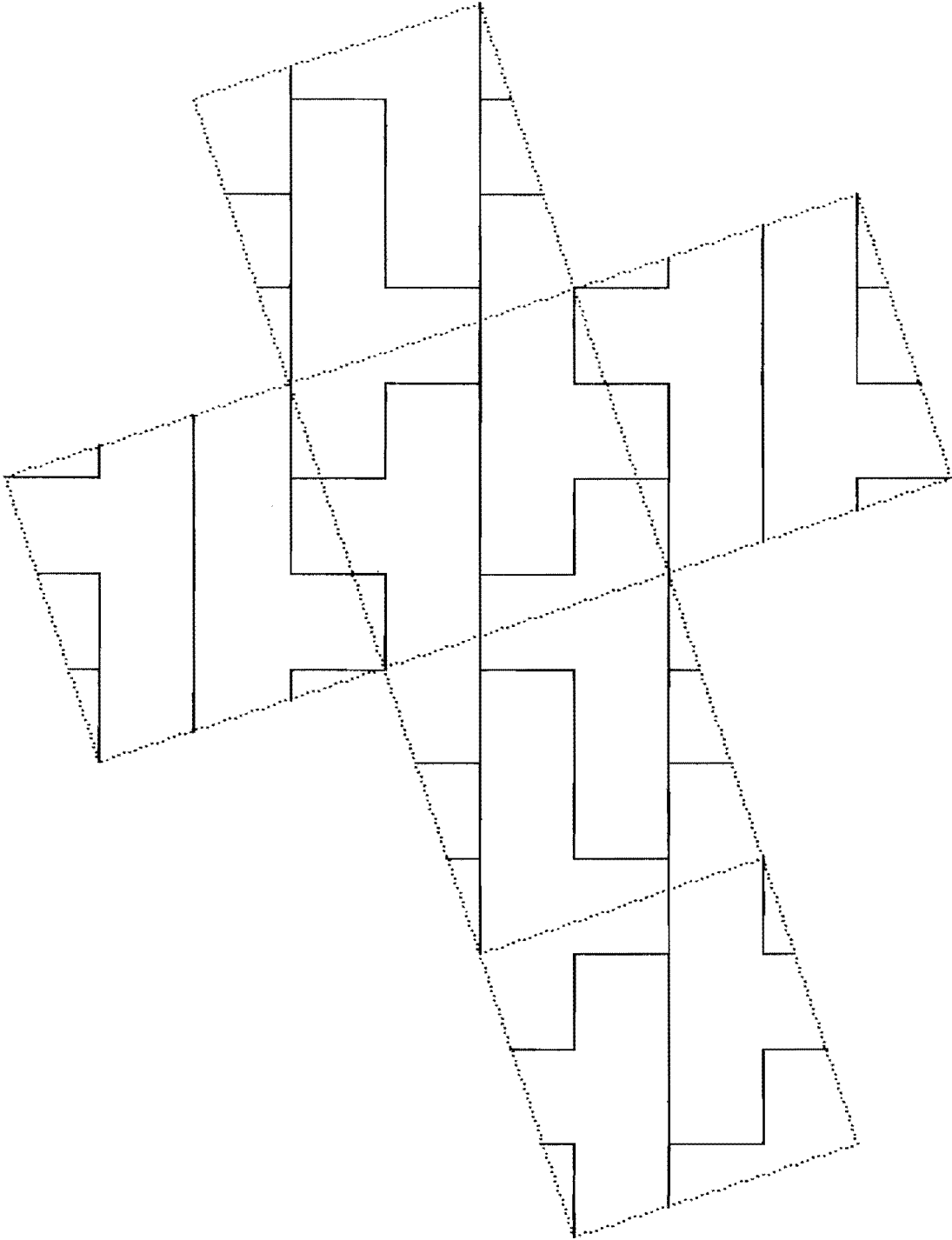
2235 2346 2353 2363 2371 2431 2482 2526 2535 2545 2646 2650

One axis order 4 (Z)



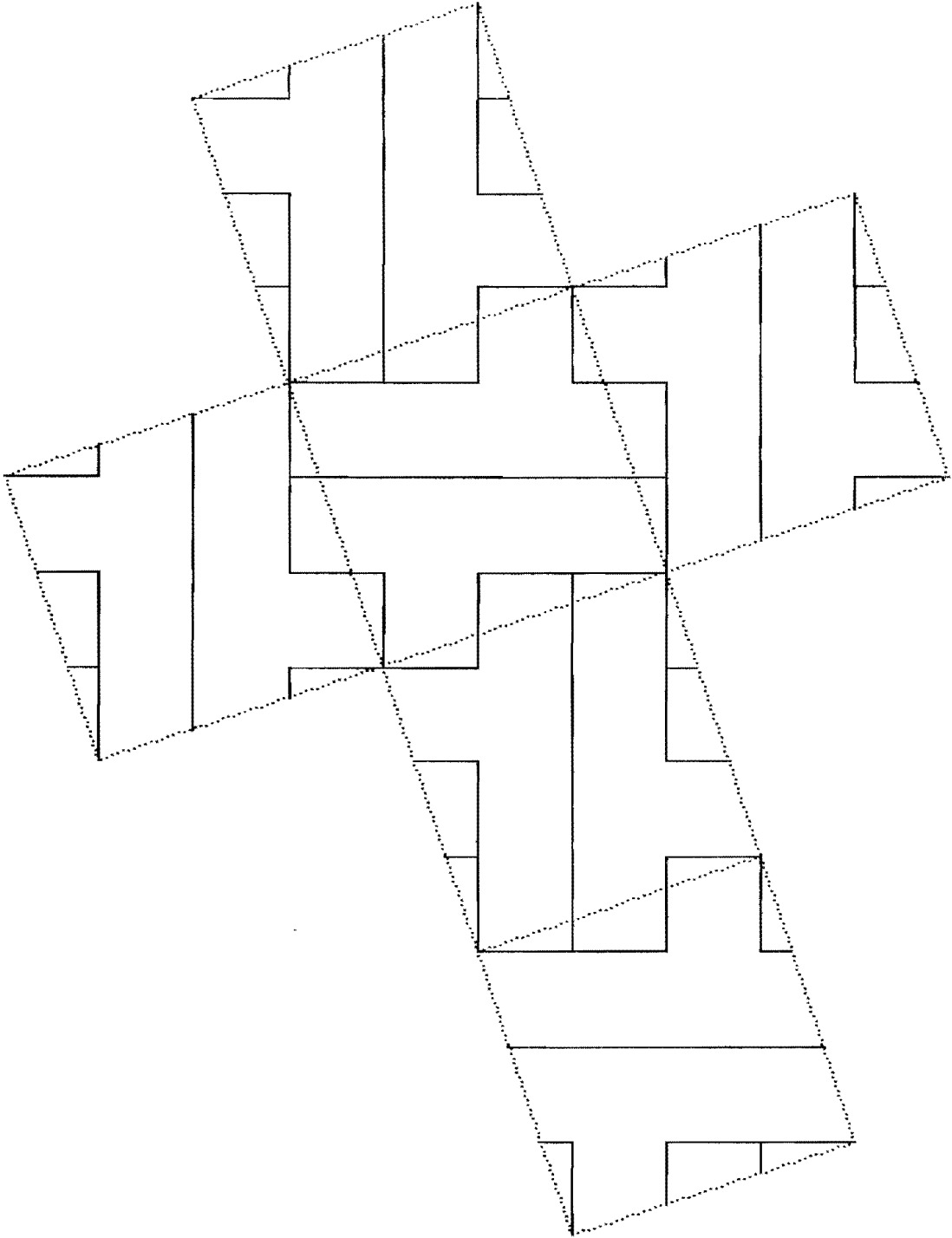
2241 2308 2326 2356 2415 2433 2496 2518 2535 2552 2596 2644

One axis order 2 (Y)



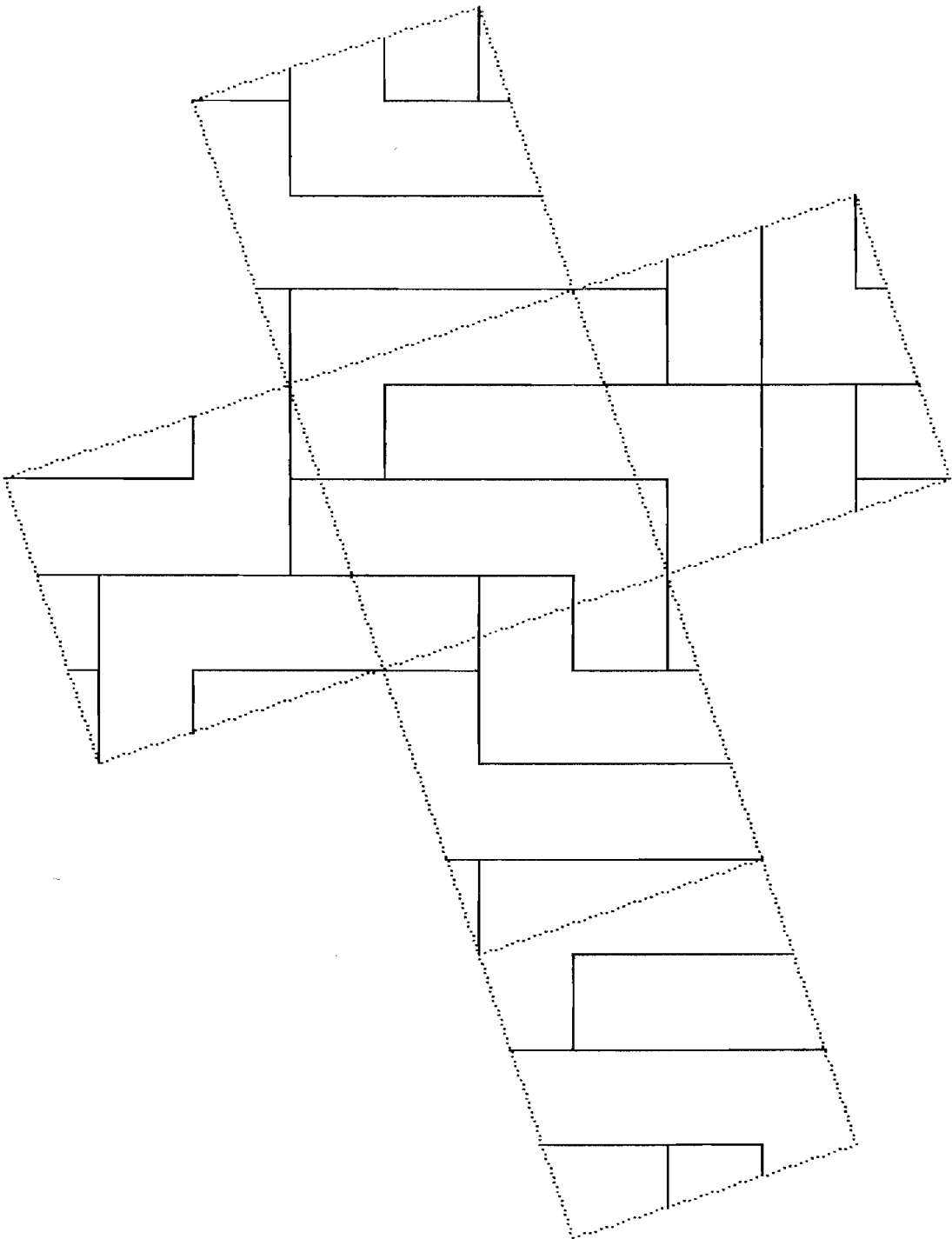
2242 2308 2326 2353 2411 2433 2480 2496 2535 2552 2593 2640

Three axes order 2 (X Y Z)

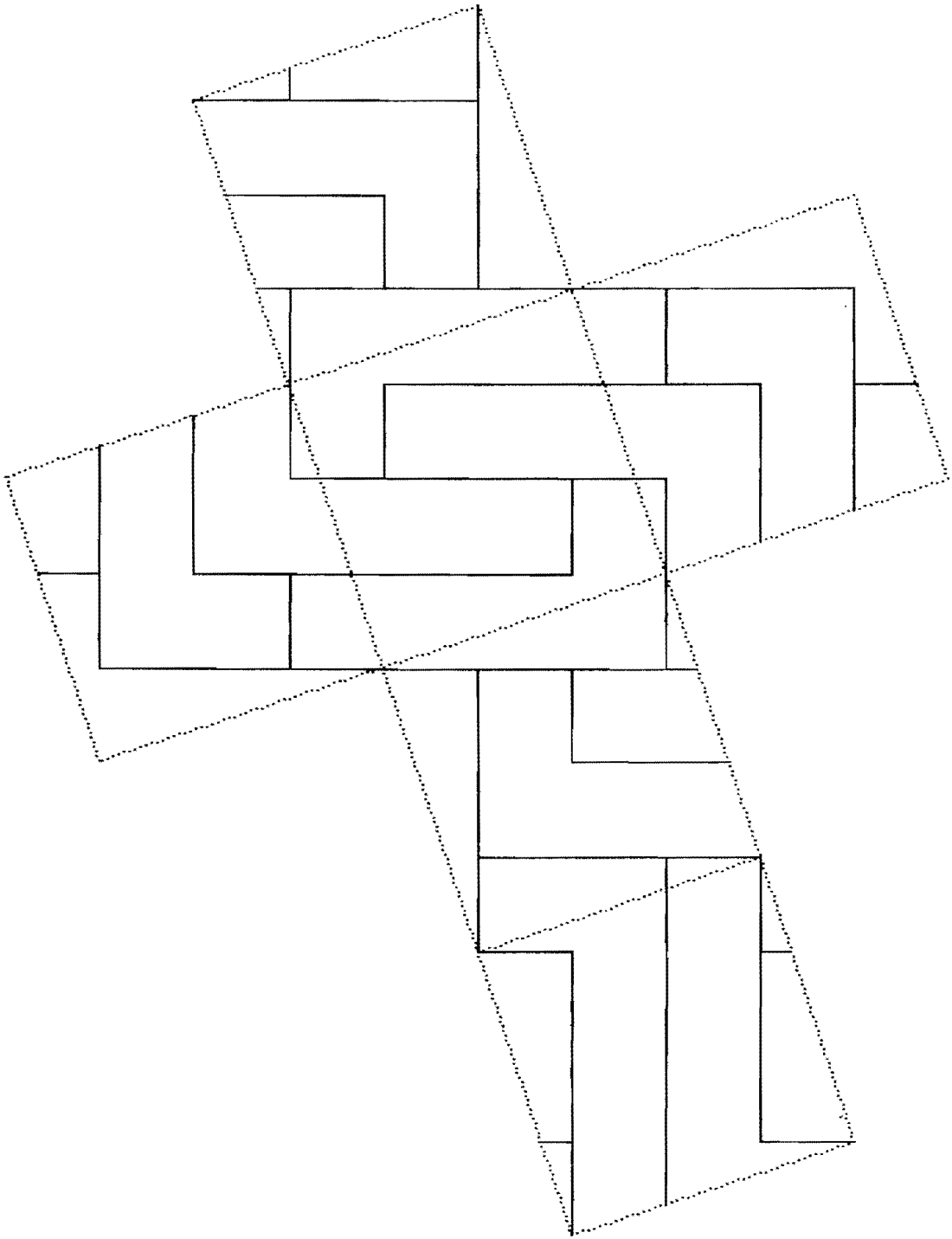


2262 2291 2303 2326 2377 2433 2445 2456 2496 2552 2569 2612

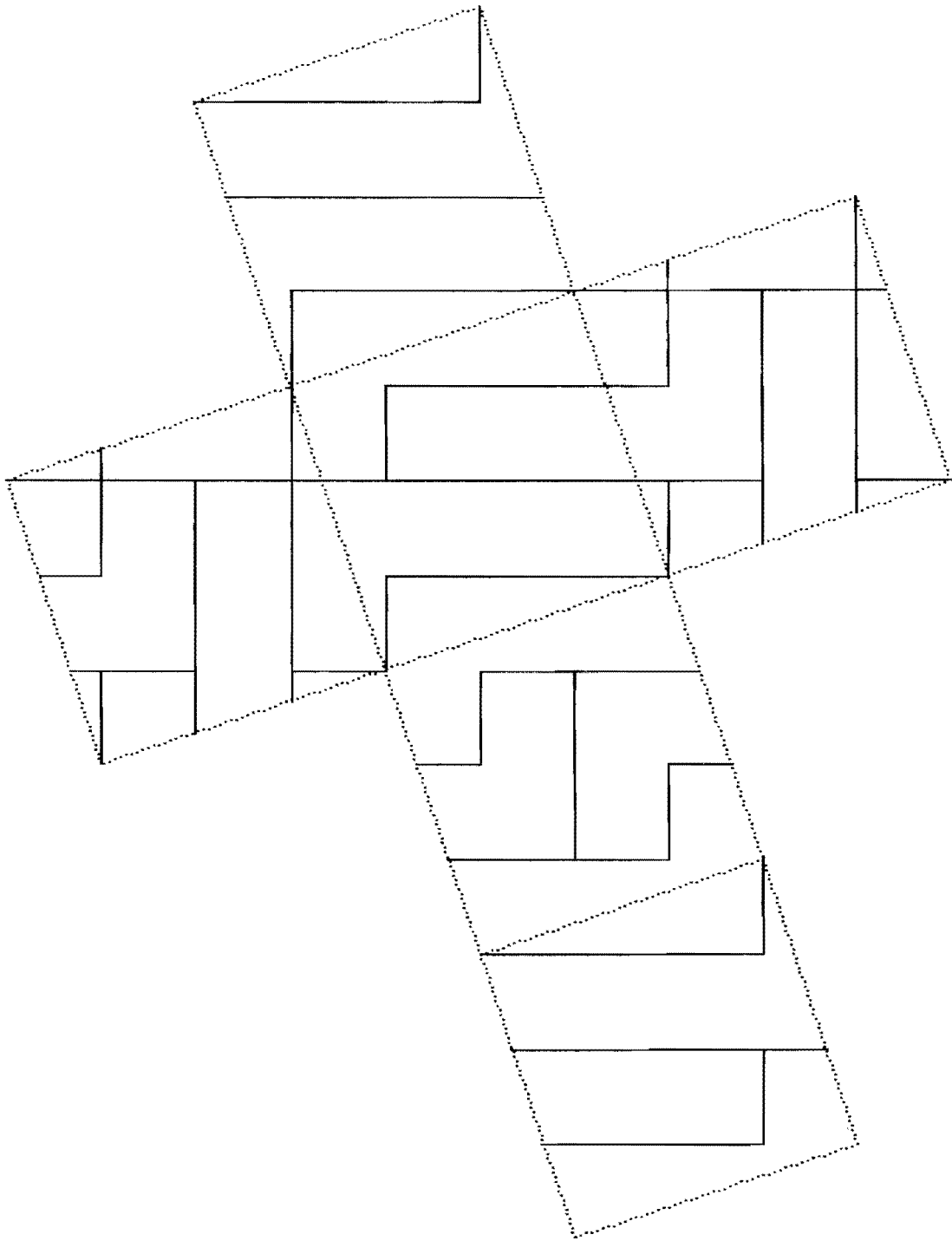
Four axes order 3 (D1 D2 D3 D4) and three axes order 2 (X Y Z)



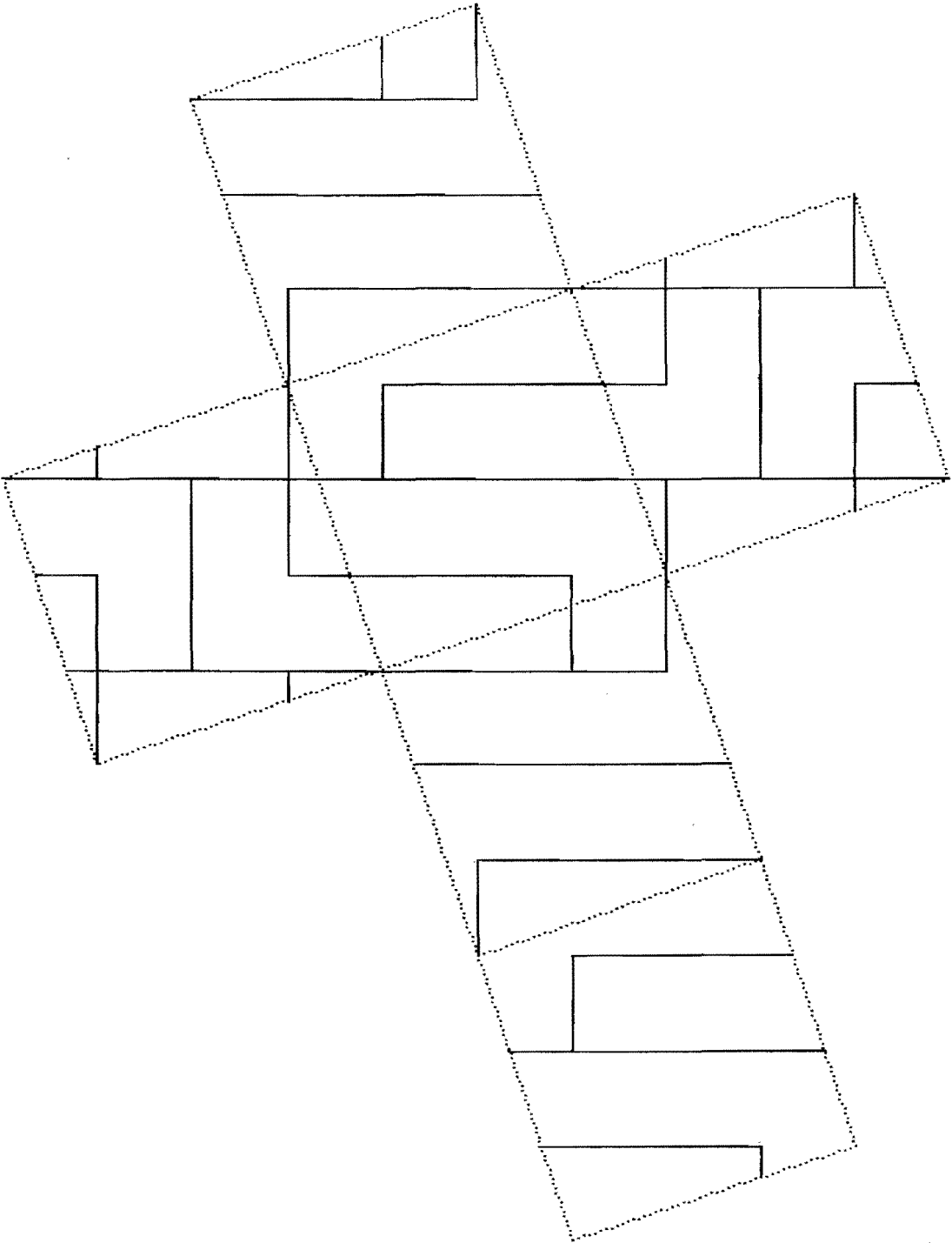
2689 2816 2844 2898 2919 2942 3034 3068 3092 3138 3151 3158



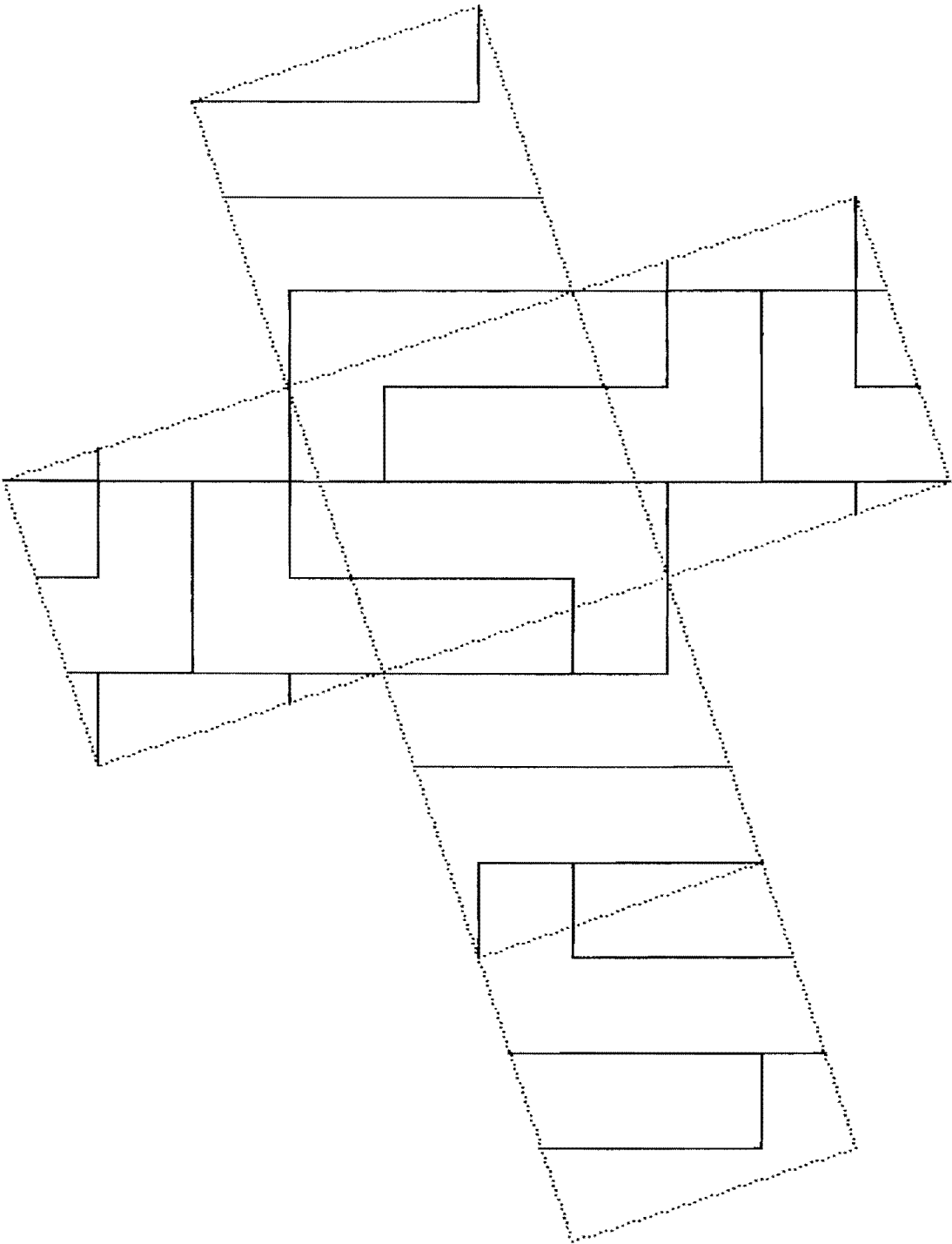
2689 2816 2846 2880 2944 2952 2959 3037 3054 3089 3141 3168



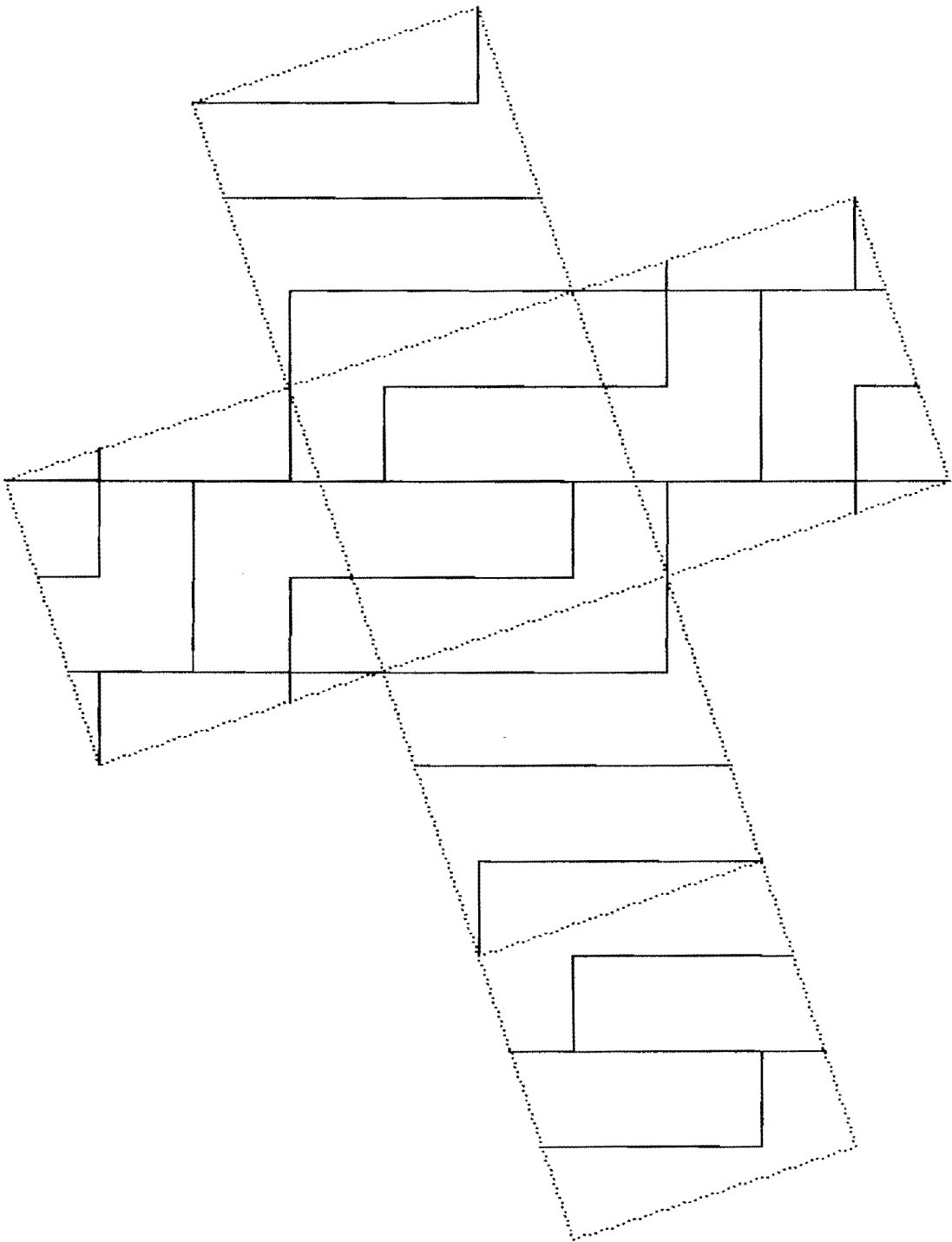
2689 2817 2843 2908 2951 2956 3007 3055 3094 3122 3155 3166



2689 2817 2844 2894 2929 2973 3034 3069 3092 3134 3155 3166

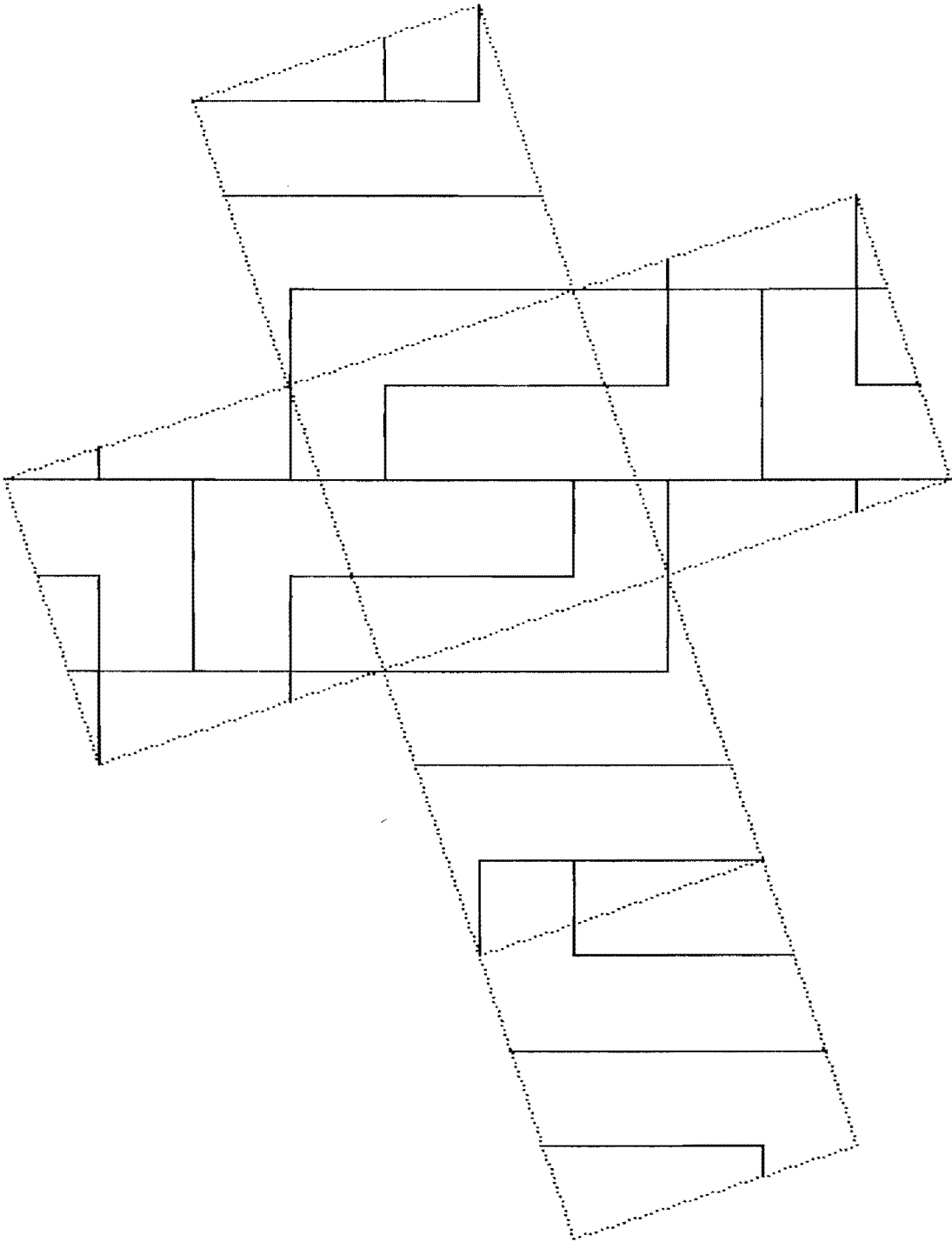


2689 2817 2844 2894 2929 2973 3038 3042 3094 3122 3155 3166

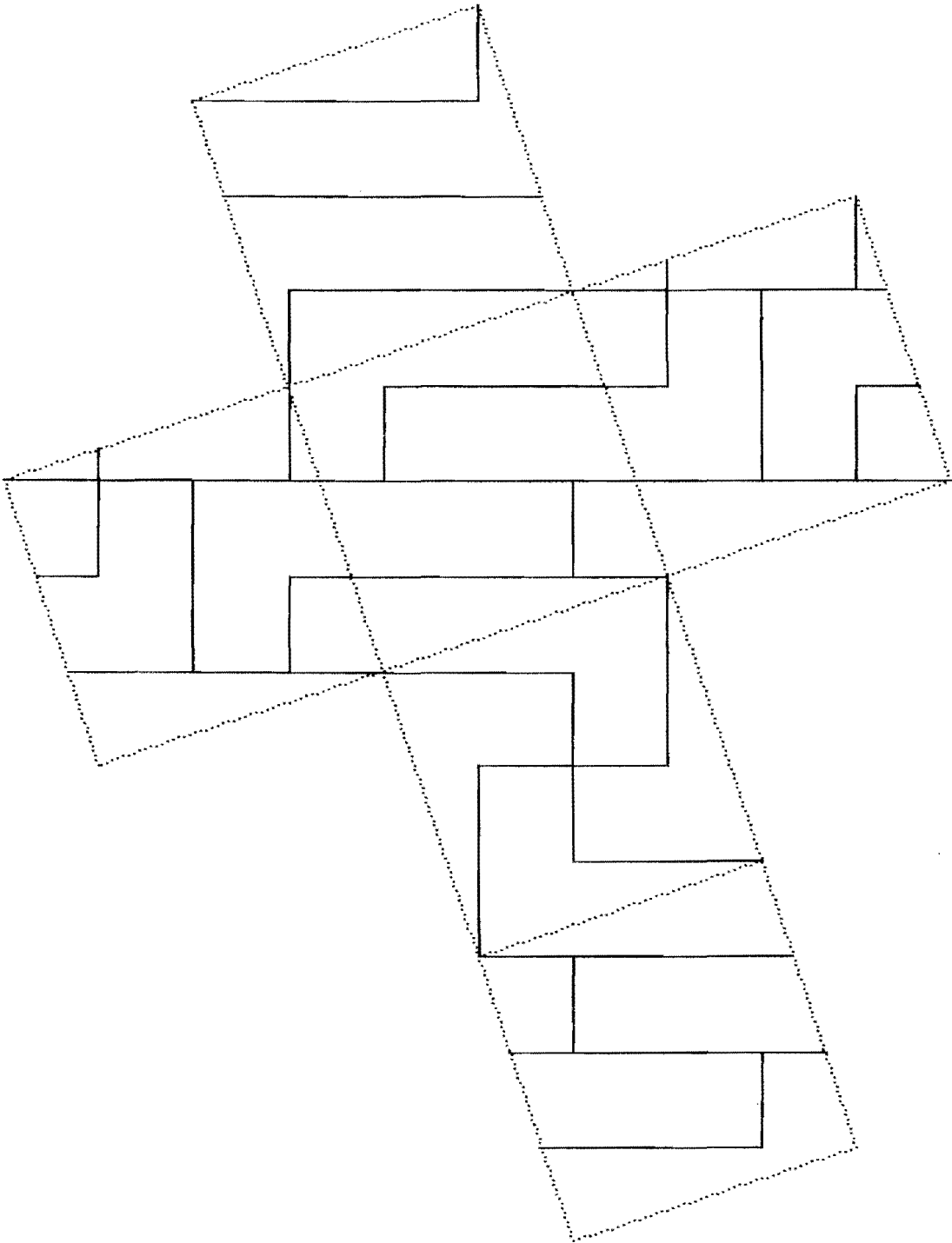


2689 2817 2847 2880 2929 2973 3034 3069 3094 3122 3155 3166

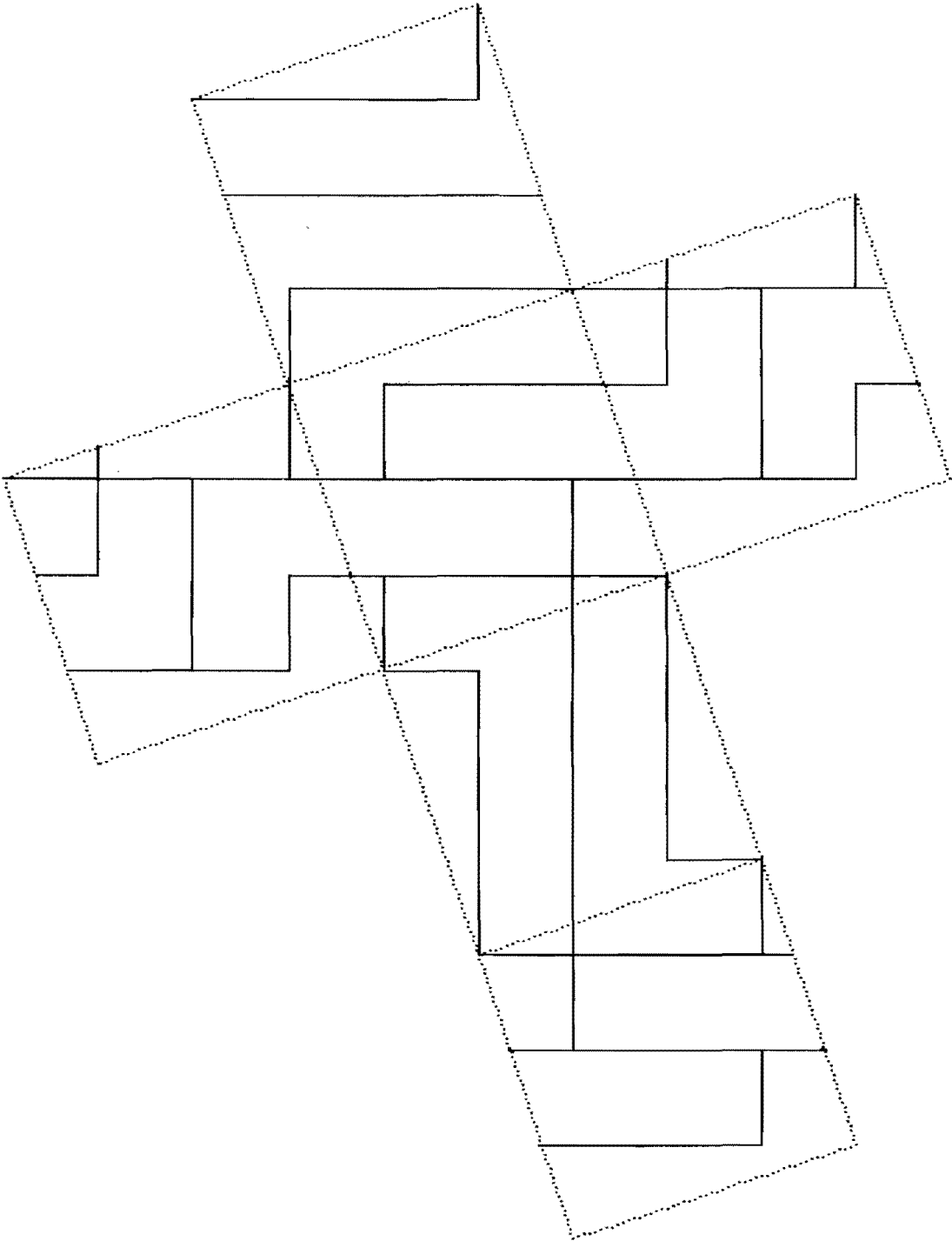
Three axes order 2 (X Y Z)



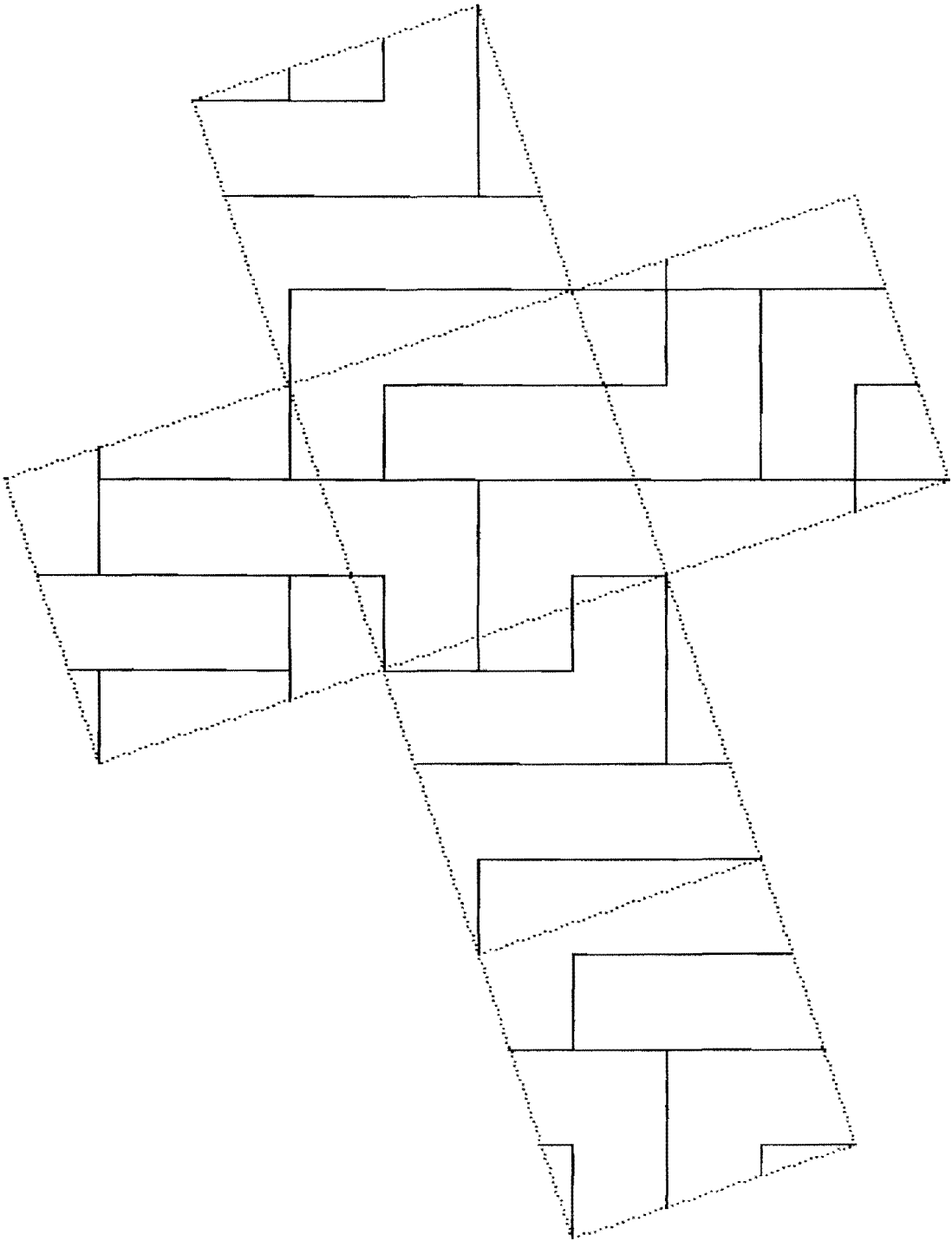
2689 2817 2847 2880 2929 2973 3038 3042 3092 3134 3155 3166



2689 2817 2847 2887 2891 2940 2995 3069 3094 3122 3155 3166

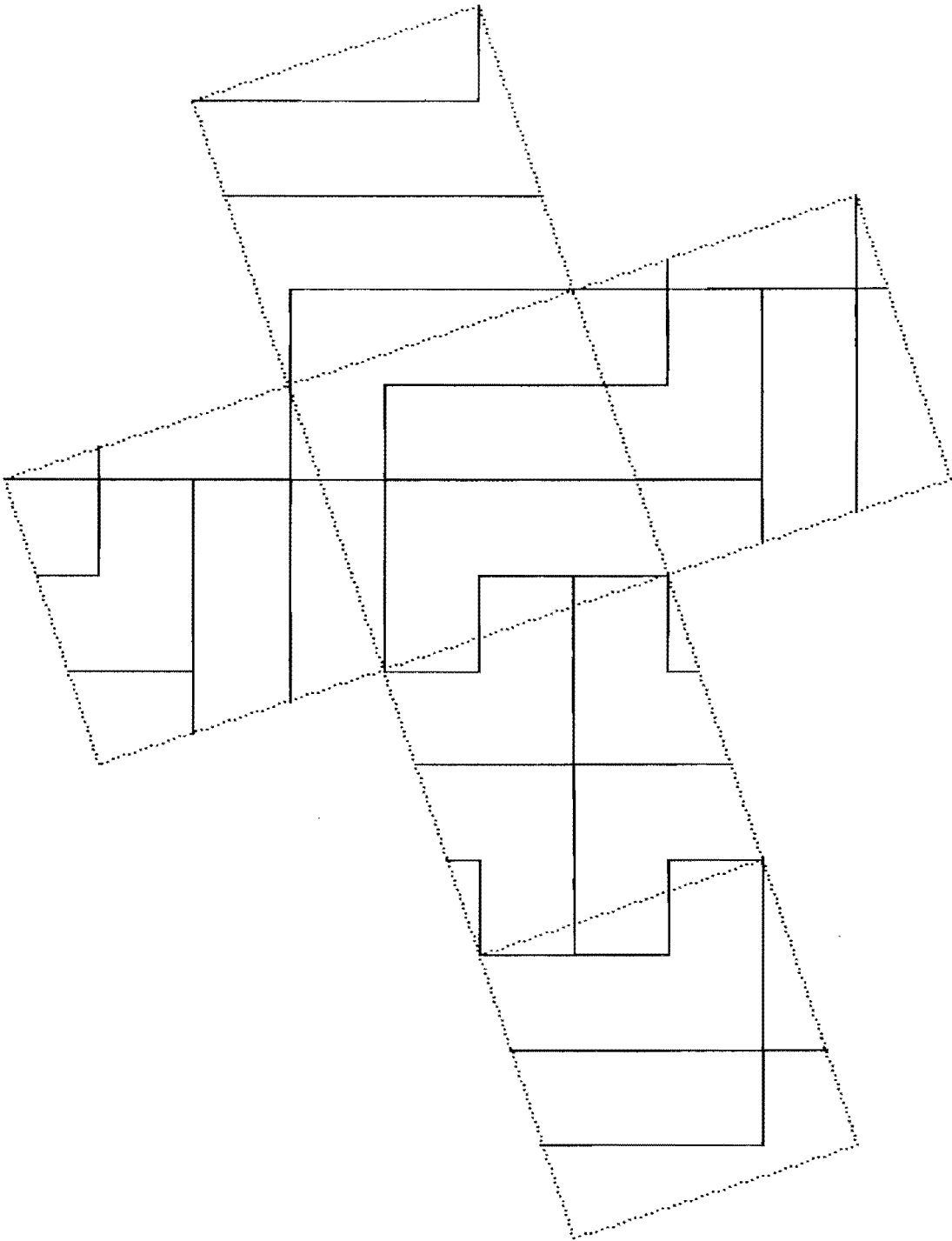


2689 2817 2847 2888 2903 2911 2928 3069 3094 3122 3155 3166

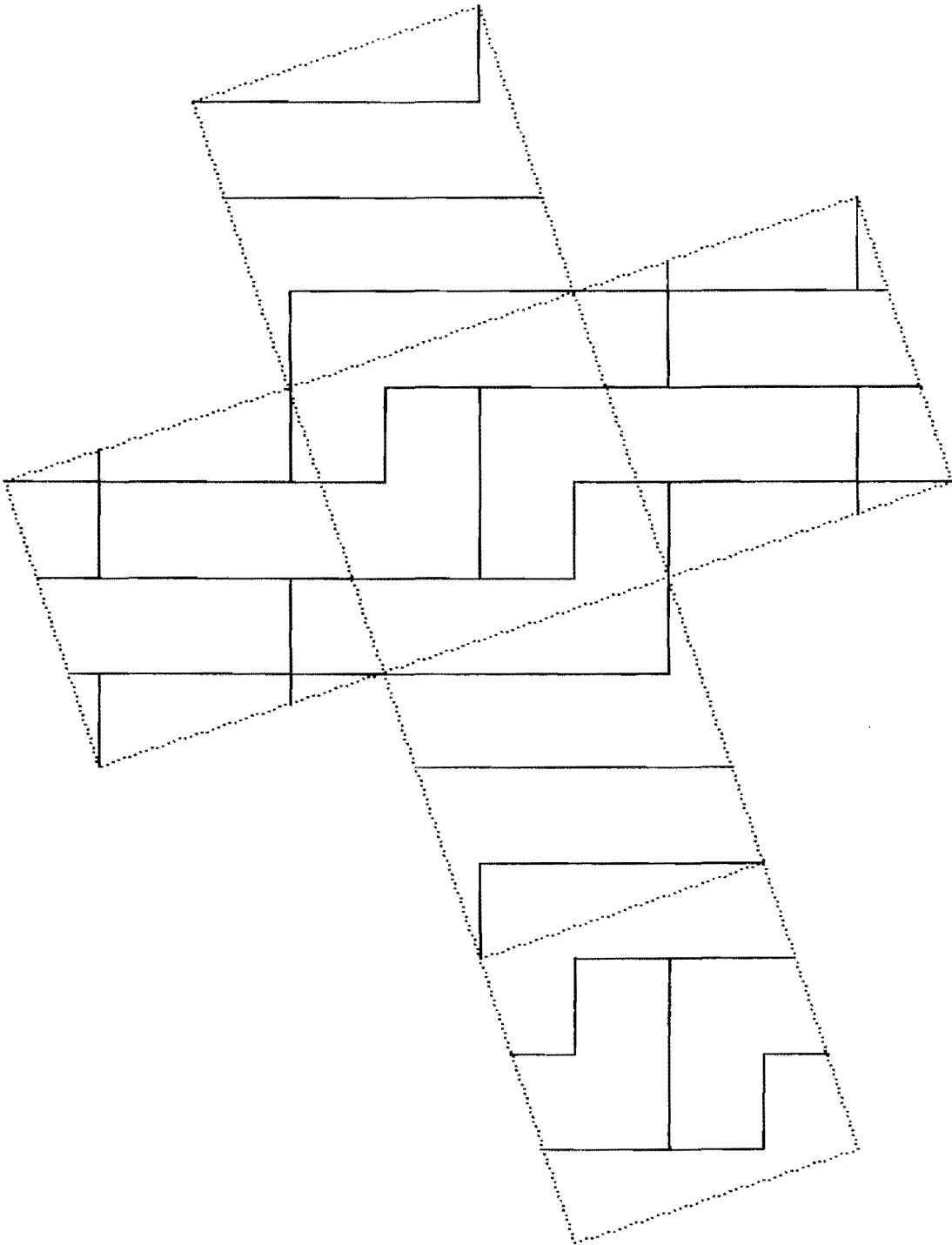


2689 2817 2850 2871 2900 2973 3034 3069 3097 3113 3140 3166

One axis order 2 (Y)

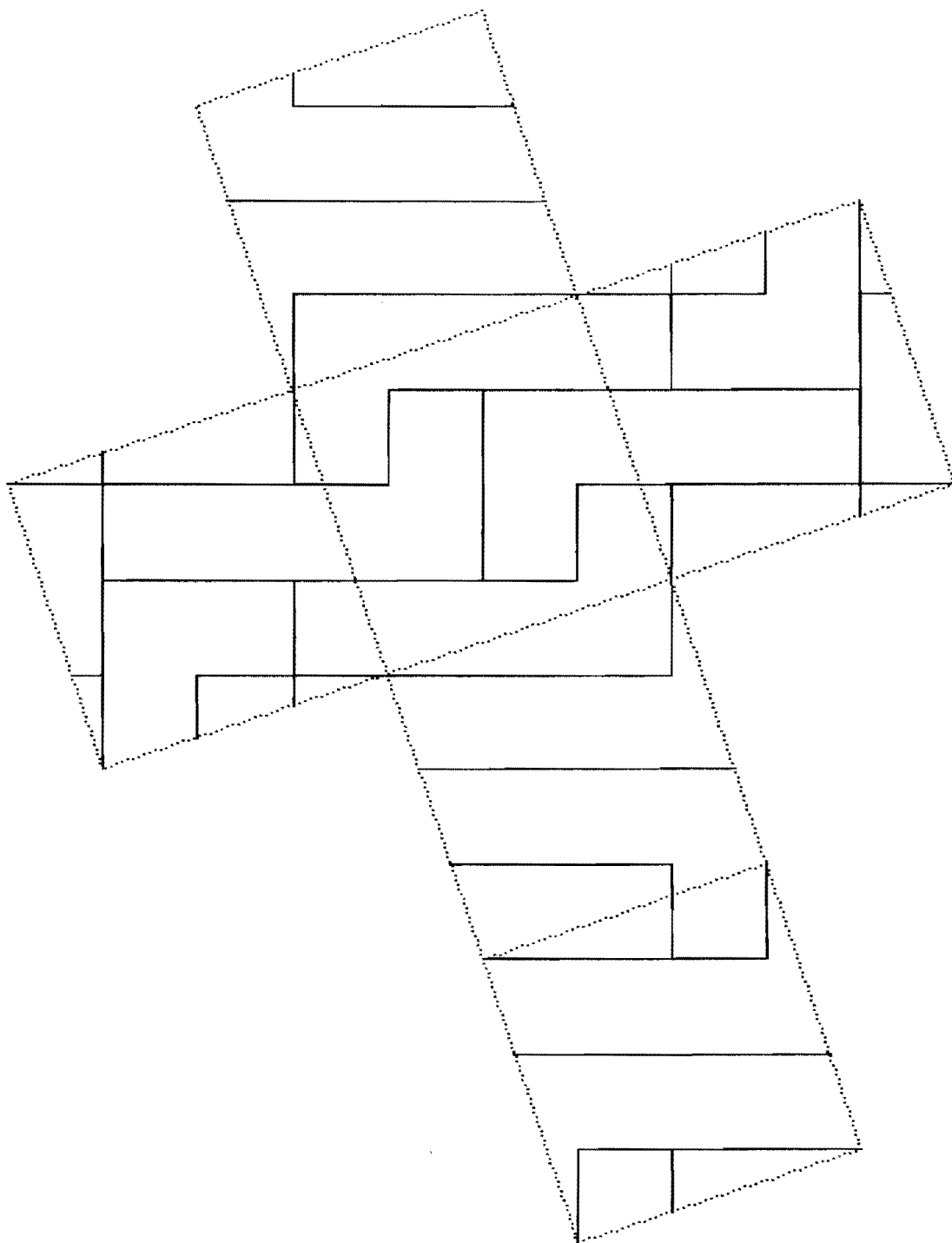


2689 2817 2854 2862 2926 2979 2998 3026 3094 3122 3155 3166



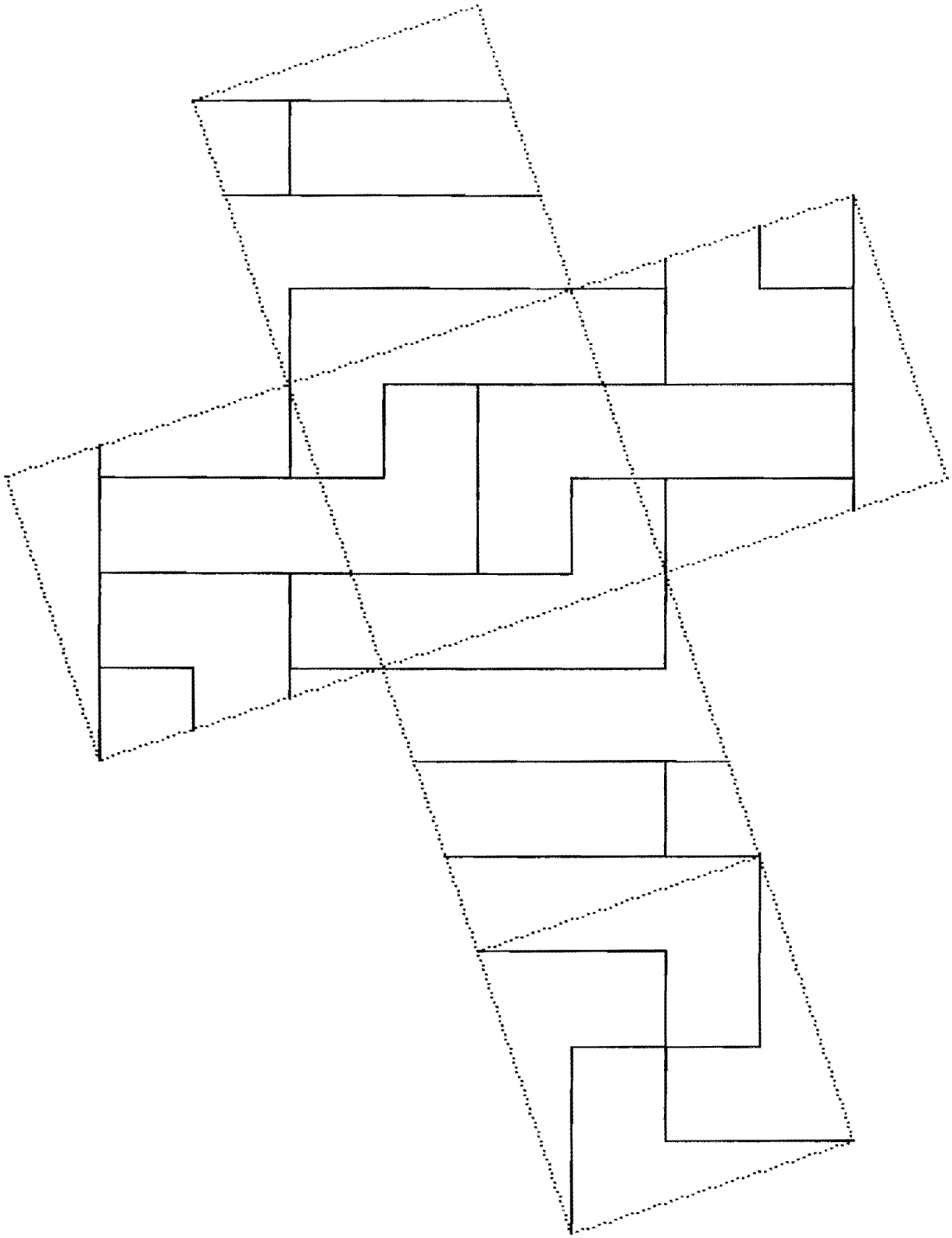
2689 2820 2825 2880 2929 2973 3034 3072 3077 3122 3155 3166

Three axes order 2 (X Y Z)



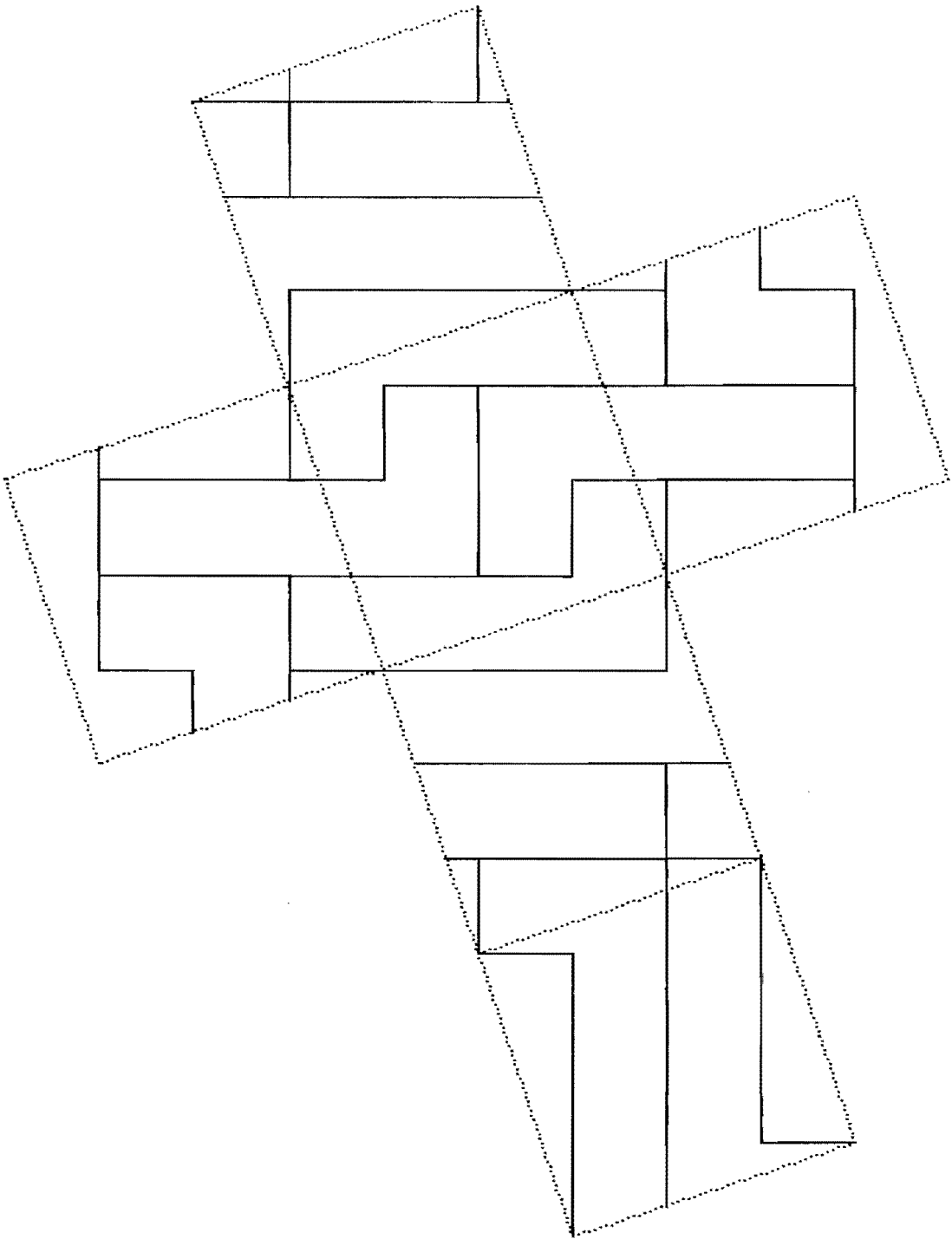
2689 2820 2825 2880 2929 2974 3020 3055 3091 3147 3150 3166

One axis order 2 .(X)

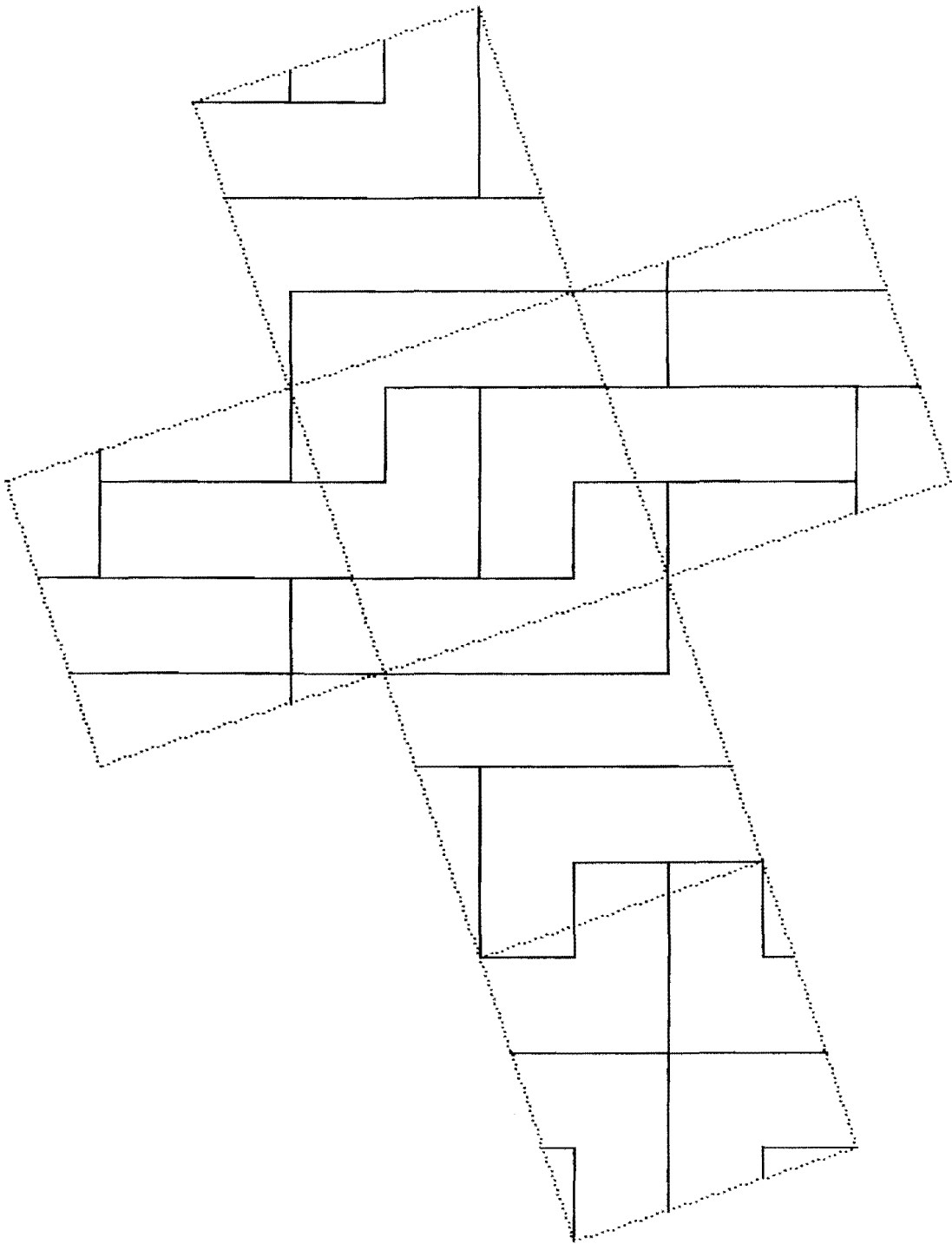


2689 2820 2825 2880 2929 2976 3014 3017 3062 3110 3160 3166

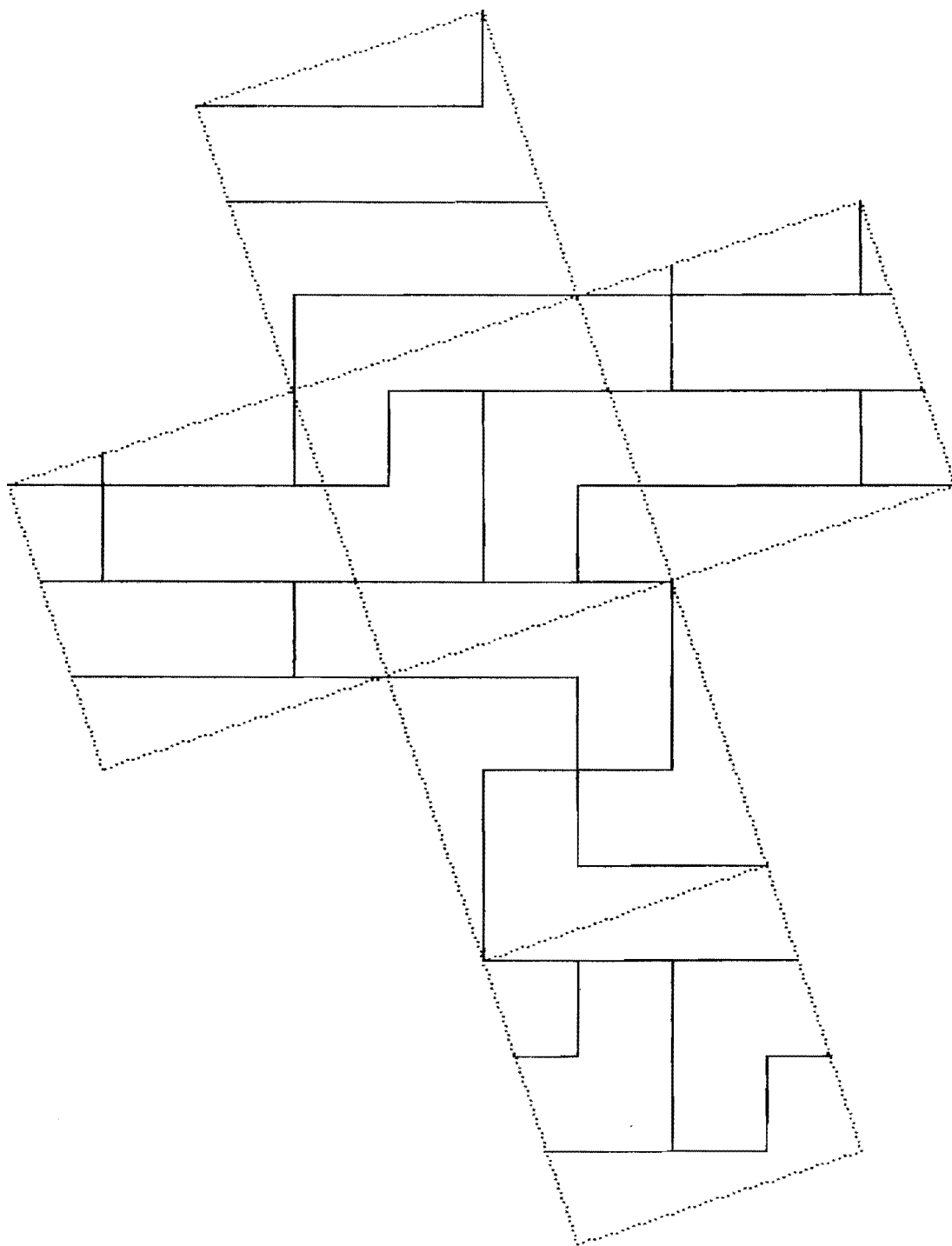
One axis order 2 (X)



2689 2820 2825 2880 2929 2976 3015 3029 3037 3054 3160 3166

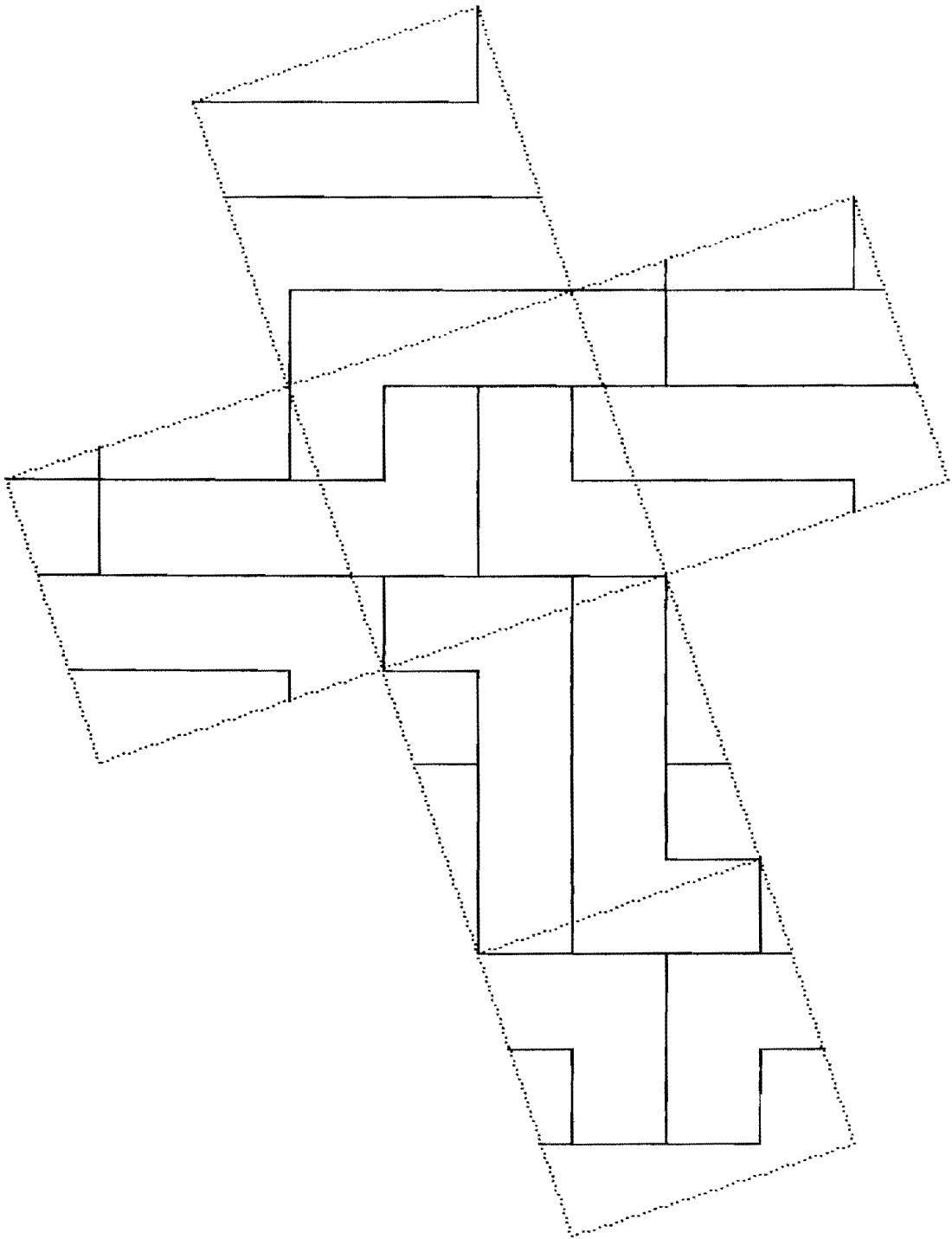


2689 2820 2825 2880 2929 2982 2989 3052 3097 3113 3140 3166



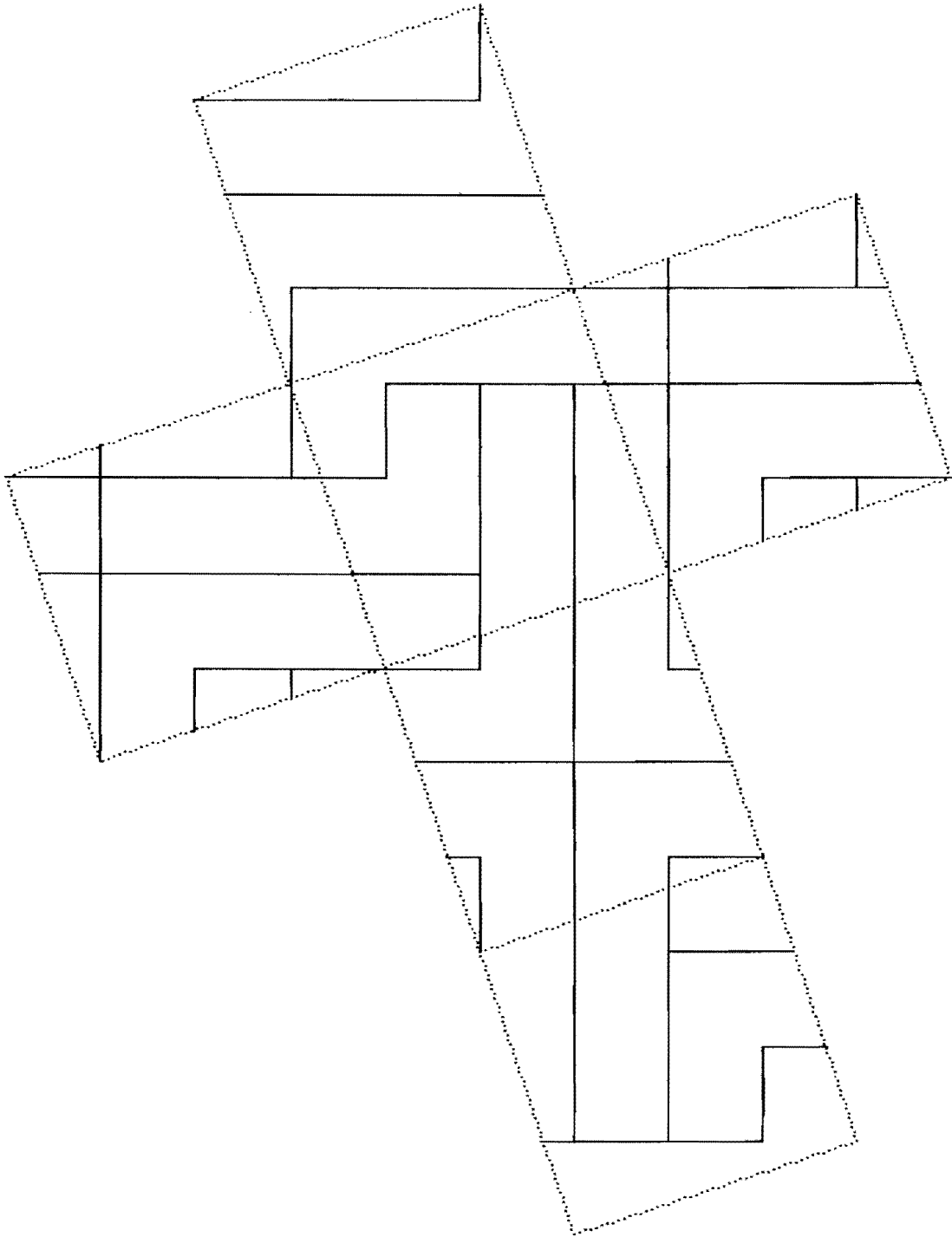
2689 2820 2825 2887 2891 2940 2995 3072 3077 3122 3155 3166

One axis order 2 (Z)

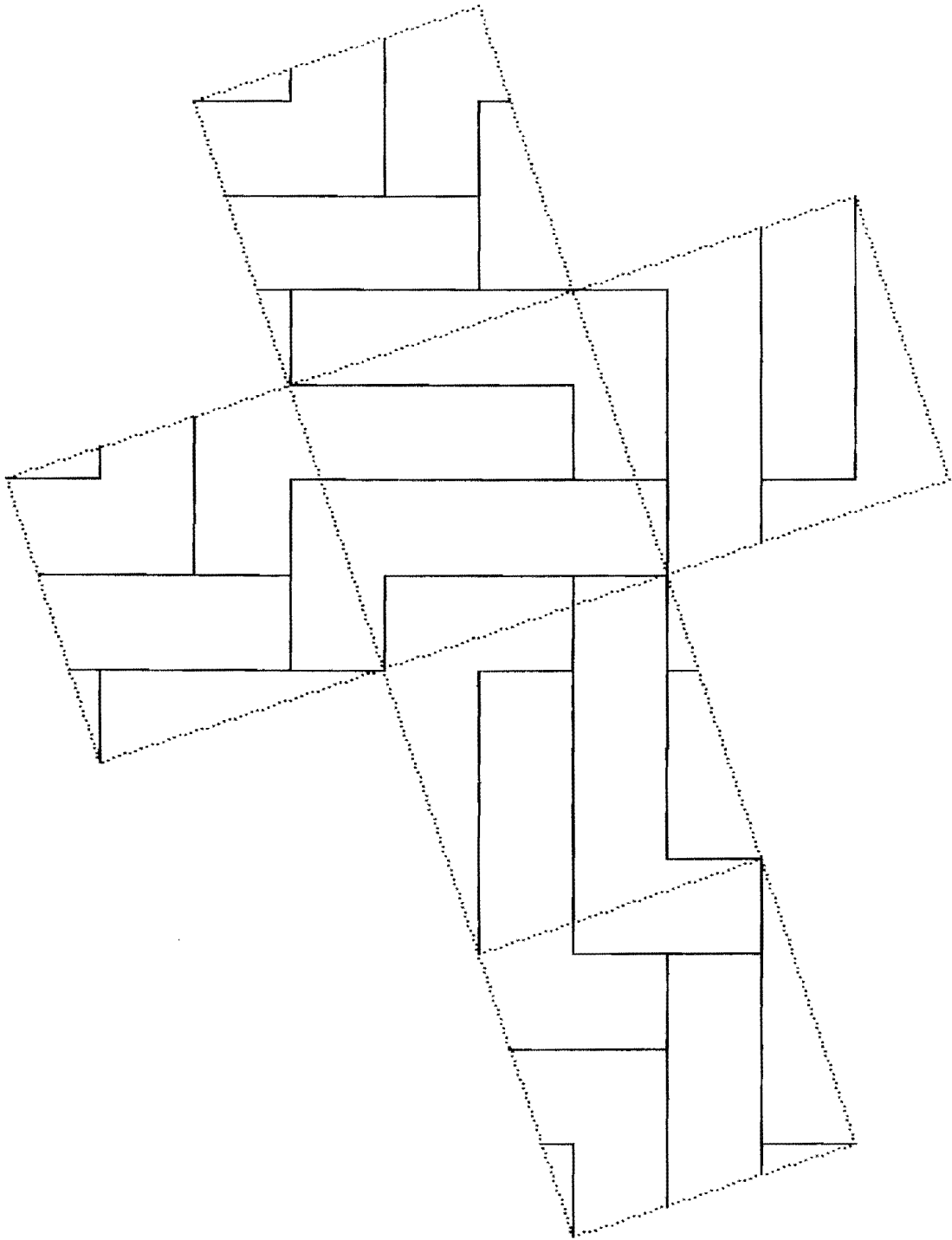


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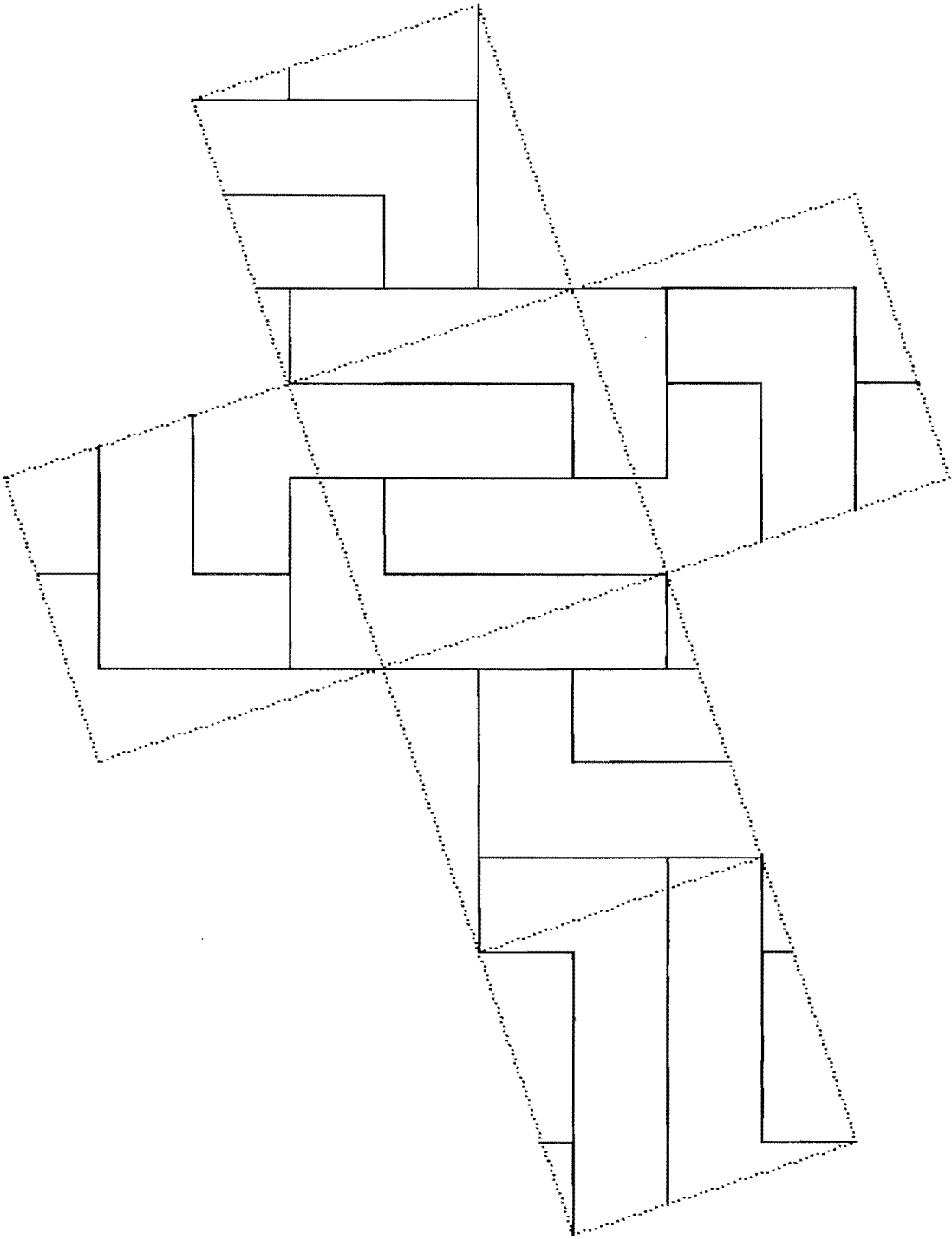
One axis order 2 (Z)



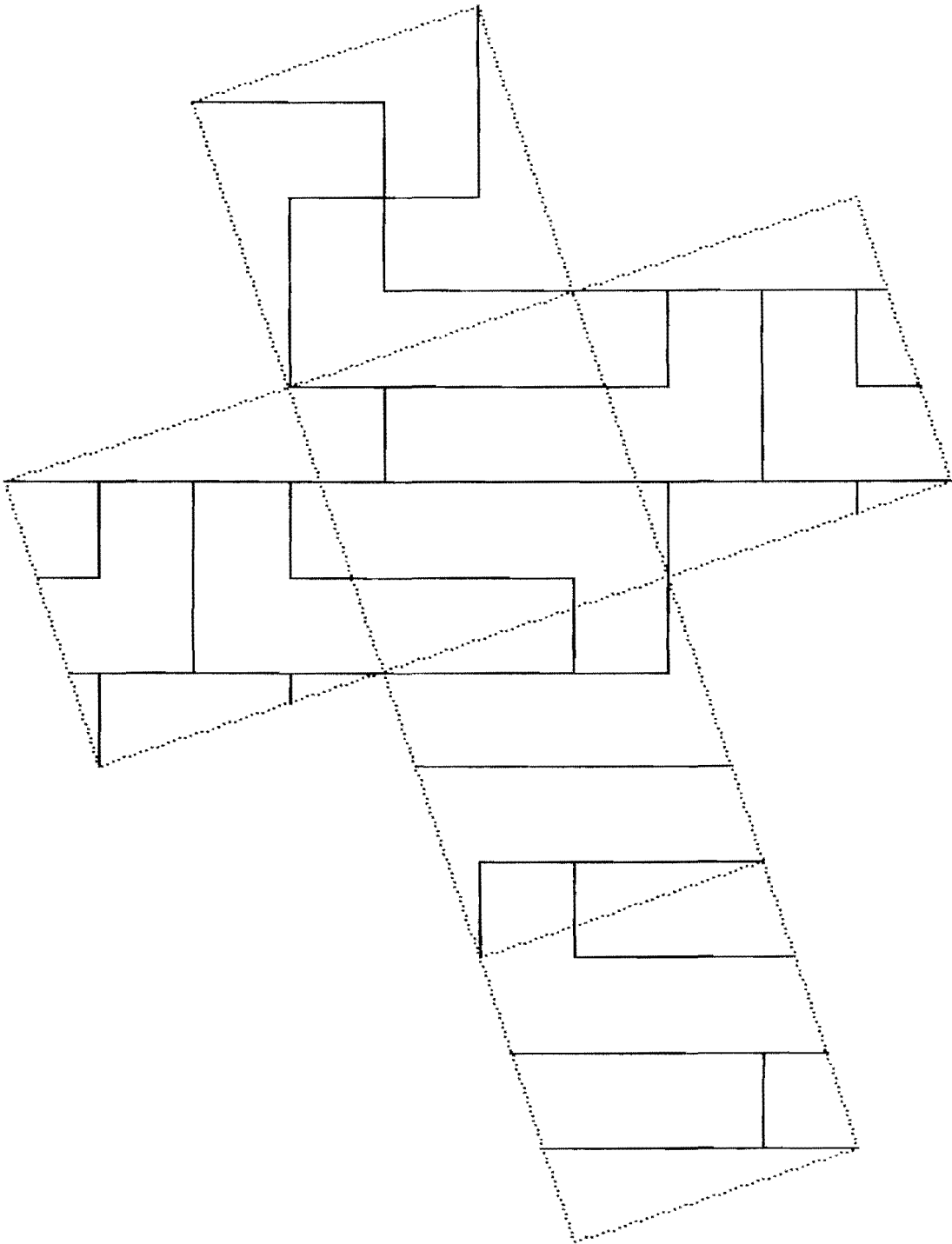
2689 2820 2831 2835 2898 2935 2978 2999 3077 3122 3155 3166



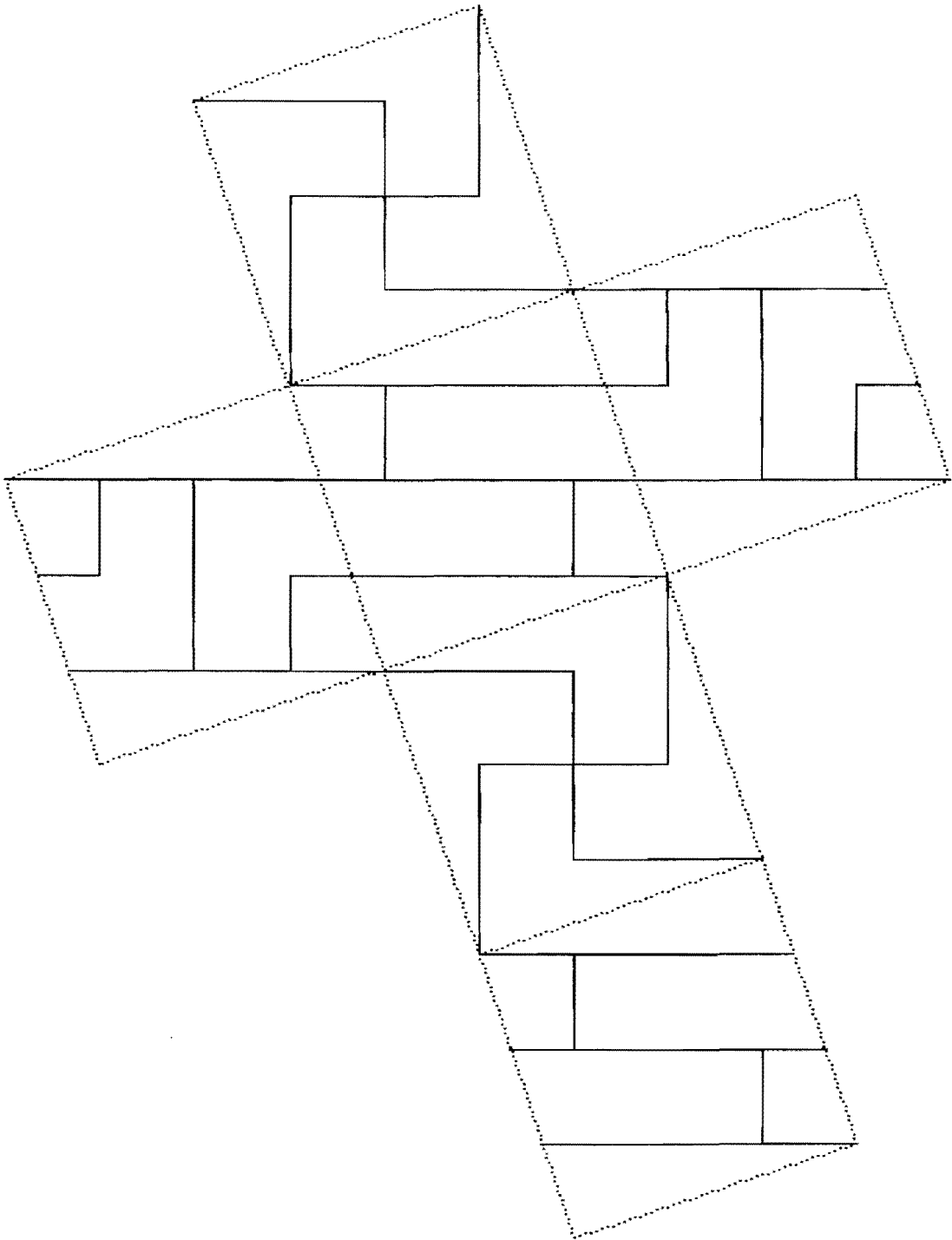
2690 2800 2843 2910 2928 2937 2954 2968 3083 3097 3143 3152



2690 2800 2853 2864 2944 2952 2959 3037 3054 3089 3141 3168

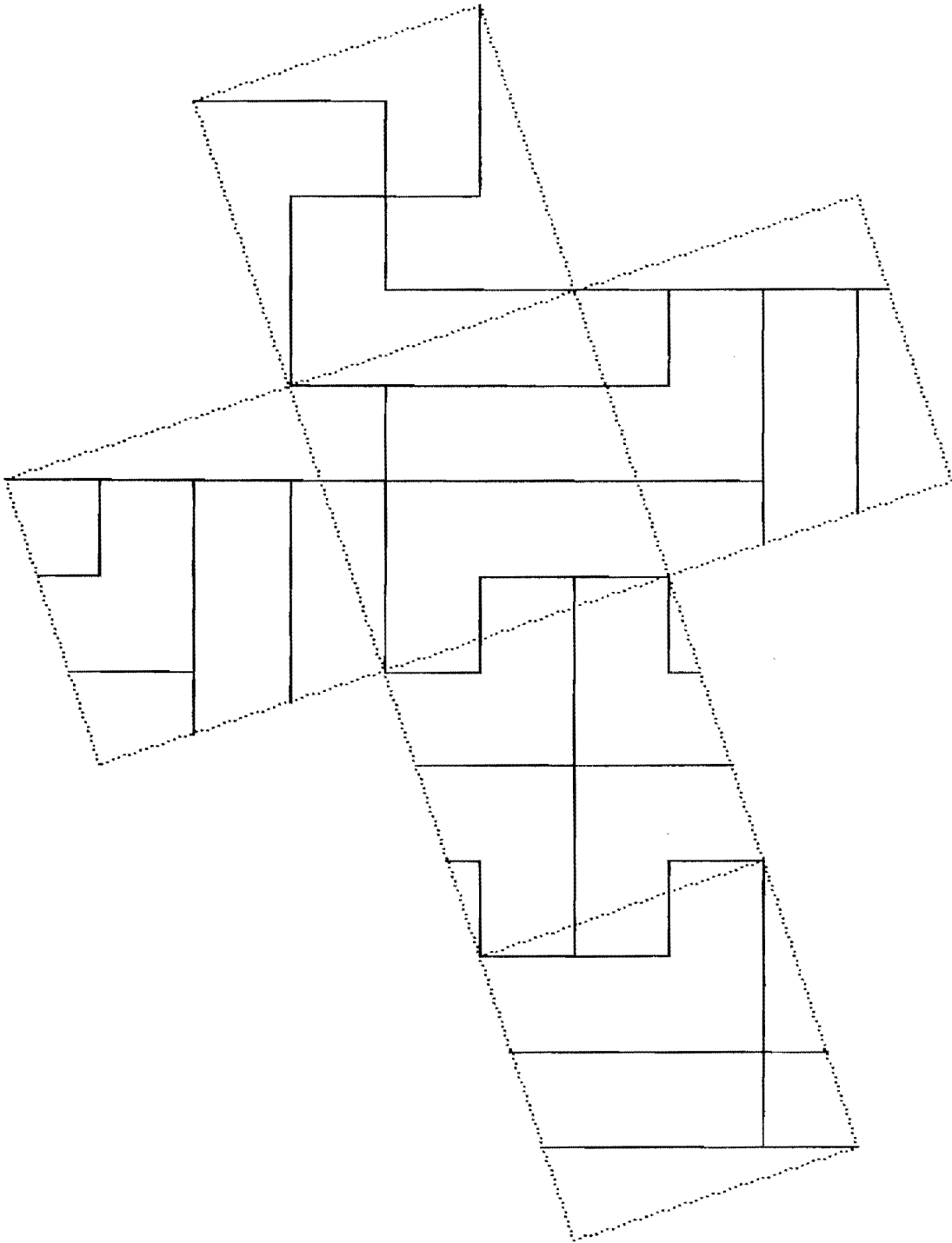


2691 2814 2817 2844 2894 2929 2973 3038 3042 3094 3129 3131

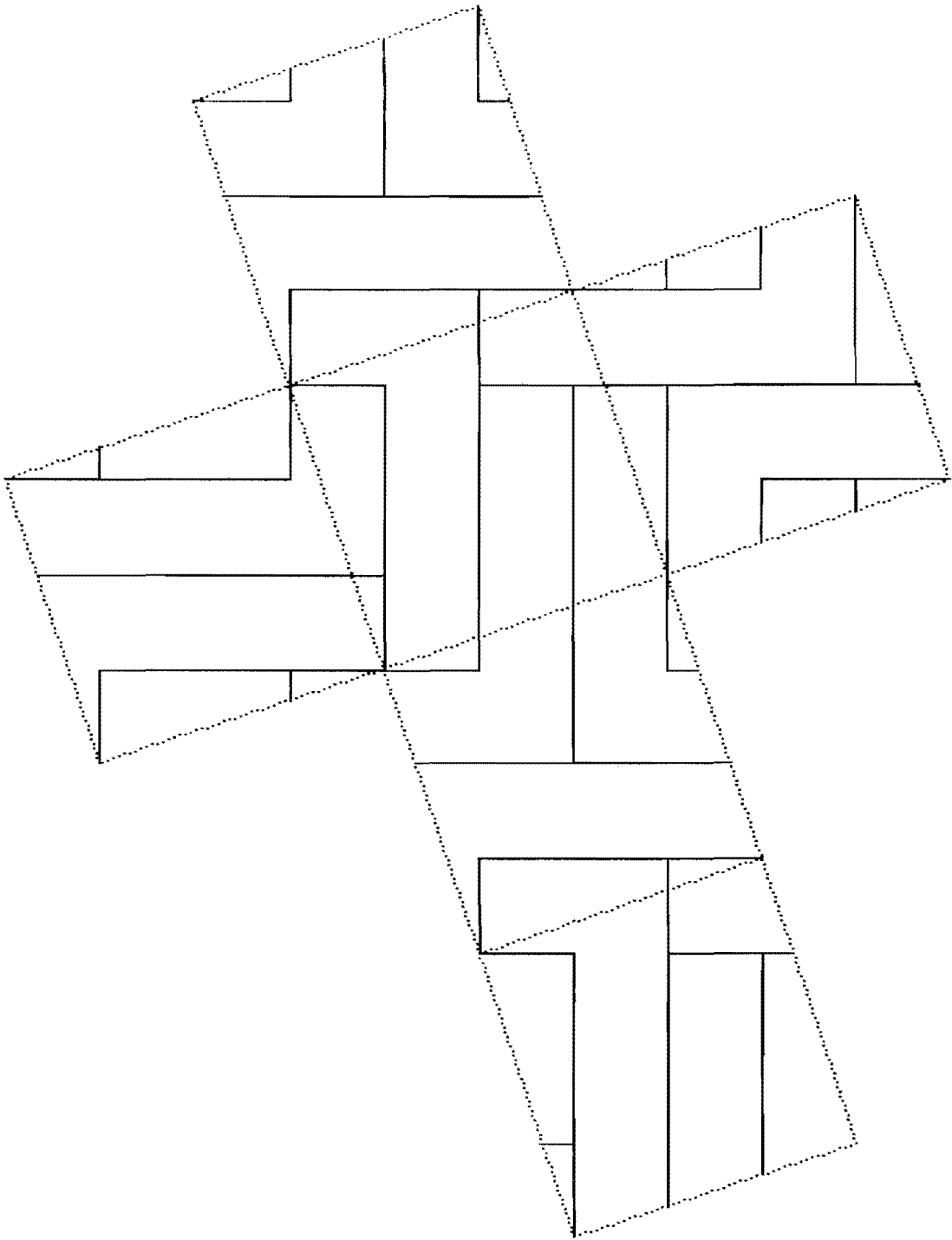


2691 2814 2817 2847 2887 2891 2940 2995 3069 3094 3129 3131

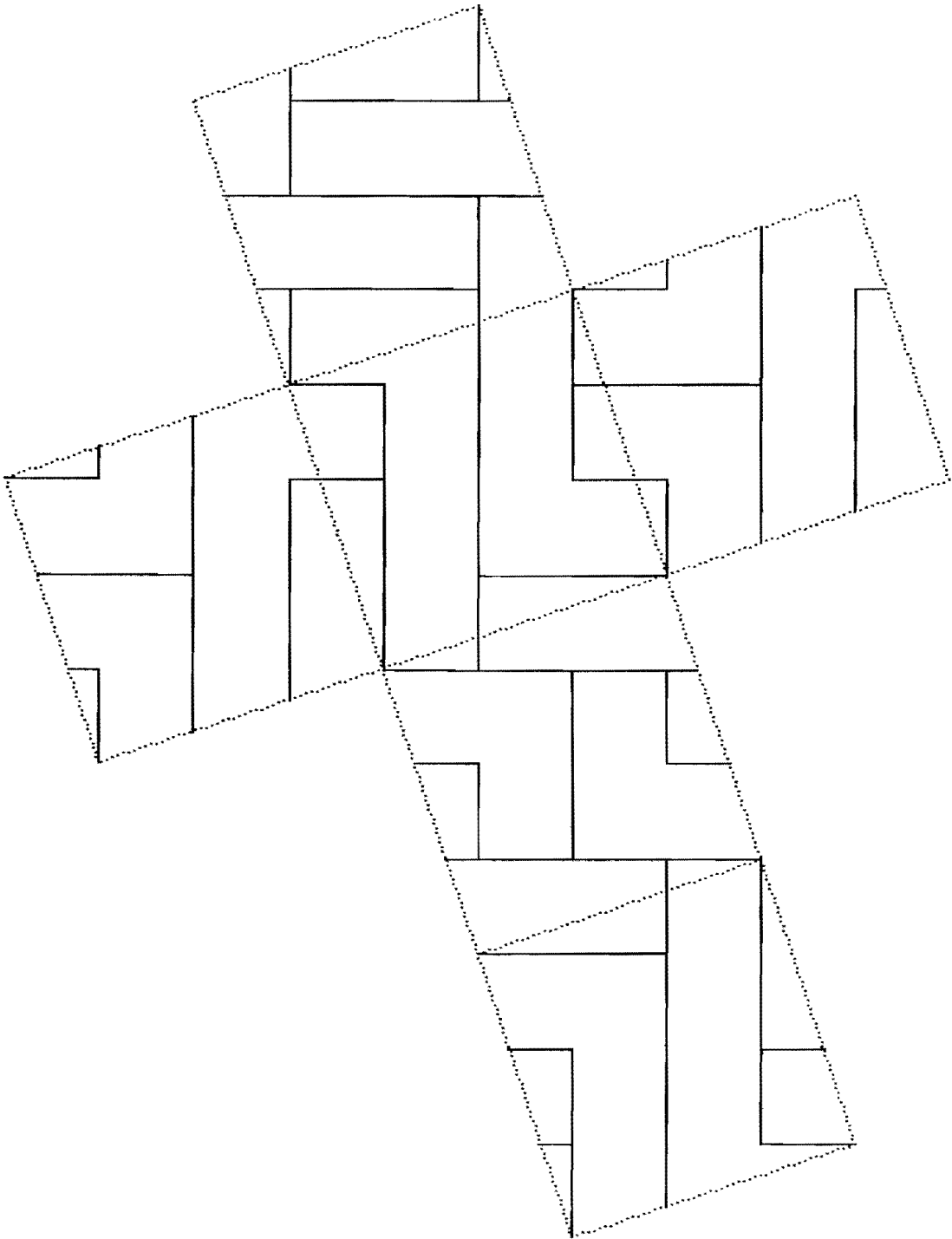
Three axes order 2 (X Y Z)



2691 2814 2817 2854 2862 2926 2979 2998 3026 3094 3129 3131

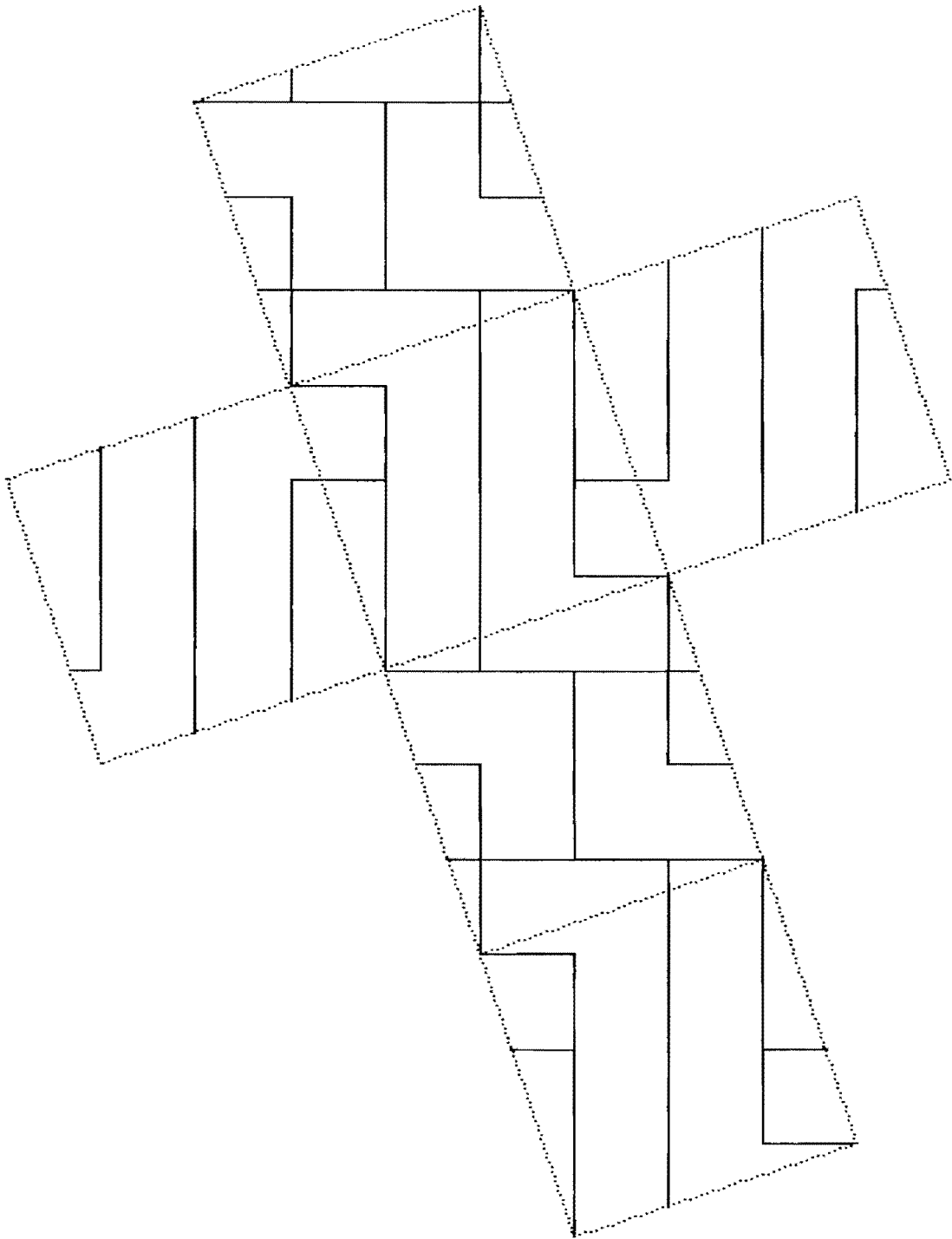


2698 2759 2810 2831 2835 2906 2935 2973 3037 3083 3087 3166



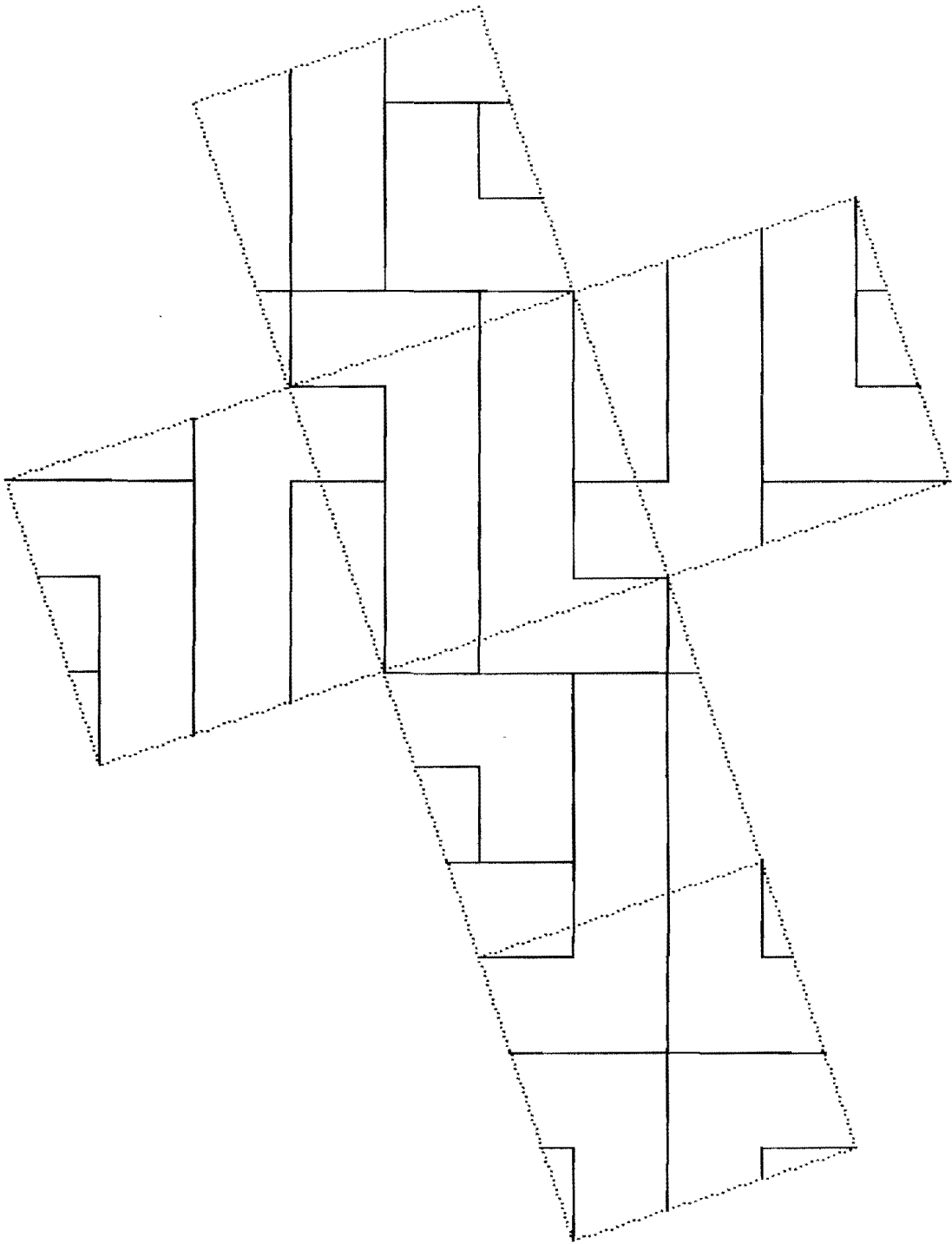
2698 2768 2794 2812 2839 2855 2961 2971 3019 3054 3063 3143

One axis order 2 (Z)



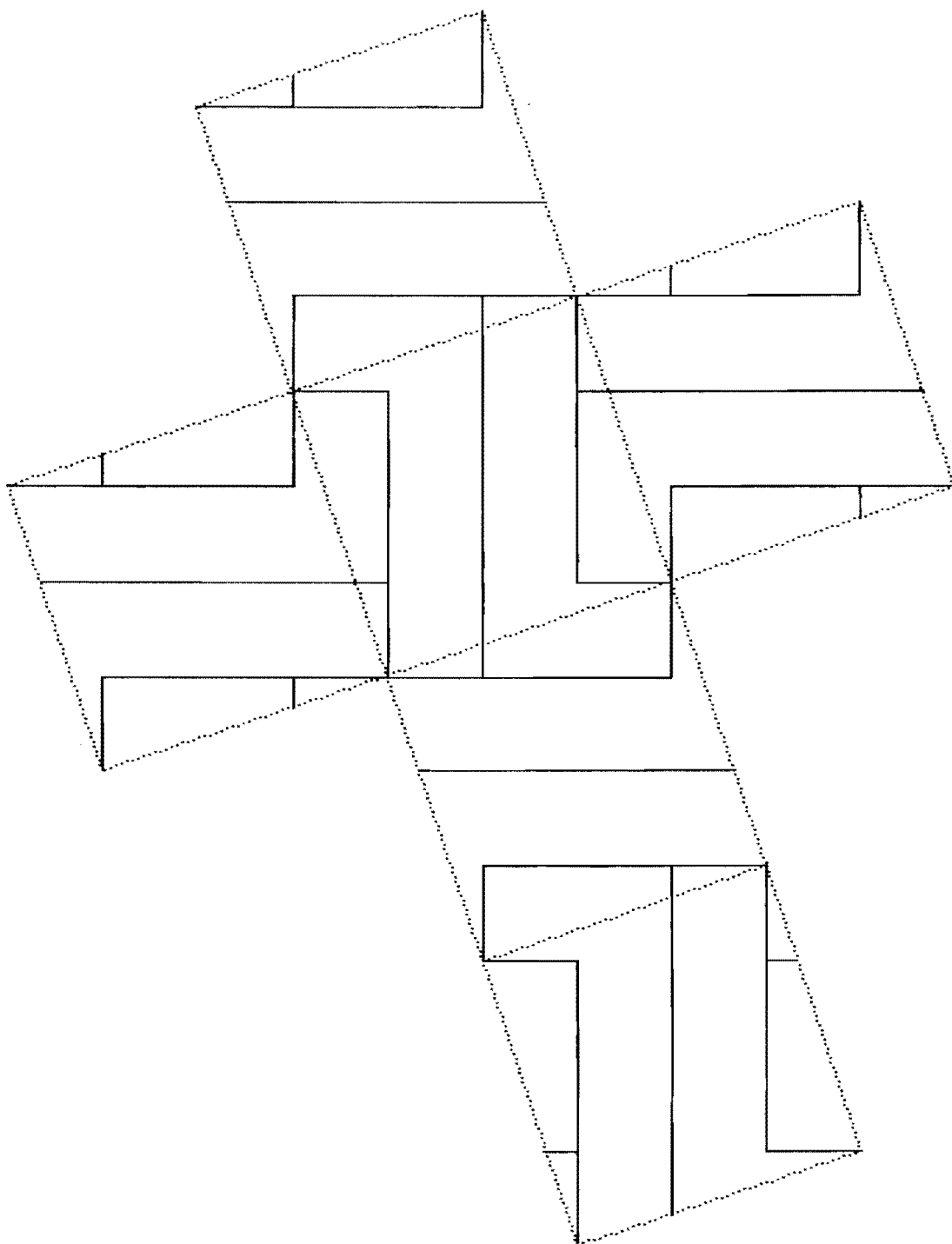
2698 2770 2787 2812 2855 2889 2961 2971 3030 3037 3054 3101

Three axes order 2 (X Y Z)



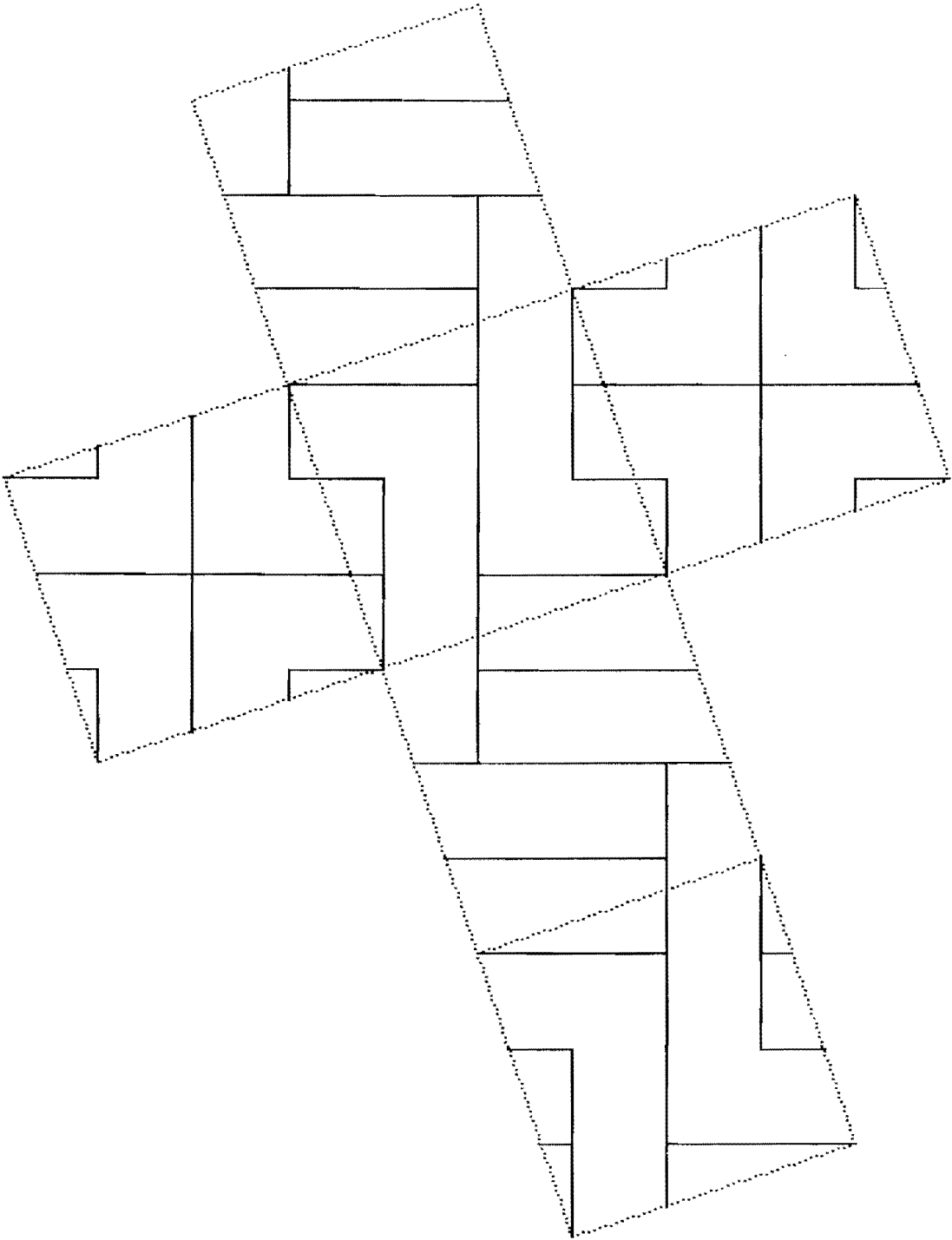
2698 2770 2787 2812 2855 2889 2962 2966 3024 3059 3096 3114

One axis order 2 (X)



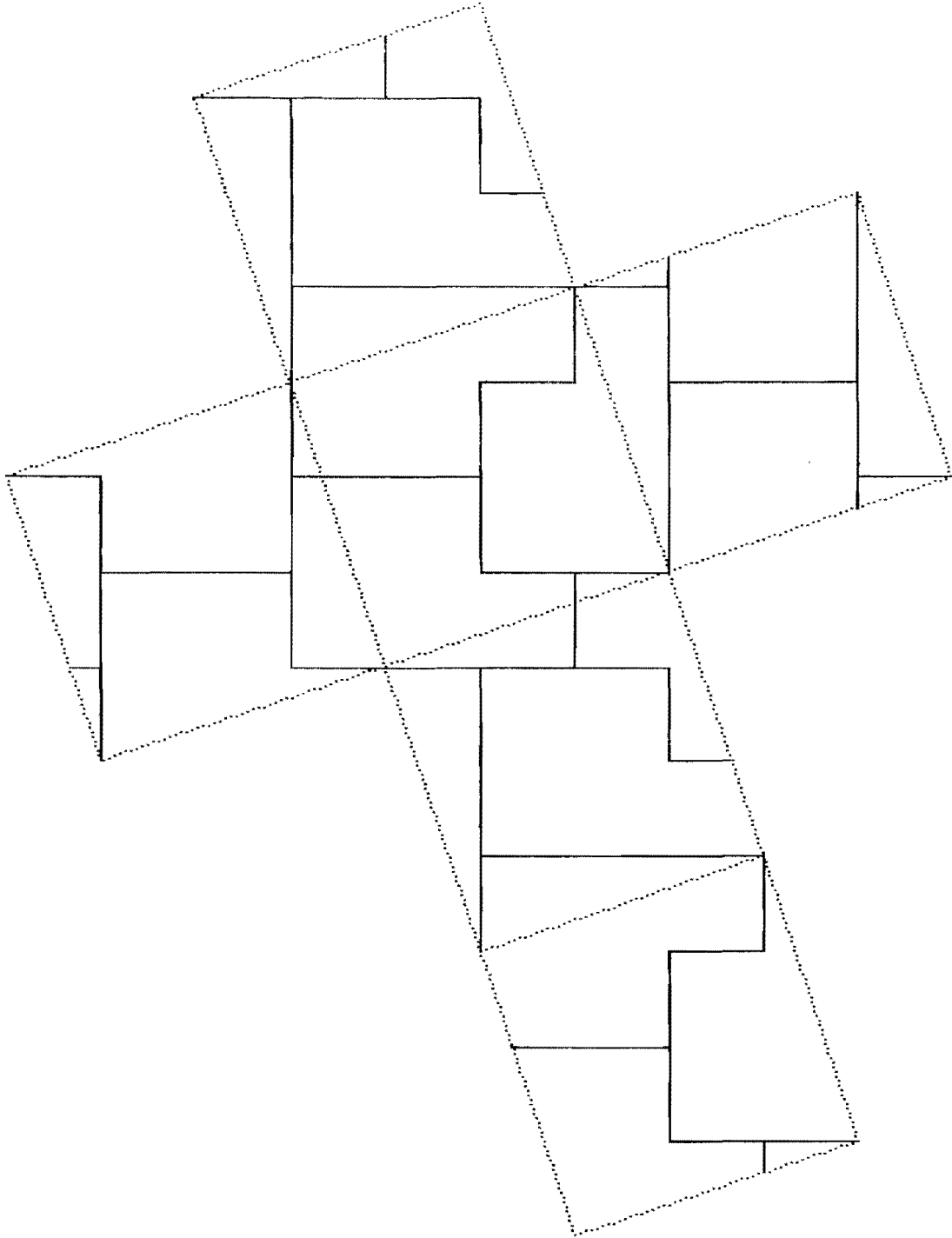
2698 2770 2791 2810 2838 2906 2929 2973 3037 3054 3155 3166

Four axes order 3 (D1 D2 D3 D4) and three axes order 2 (X Y Z)



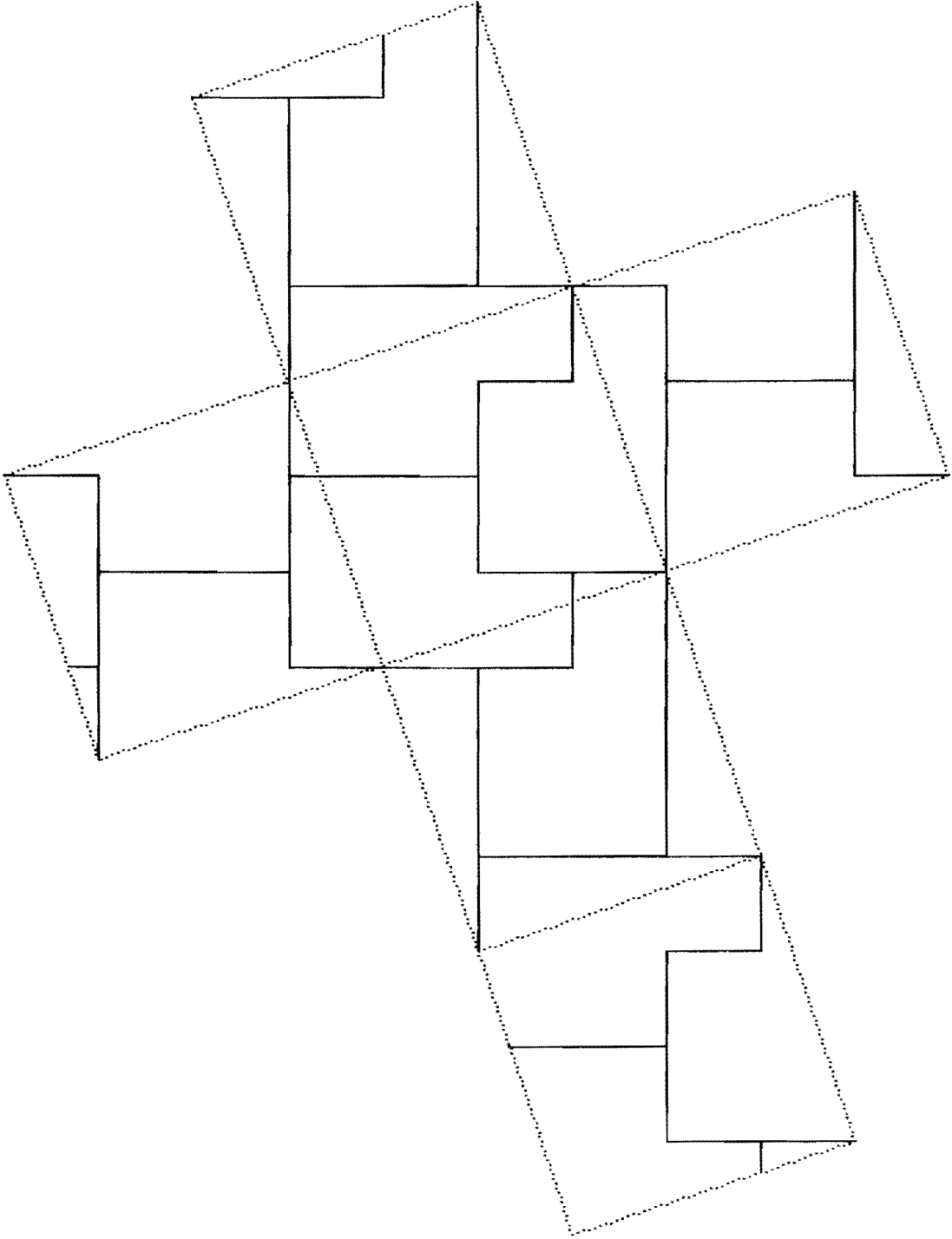
2700 2768 2794 2802 2839 2905 2947 3013 3019 3063 3090 3143

Three axes order 2 (X Y Z)



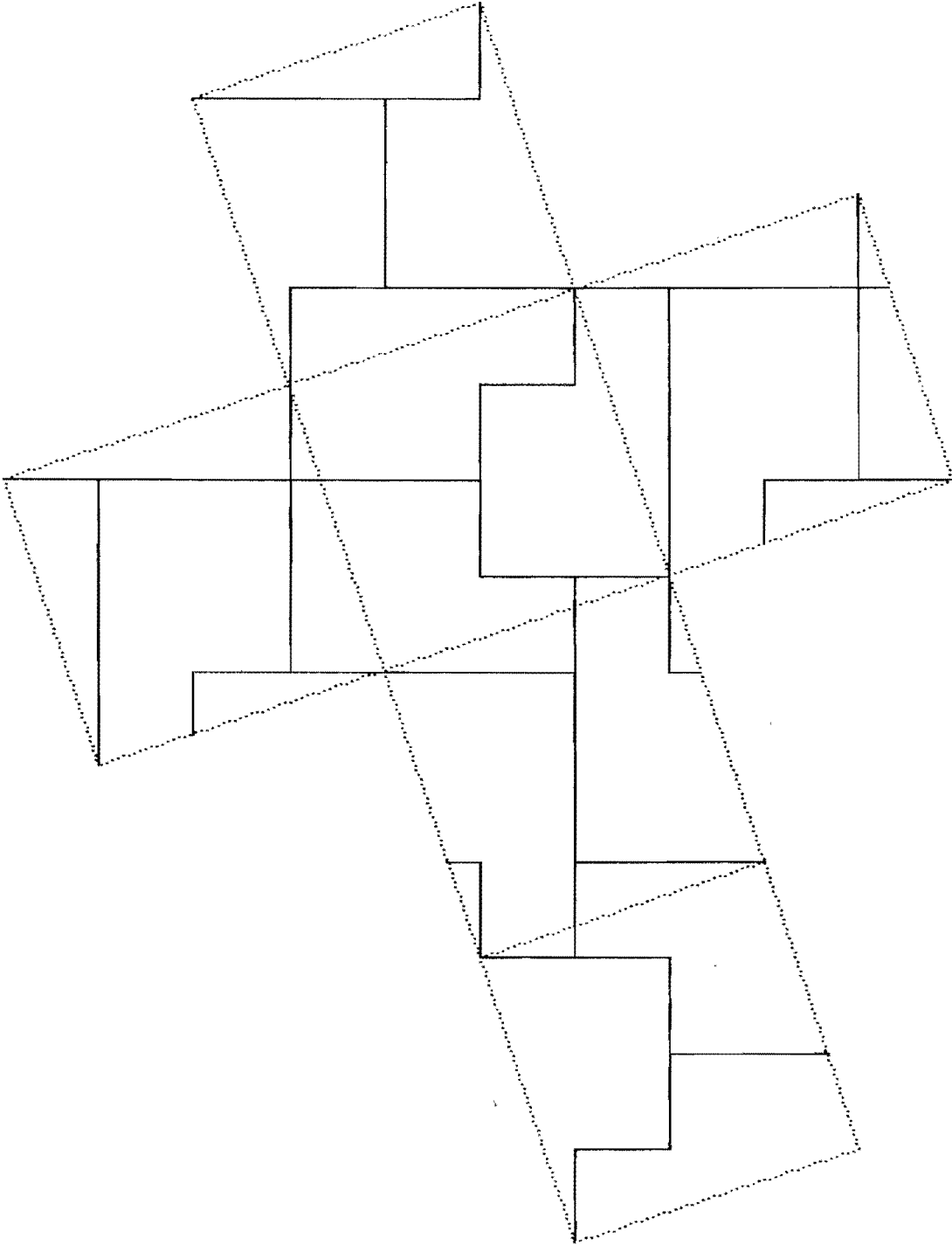
3169 3227 3287 3346 3364 3368 3429 3451 3493 3550 3564 3568

One axis order 2 (Y)



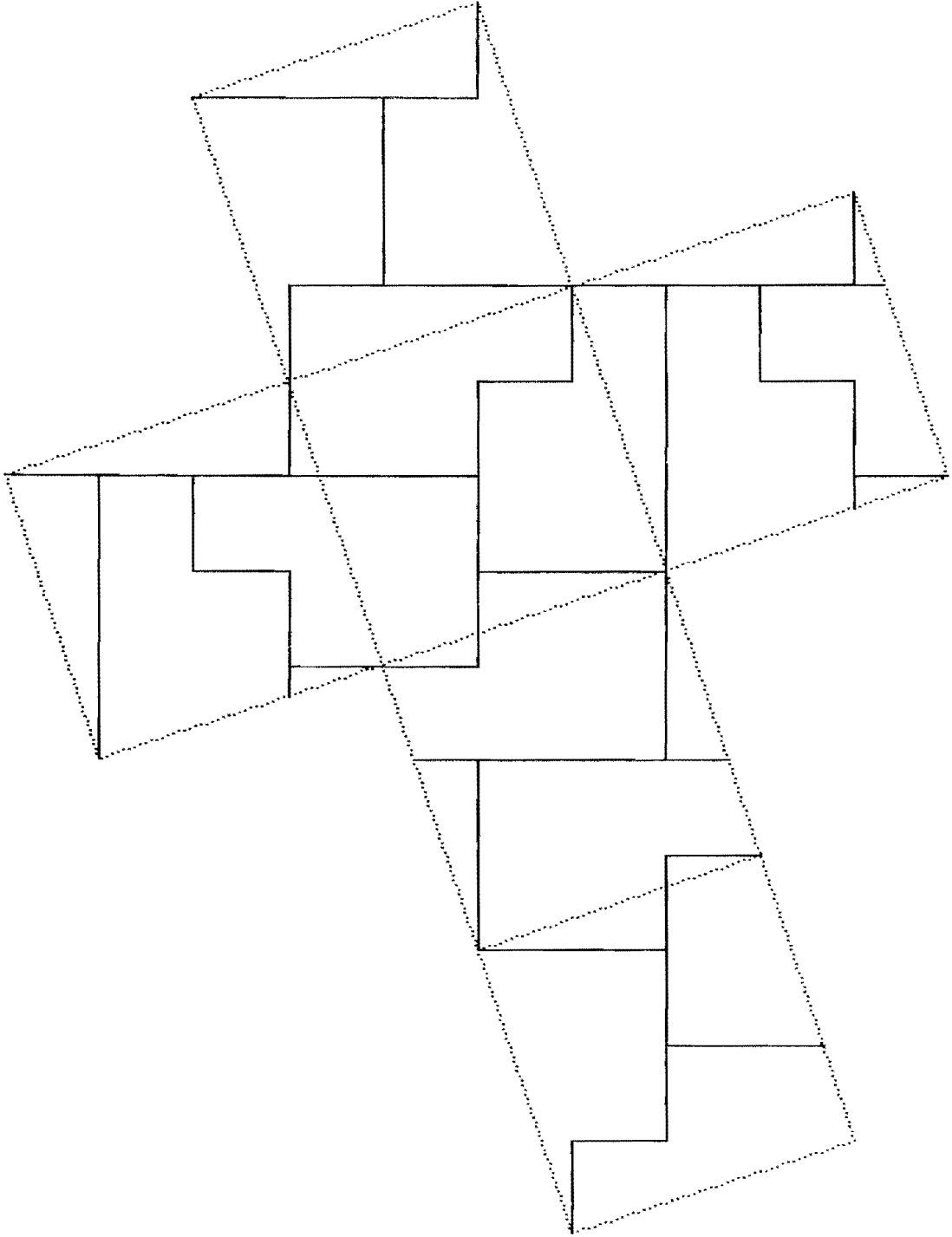
3169 3227 3287 3347 3352 3364 3429 3451 3493 3551 3555 3564

One axis order 2 (Y)



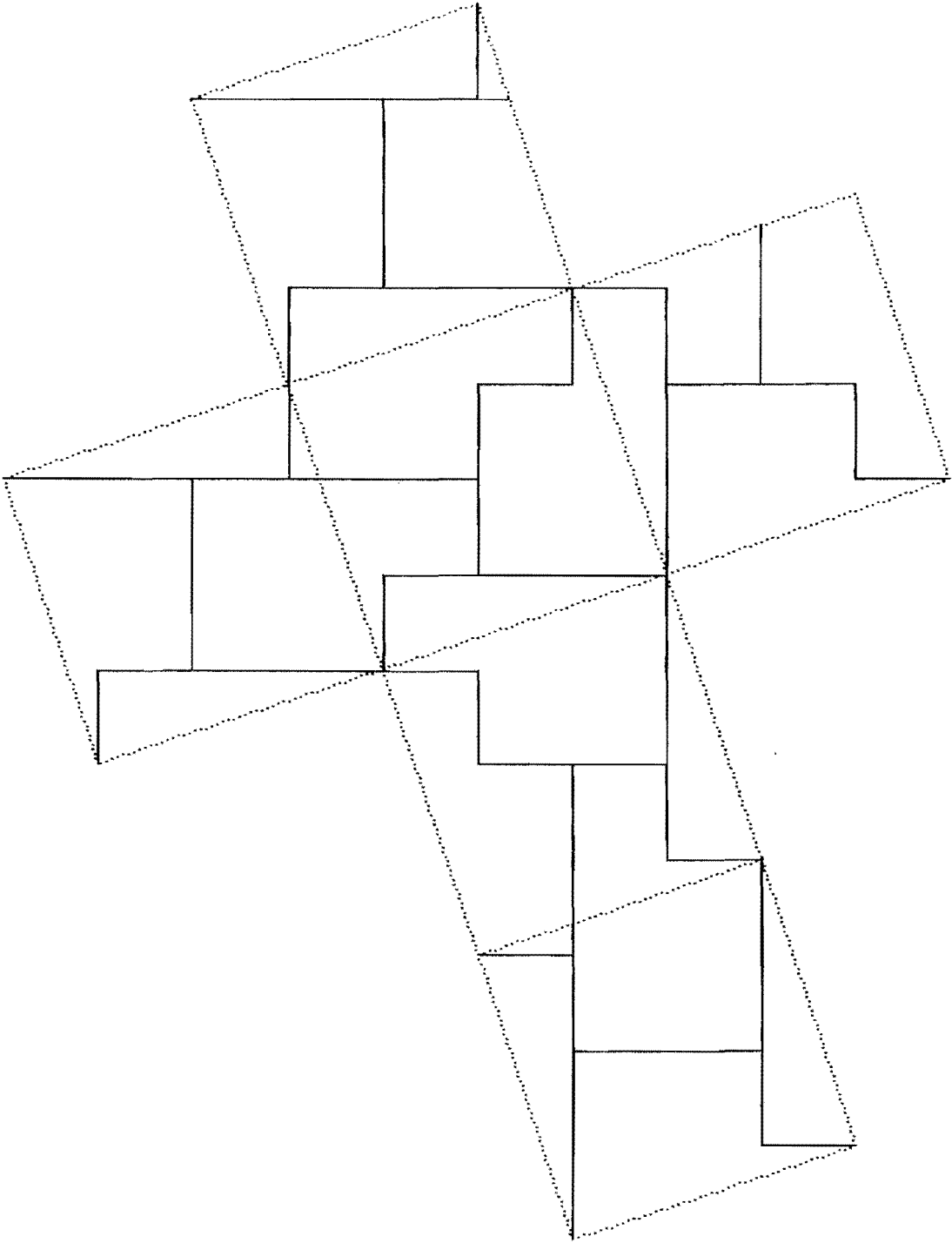
3169 3227 3287 3348 3356 3361 3428 3436 3462 3512 3553 3562

One axis order 2 (Z)



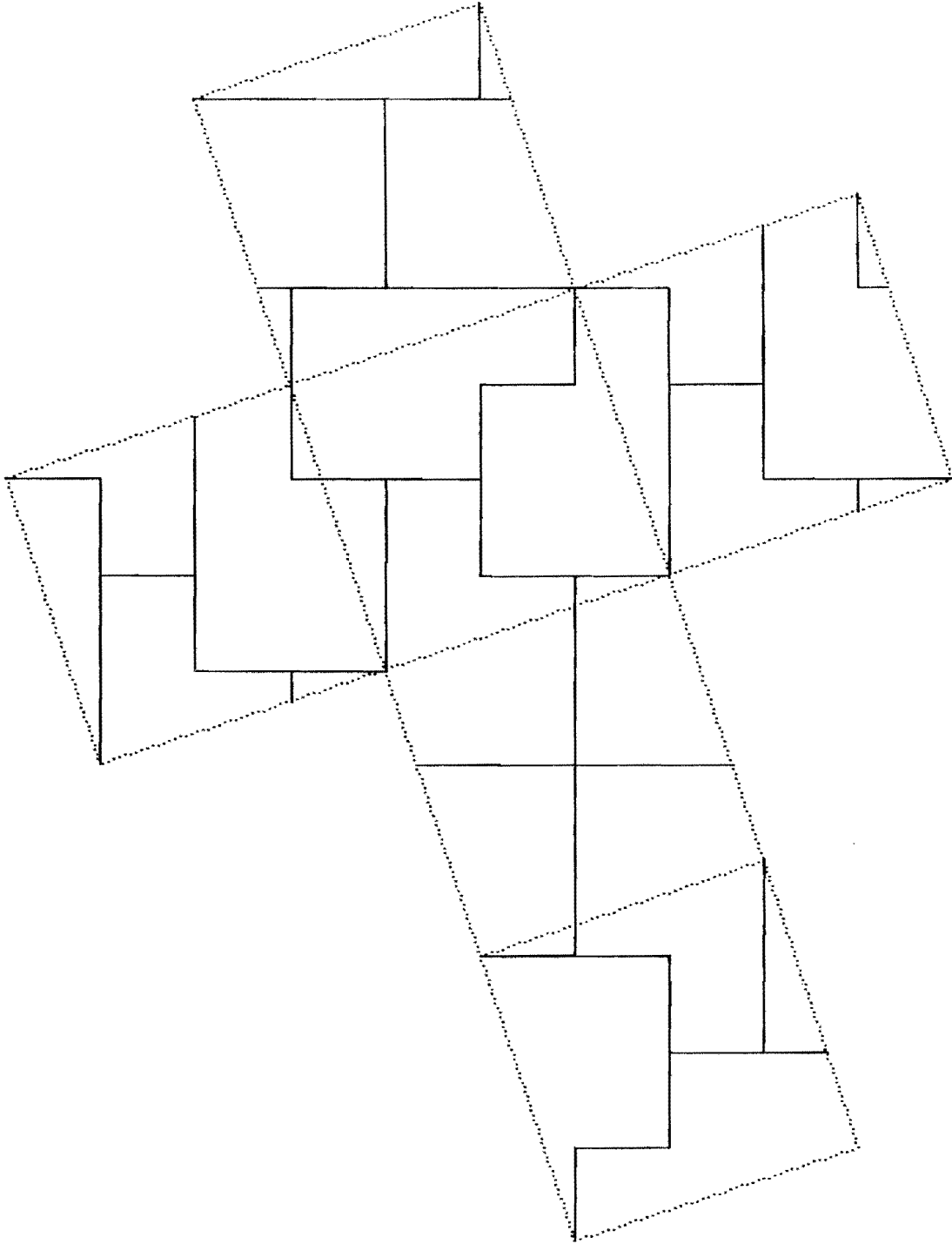
3169 3227 3289 3335 3354 3401 3403 3447 3462 3512 3553 3562

One axis order 2 (Z)



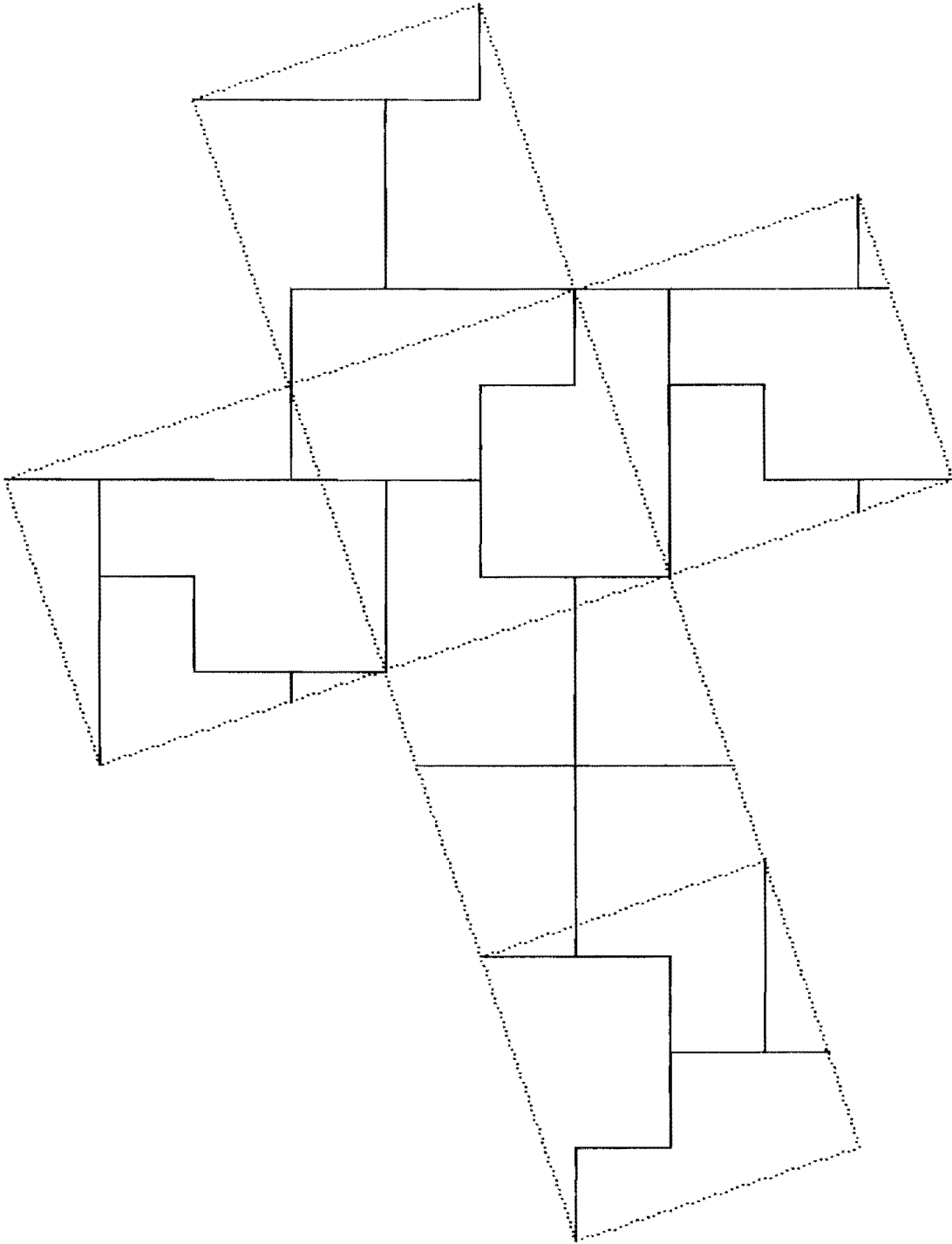
3169 3227 3291 3328 3352 3363 3415 3453 3466 3508 3562 3570

One axis order 3 (D3)



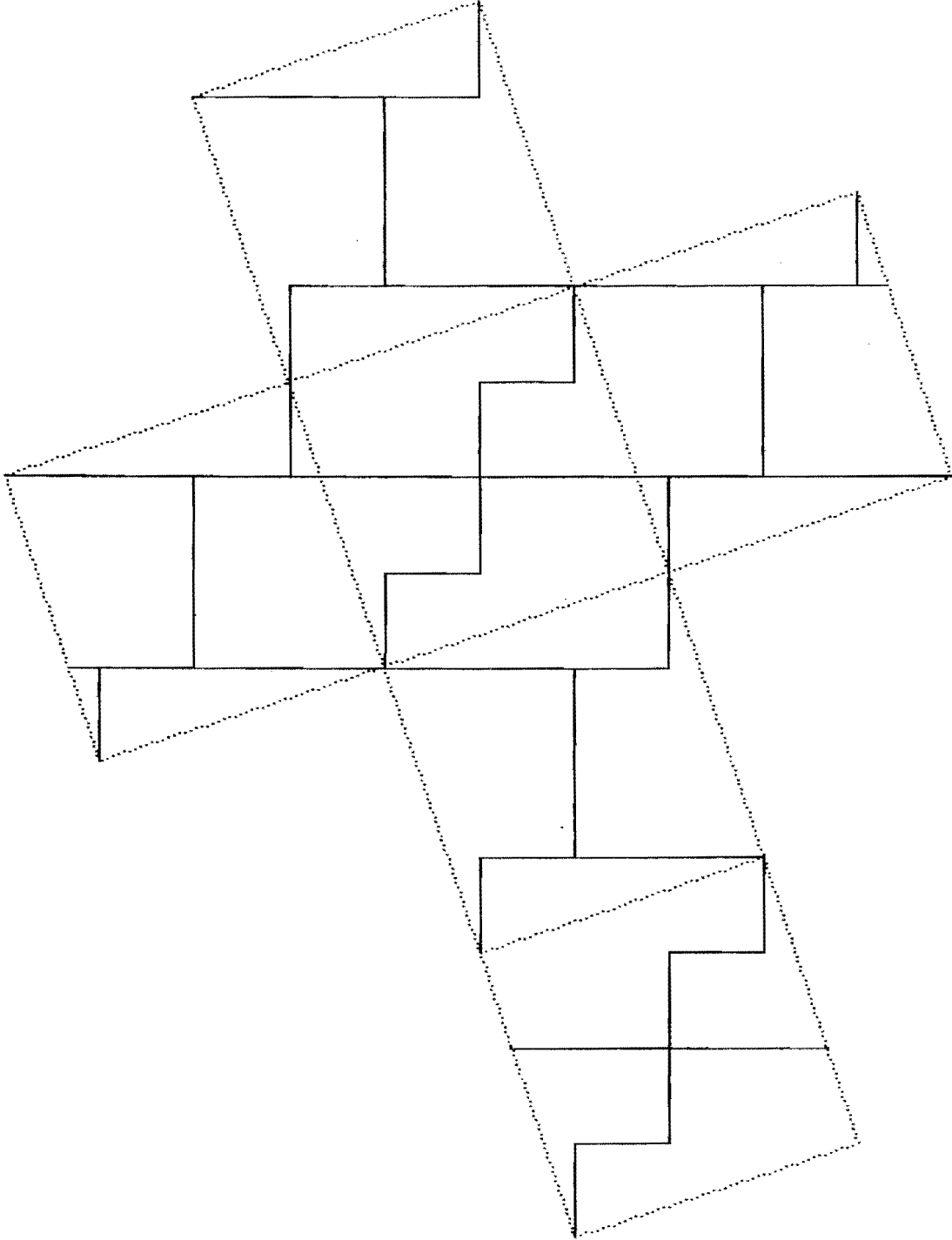
3169 3227 3296 3307 3344 3394 3413 3455 3462 3512 3563 3570

One axis order 2 (Z)



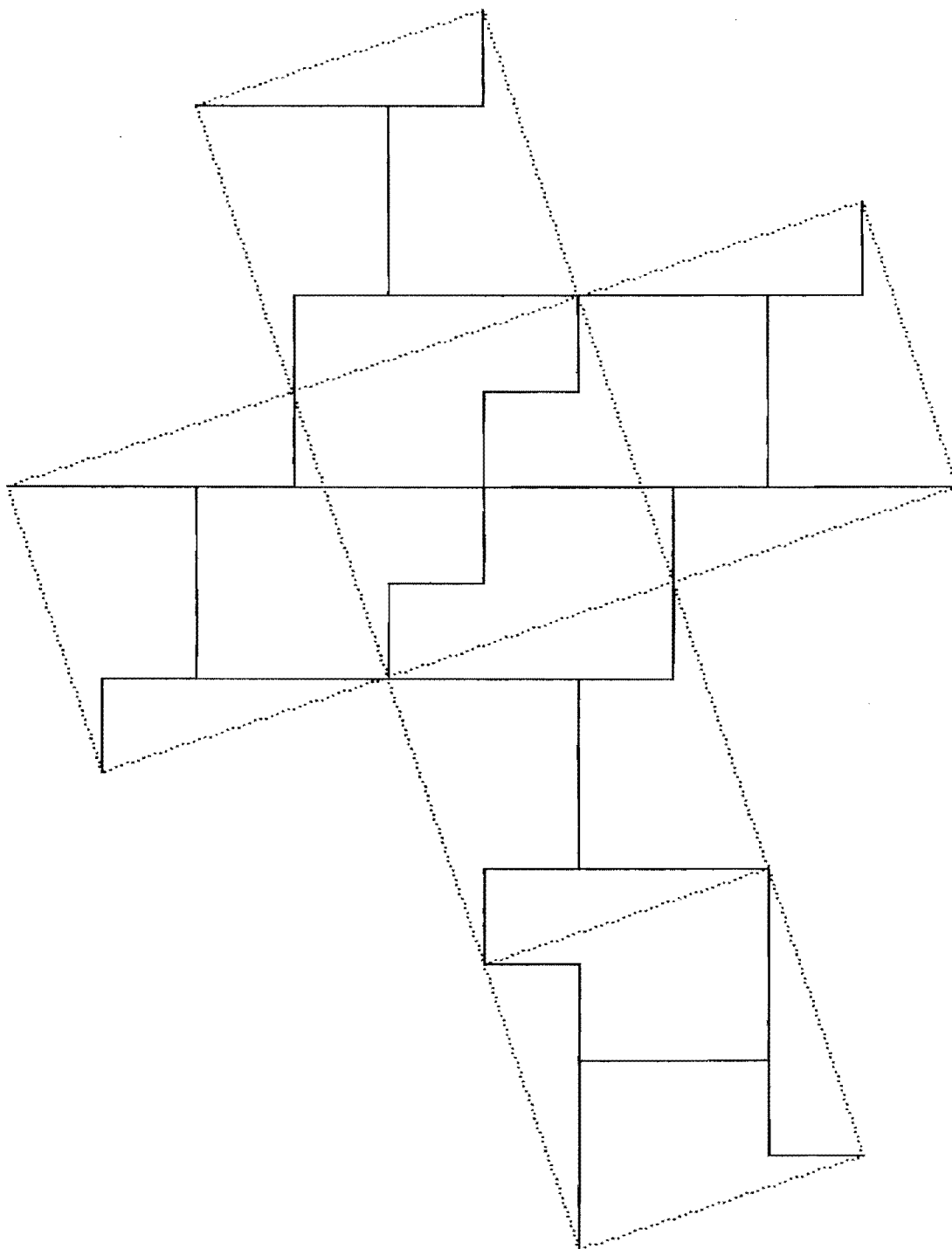
3169 3227 3297 3307 3344 3394 3413 3457 3462 3512 3553 3562

One axis order 2 (Z)



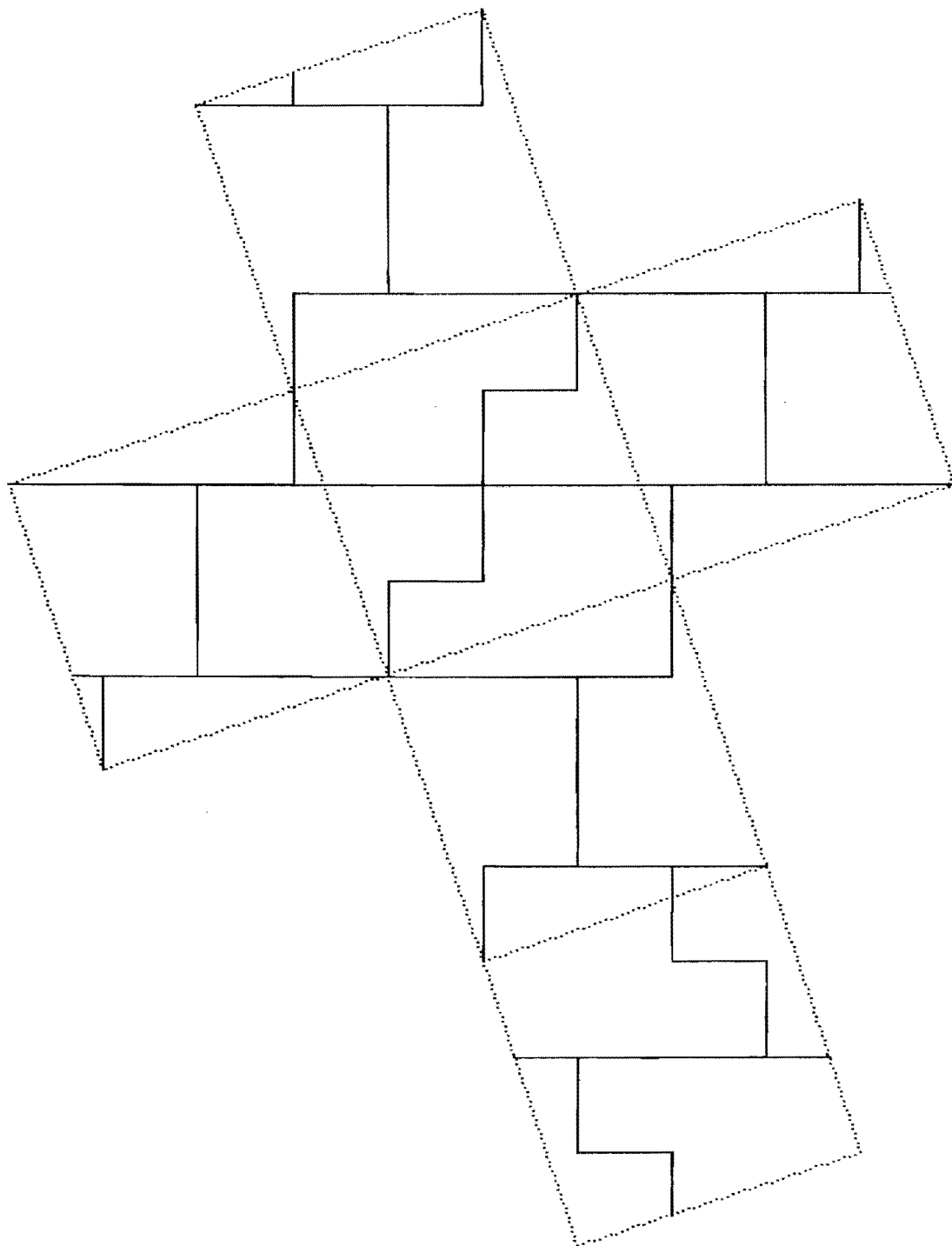
3169 3228 3291 3308 3349 3360 3429 3452 3497 3512 3553 3562

Three axes order 2 (X Y Z)



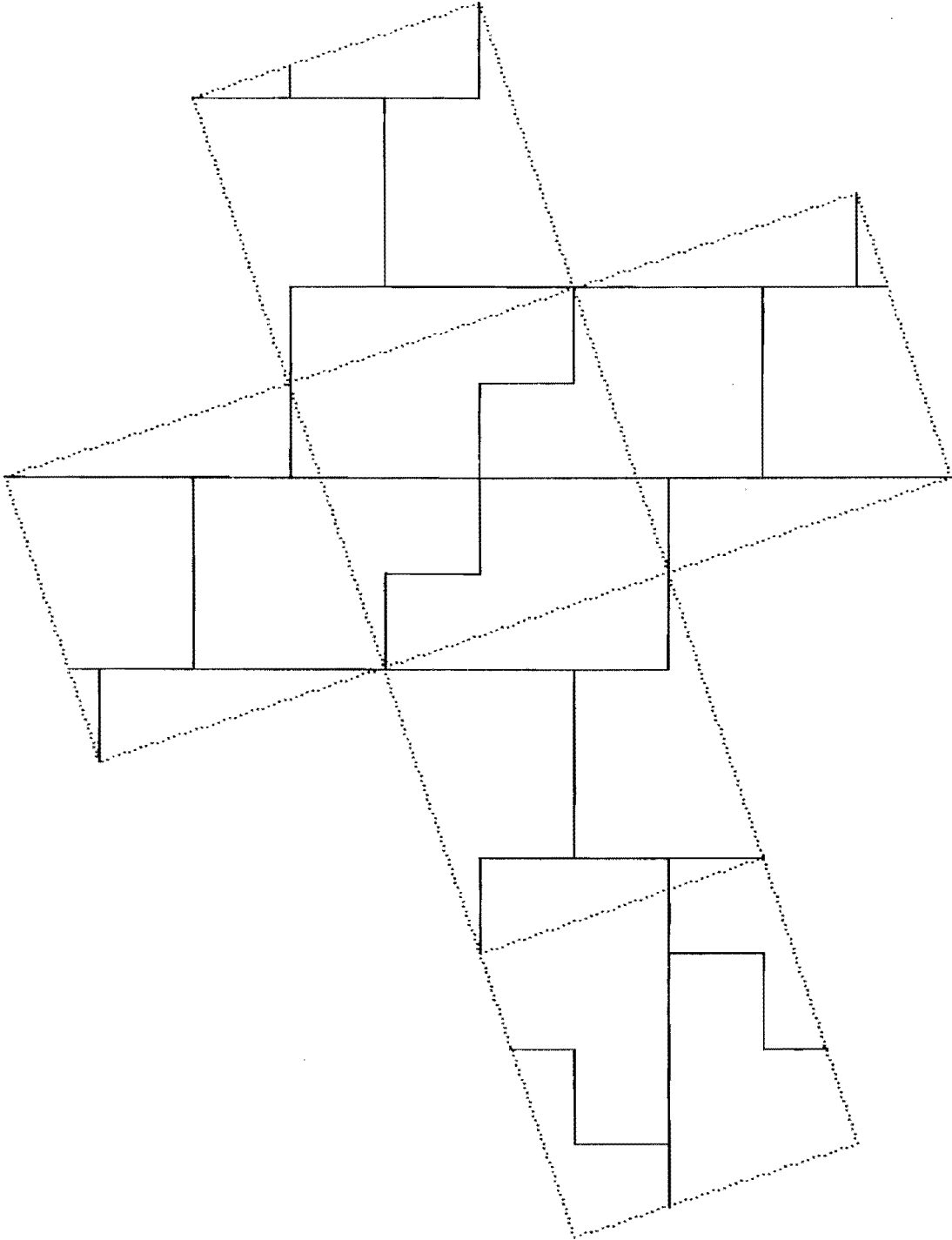
3169 3228 3291 3308 3349 3360 3430 3454 3466 3508 3553 3562

One axis order 2 (X)



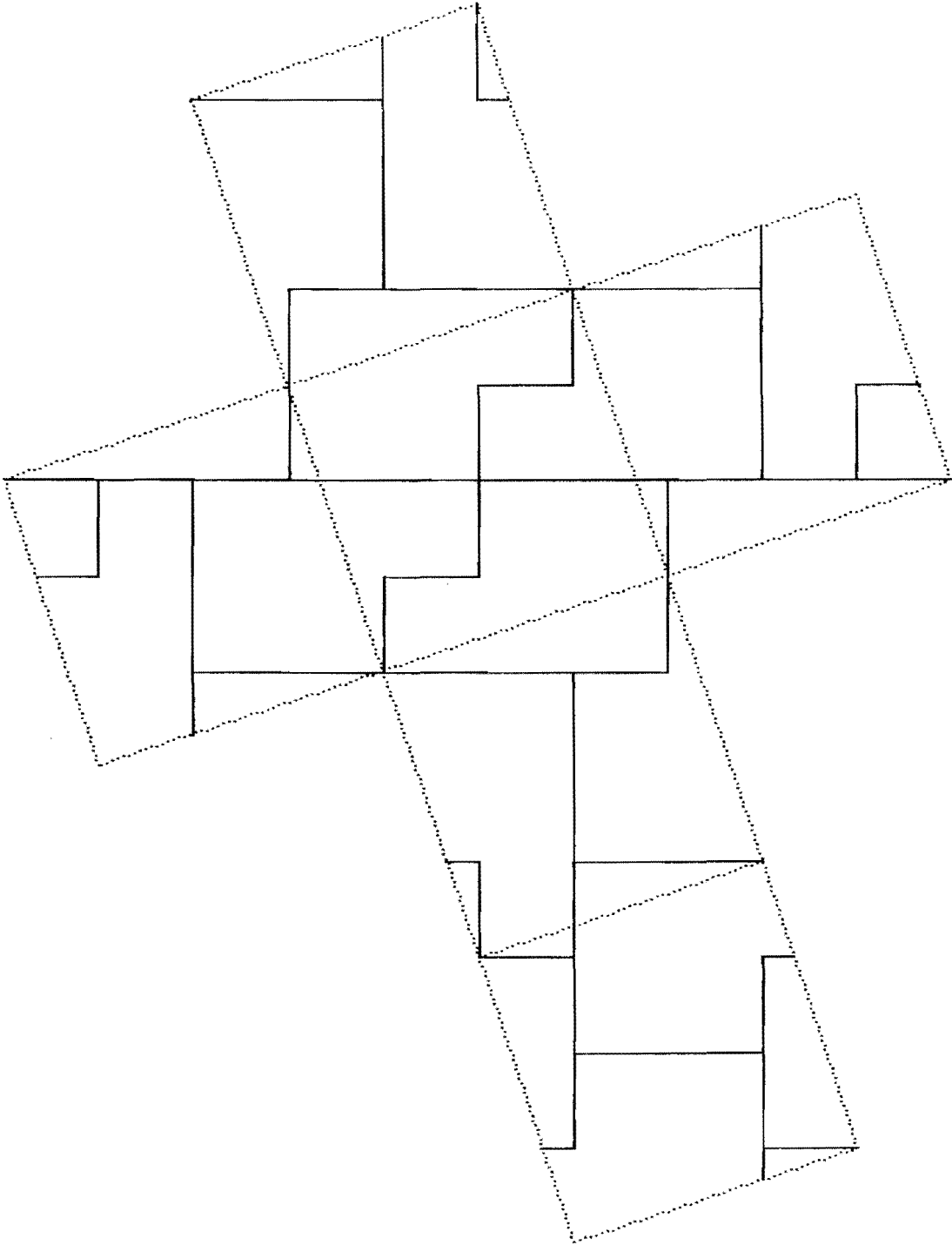
3169 3228 3291 3308 3349 3360 3431 3448 3498 3506 3553 3562

One axis order 2 (X)



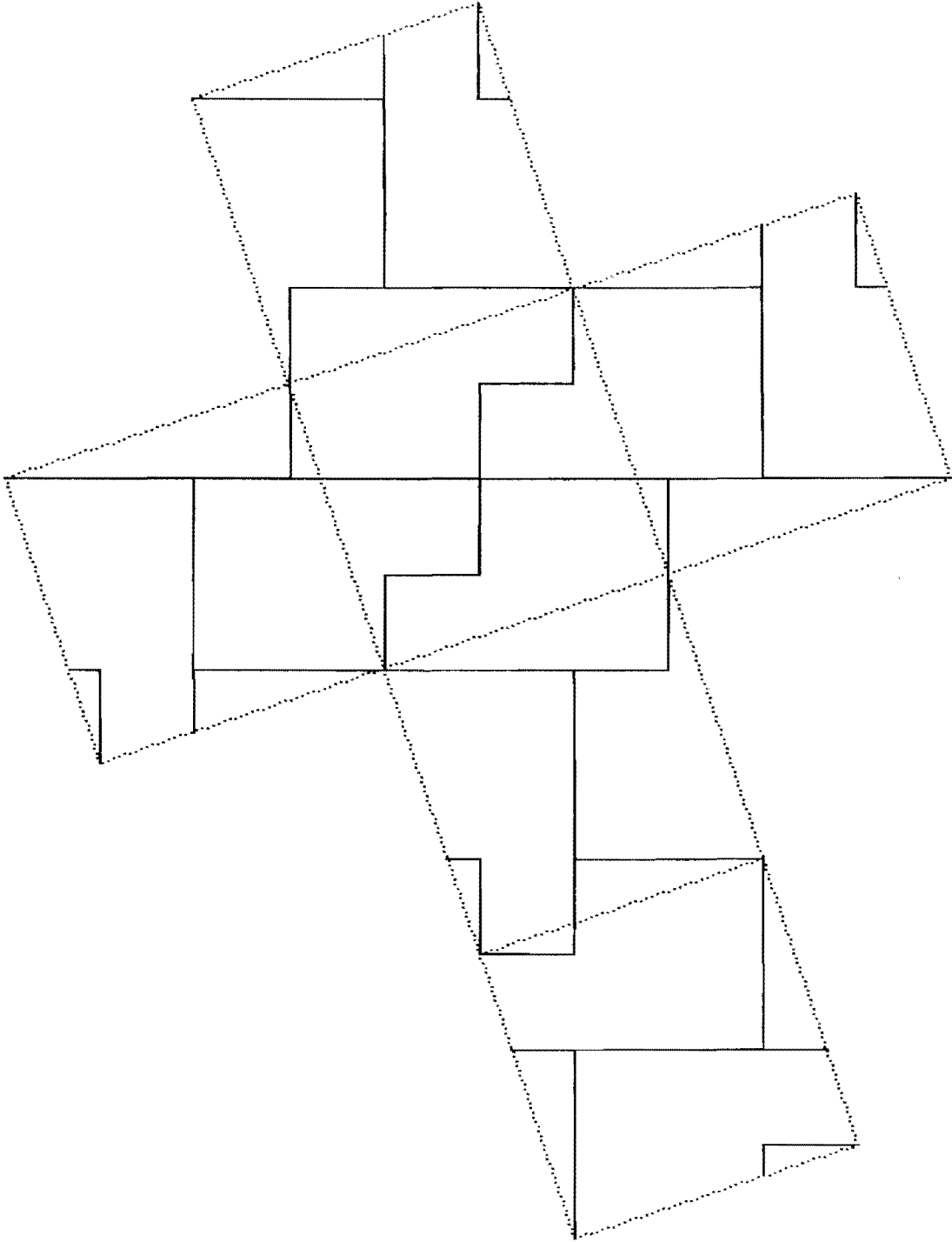
3169 3228 3291 3308 3349 3360 3433 3448 3482 3498 3553 3562

One axis order 2 (X)



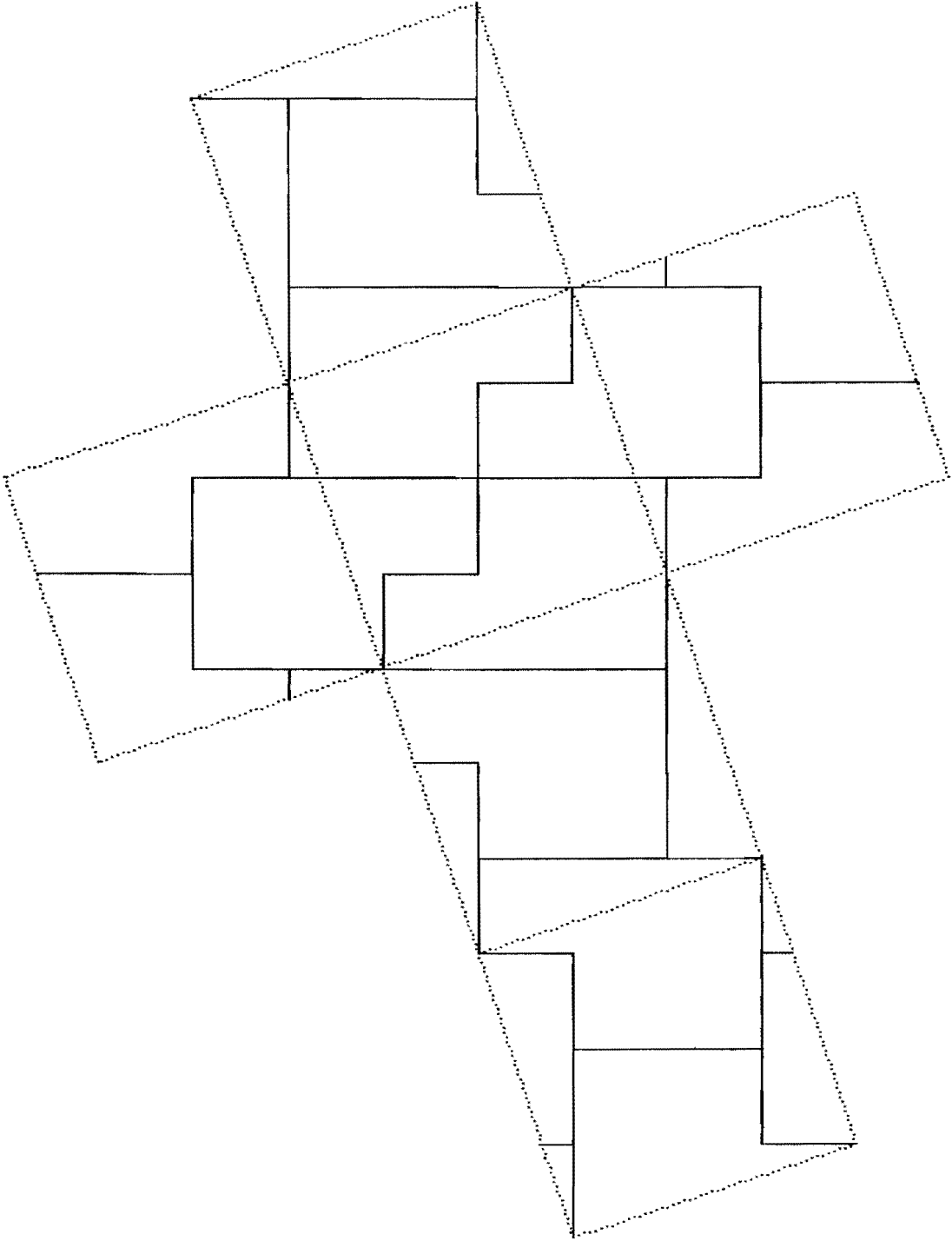
3169 3228 3291 3308 3349 3361 3426 3435 3488 3507 3552 3562

One axis order 2 (X)



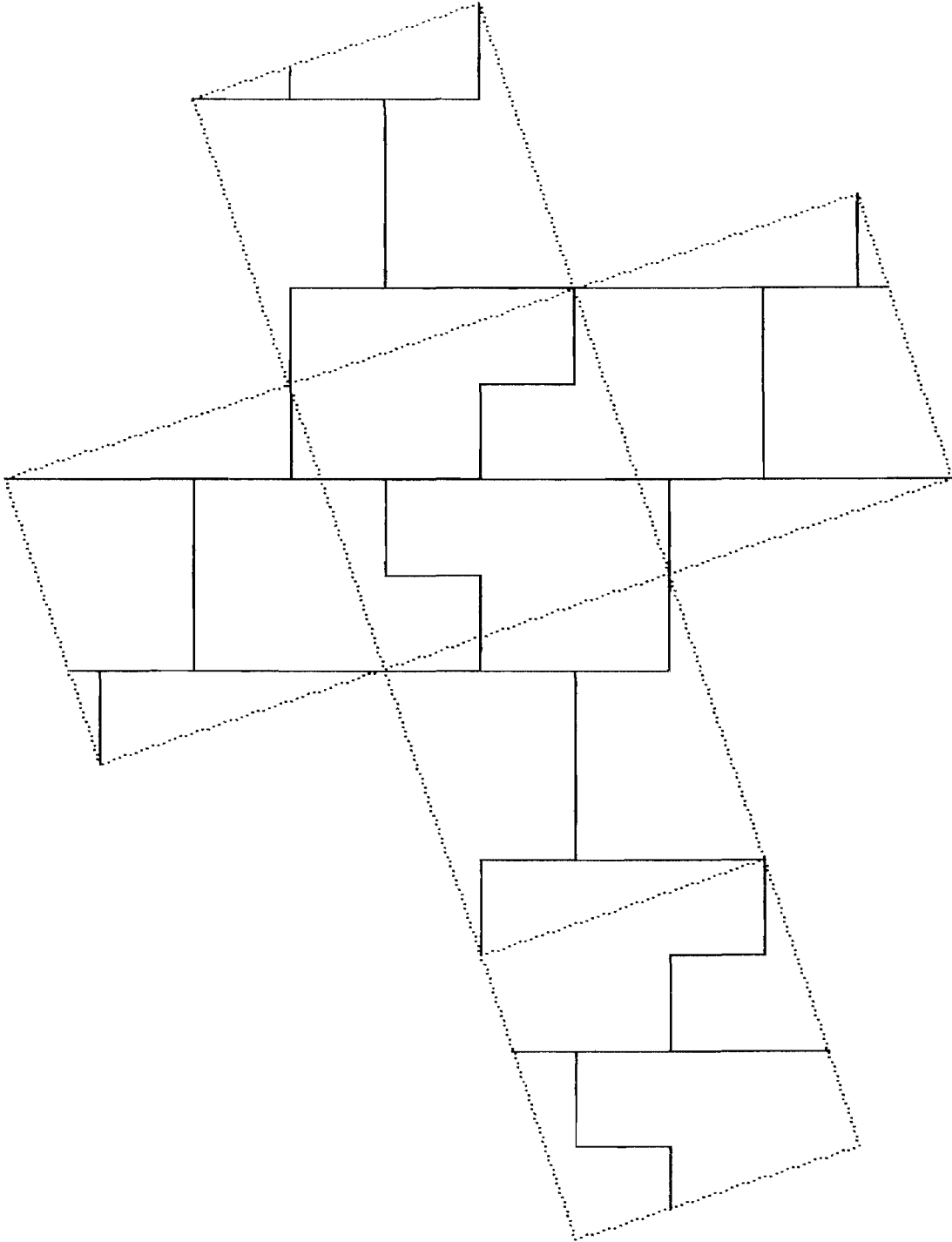
3169 3228 3291 3308 3349 3361 3427 3437 3455 3505 3552 3562

One axis order 2 (X)



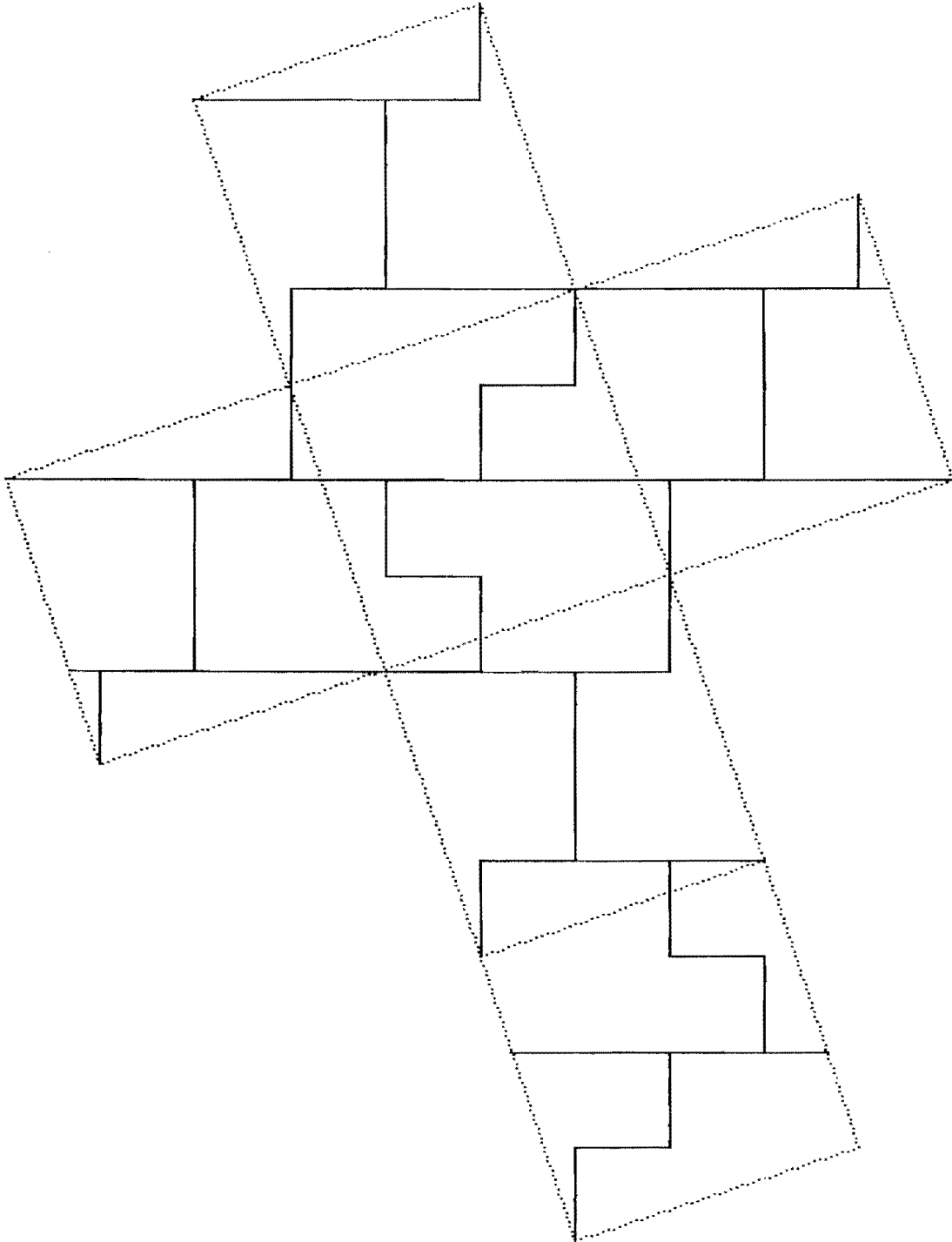
3169 3228 3291 3308 3351 3358 3397 3430 3486 3508 3526 3568

One axis order 2 (X)



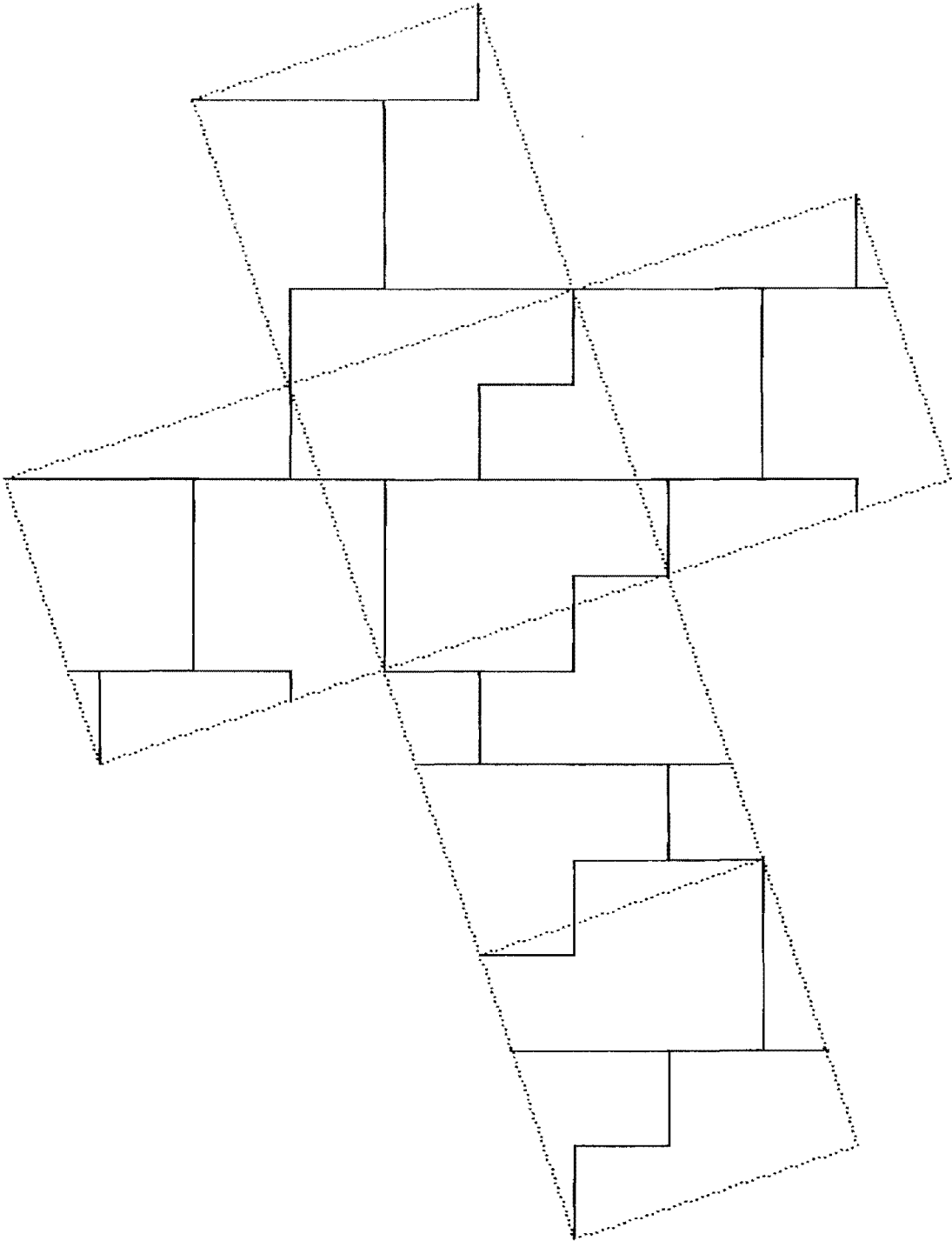
3169 3228 3292 3302 3349 3360 3429 3452 3498 3506 3553 3562

One axis order 2 (Y)



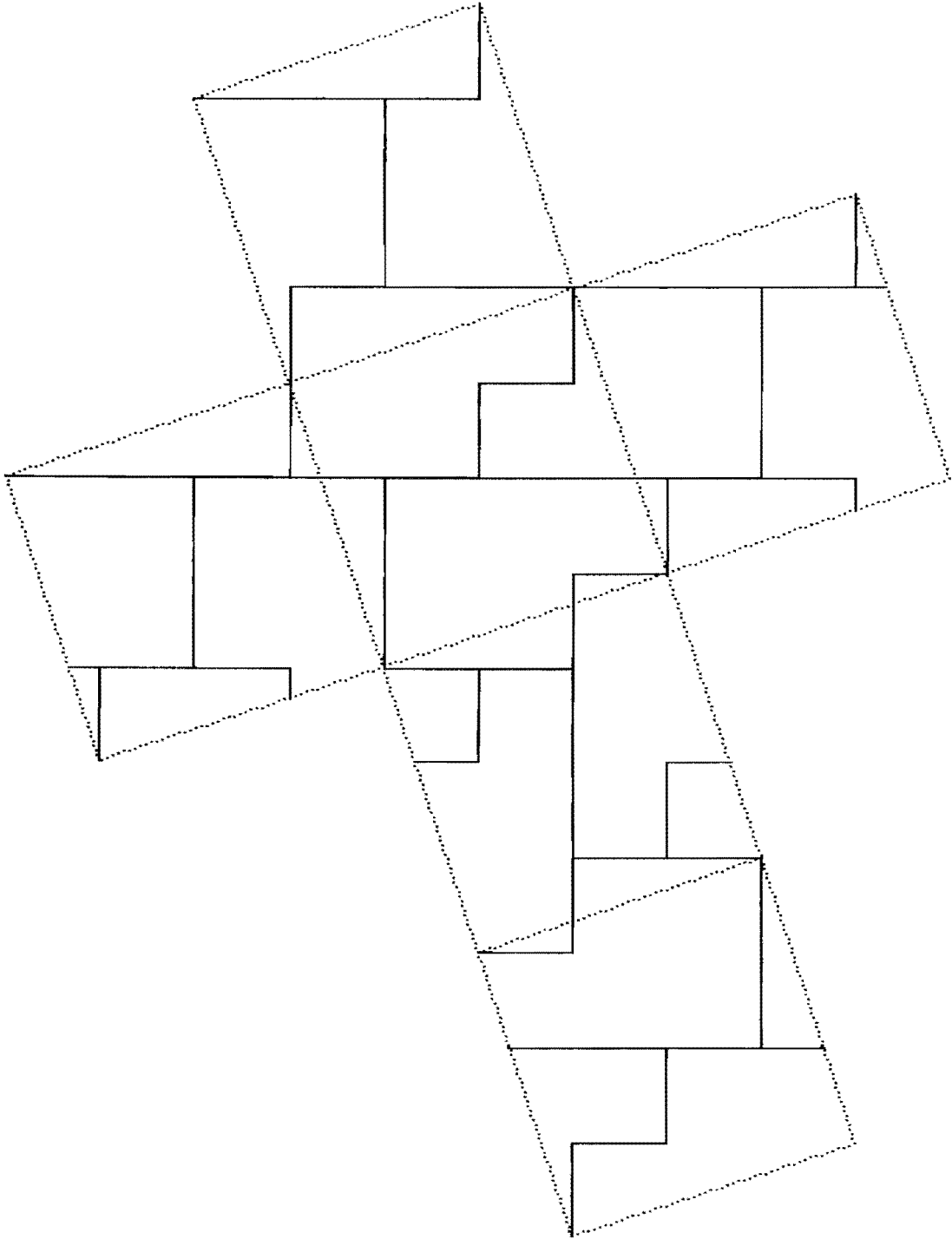
3169 3228 3292 3302 3349 3360 3431 3448 3497 3512 3553 3562

One axis order 2 (Z)



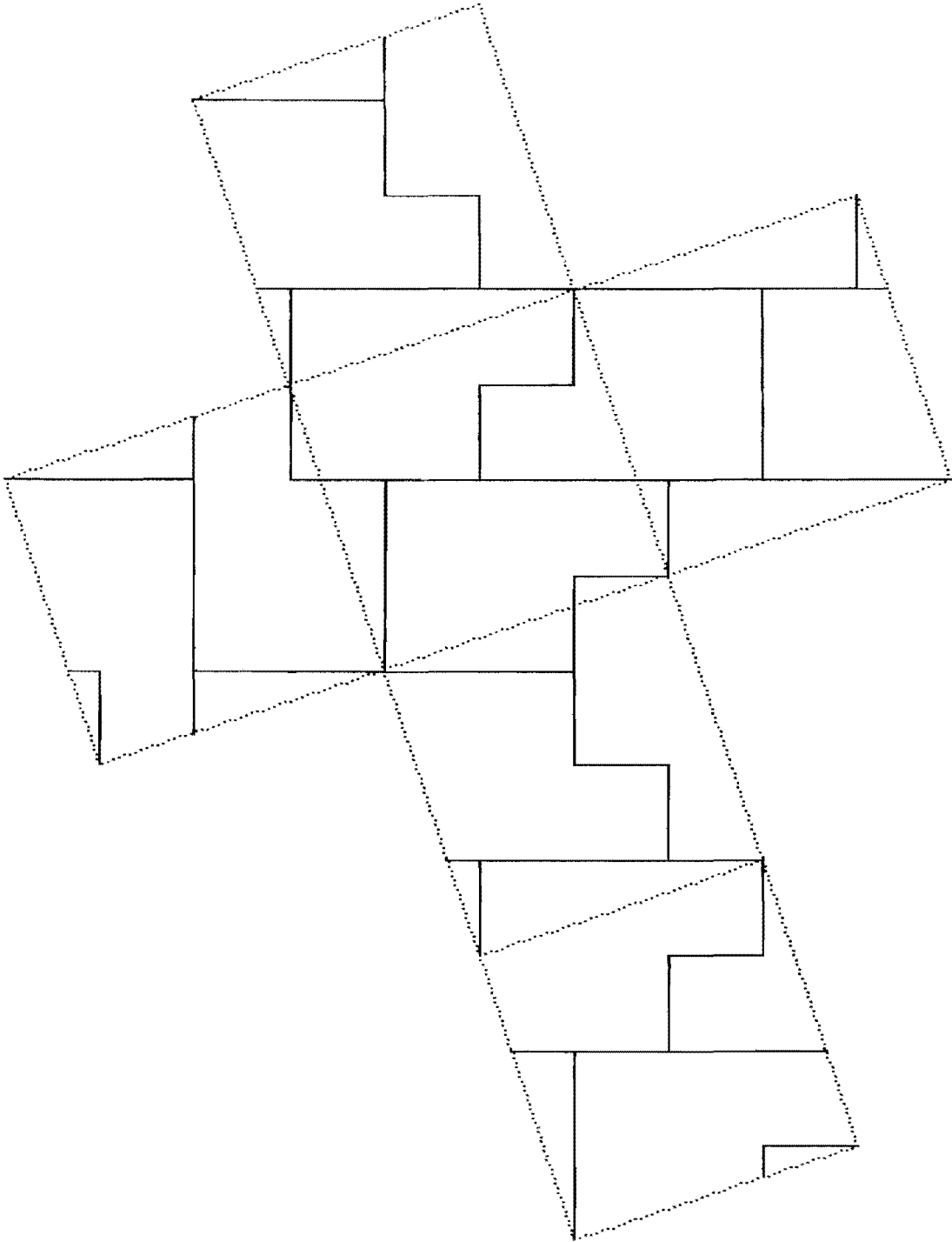
3169 3228 3294 3301 3341 3389 3418 3437 3497 3512 3553 3562

One axis order 2 (Z)



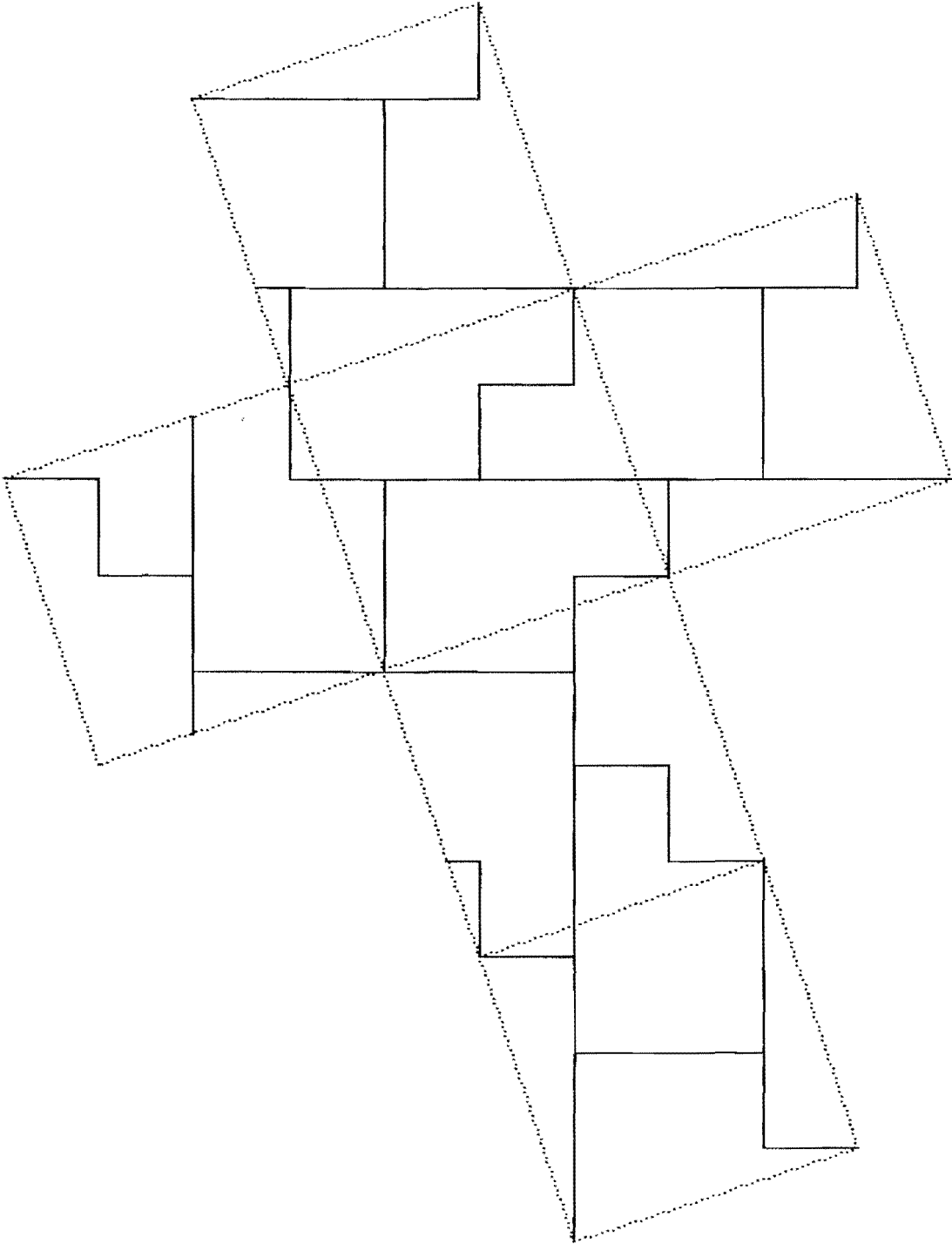
3169 3228 3294 3301 3342 3371 3418 3437 3497 3512 3553 3562

One axis order 2 (Z)



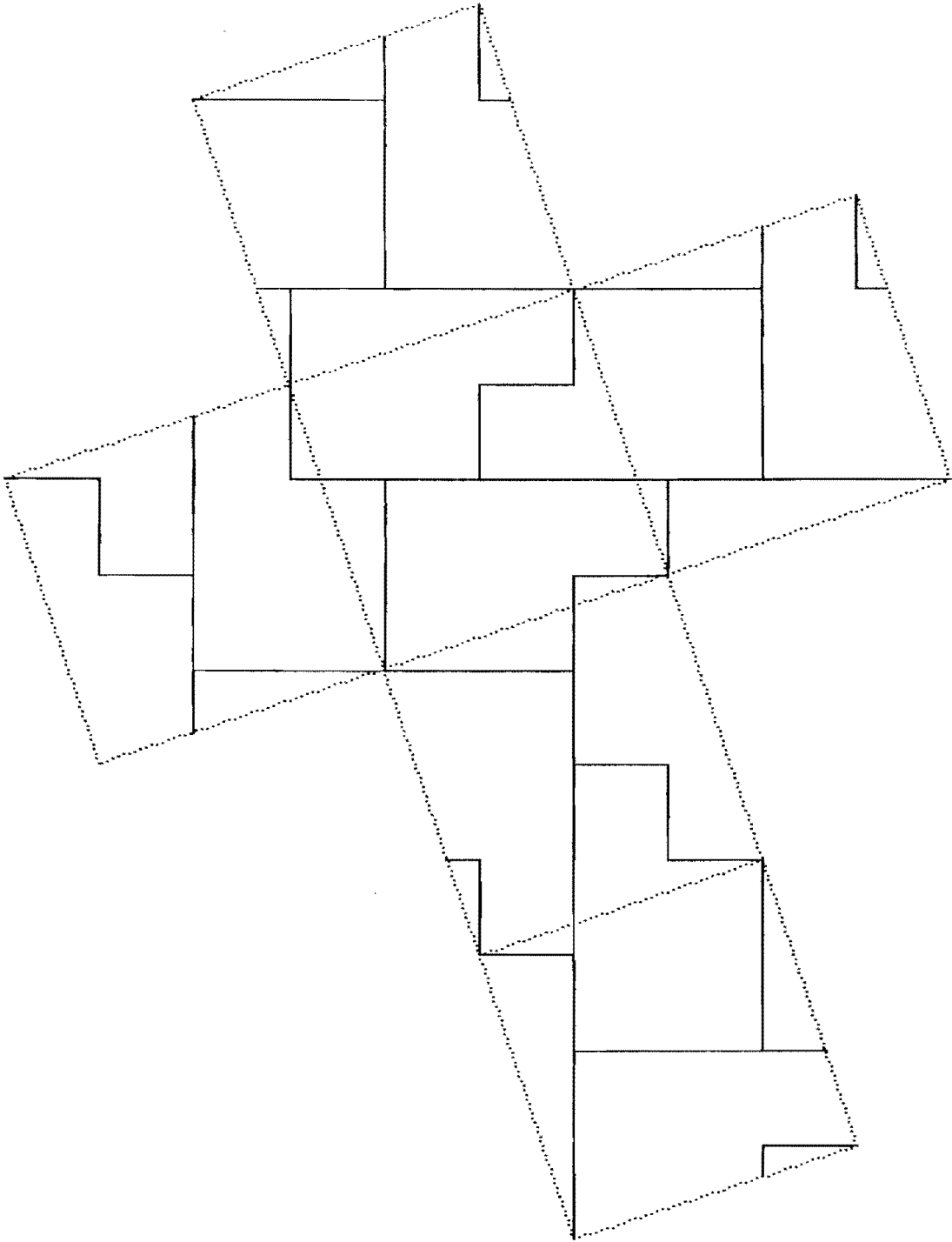
3169 3228 3296 3301 3343 3359 3427 3429 3452 3505 3547 3561

One axis order 2 (Y)



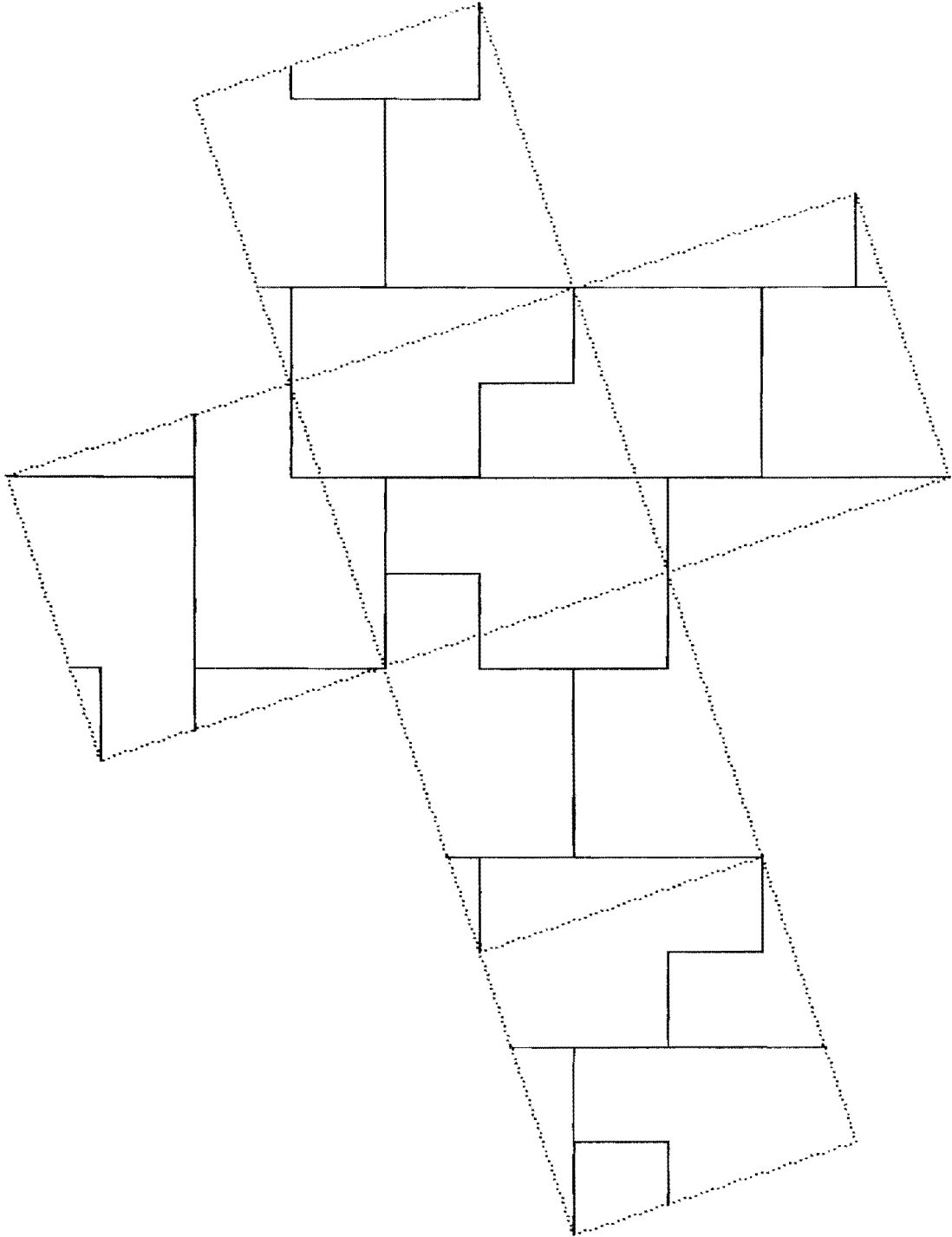
3169 3228 3296 3301 3343 3361 3415 3424 3454 3508 3553 3563

One axis order 3 (D4)



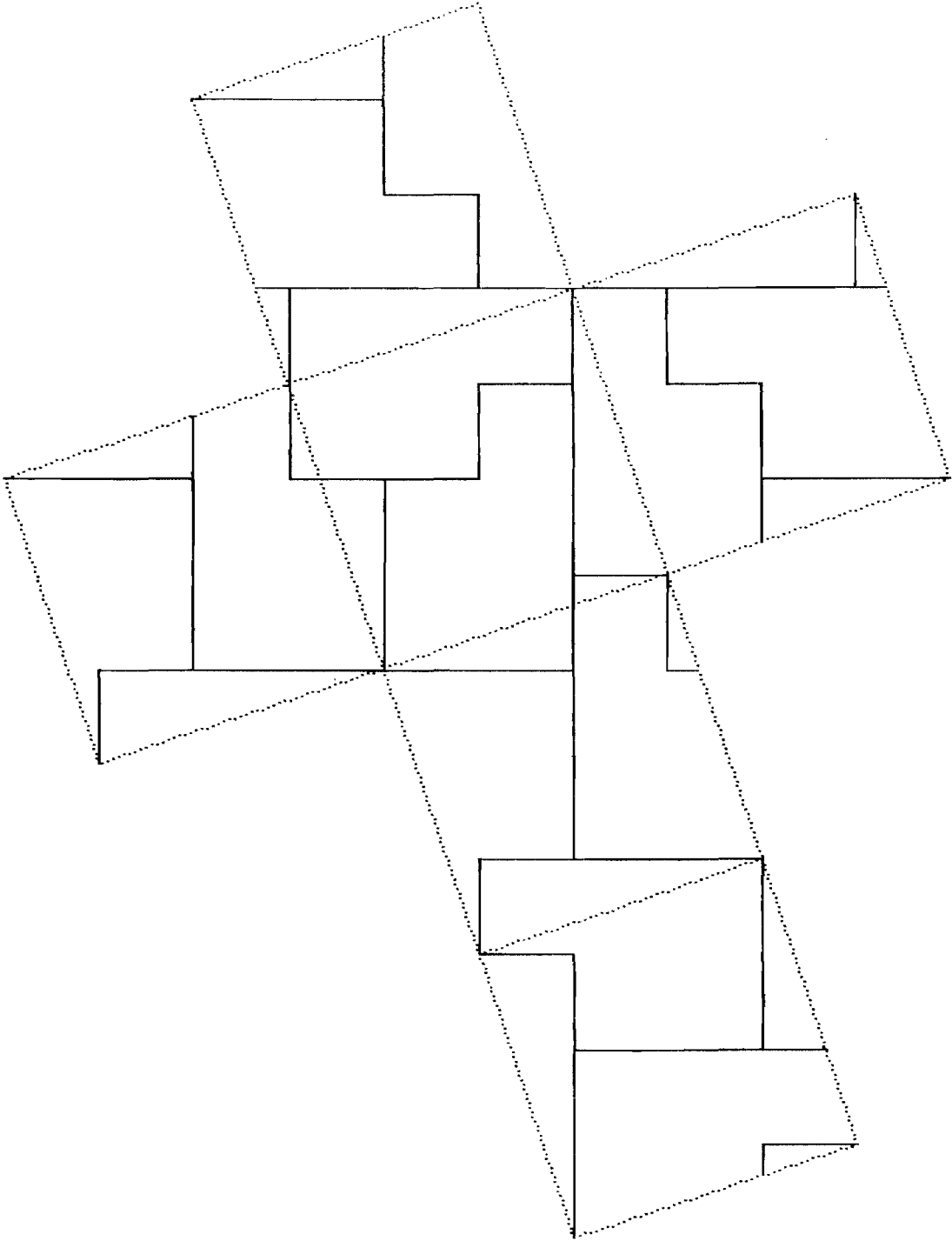
3169 3228 3296 3301 3343 3361 3415 3424 3455 3505 3552 3563

One axis order 3 (D4)



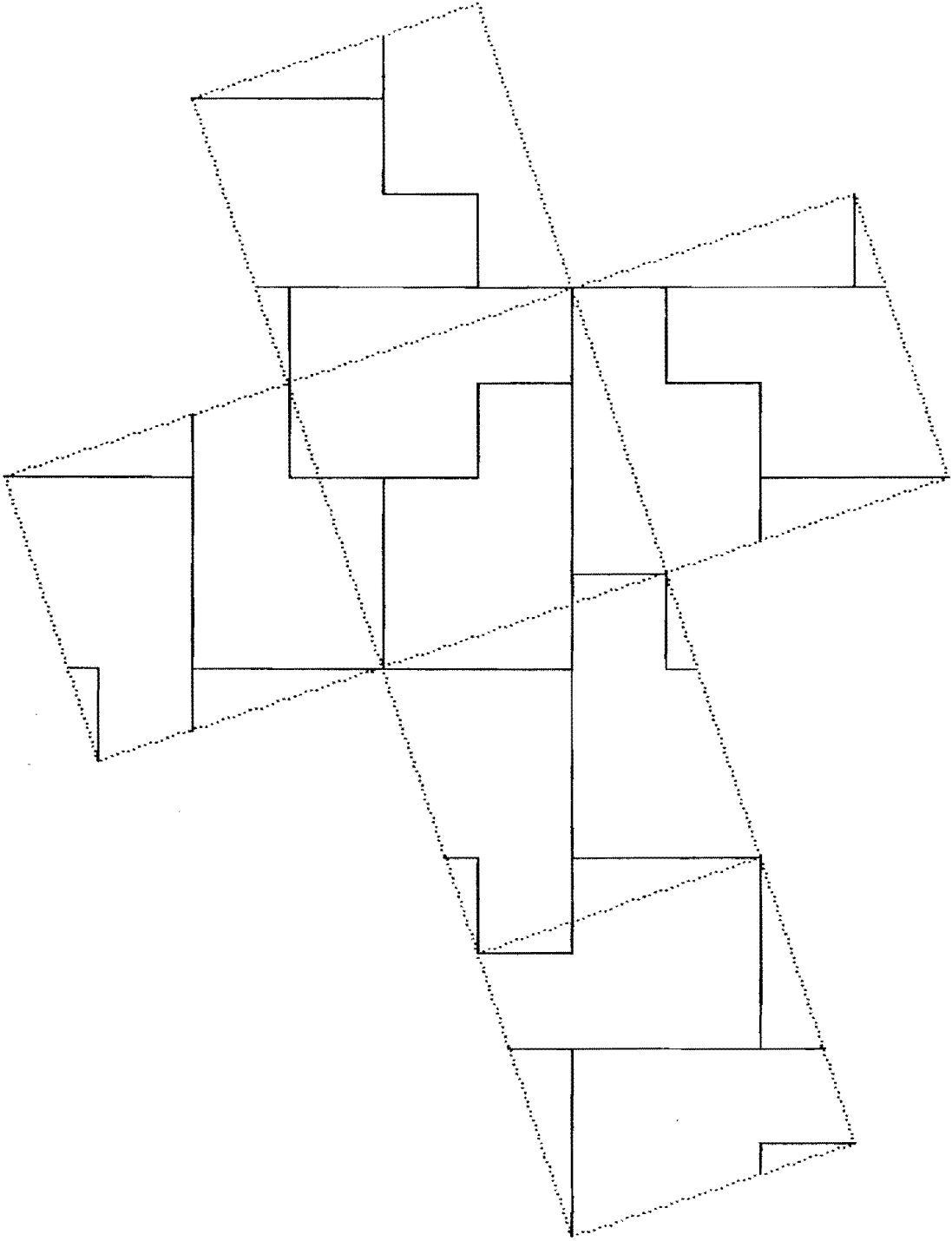
3169 3228 3296 3302 3332 3349 3427 3429 3452 3506 3536 3553

One axis order 2 (Y)



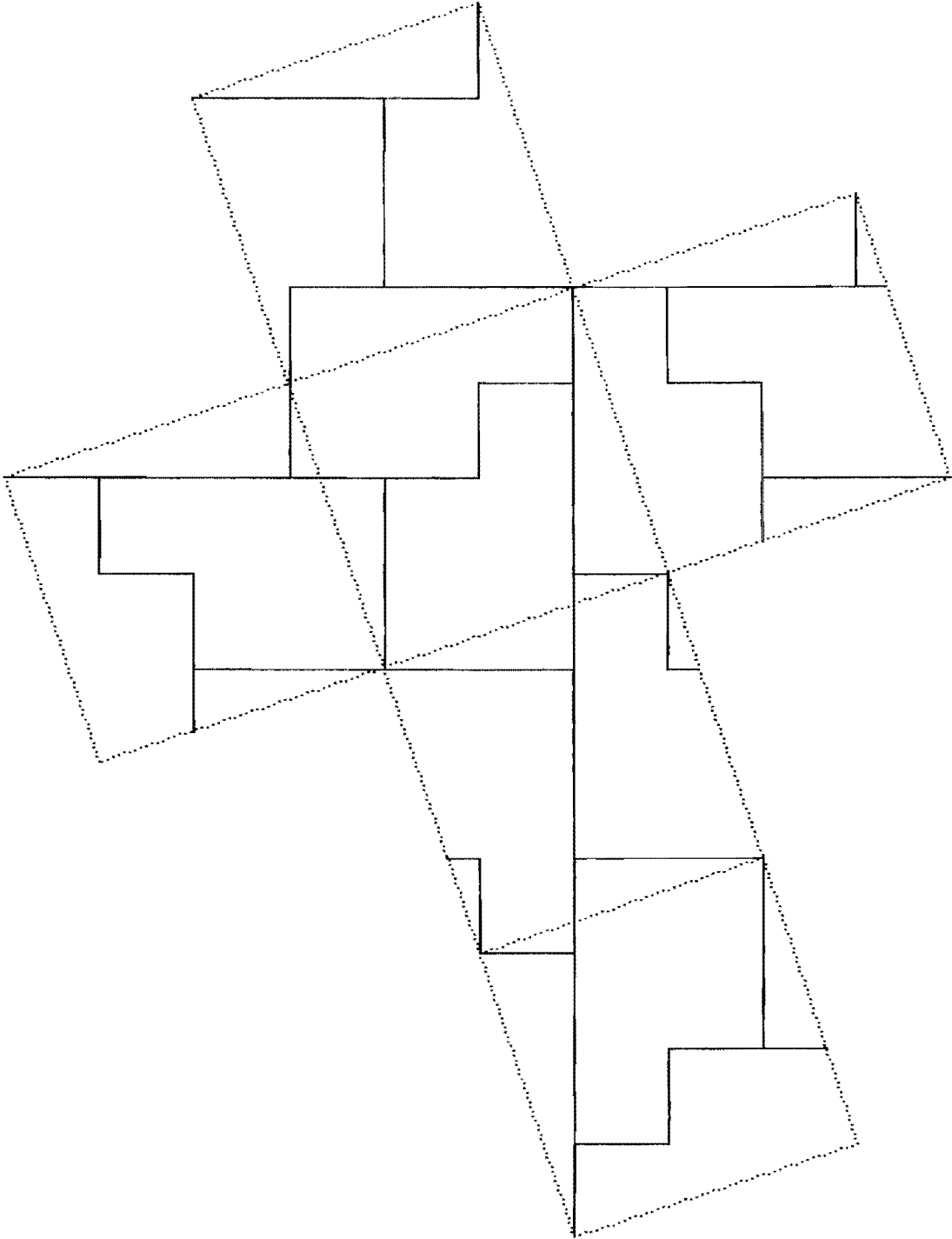
3169 3229 3275 3296 3348 3360 3430 3457 3466 3505 3547 3561

One axis order 3 (D2)



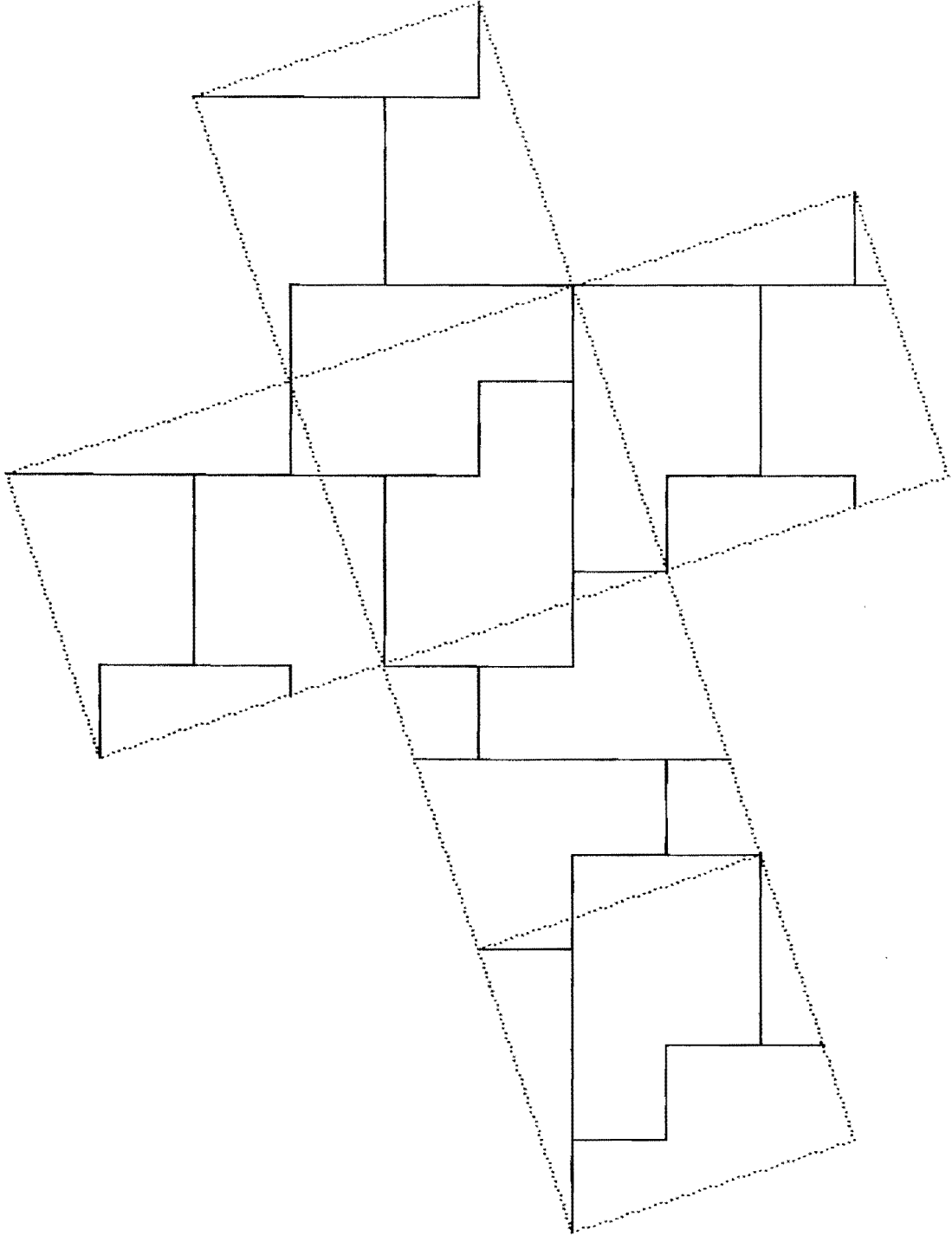
3169 3229 3275 3296 3348 3361 3427 3437 3457 3505 3547 3561

One axis order 3 (D2)



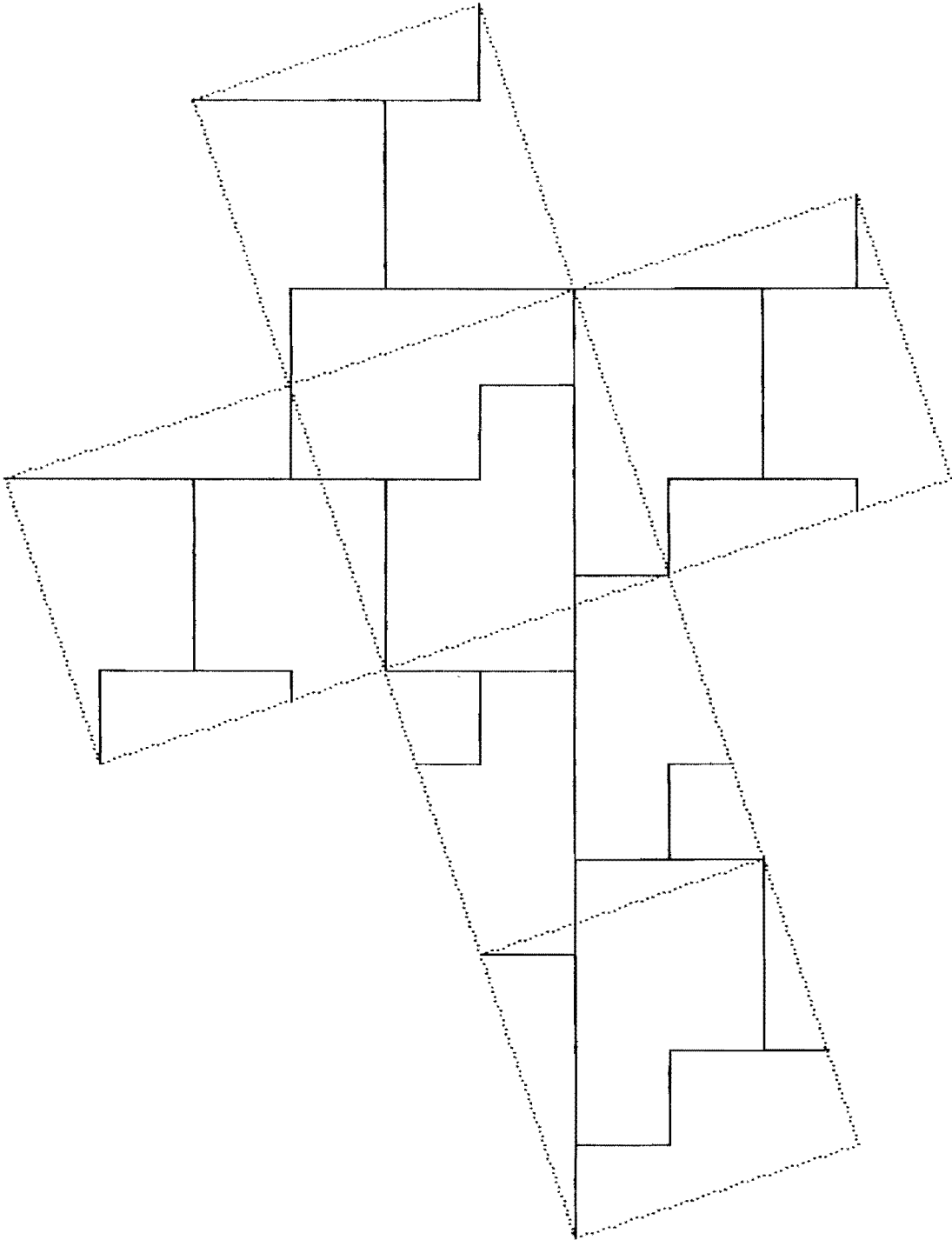
3169 3229 3275 3297 3348 3361 3424 3439 3457 3512 3553 3562

One axis order 2 (Z)



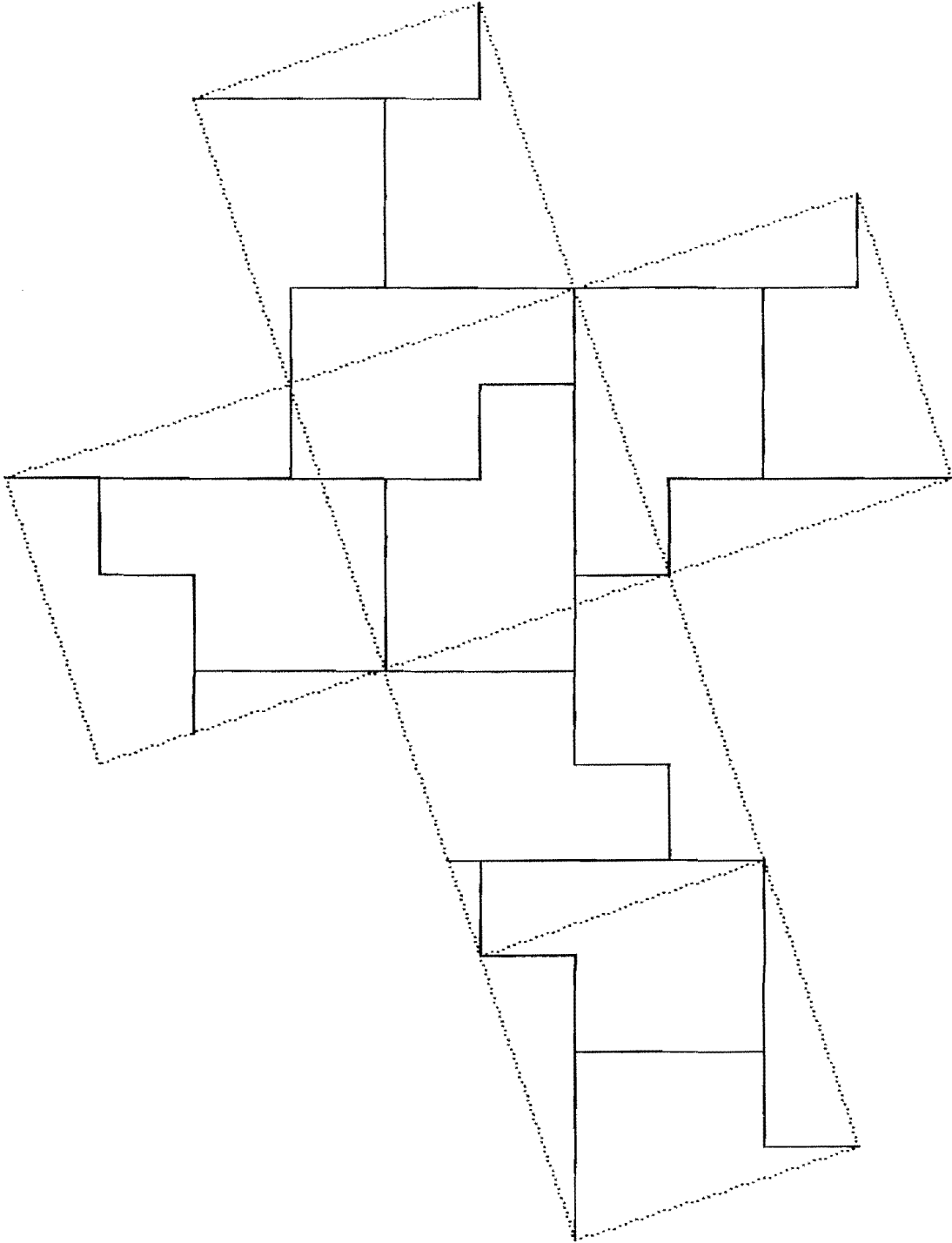
3169 3230 3275 3294 3341 3389 3418 3439 3466 3512 3553 3562

One axis order 2 (Z)



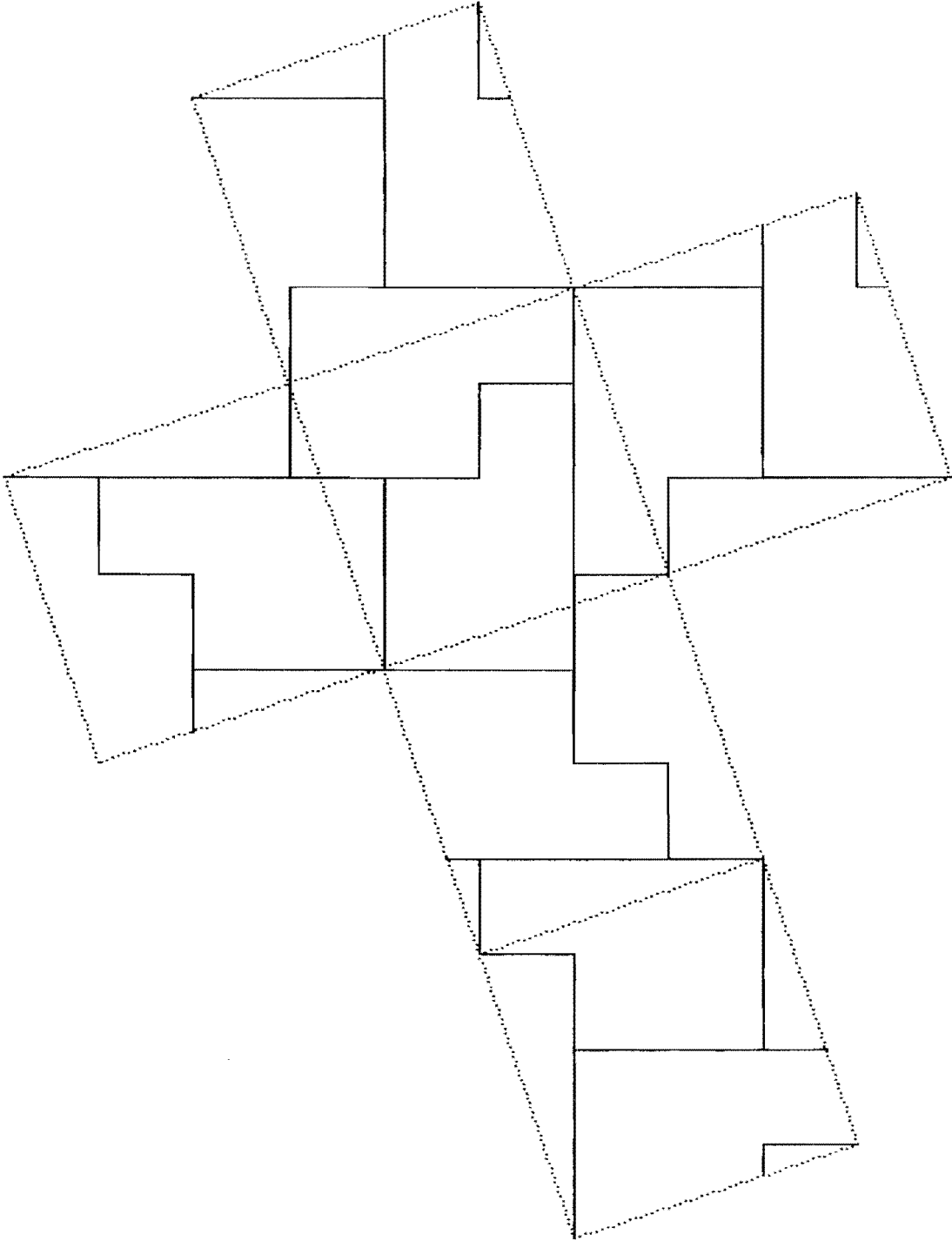
3169 3230 3275 3294 3342 3371 3418 3439 3466 3512 3553 3562

One axis order 2 (Z)



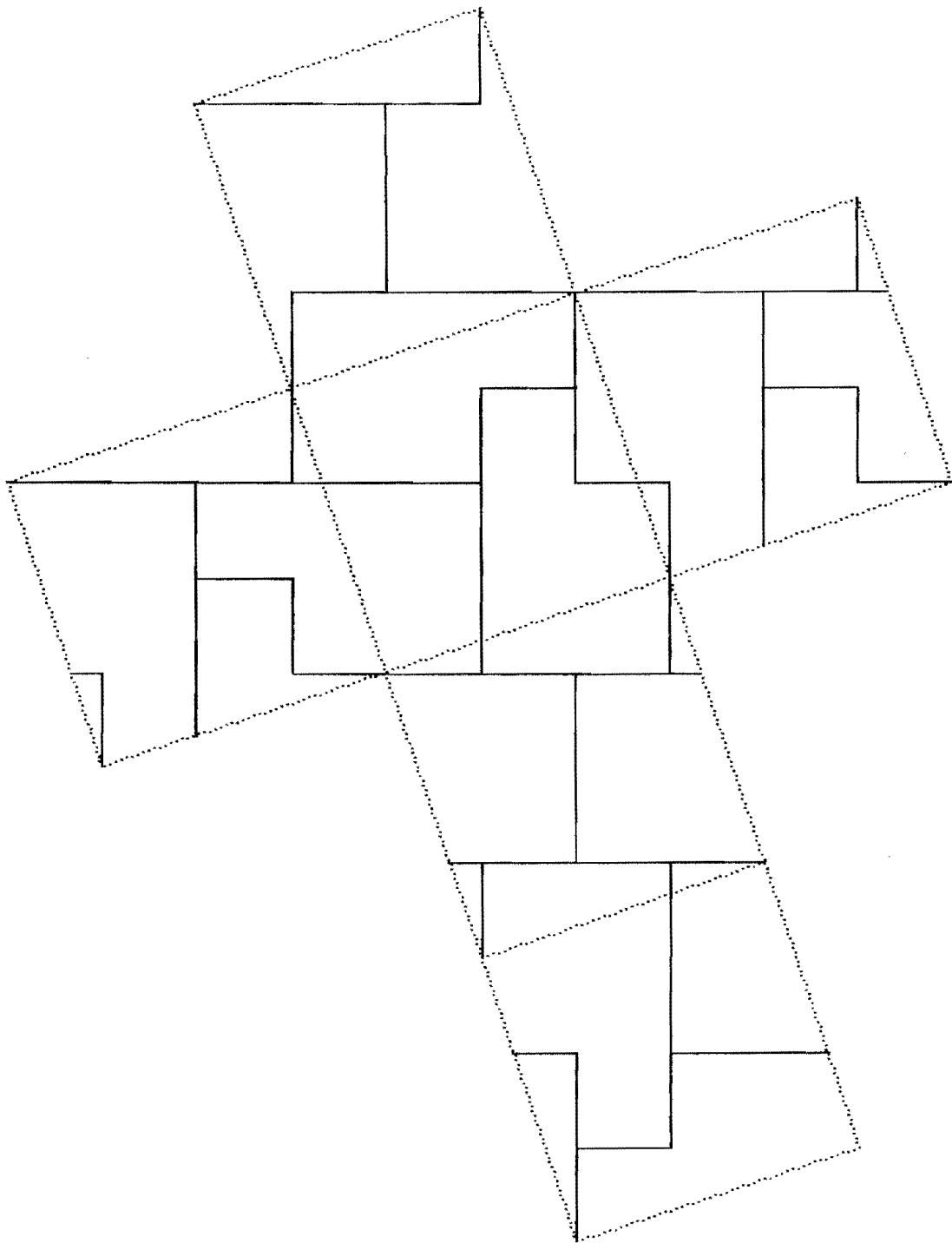
3169 3230 3275 3297 3343 3359 3424 3430 3454 3508 3553 3562

One axis order 3 (D4)



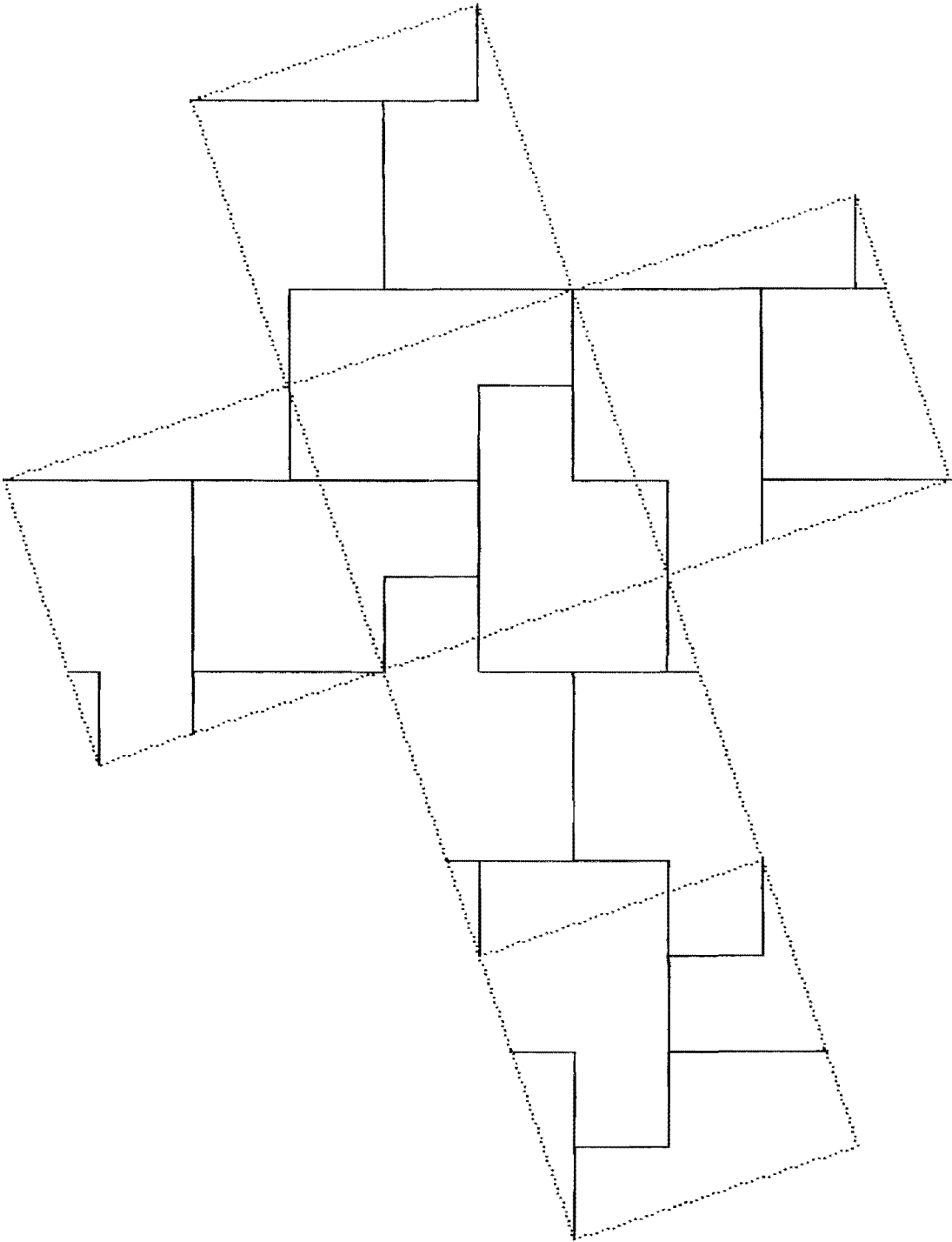
3169 3230 3275 3297 3343 3359 3424 3430 3455 3505 3552 3562

One axis order 3 (D4)



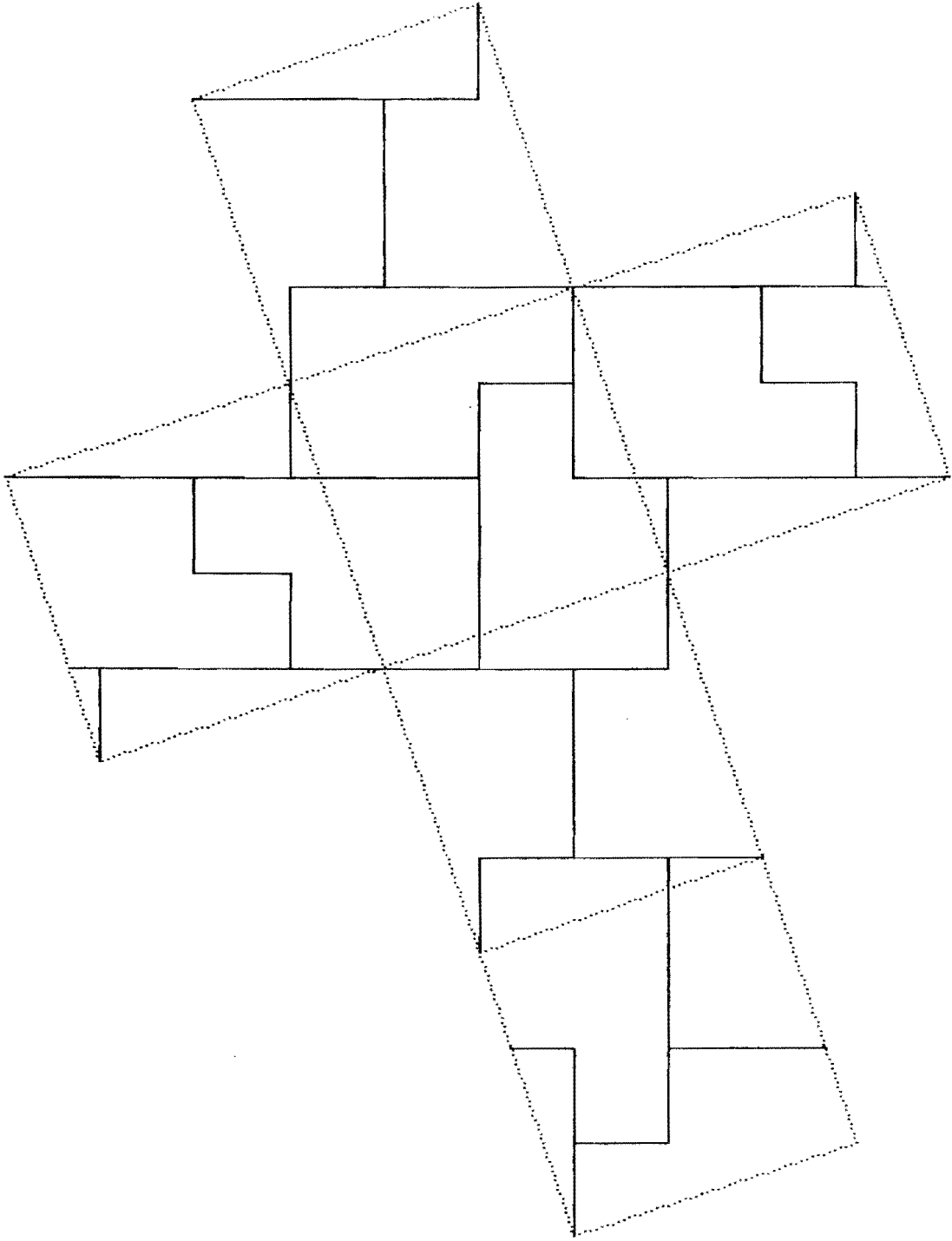
3169 3231 3276 3289 3362 3377 3427 3433 3447 3512 3553 3562

One axis order 2 (Z)

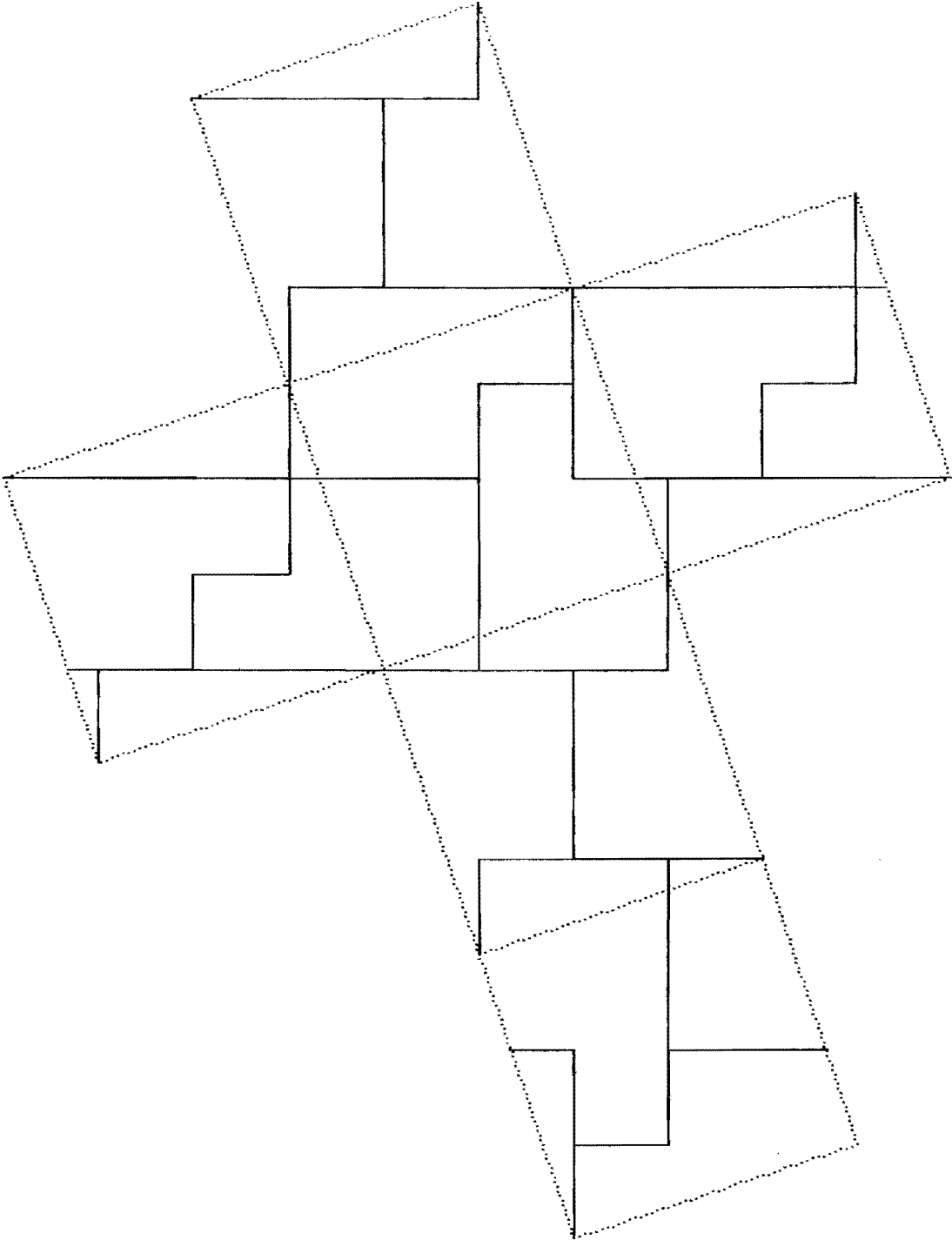


3169 3231 3276 3291 3332 3375 3427 3433 3452 3512 3553 3562

One axis order 2 (Z)

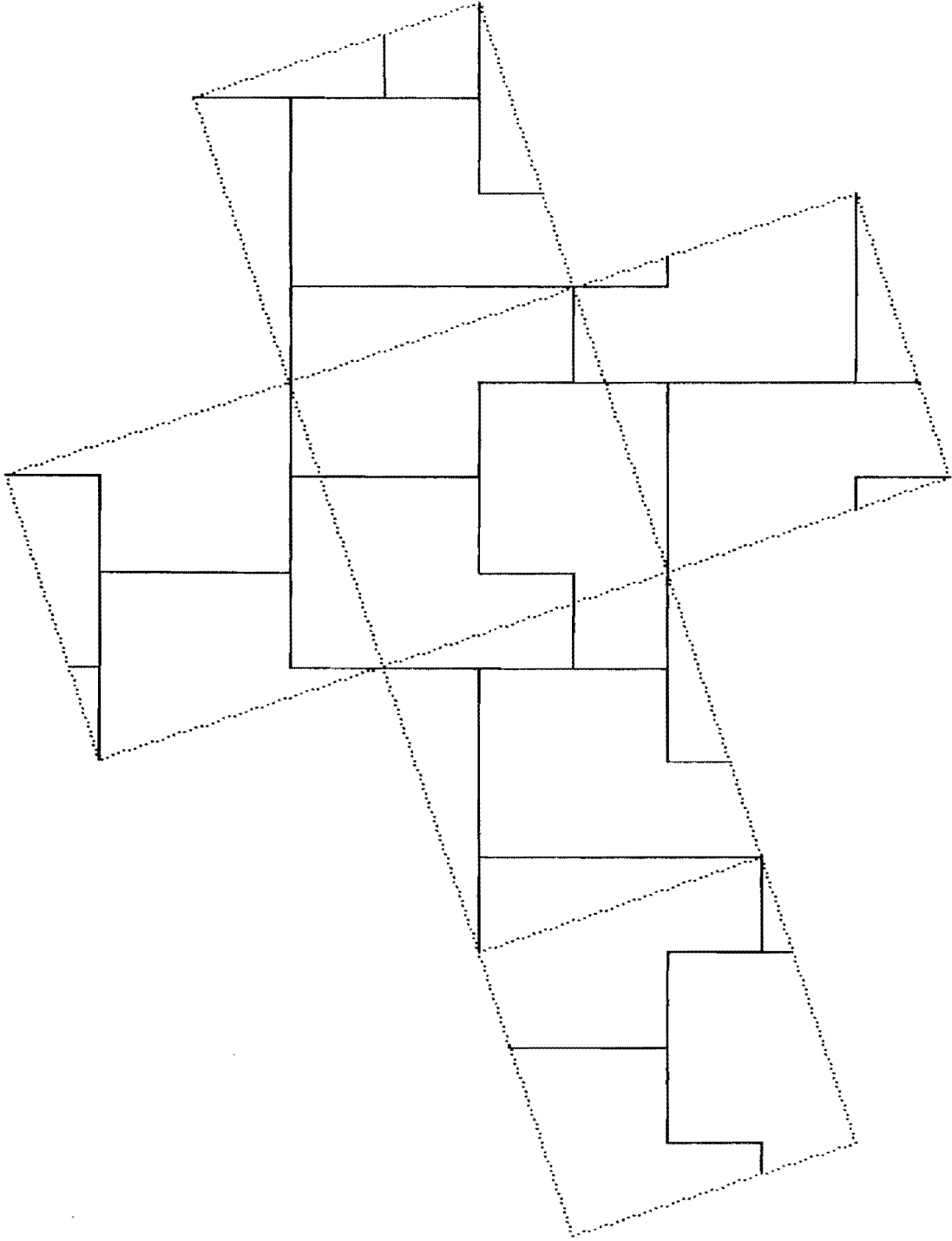


3169 3235 3276 3289 3349 3360 3433 3447 3503 3512 3553 3562



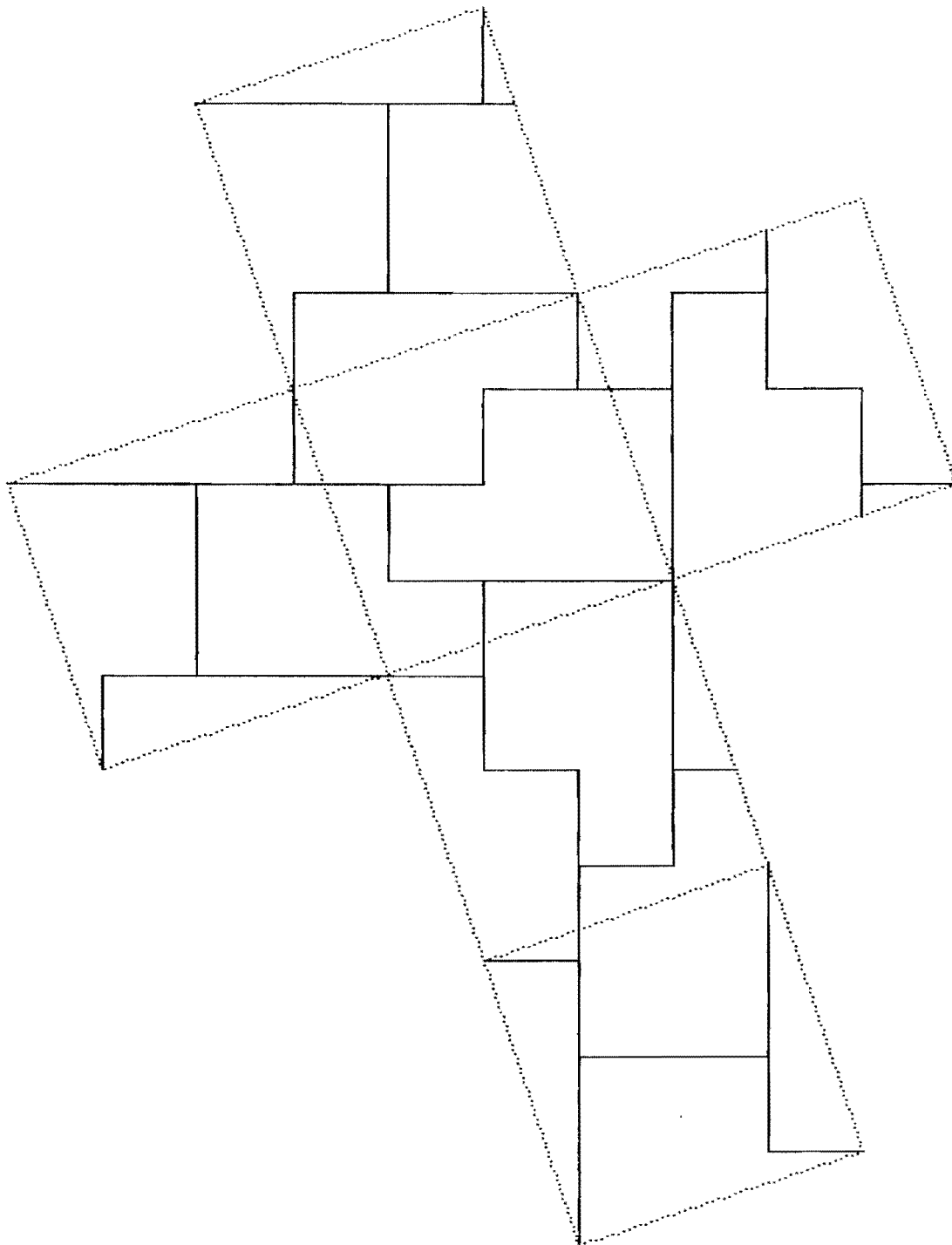
3169 3236 3276 3290 3349 3360 3433 3446 3502 3512 3553 3562

One axis order 2 (Z)



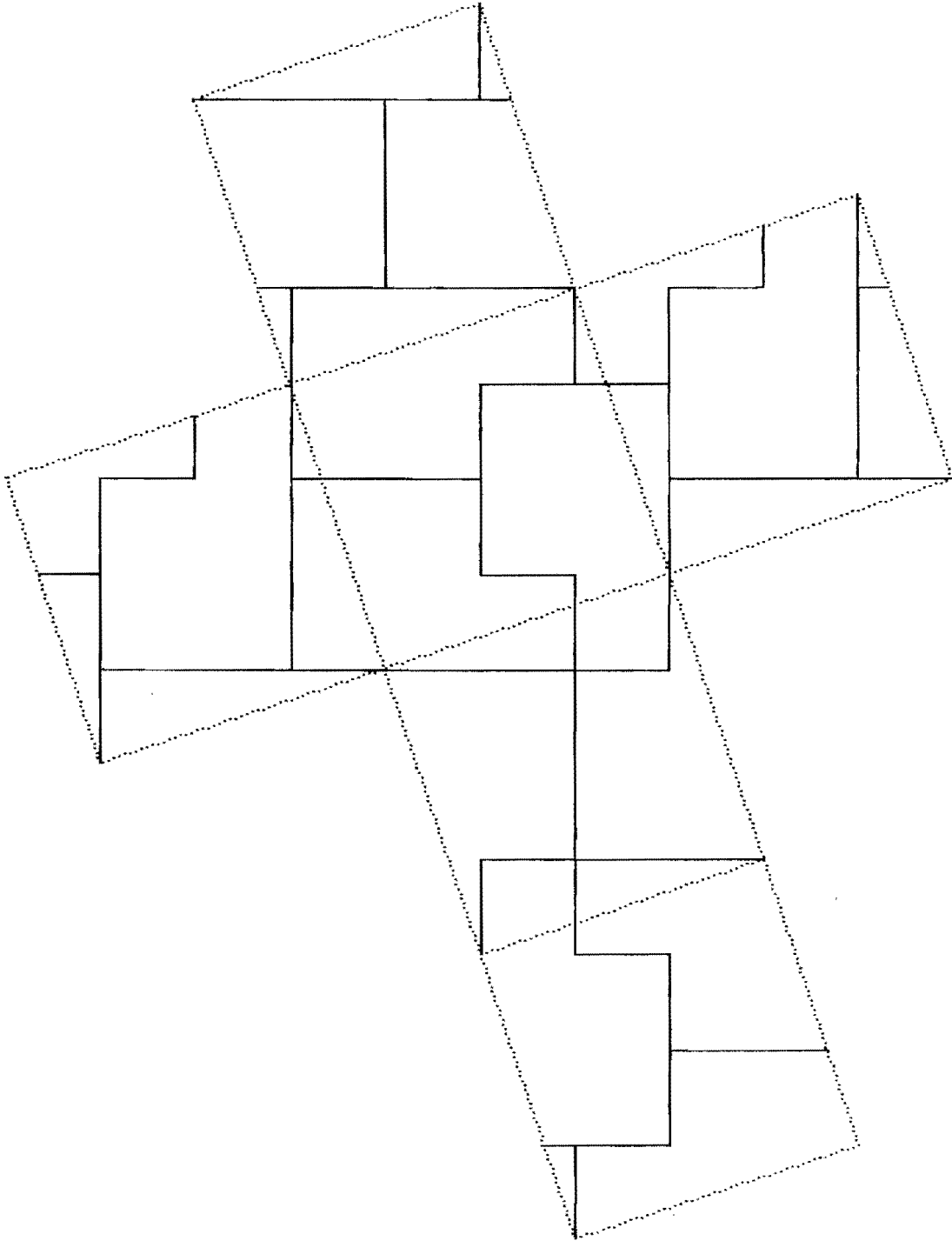
3169 3238 3271 3287 3353 3364 3368 3429 3477 3493 3564 3568

One axis order 2 (Y)



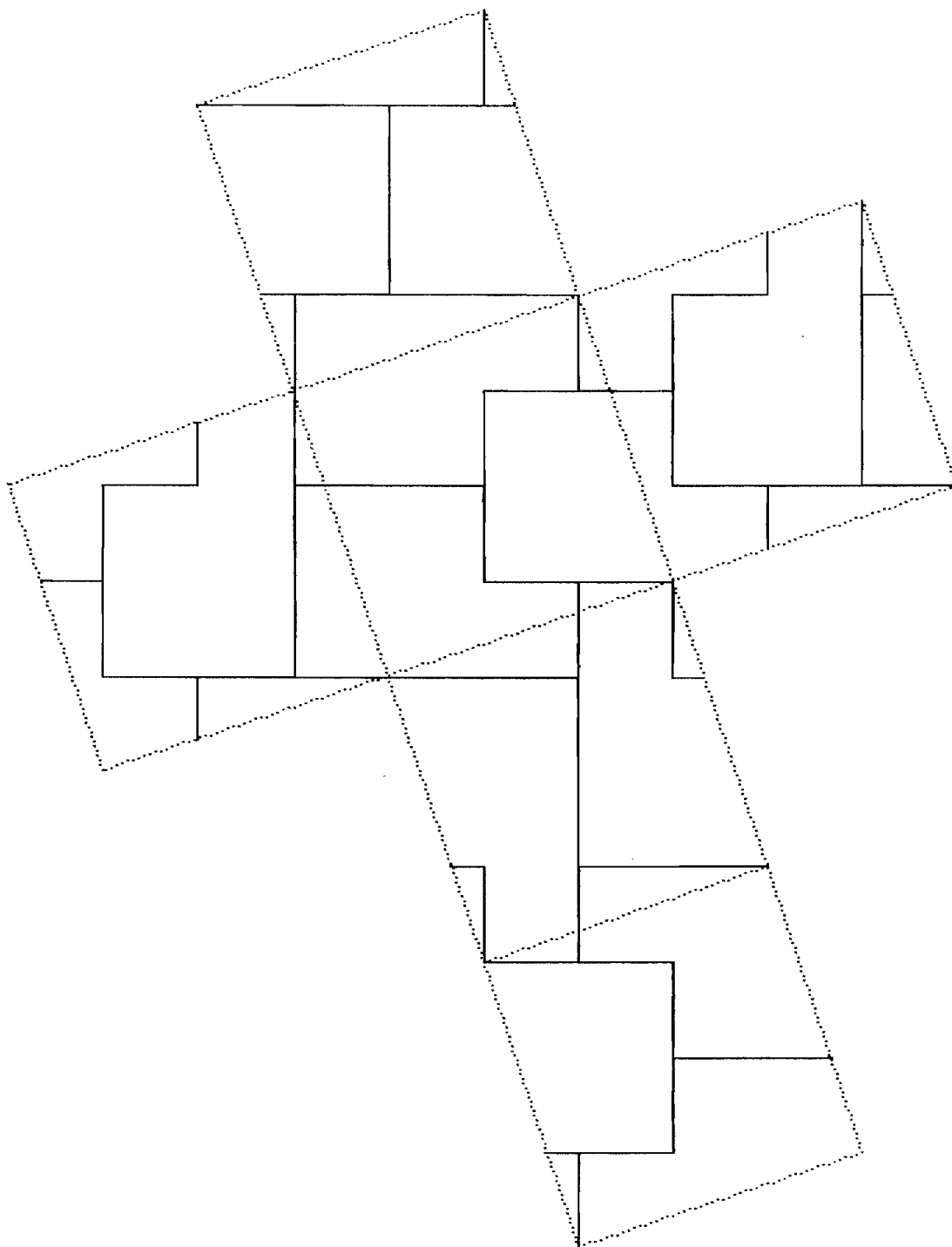
3169 3239 3269 3292 3338 3354 3363 3416 3453 3466 3508 3562

One axis order 3 (D3)



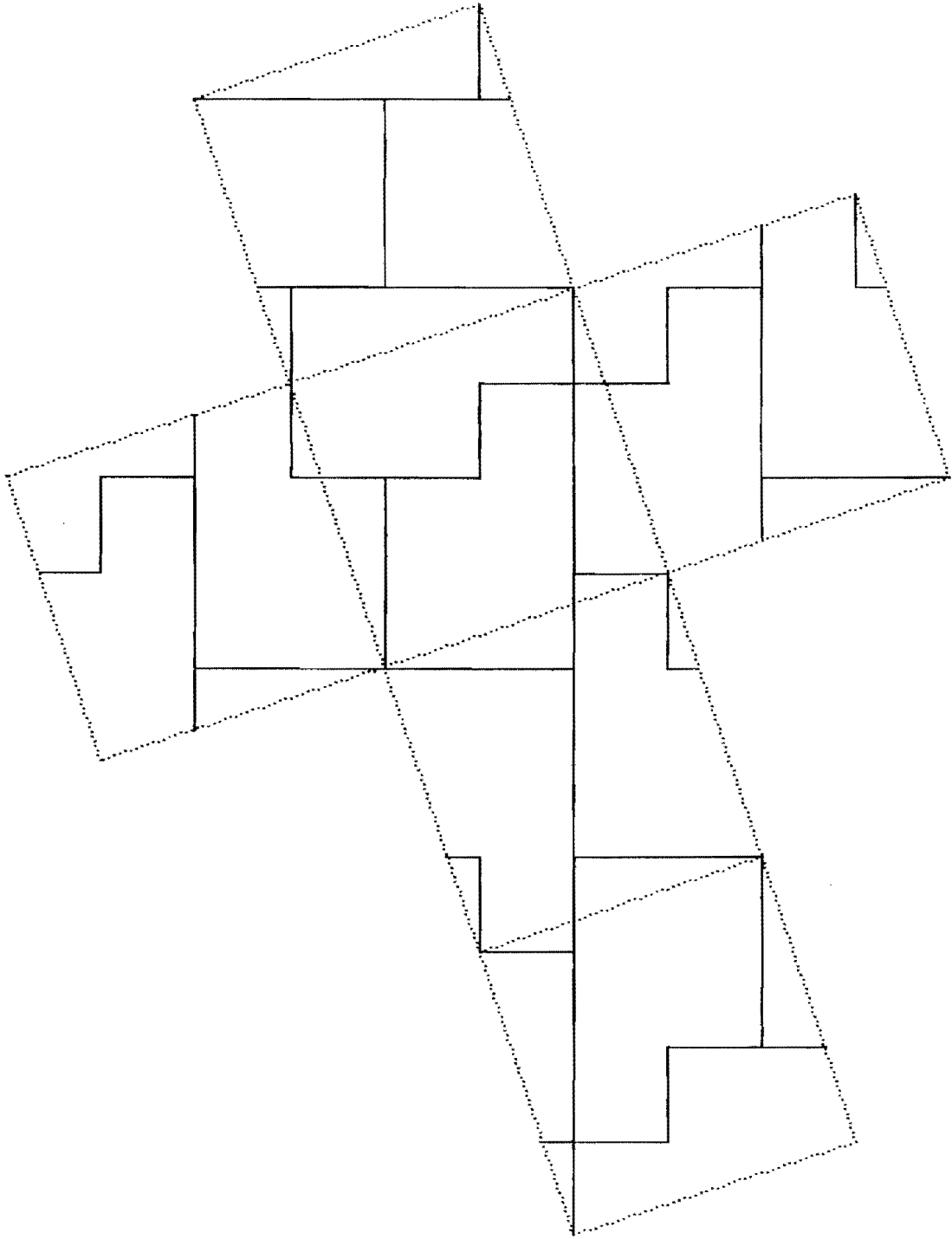
3169 3239 3271 3287 3349 3360 3434 3436 3512 3523 3558 3576

One axis order 2 (Z)



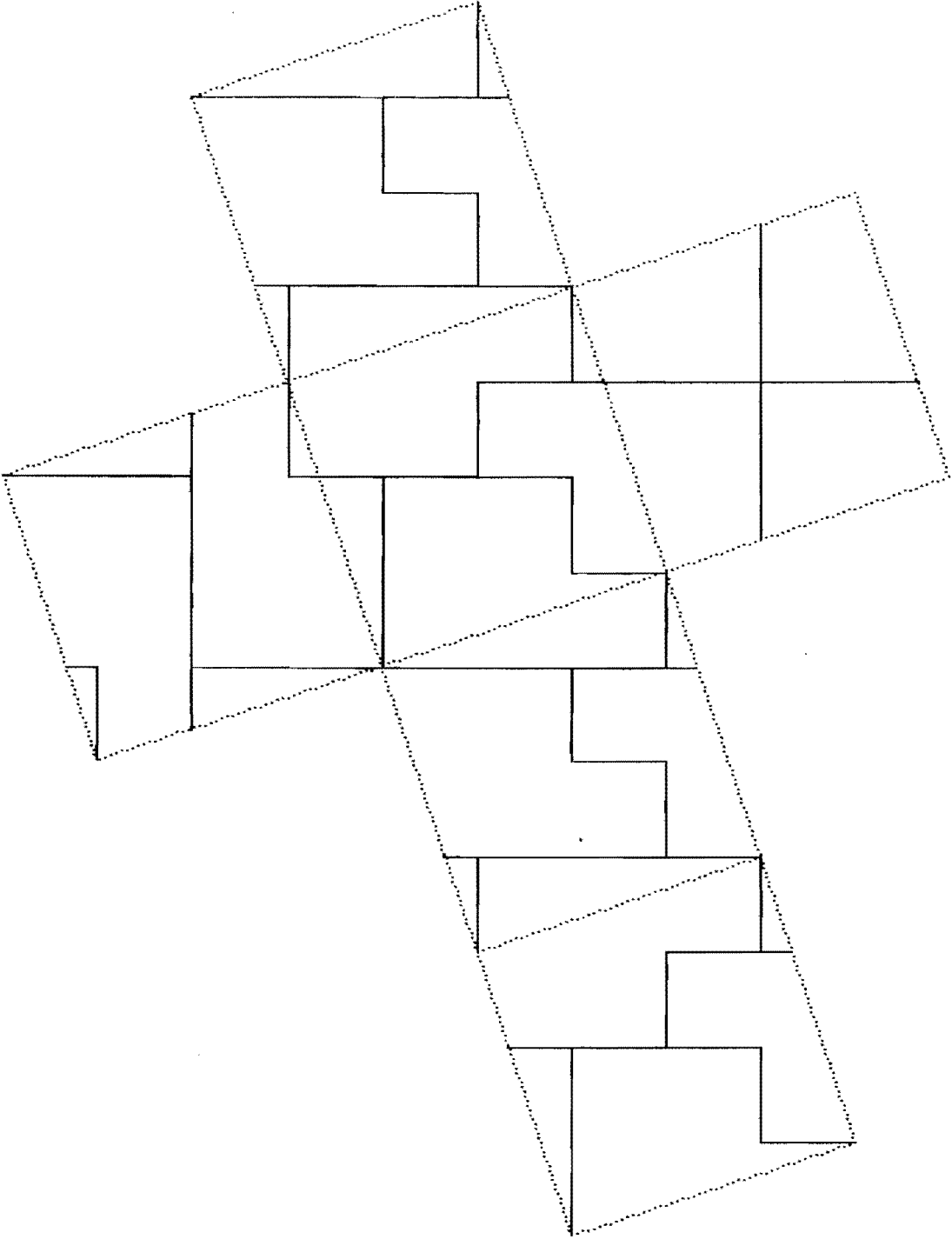
3169 3239 3272 3287 3348 3361 3421 3436 3512 3523 3558 3576

One axis order 2 (Z)



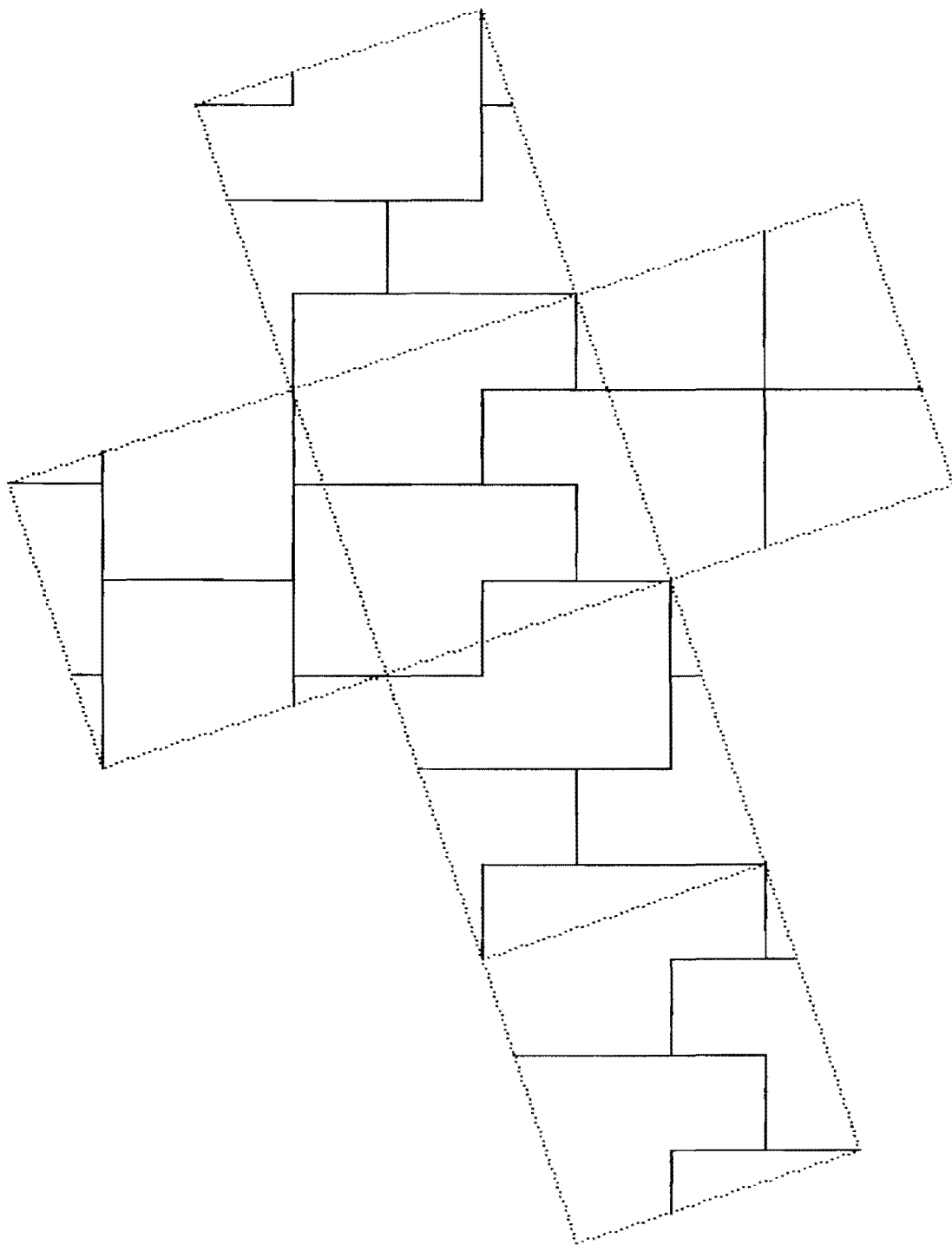
3169 3239 3275 3281 3296 3348 3361 3426 3439 3455 3512 3523

One axis order 2 (Z)



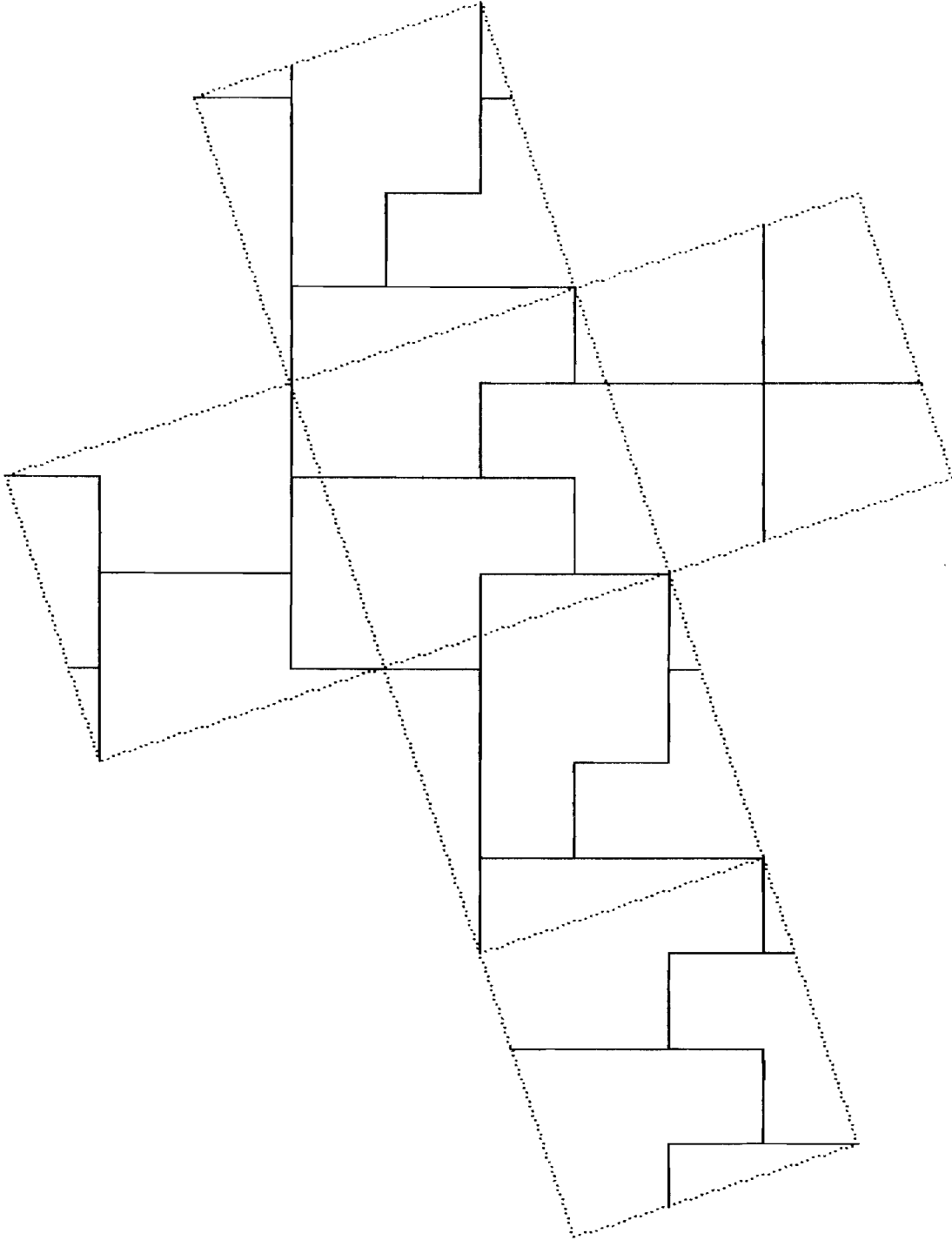
3169 3240 3274 3296 3304 3359 3378 3427 3429 3480 3508 3561

One axis order 2 (Y)



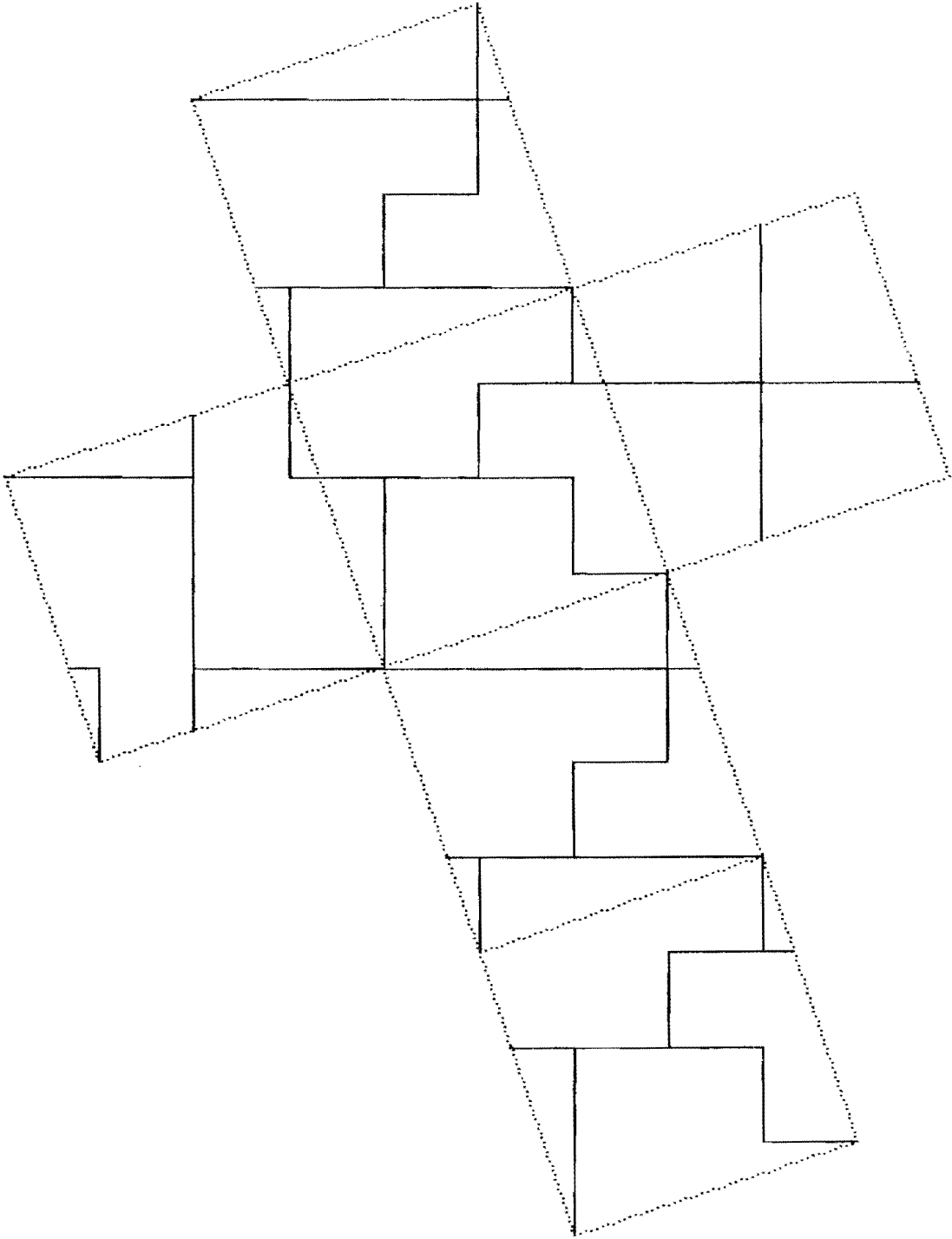
3169 3241 3274 3285 3335 3382 3395 3429 3480 3491 3539 3572

One axis order 2 (Y)



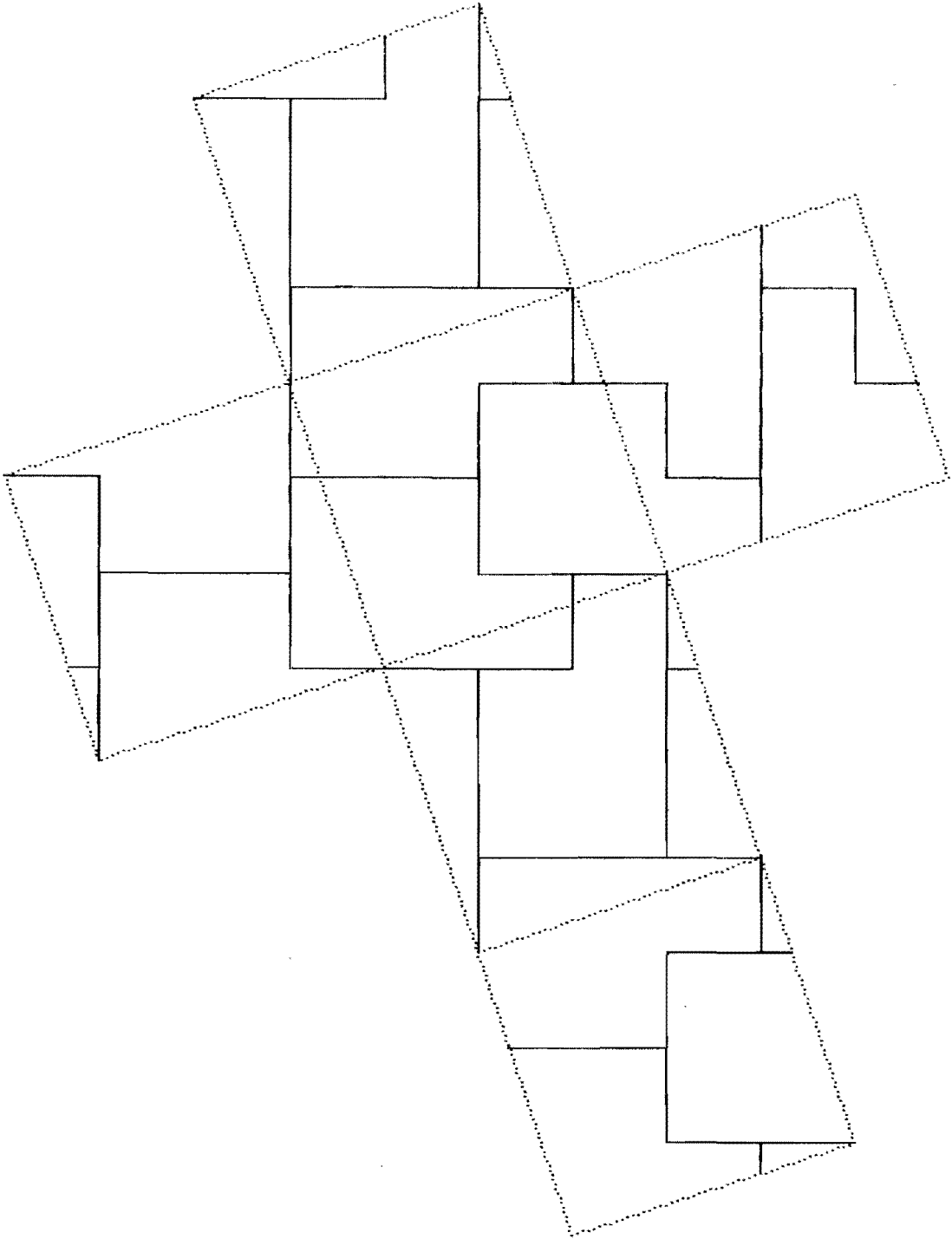
3169 3241 3274 3285 3337 3364 3382 3429 3480 3491 3541 3564

One axis order 2 (Y)



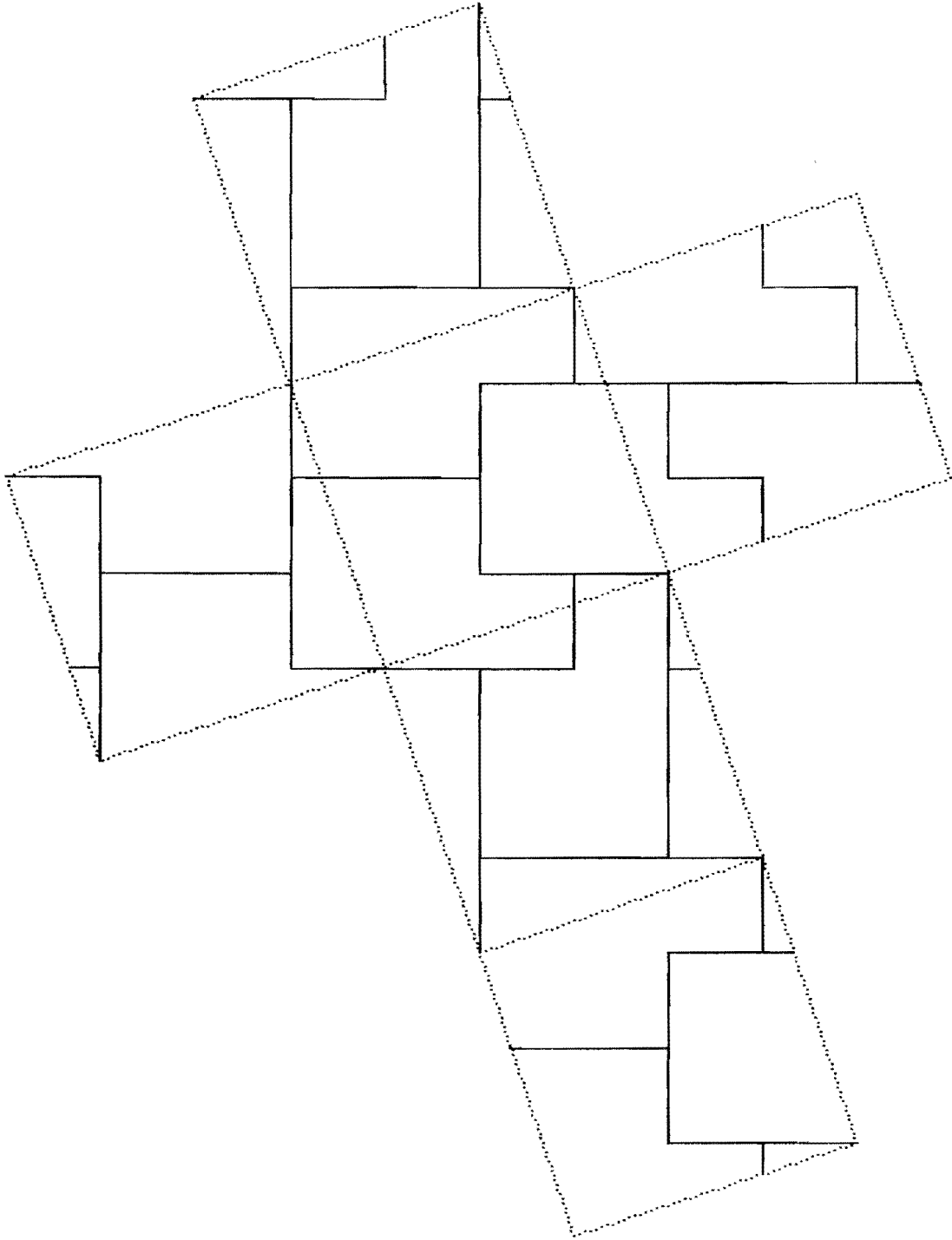
3169 3241 3274 3296 3304 3357 3382 3427 3429 3480 3508 3559

One axis order 2 (Y)



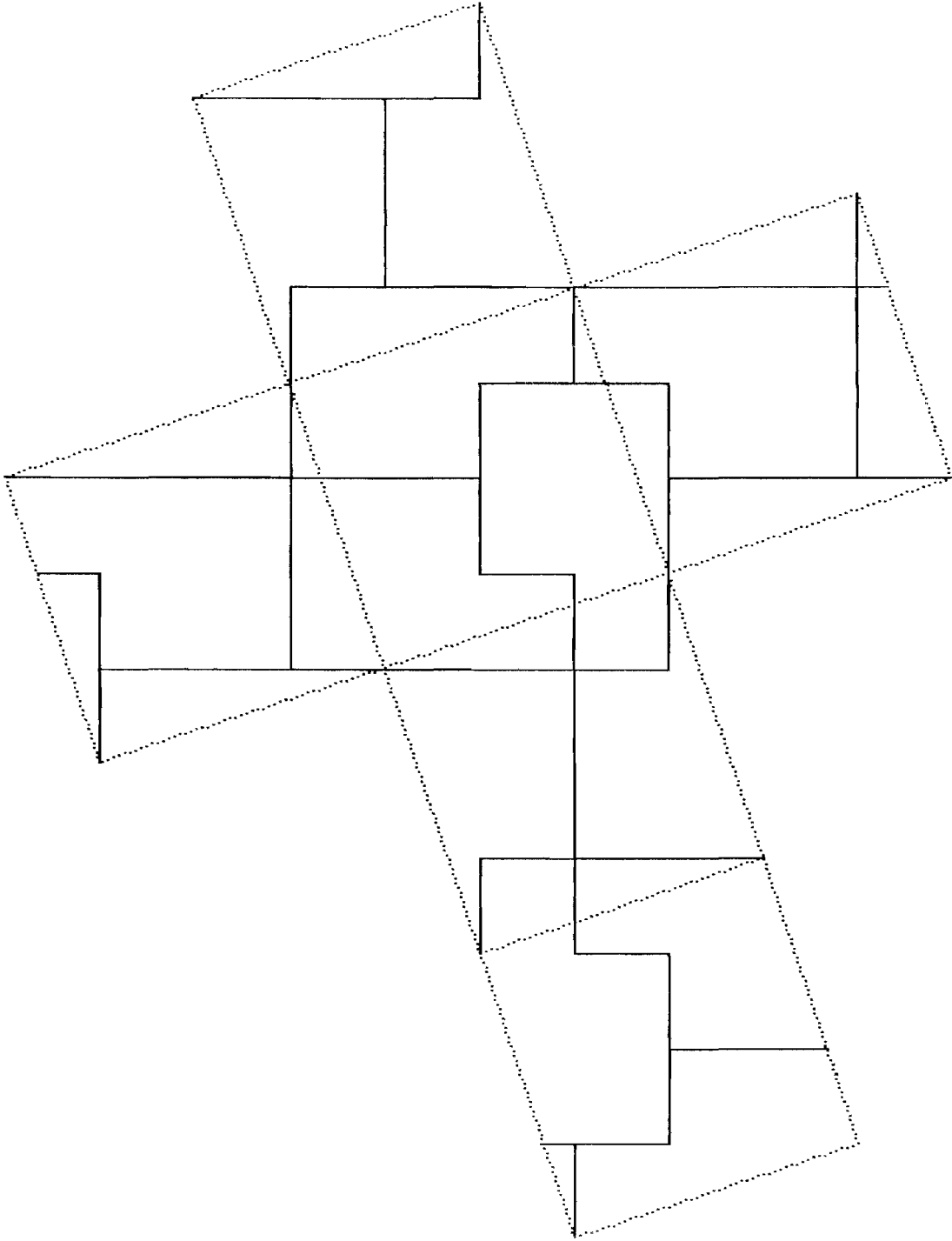
3169 3242 3272 3287 3347 3364 3386 3429 3478 3493 3551 3564

One axis order 2 (Y)



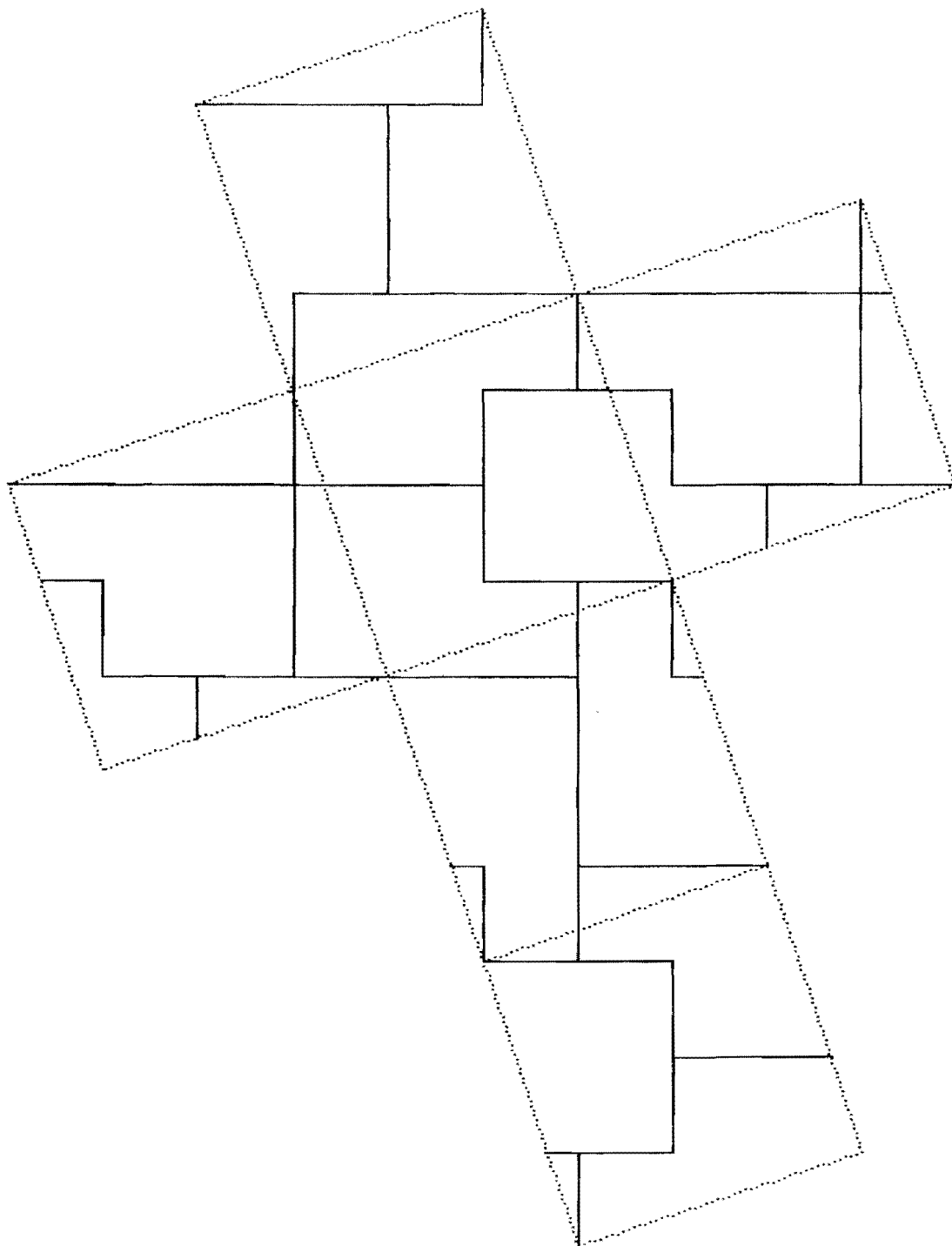
3169 3243 3272 3287 3347 3364 3385 3429 3478 3493 3551 3564

One axis order 2 (Y)

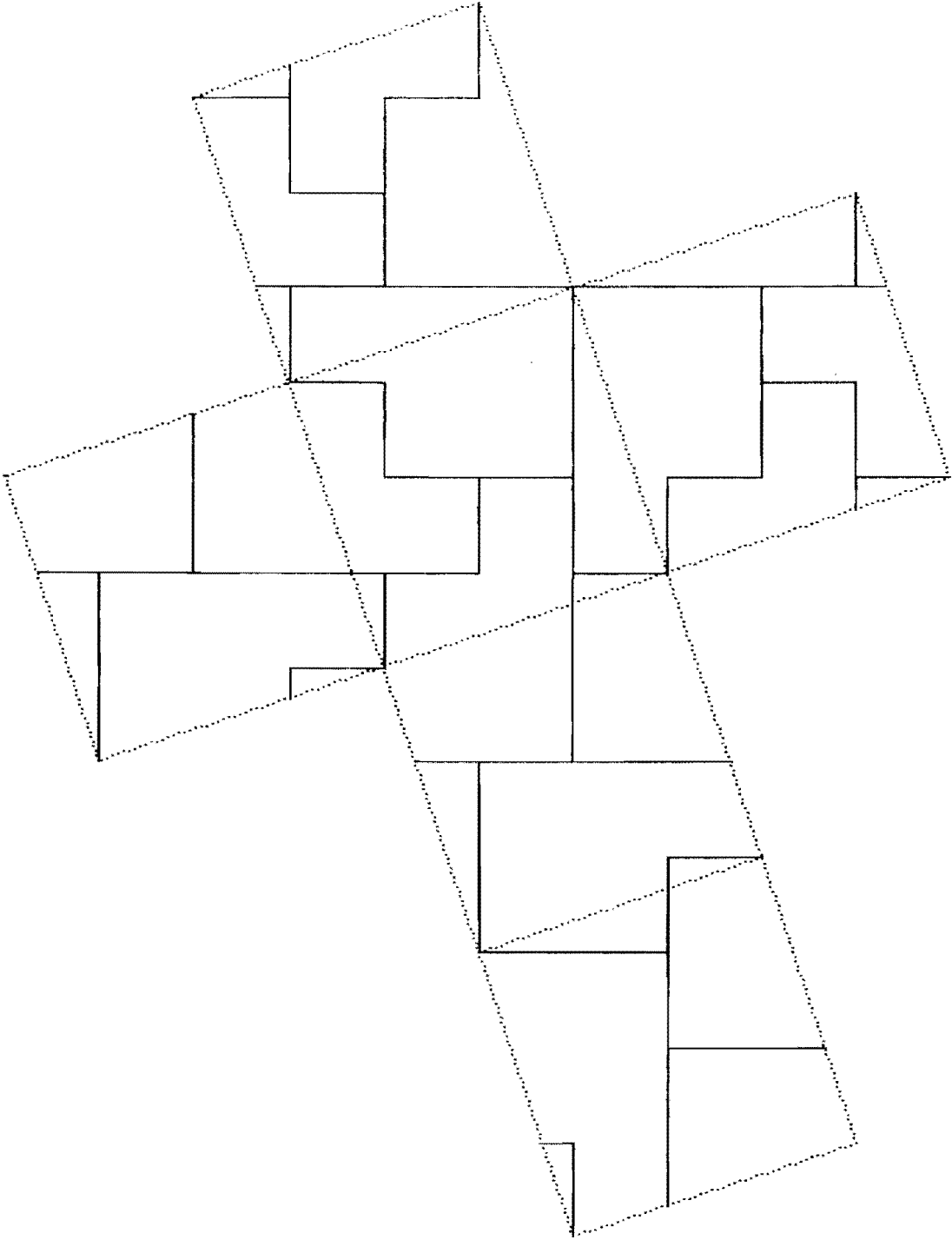


3169 3244 3271 3287 3349 3360 3434 3436 3512 3530 3553 3562

One axis order 2 (Z)

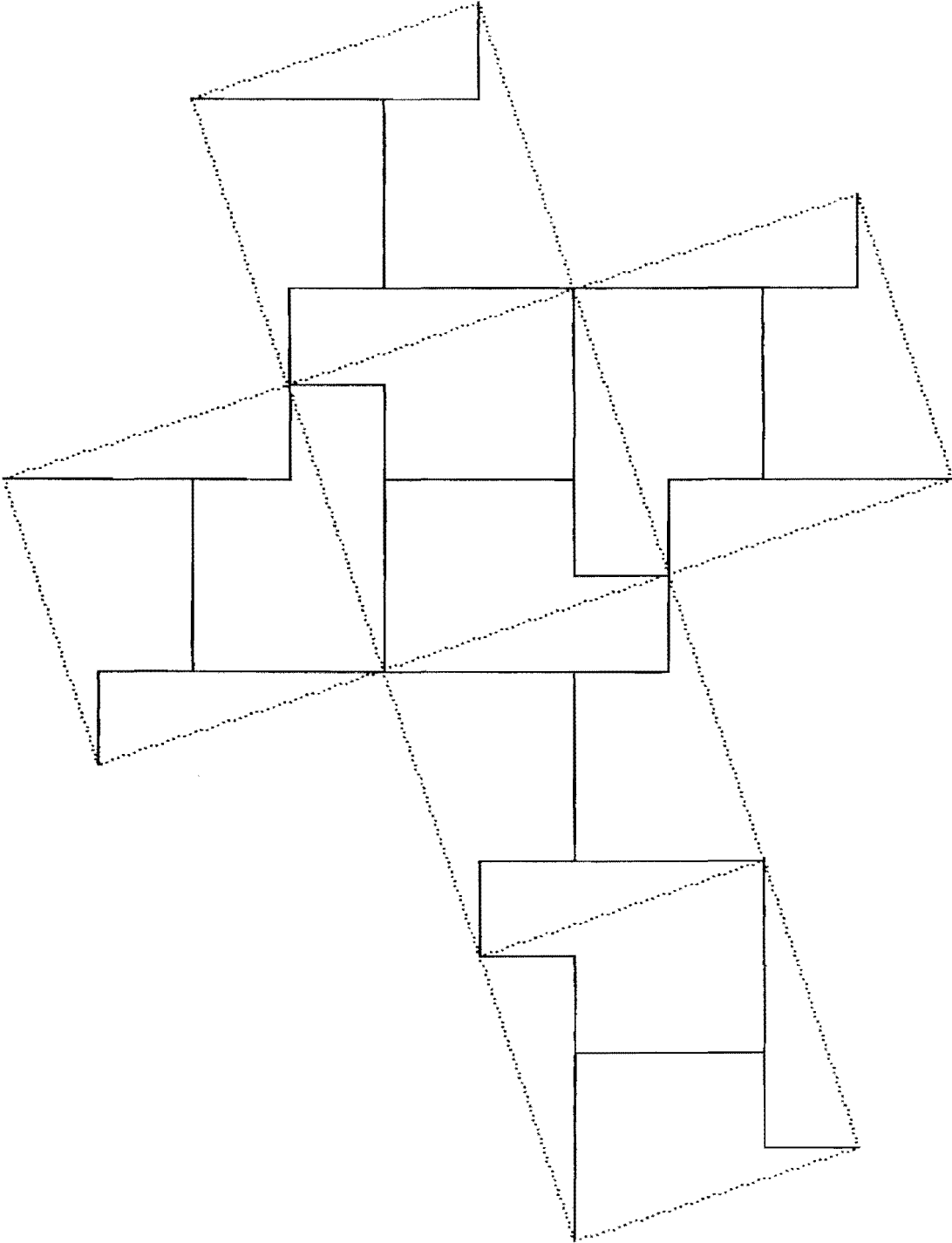


3169 3244 3272 3287 3348 3361 3421 3436 3512 3530 3553 3562



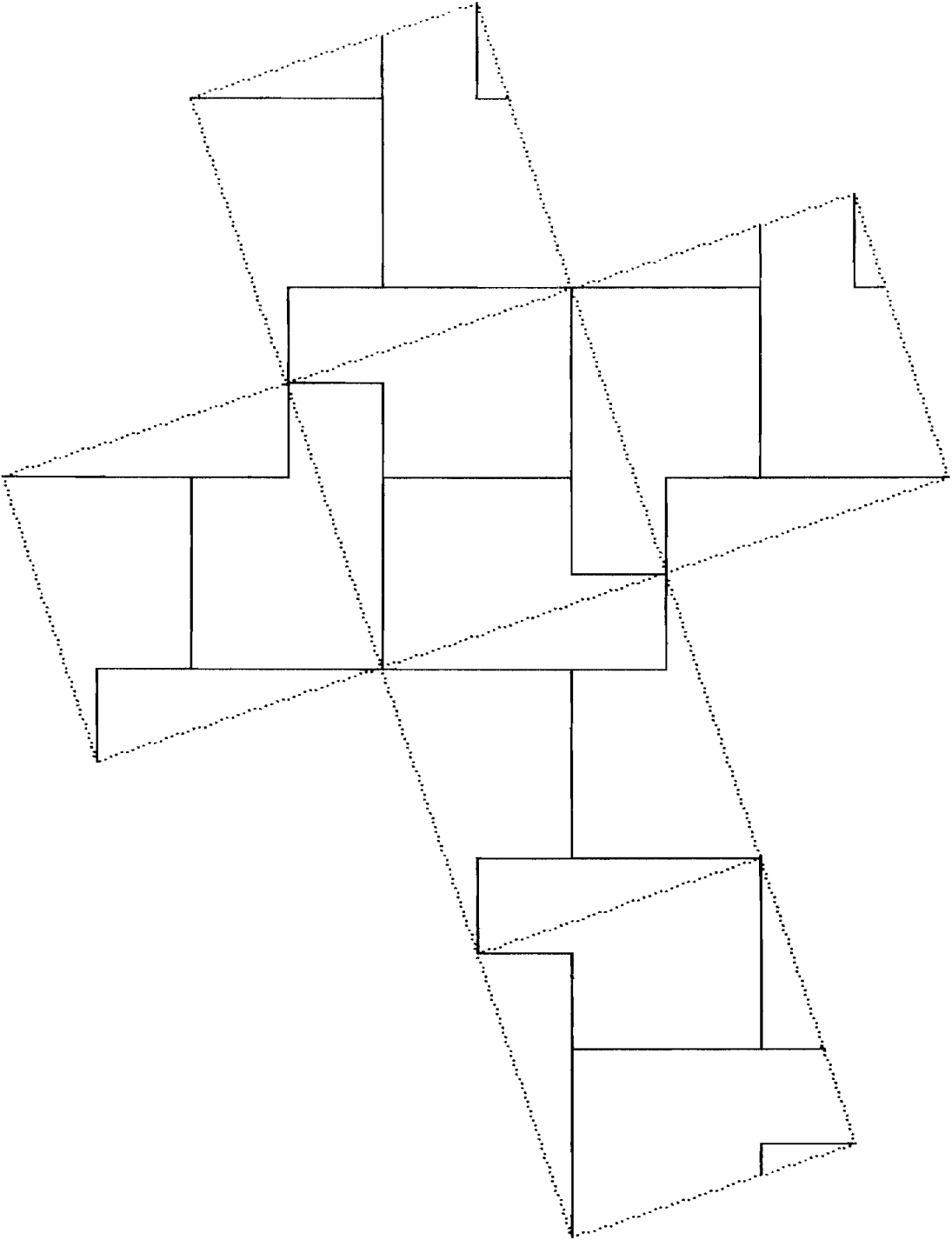
3170 3230 3254 3312 3325 3345 3403 3447 3463 3514 3525 3553

One axis order 3 (D2)



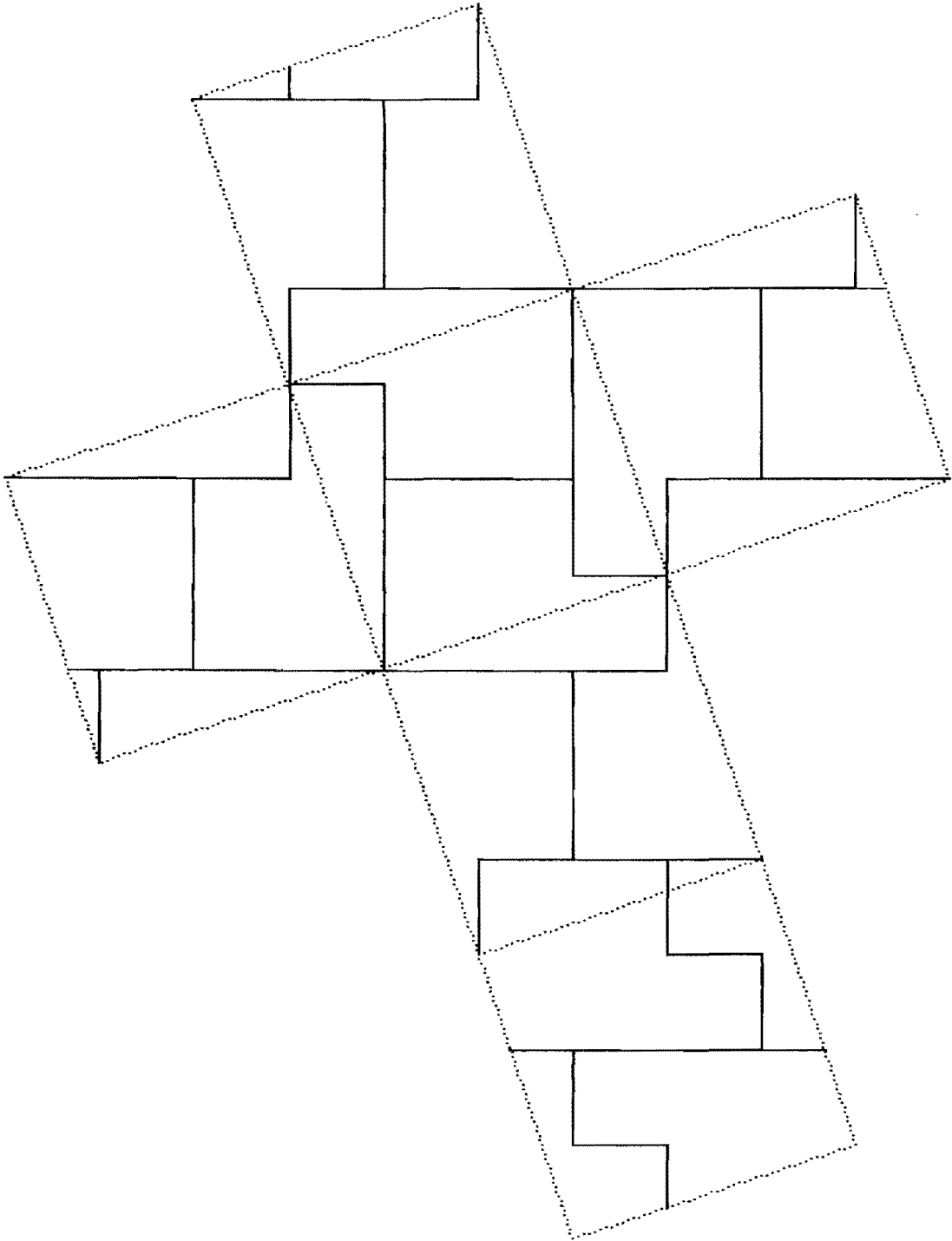
3170 3230 3256 3304 3349 3360 3430 3454 3466 3508 3553 3562

Four axes order 3 (D1 D2 D3 D4) and three axes order 2 (X Y Z)



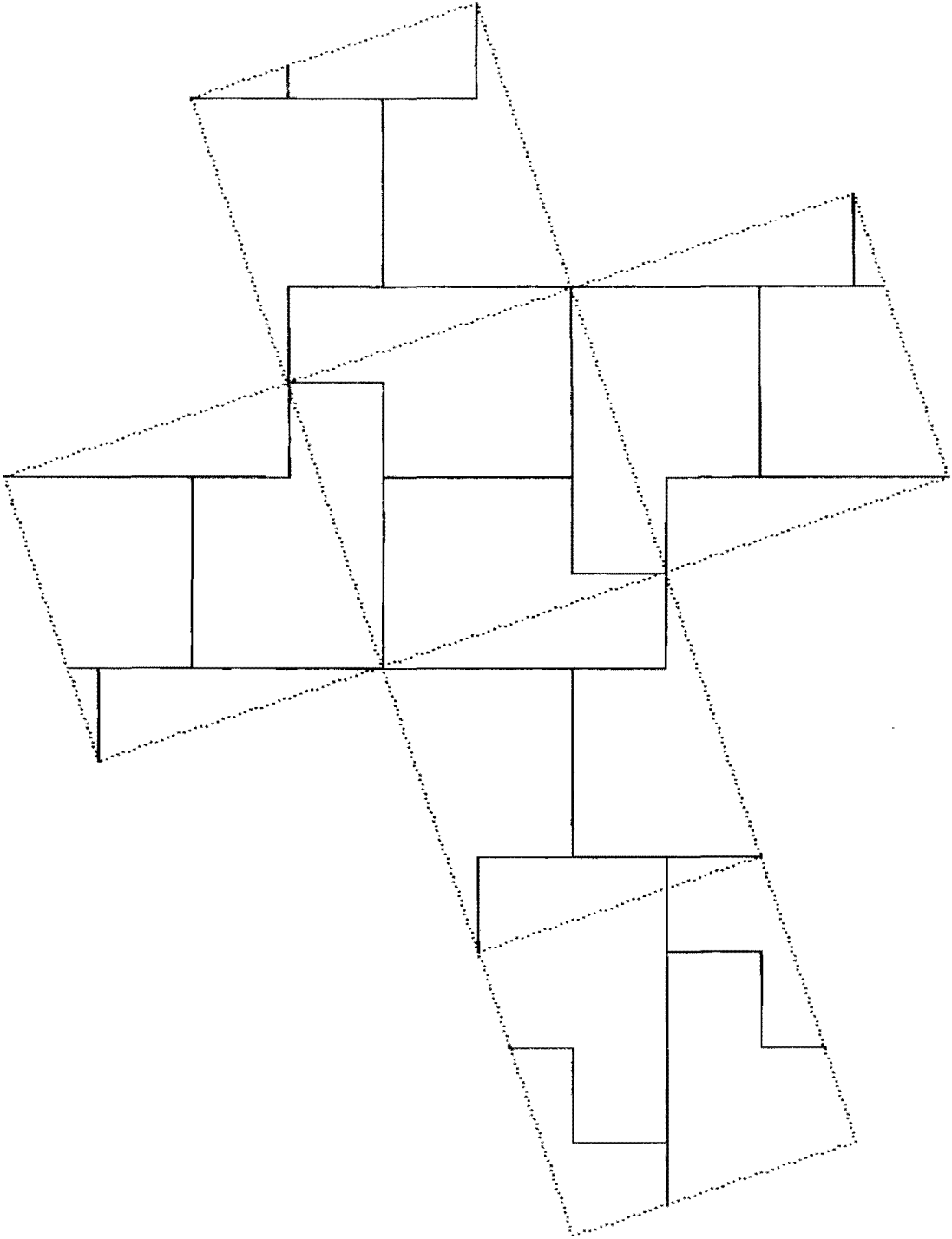
3170 3230 3256 3304 3349 3360 3430 3455 3466 3505 3552 3562

One axis order 3 (D4)



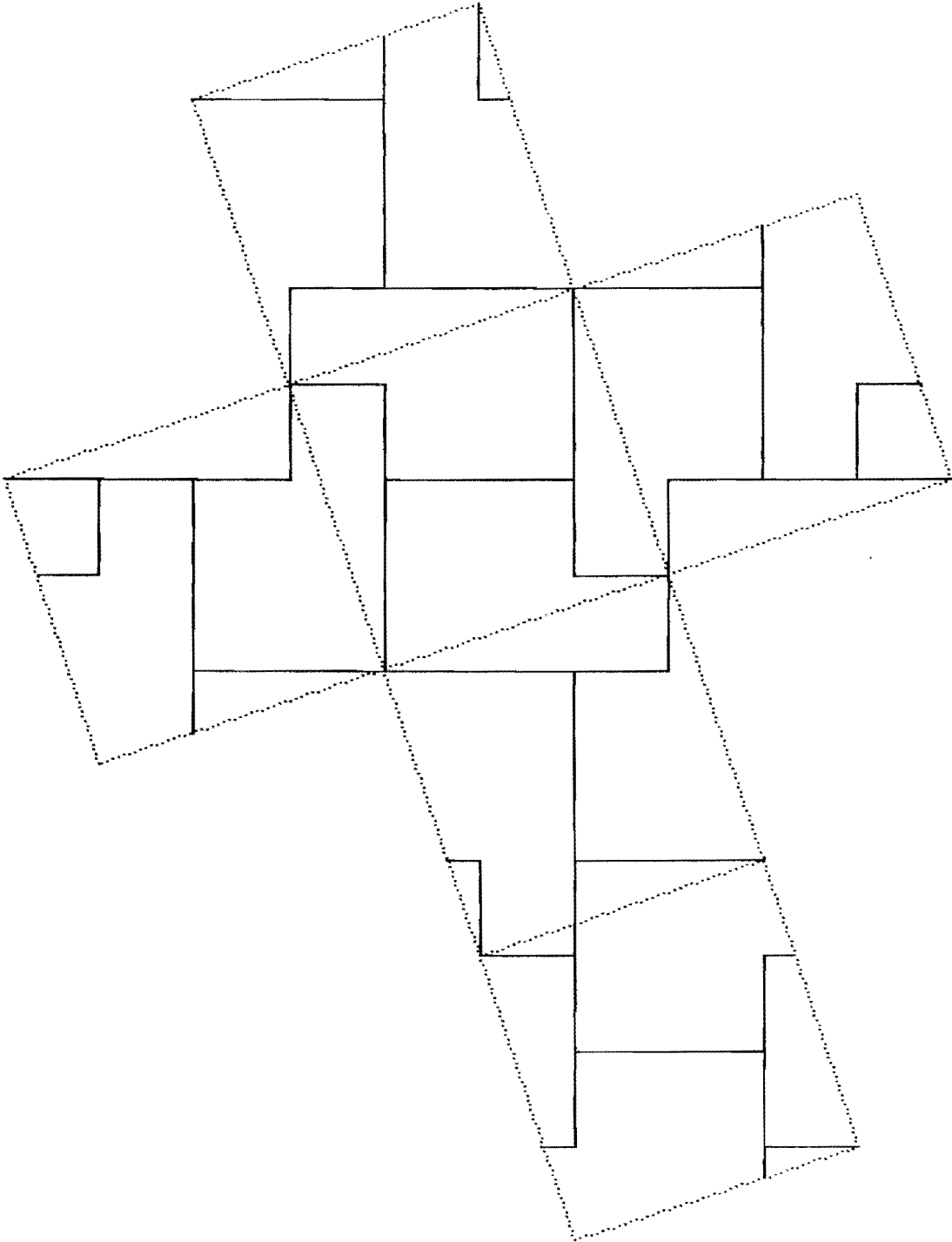
3170 3230 3256 3304 3349 3360 3431 3448 3498 3506 3553 3562

One axis order 2 (X)



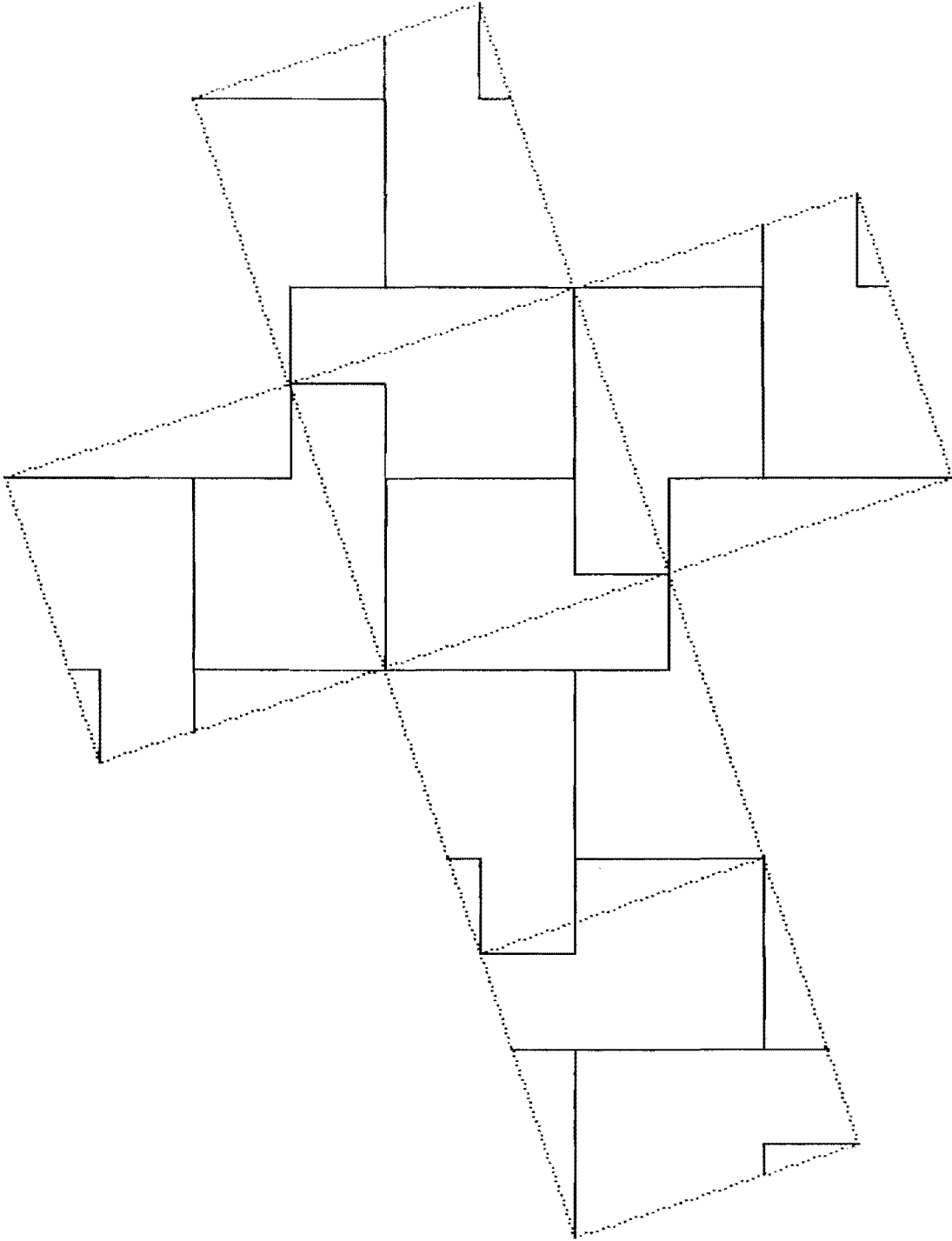
3170 3230 3256 3304 3349 3360 3433 3448 3482 3498 3553 3562

One axis order 2 (X)



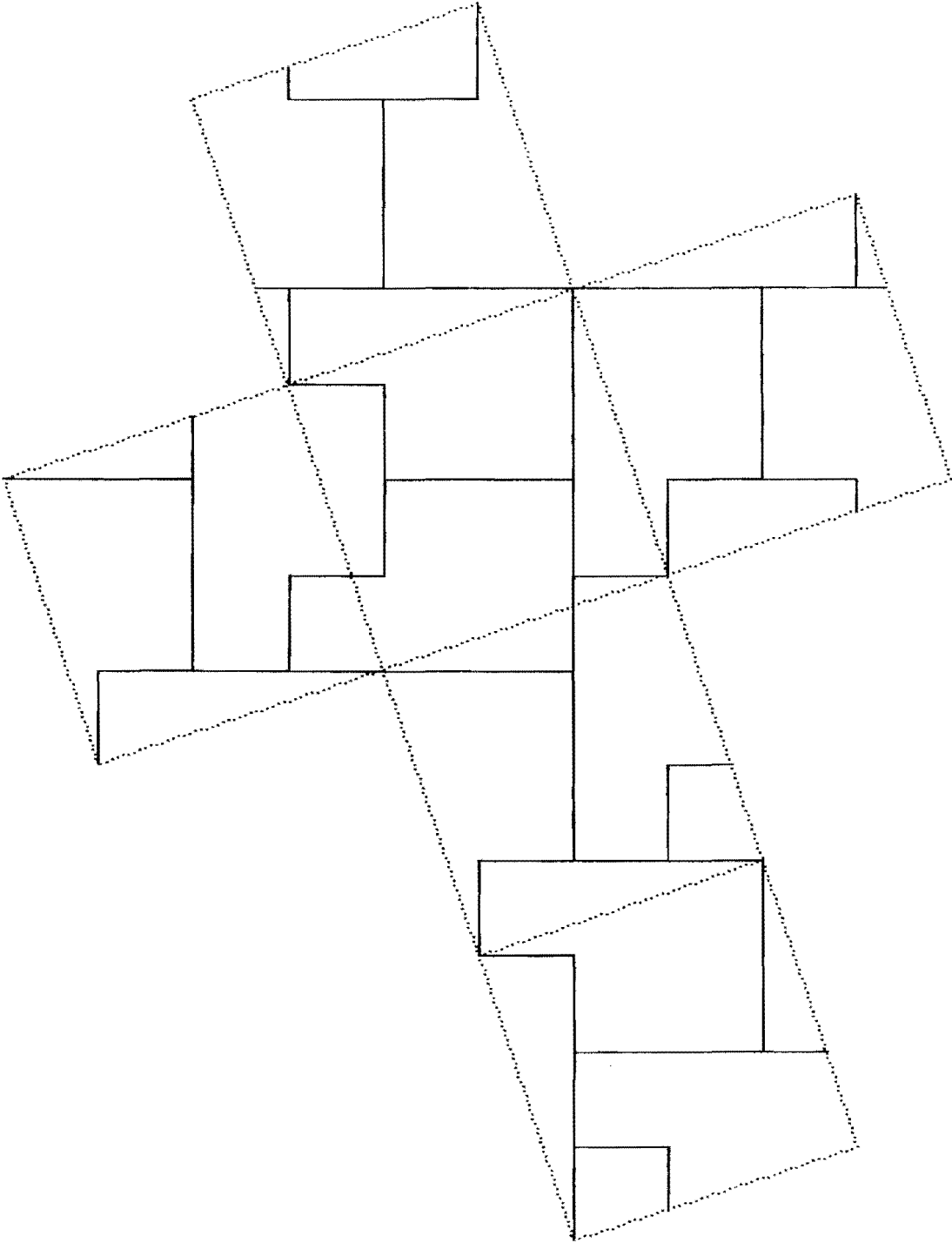
3170 3230 3256 3304 3349 3361 3426 3435 3488 3507 3552 3562

One axis order 2 (X)



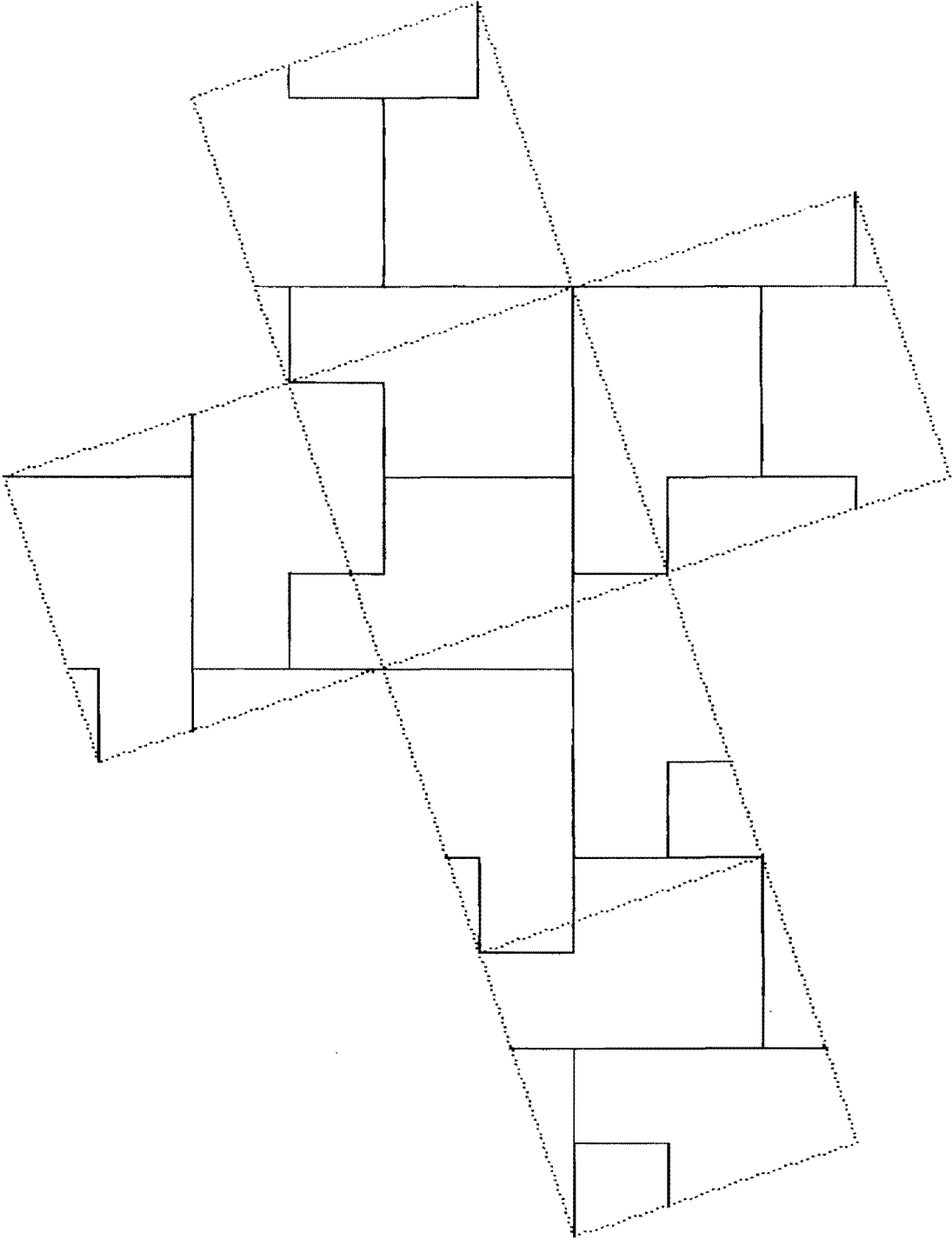
3170 3230 3256 3304 3349 3361 3427 3437 3455 3505 3552 3562

One axis order 2 (X)



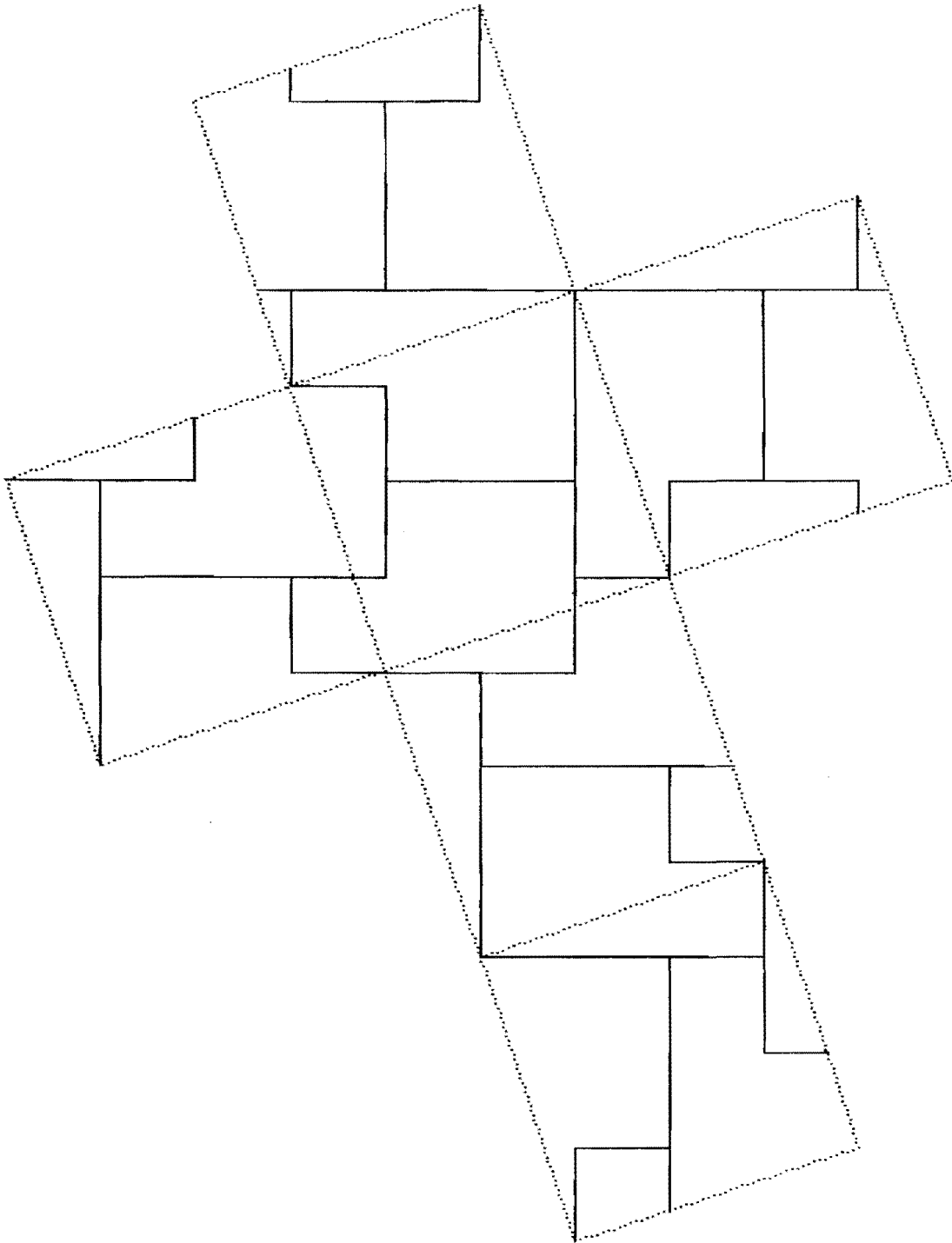
3170 3230 3258 3303 3342 3360 3418 3430 3466 3506 3536 3553

One axis order 3 (D2)



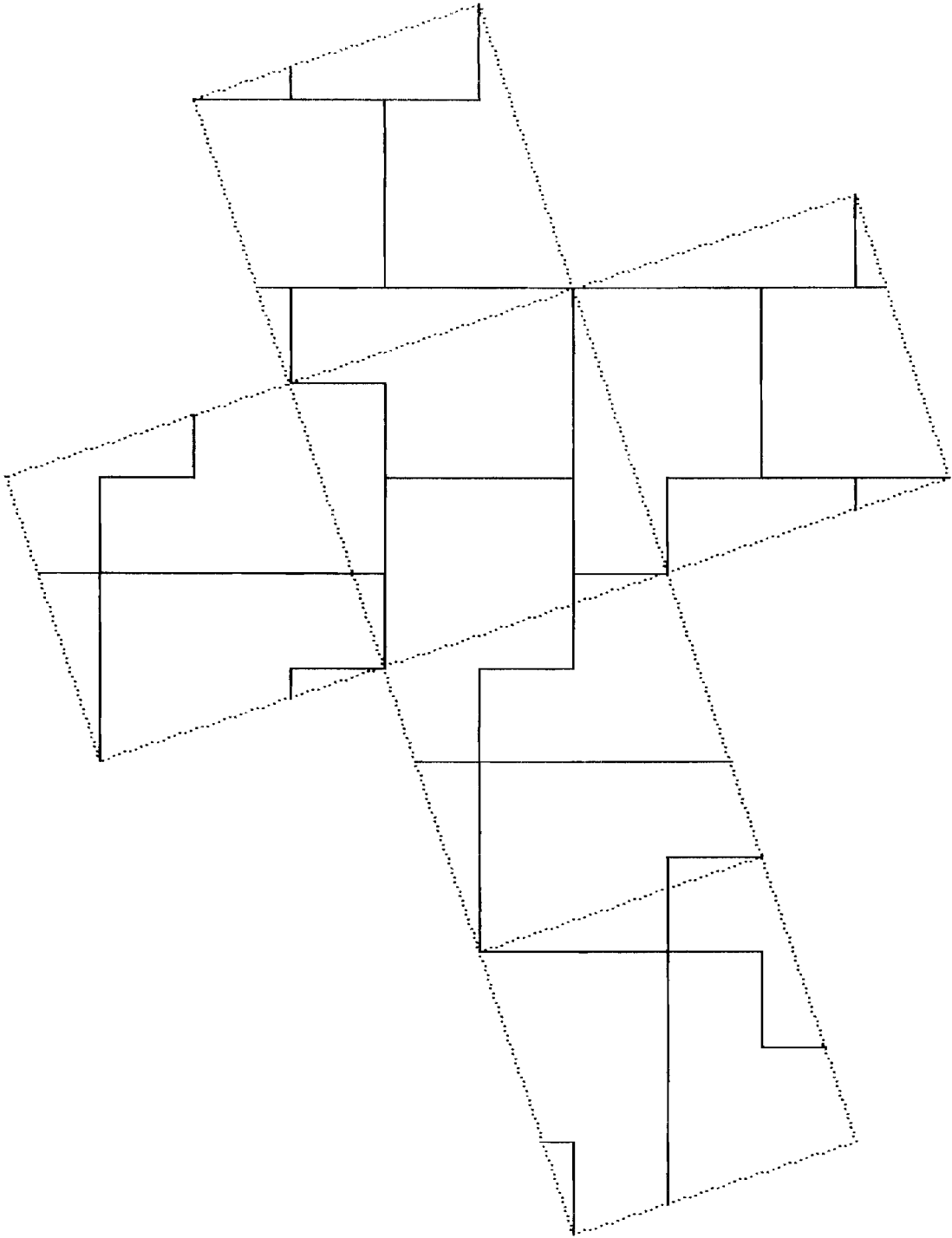
3170 3230 3258 3303 3342 3361 3418 3427 3437 3506 3536 3553

One axis order 3 (D2)



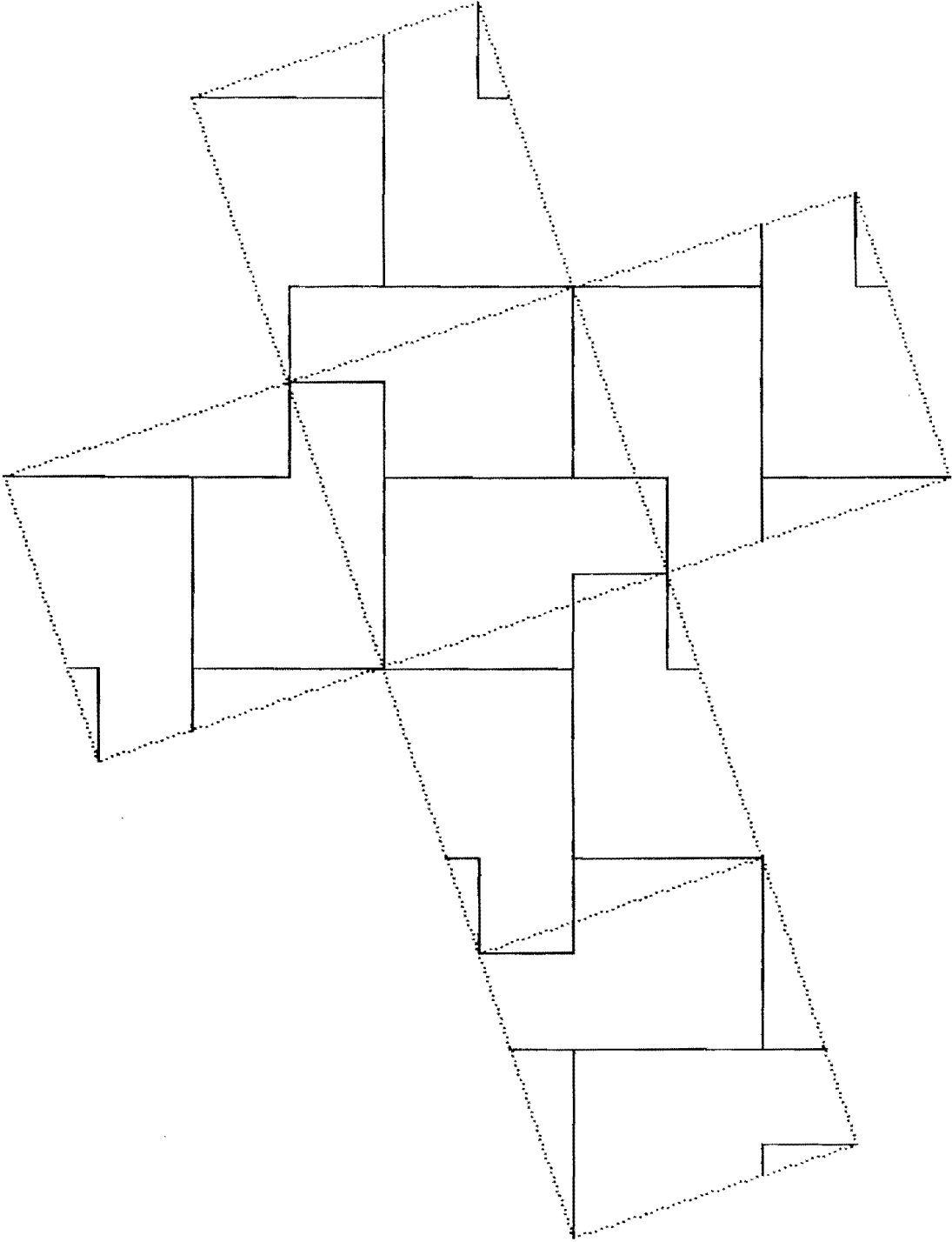
3170 3230 3259 3303 3341 3364 3406 3418 3462 3482 3536 3553

One axis order 3 (D2)



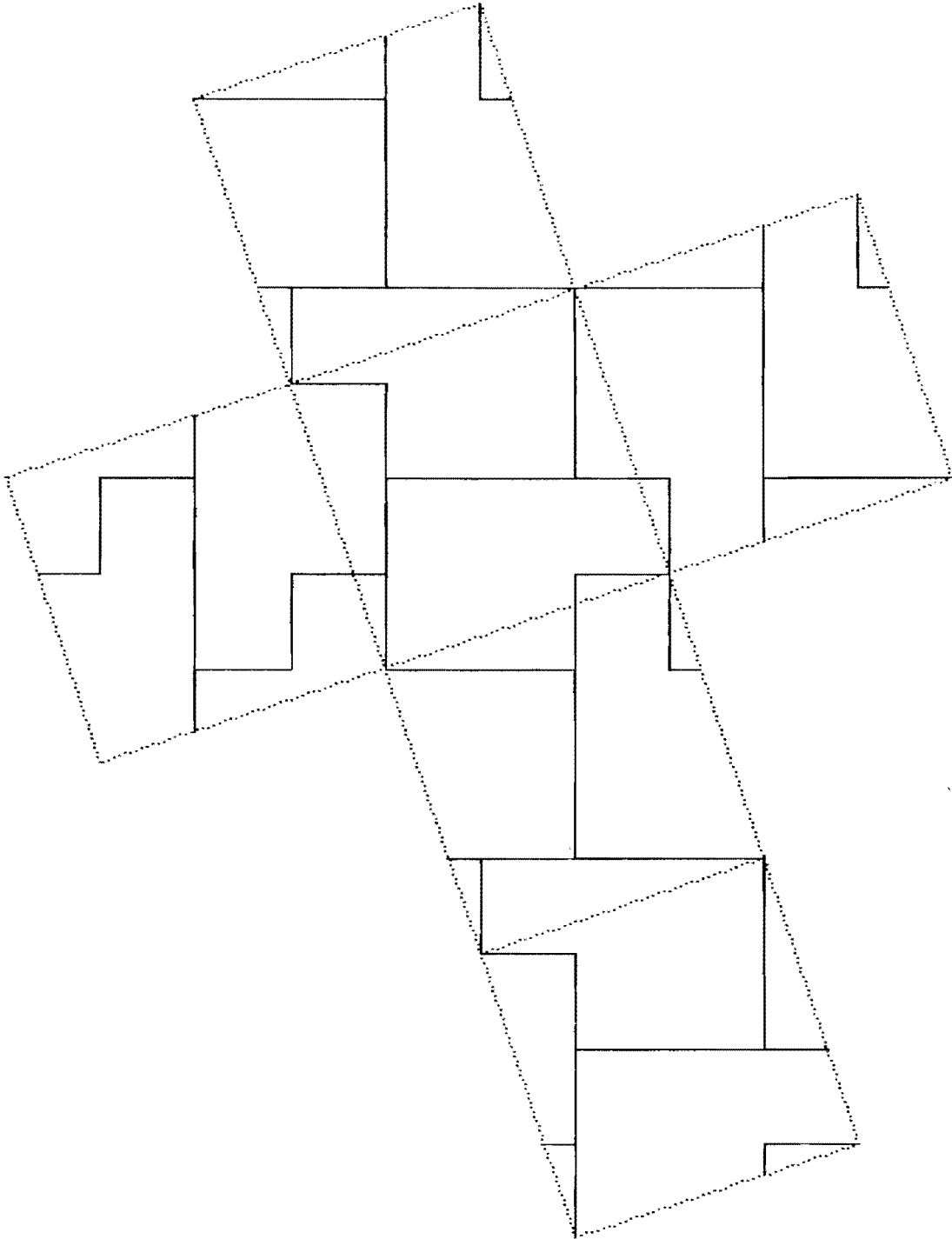
3170 3230 3259 3305 3325 3341 3403 3448 3463 3482 3523 3553

One axis order 3 (D2)



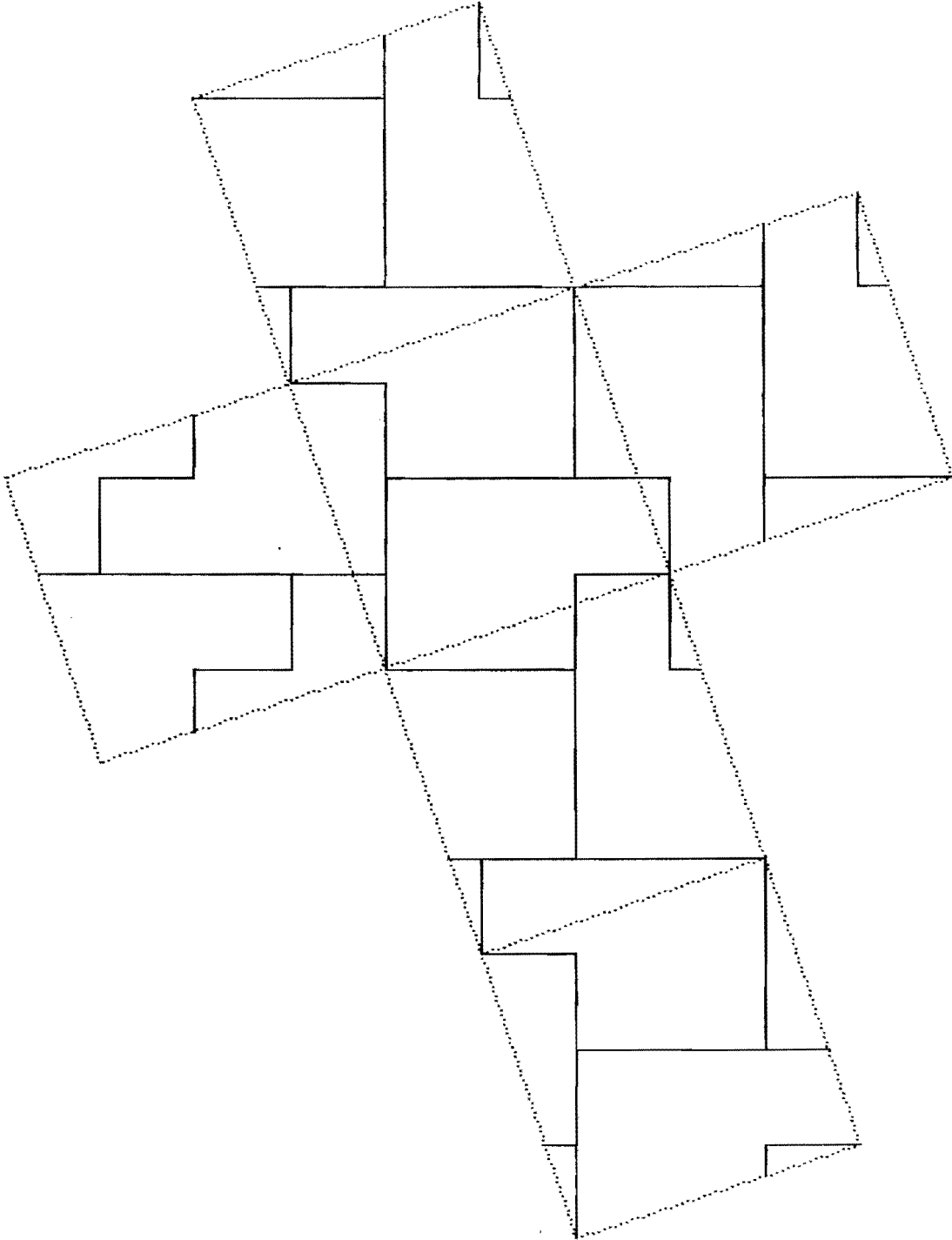
3170 3231 3256 3301 3348 3361 3427 3437 3455 3505 3552 3562

One axis order 3 (D1)



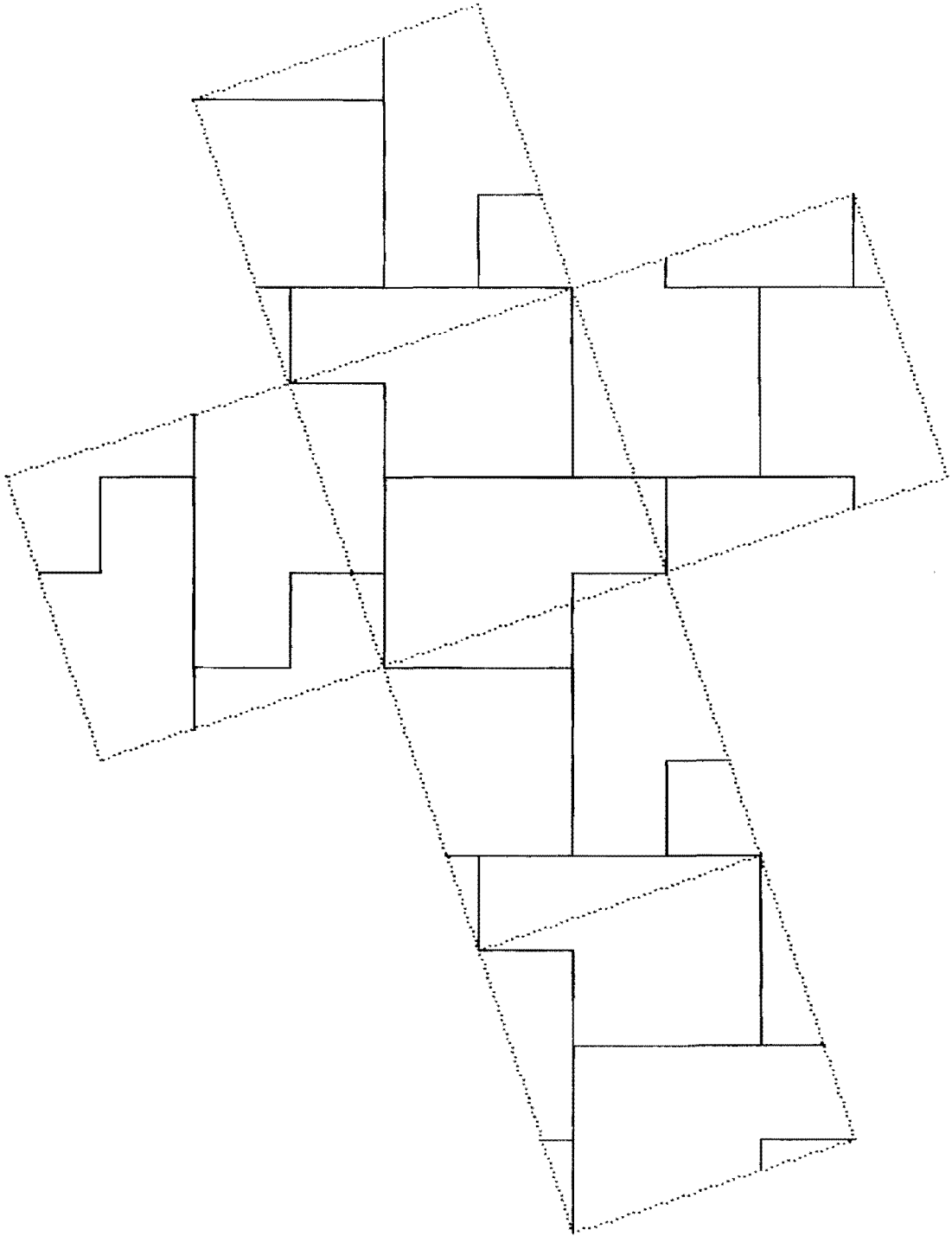
3170 3231 3258 3301 3319 3348 3426 3430 3455 3505 3523 3552

One axis order 2 (Y)



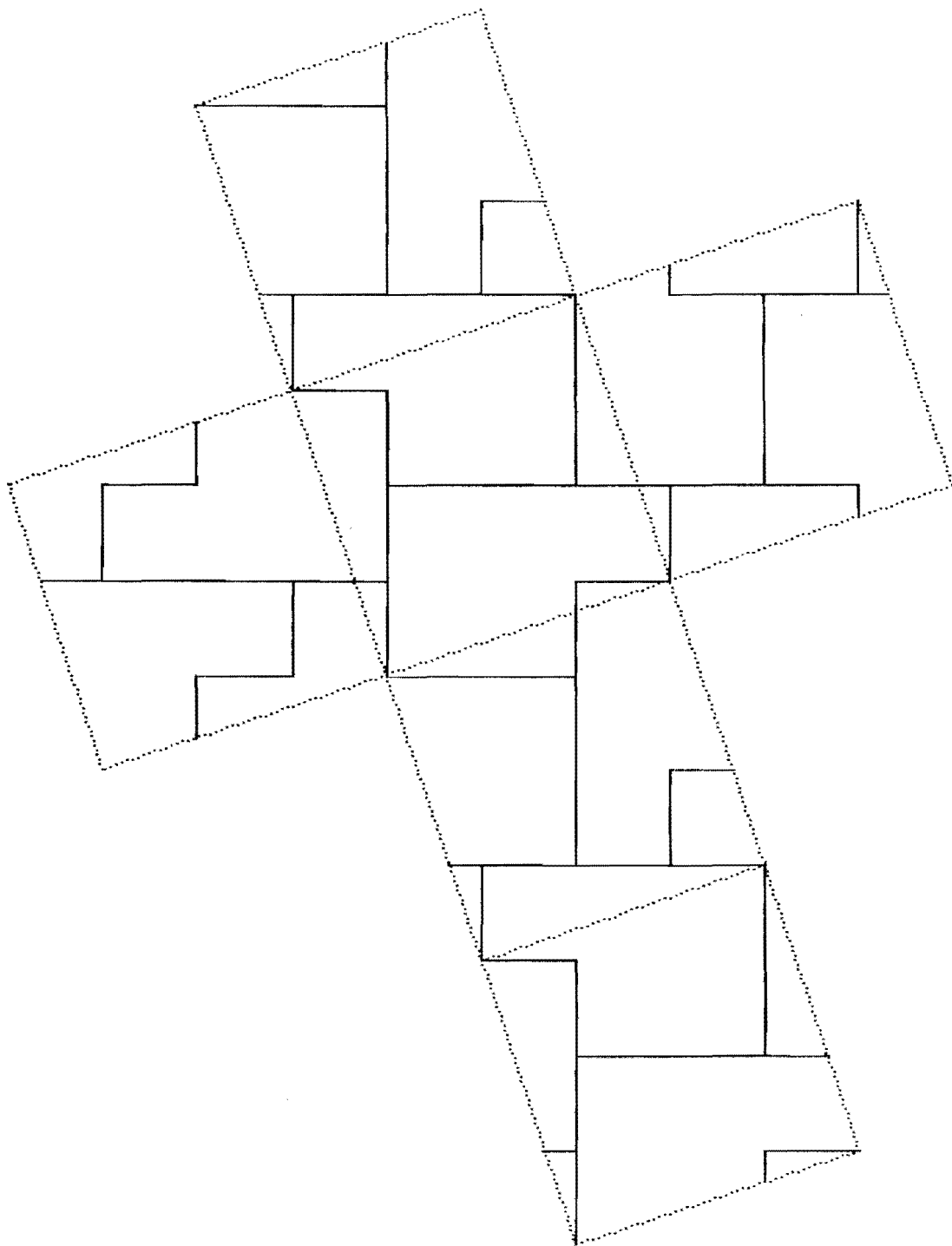
3170 3231 3259 3301 3319 3348 3425 3430 3455 3505 3523 3552

One axis order 2 (Y)



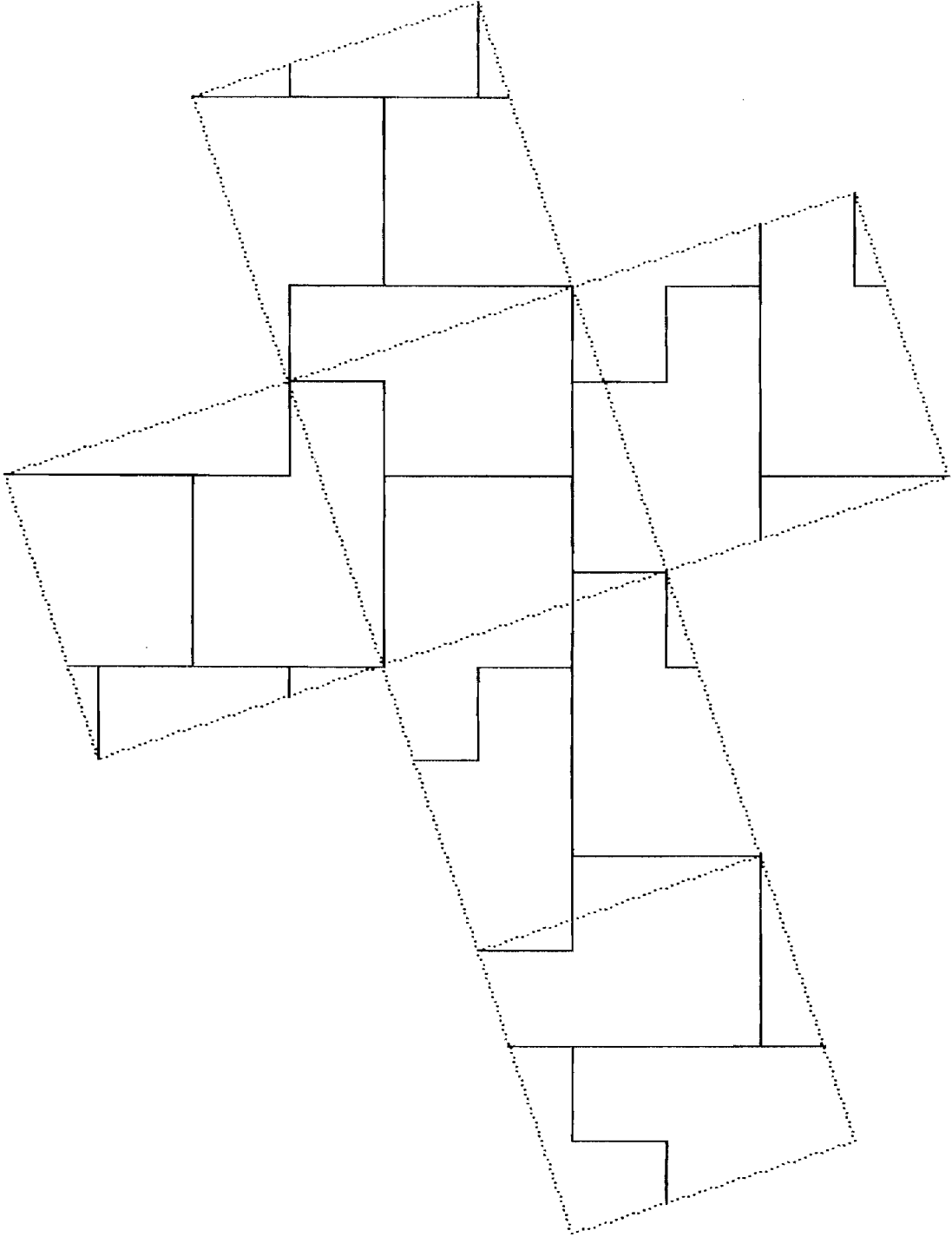
3170 3234 3258 3301 3319 3342 3418 3426 3430 3505 3523 3546

One axis order 2 (Y)



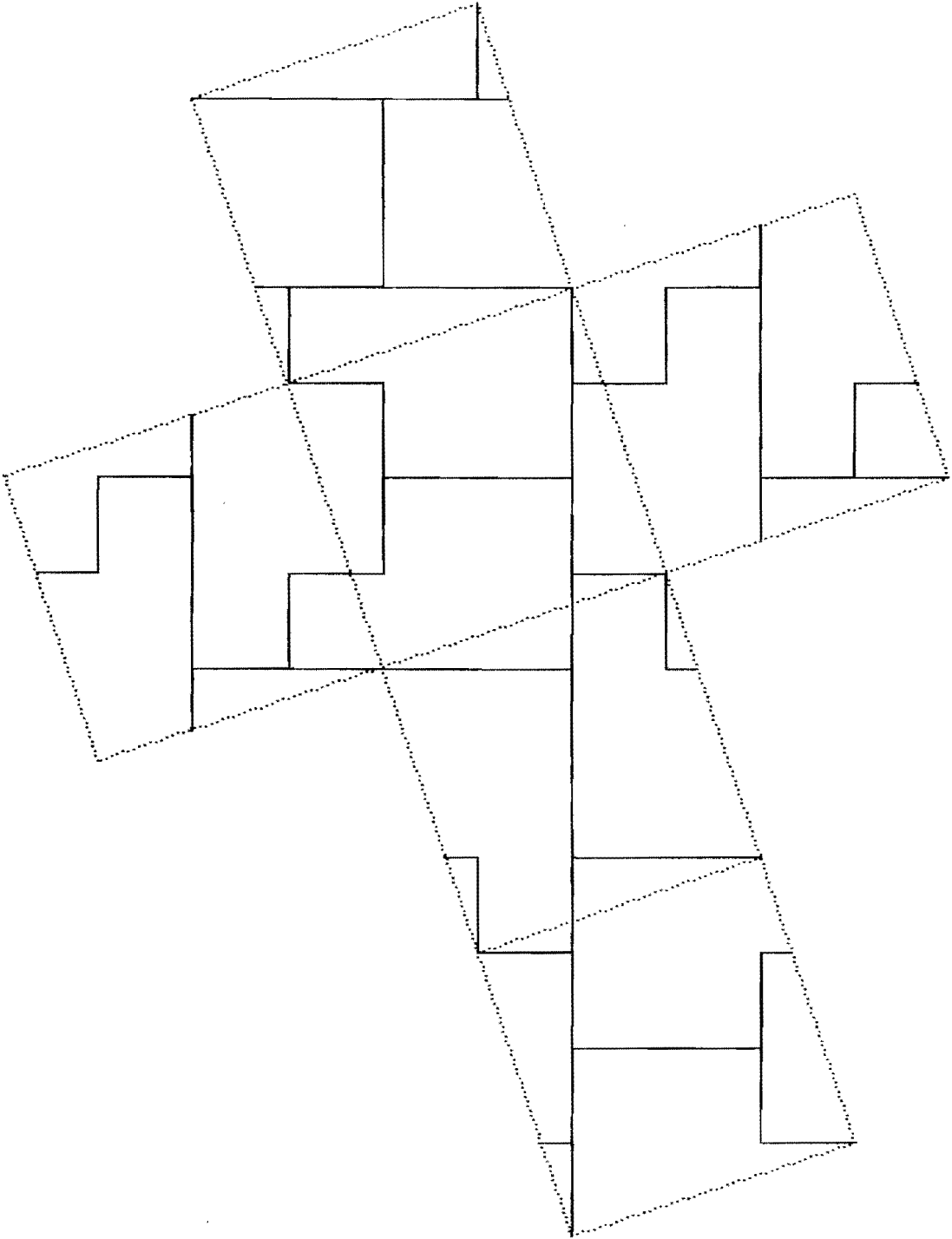
3170 3234 3259 3301 3319 3342 3418 3425 3430 3505 3523 3546

One axis order 2 (Y)



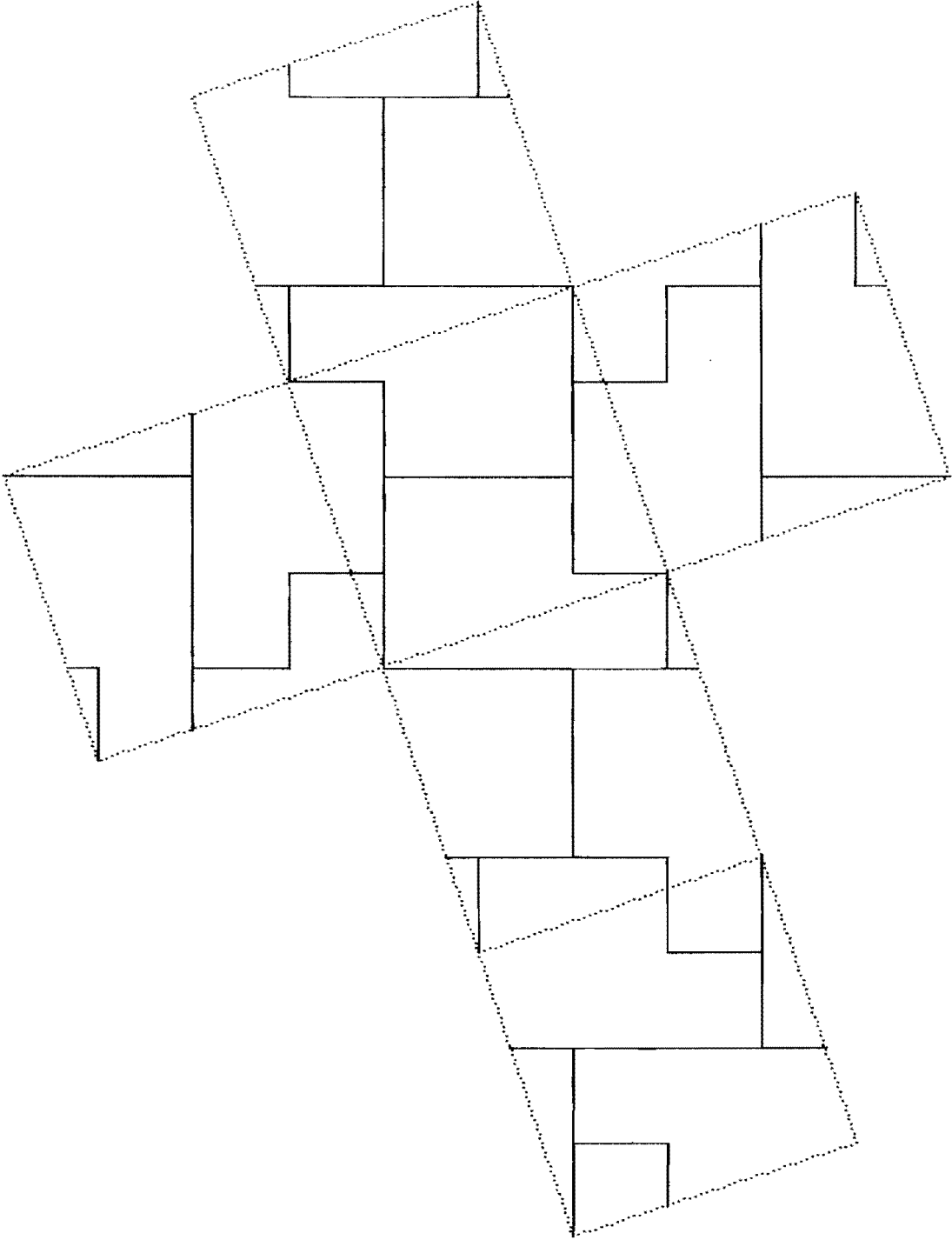
3170 3239 3256 3281 3305 3348 3371 3437 3455 3498 3506 3562

One axis order 3 (D1)



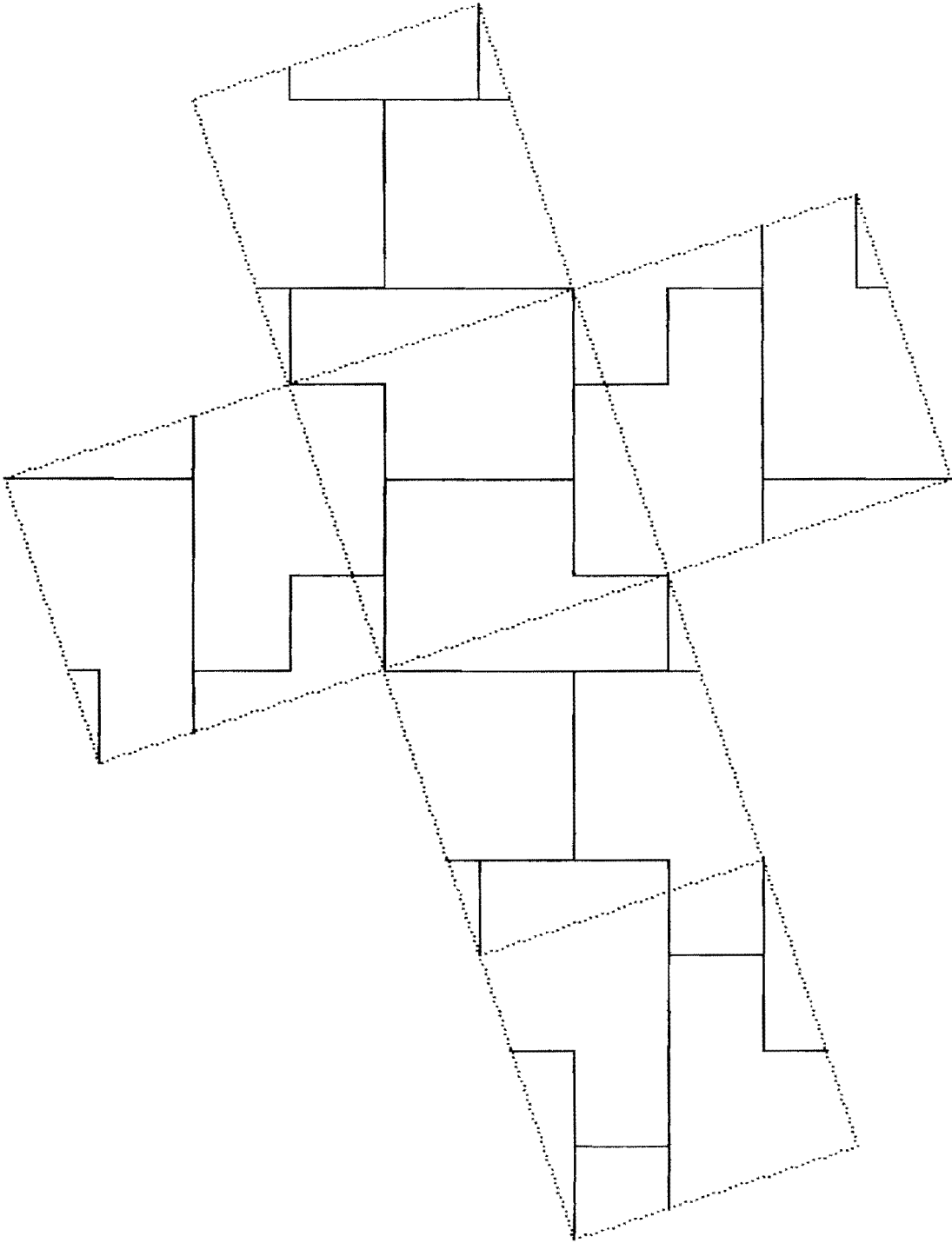
3170 3239 3258 3281 3303 3348 3361 3426 3435 3488 3508 3523

One axis order 2 (Z)



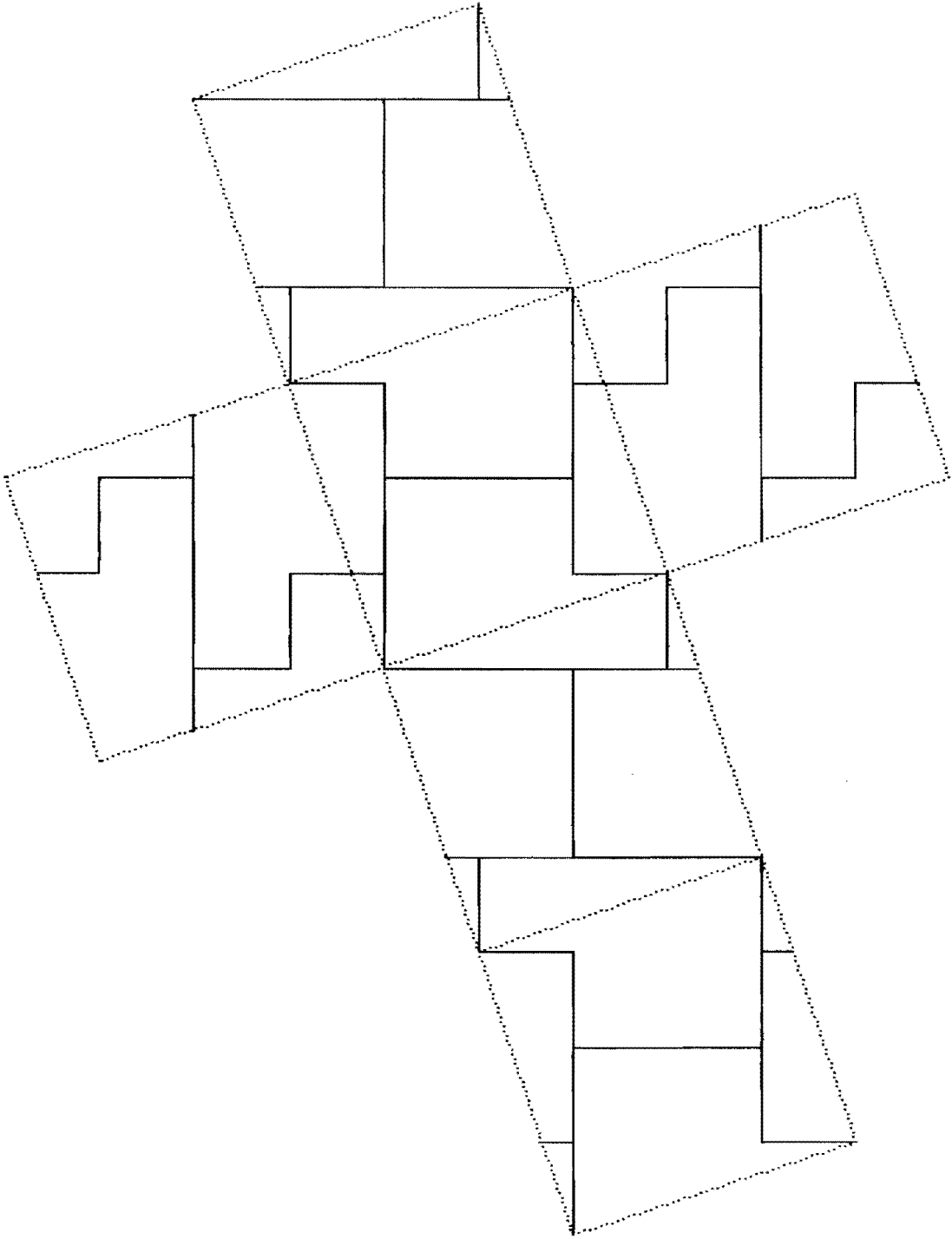
3170 3239 3258 3281 3304 3319 3375 3427 3431 3455 3506 3536

One axis order 2 (X)



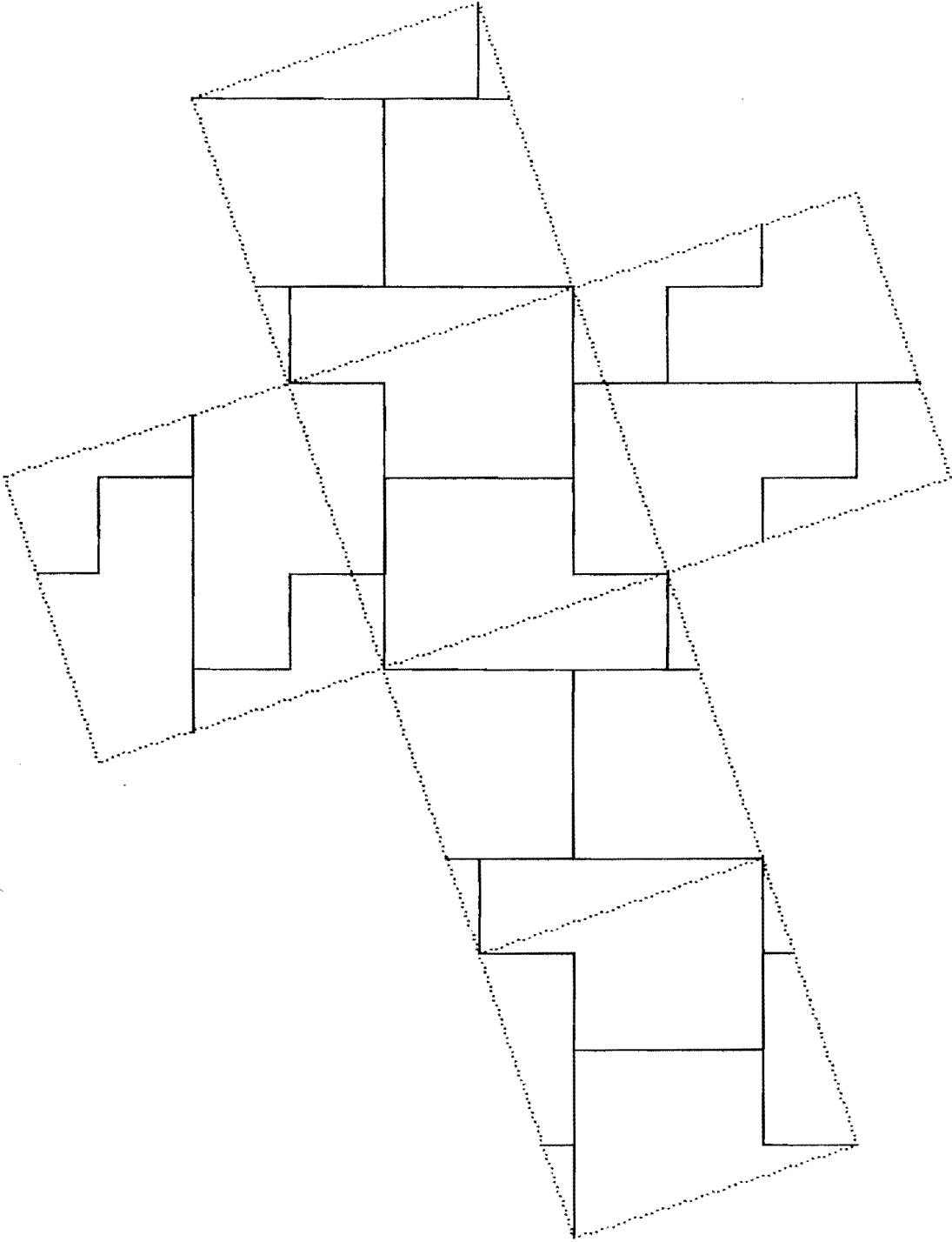
3170 3239 3258 3281 3304 3319 3375 3427 3433 3455 3482 3536

One axis order 2 (X)



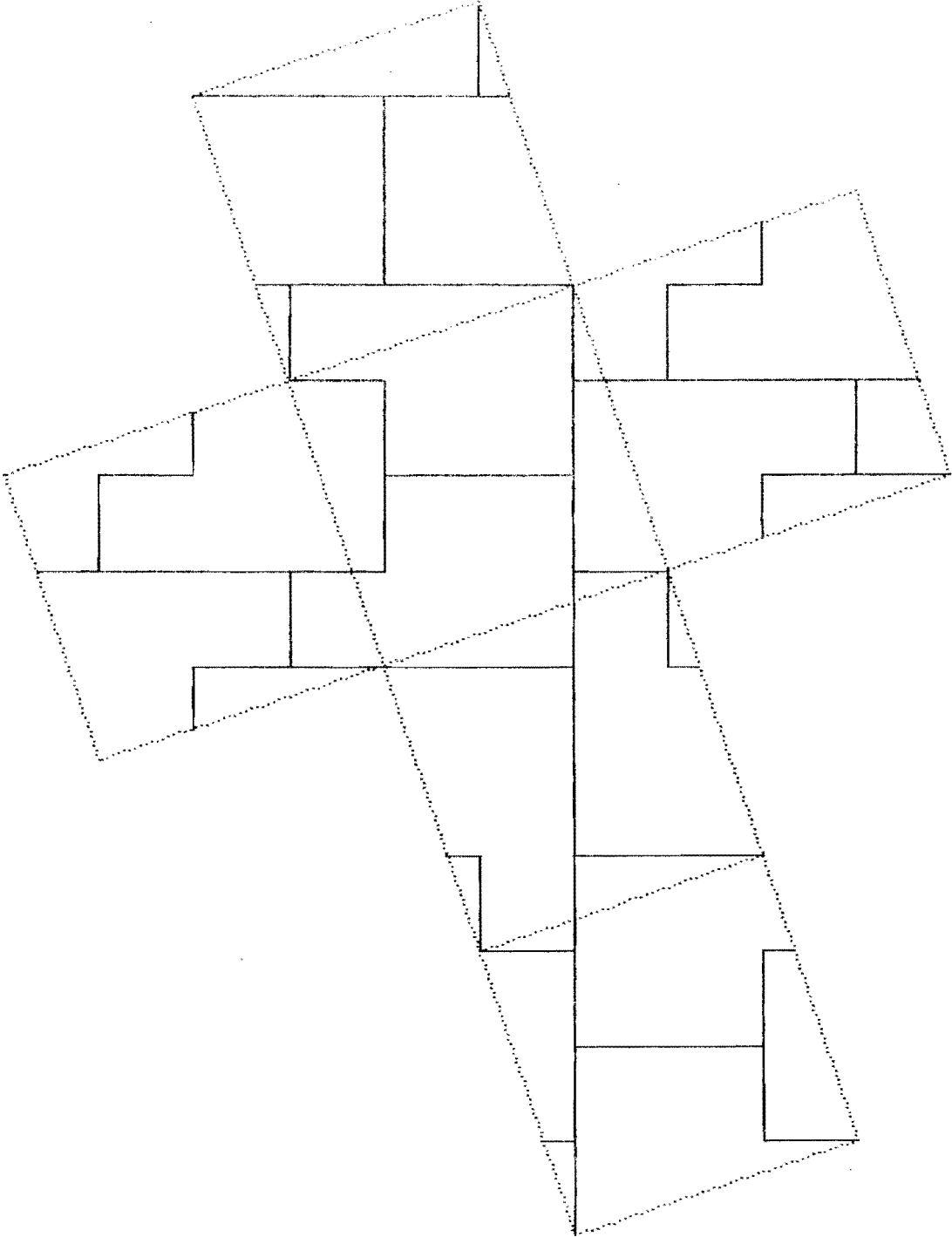
3170 3239 3258 3281 3304 3319 3376 3426 3430 3488 3508 3523

Three axes order 2 (X Y Z)



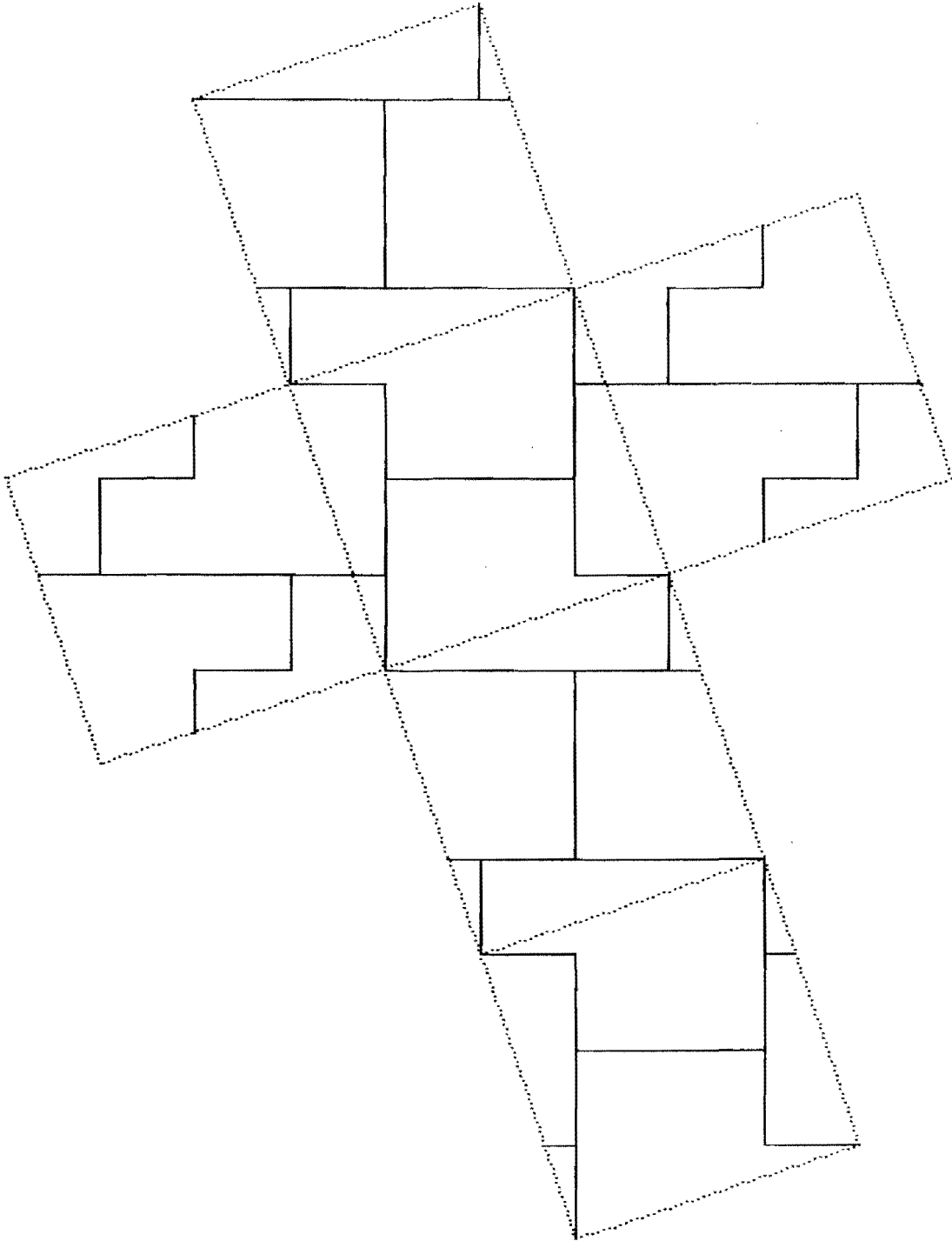
3170 3239 3258 3282 3304 3319 3376 3426 3430 3487 3508 3523

One axis order 2 (Y)



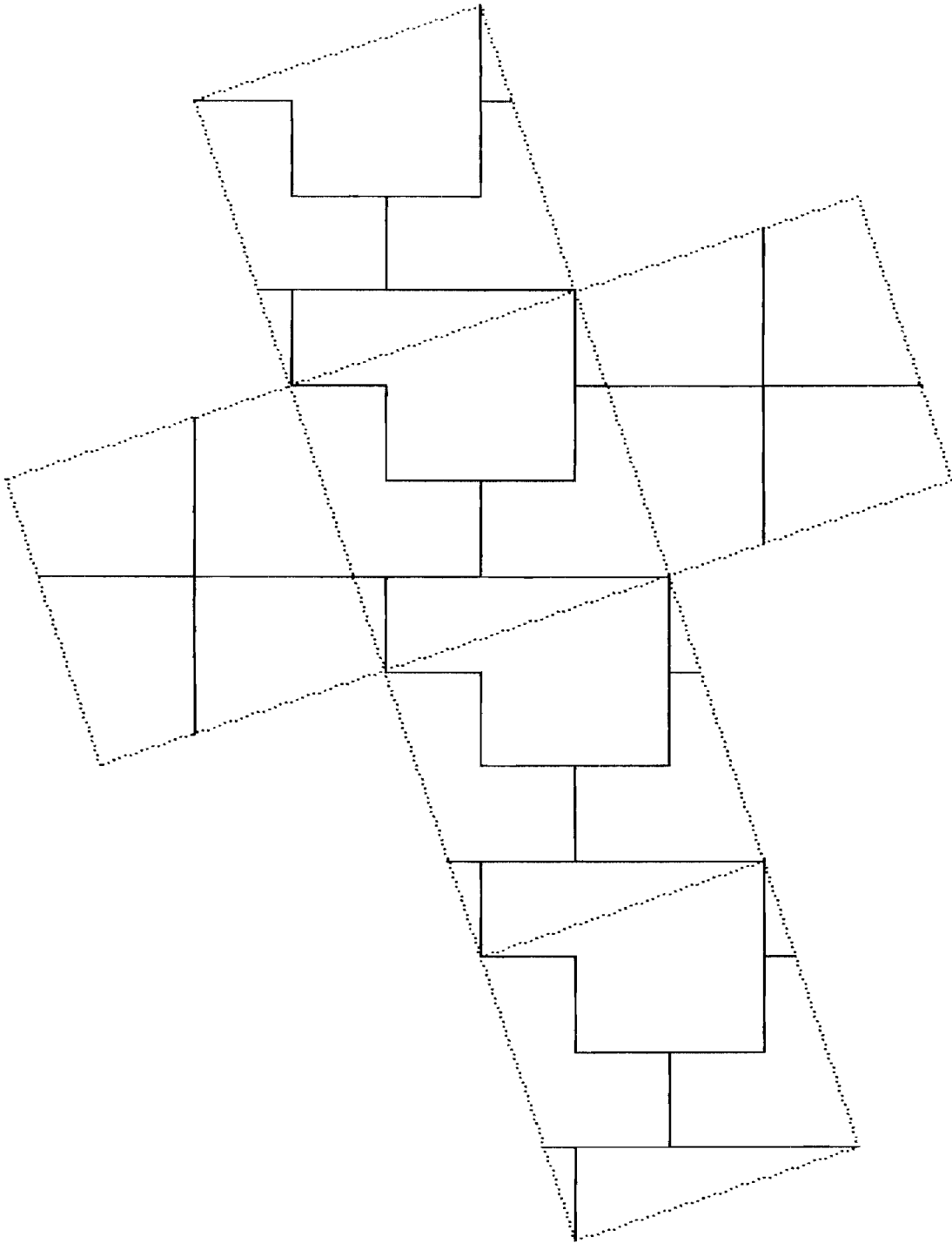
3170 3239 3259 3282 3303 3348 3361 3425 3435 3487 3508 3523

One axis order 2 (Z)

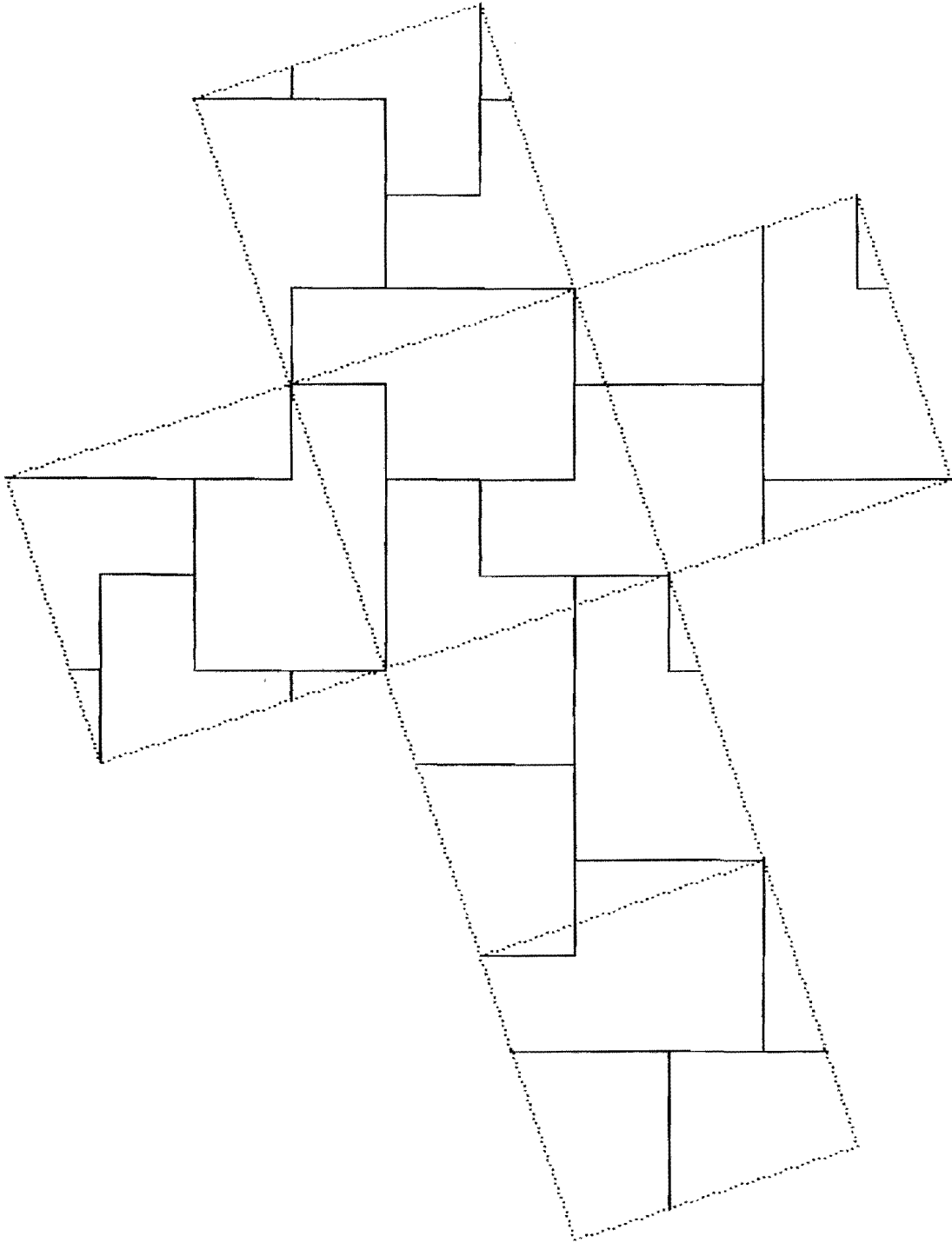


3170 3239 3259 3282 3304 3319 3376 3425 3430 3487 3508 3523

Three axes order 2 (X Y Z)

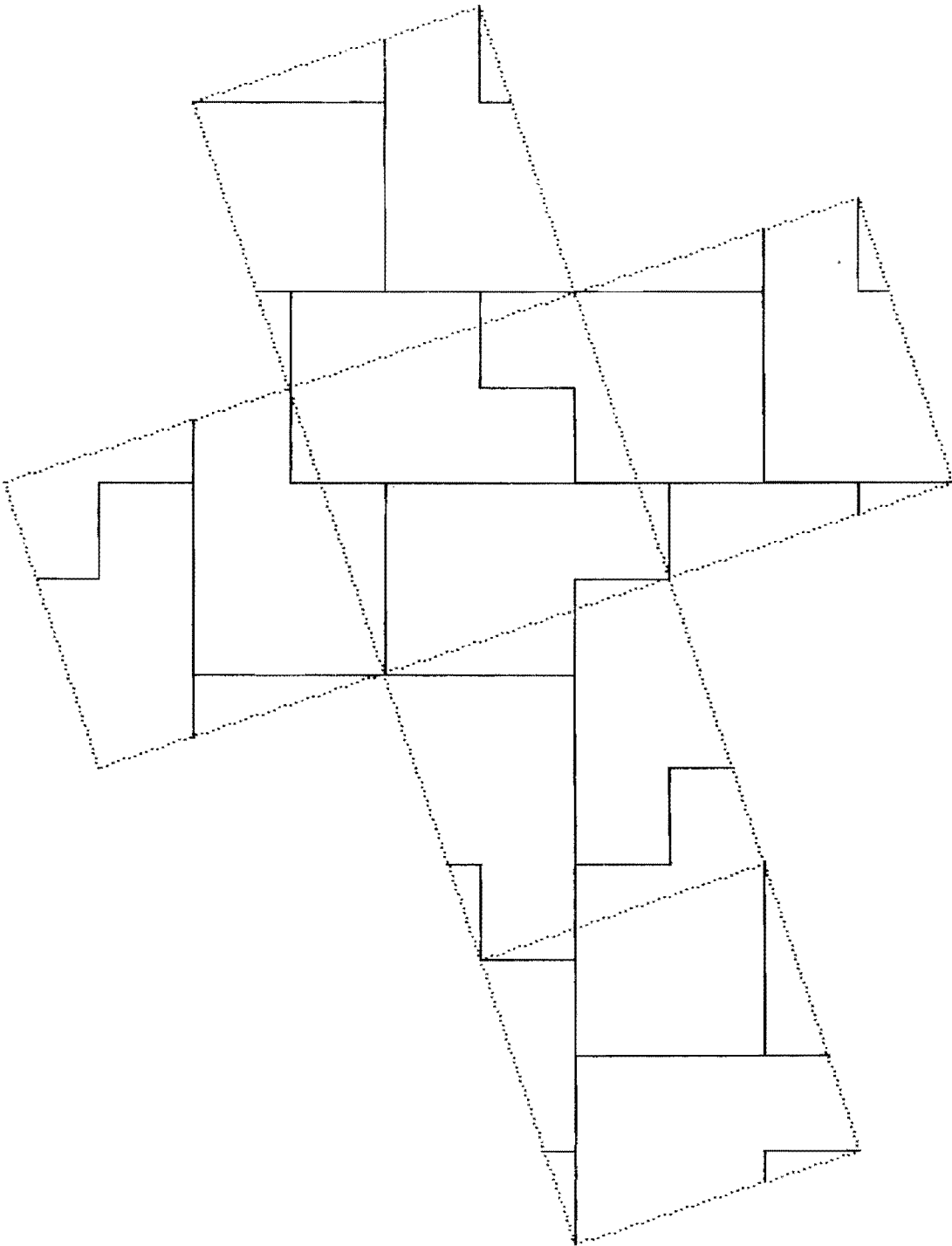


3170 3241 3254 3278 3321 3328 3382 3423 3430 3484 3525 3532



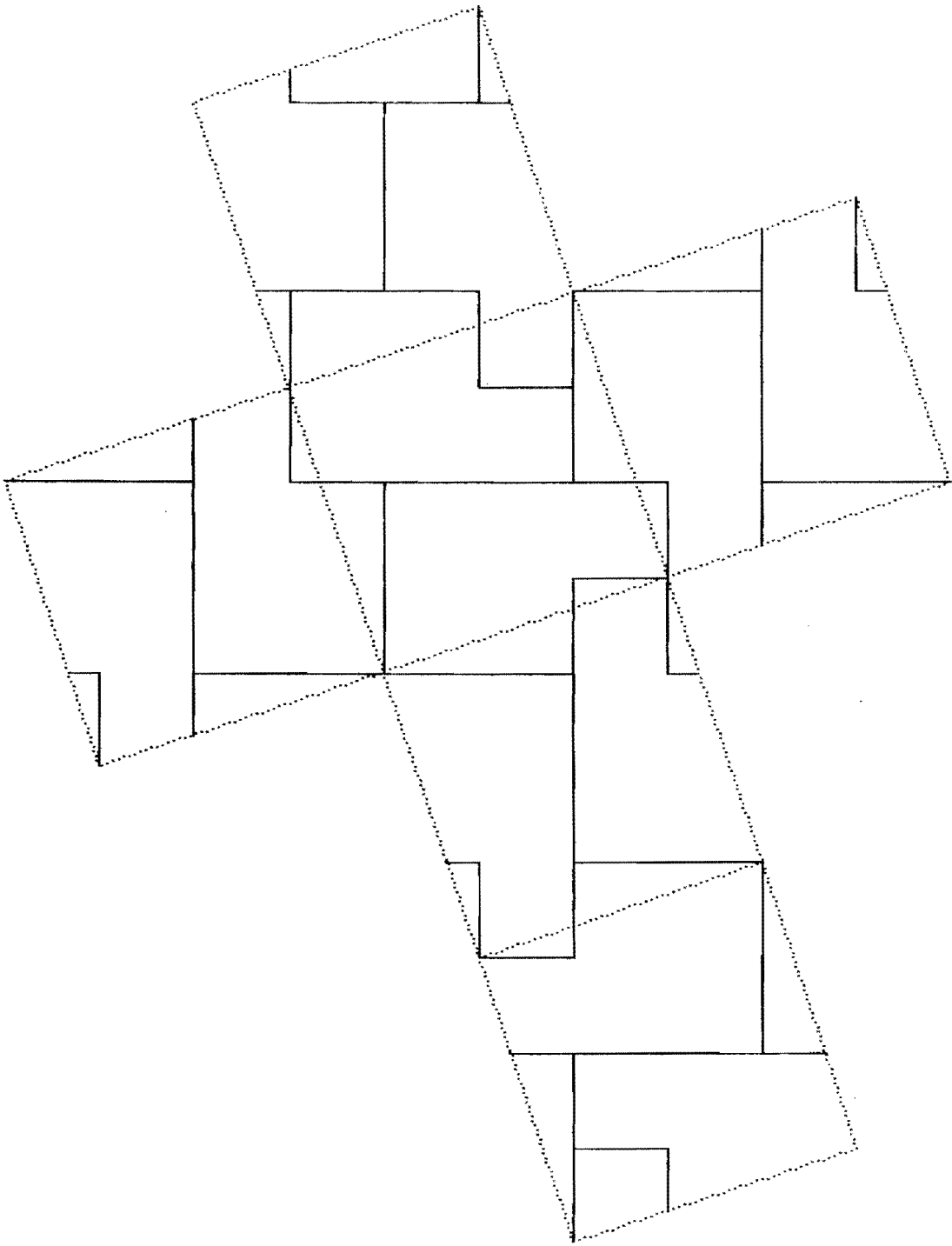
3170 3241 3256 3278 3307 3348 3394 3437 3455 3495 3515 3562

One axis order 3 (D1)



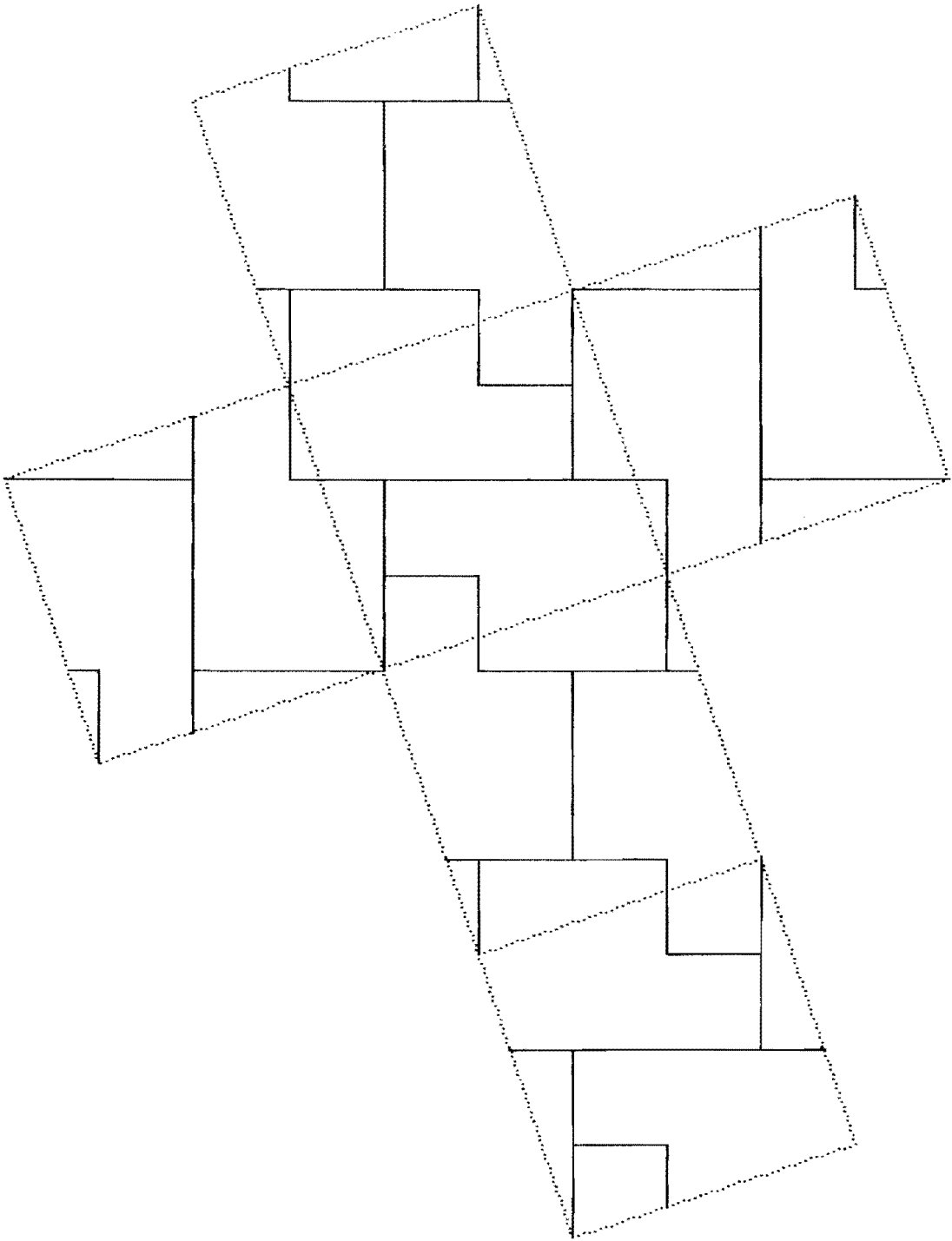
3173 3222 3296 3301 3342 3361 3416 3426 3455 3505 3523 3552

One axis order 3 (D4)



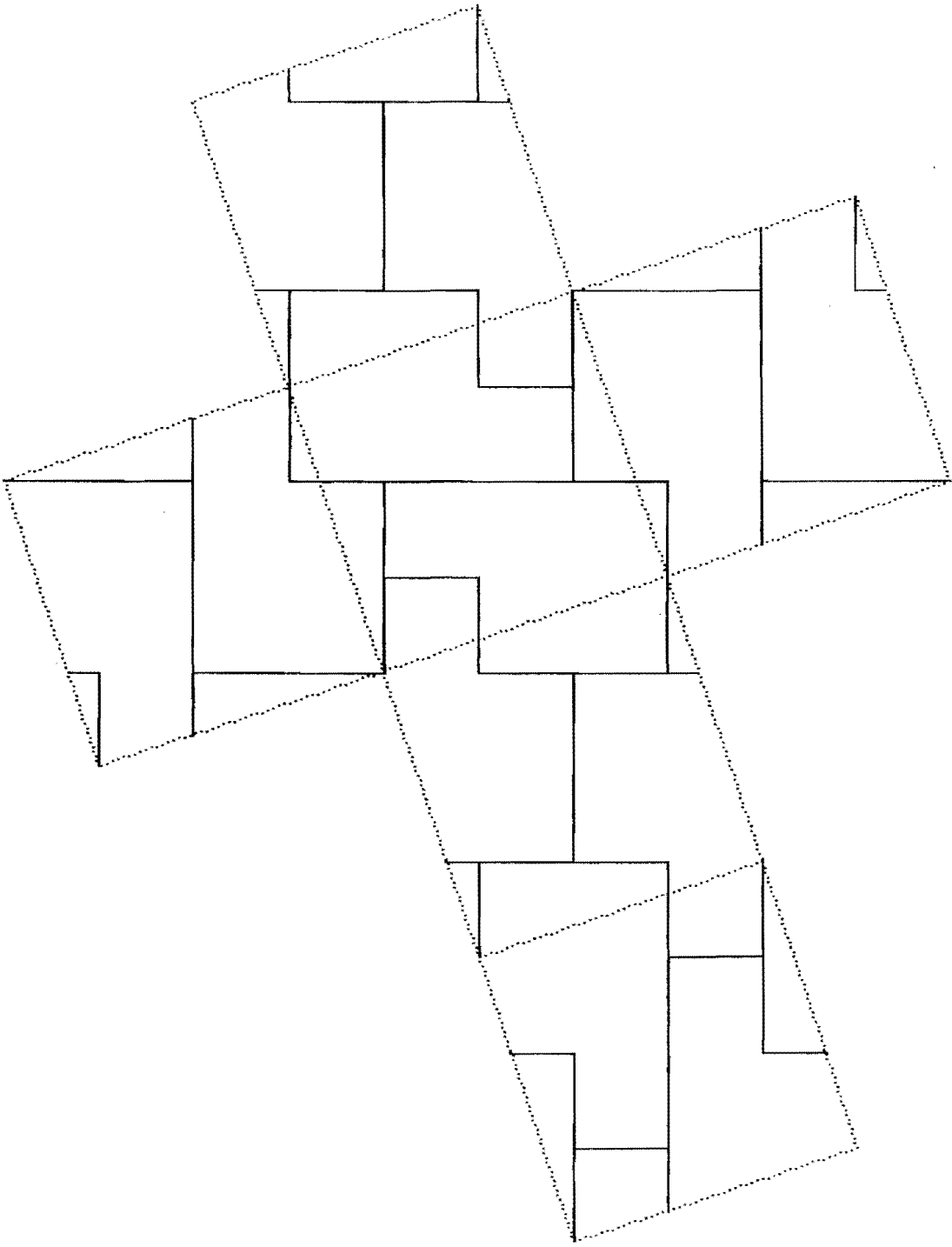
3173 3226 3231 3296 3301 3348 3361 3427 3437 3455 3506 3536

One axis order 2 (Z)



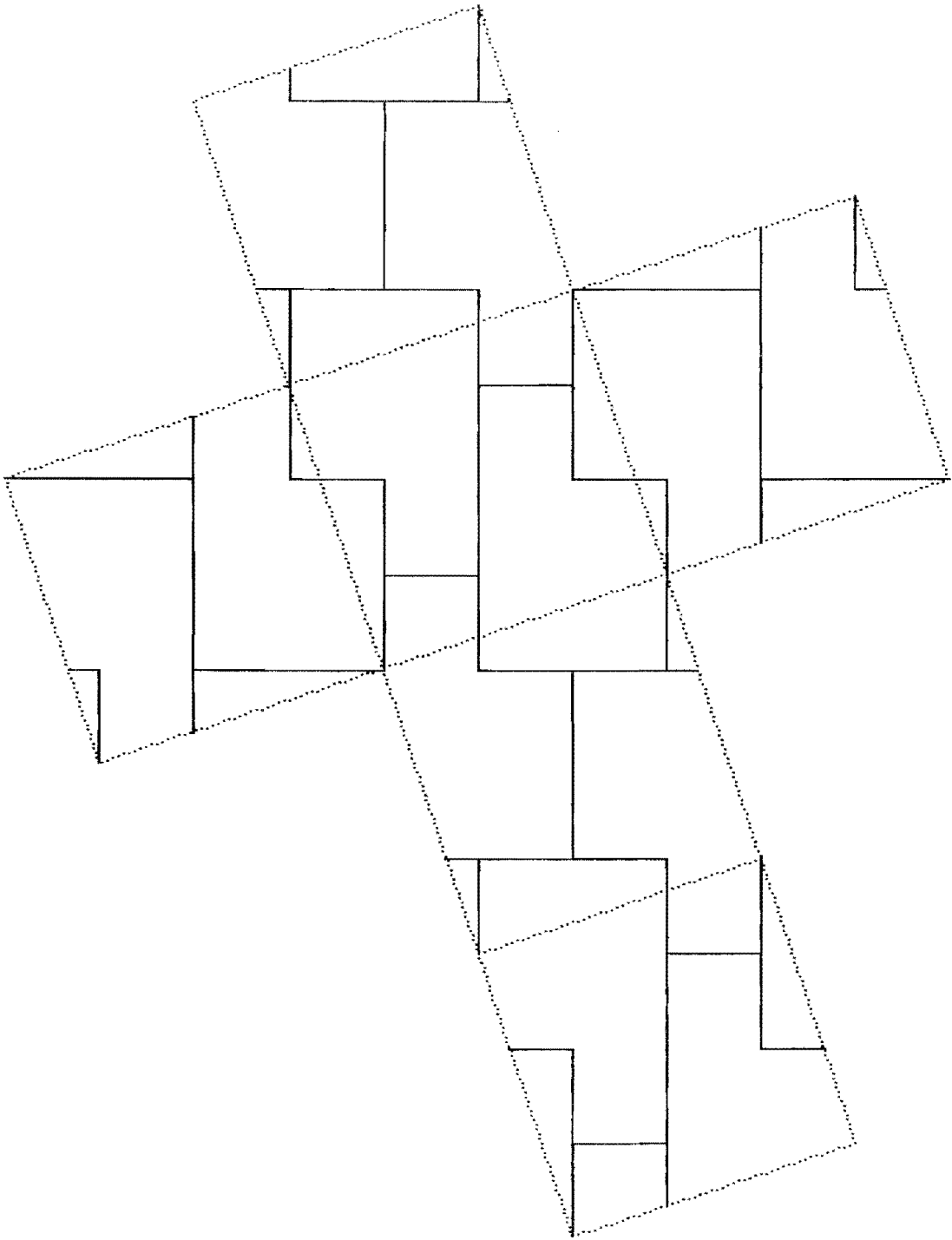
3173 3226 3231 3296 3302 3332 3375 3427 3431 3455 3506 3536

Three axes order 2 (X Y Z)



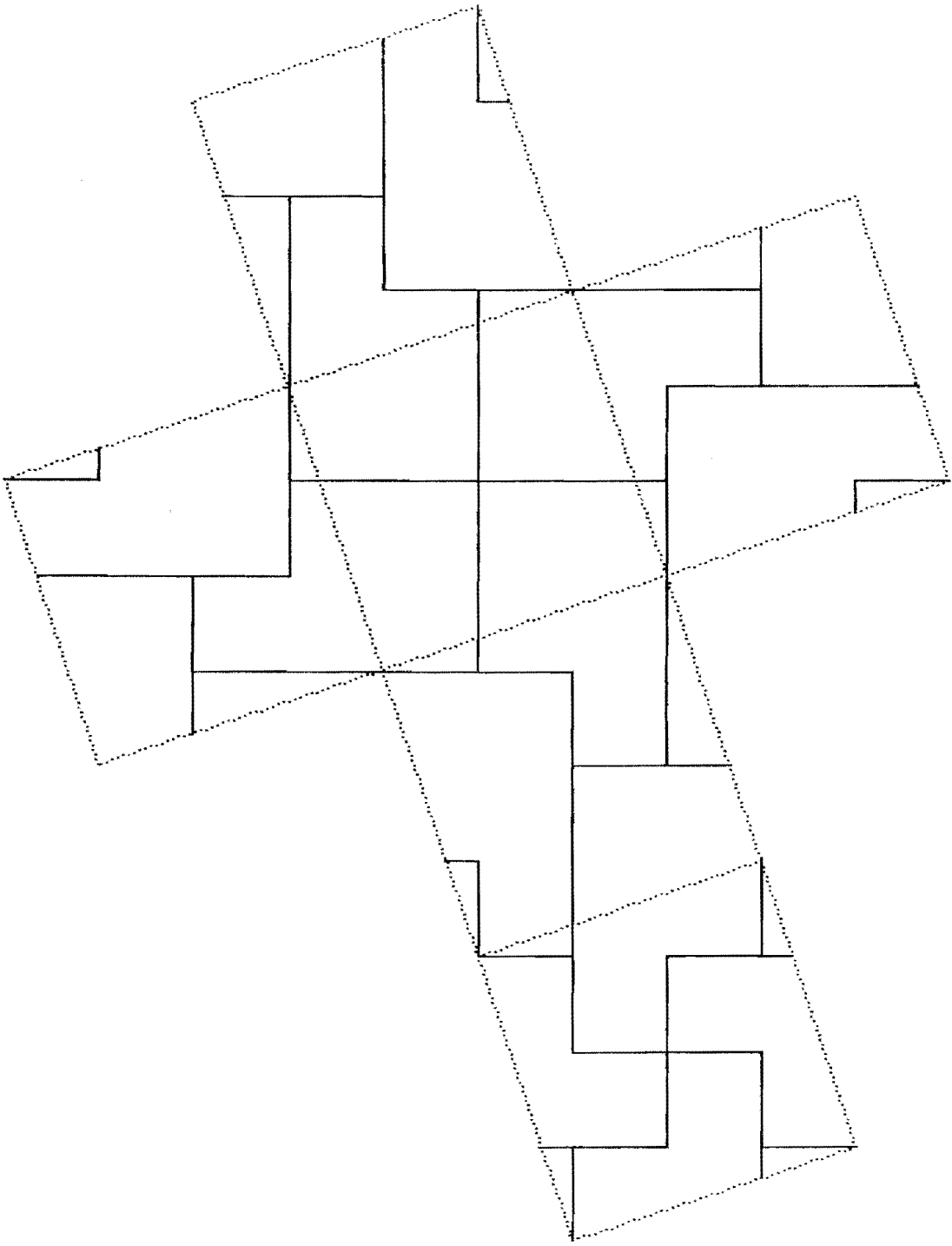
3173 3226 3231 3296 3302 3332 3375 3427 3433 3455 3482 3536

One axis order 2 (X)



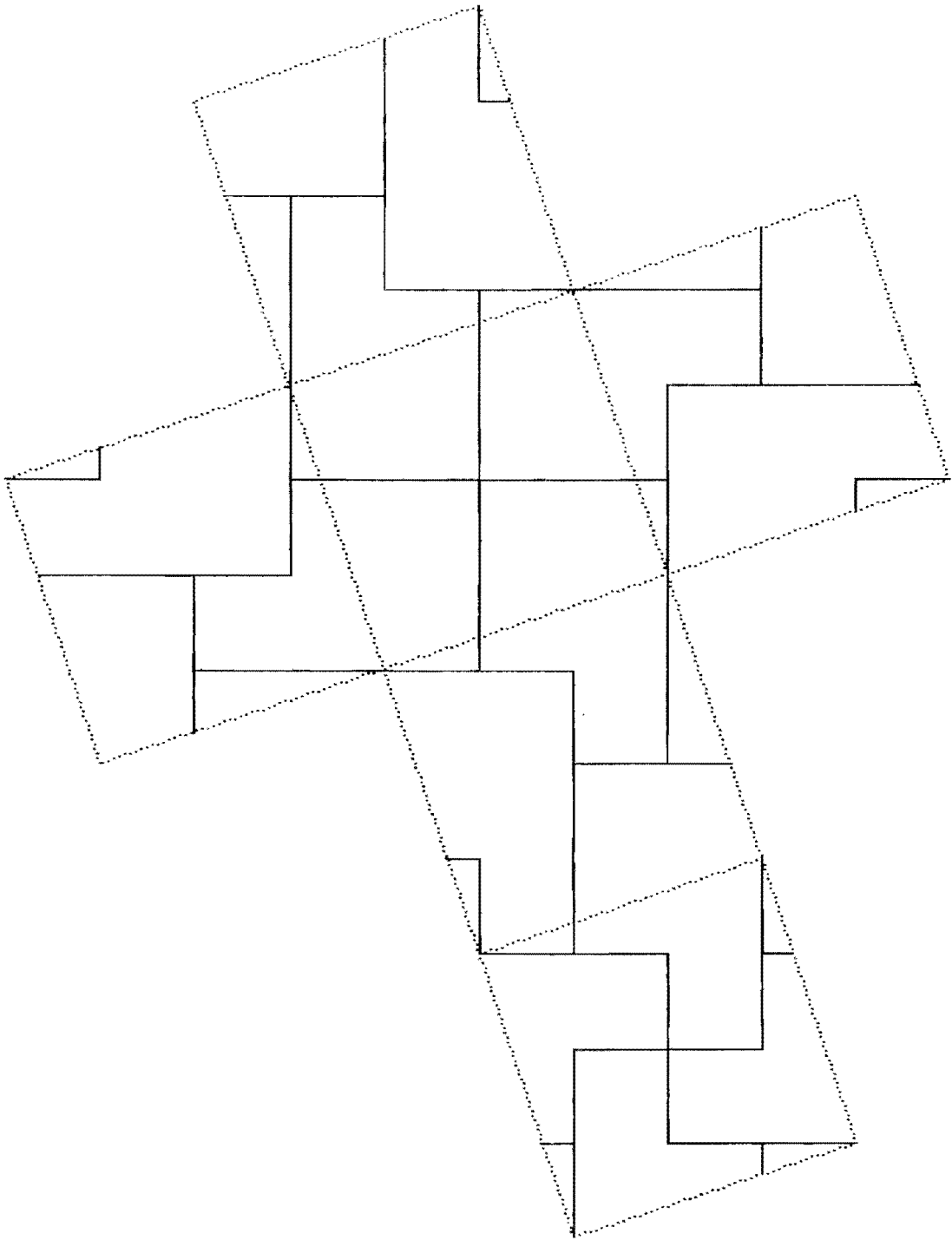
3175 3226 3231 3276 3296 3332 3375 3427 3433 3455 3482 3536

Three axes order 2 (X Y Z)



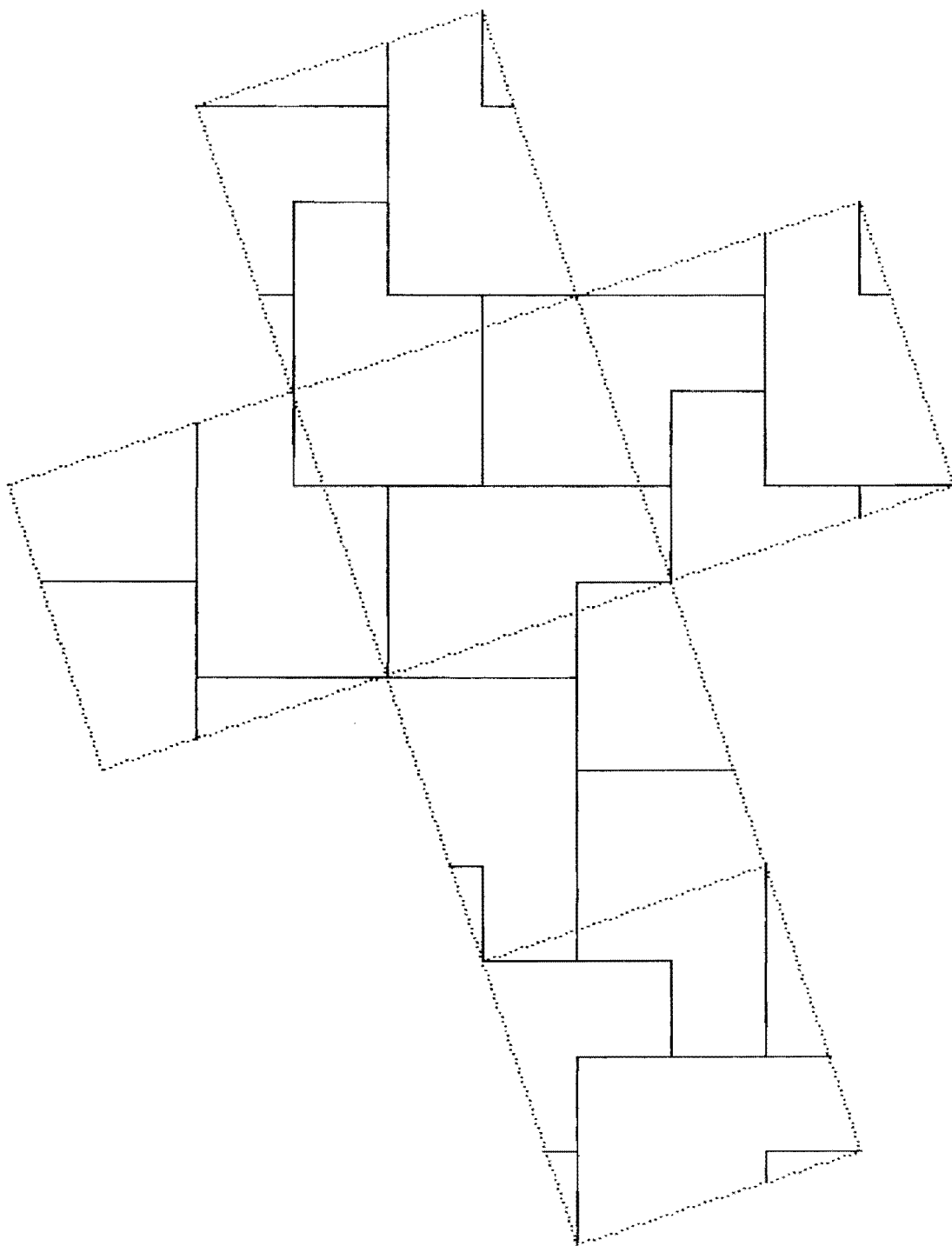
3176 3221 3290 3311 3353 3361 3412 3423 3480 3516 3529 3552

One axis order 4 (X)



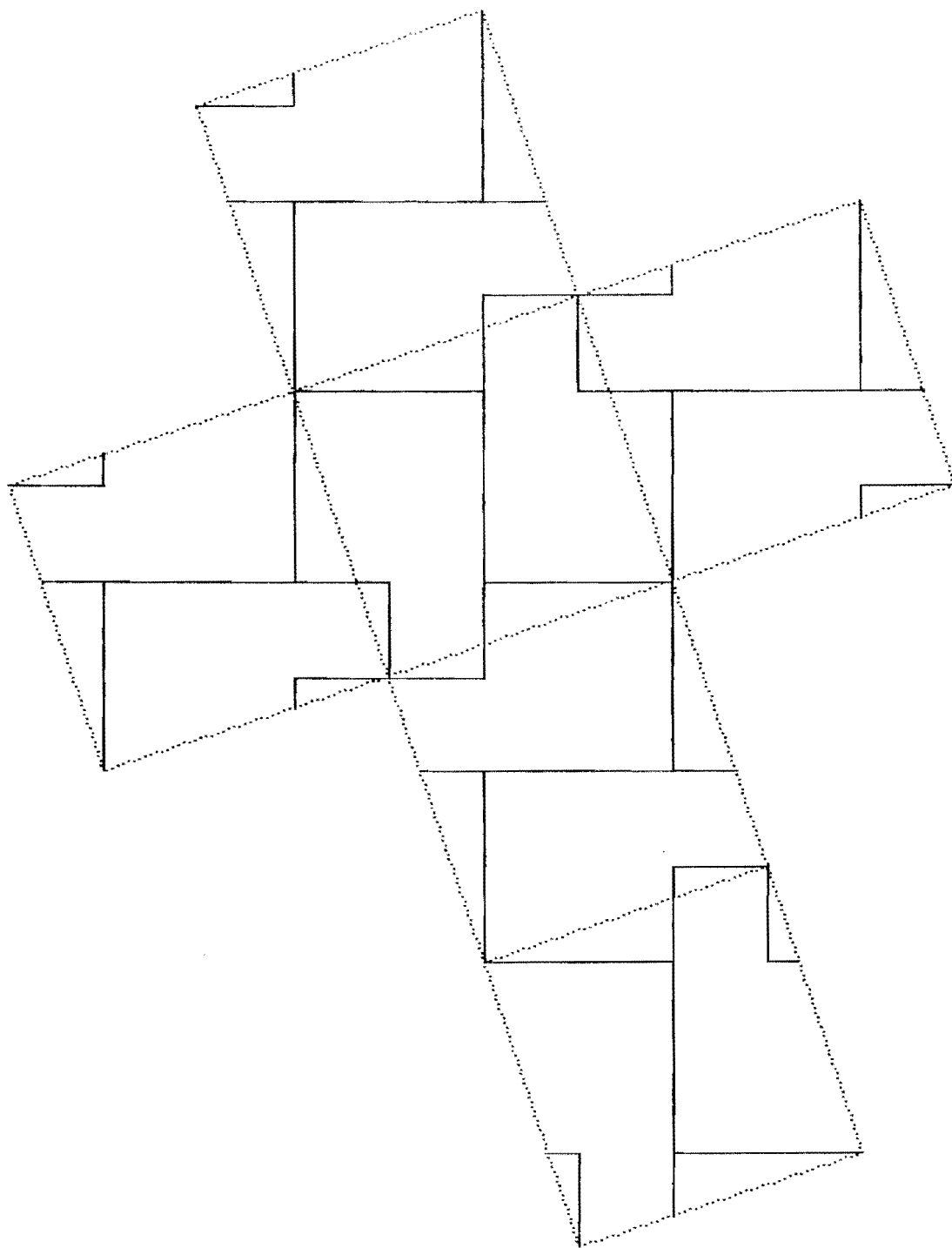
3176 3221 3290 3311 3353 3361 3413 3422 3484 3511 3529 3552

One axis order 4 (X)



3176 3221 3296 3301 3344 3361 3413 3422 3455 3505 3524 3552

One axis order 3 (D4)



3185 3225 3238 3249 3325 3335 3353 3403 3450 3463 3529 3539

Four axes order 3 (D1 D2 D3 D4) and three axes order 2 (X Y Z)