Computational results for the control of a divergent N-echelon inventory system

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Abstract

Consider a divergent multi-echelon inventory system, e.g., a distribution system or a production system. At every facility in the system orders are placed (or production is initiated) periodically. The order arrives after a fixed lead time. At the end of each period linear costs are incurred at each facility for holding inventory. Also, linear penalty costs are incurred at the most downstream facilities for backorders. The objective is to minimize the expected holding and penalty costs per period. Within a class of practically useful policies the decomposition result is used to develop an algorithm which determines the control parameters of a near cost-optimal replenishment policy. A simulation study of a divergent 3-echelon system reveals that this algorithm performs well.

Keywords: multi-echelon, inventory, allocation, rationing, divergent

1 Introduction

The research of multi-echelon models has gained importance over the last decade because integrated control of supply chains, consisting of a number of processing and distribution stages, has become feasible through modern information technology. Multi-echelon inventory systems provide a means of modeling such supply chains, thereby enabling quantitative analysis and characterization of optimal control policies (cf. Clark & Scarf [1960], Federgruen & Zipkin [1984], Rosling [1989] and Langenhoff & Zijm [1990]).

The start of research on multi-echelon inventory models is in generally allotted to Clark & Scarf [1960], who study an N-echelon serial system without lot sizing. They introduced the concept of echelon stock for a given stockpoint to prove that the optimal control policies for the N-echelon serial system with discounted penalty and holding costs, are characterized by N so-called echelon order-up-to-levels. The echelon stock of a stockpoint equals all stock at this stockpoint plus in transit to or on hand at any of its downstream stockpoints minus the backorders at its downstream stockpoints. Like Van Houtum & Zijm [1991a] and Zijm & Van Houtum [1994] we like to define the echelon inventory position of a stockpoint as its echelon stock plus all material in transfer to that stockpoint.

Although much attention has been given to divergent two-echelon systems, one seldom finds extensions to more general divergent N-echelon systems. In practice, however, large production and distribution networks are frequently encountered and therefore generalization of two-echelon policies is needed. In this paper we analyze a divergent N-echelon inventory system in which every stockpoint is allowed to hold stock. Every stockpoint places replenishment orders periodically. The order arrives after a fixed lead time, and then it is decided how much and in what way the stock is allocated among its successors. Only the unfilled demand at the end-stockpoints are backordered. Penalty costs proportional to the amount short at every end-stockpoint are incurred at the end of each period. Also holding costs proportional to the inventory on hand are incurred at the end of each period. The objective is to minimize the average costs per period on the long run.

This model can be regarded as an extension of Langenhoff & Zijm [1990] and Van Houtum & Zijm [1991b]. Langenhoff & Zijm [1990] prove exact decomposition results for a two-echelon assembly system, a two-echelon serial system and a divergent two-echelon system (which is more thoroughly analyzed in Van Houtum & Zijm [1991b]). Furthermore, Diks & De Kok [1996] prove exact decomposition results for the divergent N-echelon system given the *balance assumption*. Under this assumption the rationing rule always allocates non-negative stock quantities. In Eppen & Schrage [1981], Langenhoff & Zijm [1990] and De Kok, Lagodimos & Seidel [1994] similar assumptions are made. This balance assumption is *not* required if immediately after taking a rationing decision there is a sufficiently large 'demandless' period (e.g. week-end). Since such a period enables to transship products from the stockpoints with negative allocation quantities to those with positive allocation quantities.

Verrijdt & De Kok [1995] study a similar divergent N-echelon system, although, in their model no intermediate stock is allowed. The control parameters of the replenishment policy are determined so as to meet the pre-determined target service levels (fill rates) at the end-stockpoints. In this more 'service related' approach the main goal is to attain the target service-levels at the end-stockpoints (also see De Kok [1990] and Lagodimos [1992]), instead of the minimization of a cost-function. For an overview of most of these service related models we refer to Van der Heijden, Diks & De Kok [1996], who did a comparison study on the performance of most of these approaches.

The paper is organized as follows. In Section 2 we describe the model under consideration. In Section 3 we present an average cost analysis for the divergent N-echelon system. A near cost-optimal control policy within a class of practically useful policies is derived, given the balance assumption. In Section 4 we develop an algorithm to determine all the control parameters. This algorithm is based on the decomposition of the network. In Section 5 we present the results obtained by applying the algorithm on a 3-echelon system. These results are validated by a simulation study. For most instances the performance of the algorithm yields very good results. Finally in Section 6 we give a few concluding remarks.

2 Model description

Consider a discrete-time multi-echelon inventory system where every stockpoint is allowed to hold stock. The system has an *arborescent* structure, i.e., each location has a unique supplier. We refer to these kind of systems as divergent multi-echelon systems. Notice that a divergent multi-echelon system can be described by a directed graph (see for example Figure 1). The most upstream stockpoint can place orders at an external supplier which has an infinite capacity, which means that this supplier can always meet the demand.

The inventory in this multi-echelon system is controlled by periodic review policies. That is, every R periods the most upstream stockpoint, i say, issues a replenishment order. The replenishment order arrives after L_i periods, where L_i is a fixed, non-negative integer. Then the physical stock at stockpoint i (or part of it) is allocated immediately to its successors. There are two possibilities:

(*i*). The physical stock is sufficient to raise the echelon inventory position of each successor to its order-up-to-level. Then the required amounts are sent to the successors and excess stock is kept at stockpoint i to be allocated in the next occasion.

(*ii*). The physical stock is *not* sufficient to reach the order-up-to-levels. Then material rationing is required to allocate the available physical stock over its successors appropriately. For this purpose we introduce rationing functions.

A similar allocation procedure is applied at the other intermediate stockpoints when a replenishment order arrives.

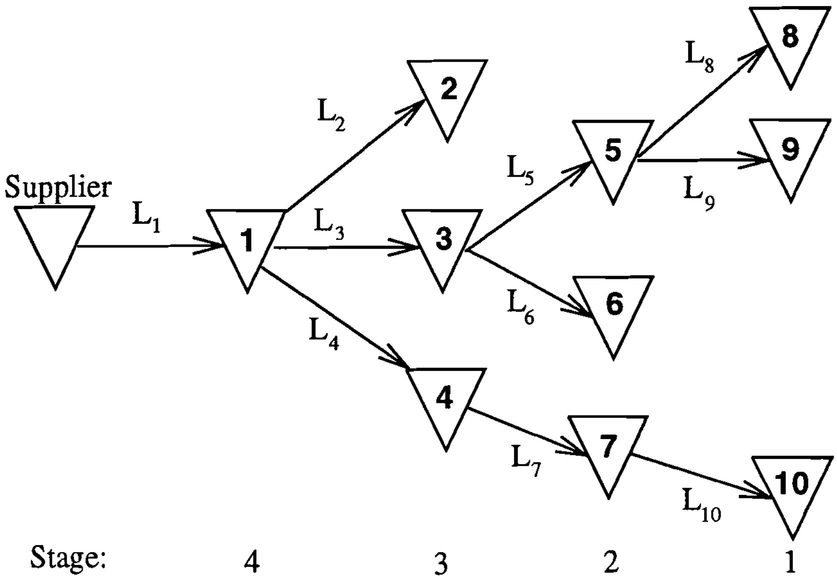


Figure 1: Schematic representation of a divergent 4-echelon inventory system.

Without loss of generality we assume that only the end-stockpoints face external customer demand. In case an intermediate stockpoint i faces external demand, we redirect this demand to a new successor j with lead time $L_j := 0$. By definition this successor j is an end-stockpoint. During one period the demand between end-stockpoints may be correlated, however, the demand in subsequent periods are i.i.d.. With respect to the customer demand process, we assume that all demand which cannot be satisfied immediately is backordered.

At the end of each period both penalty and holding costs are incurred. The penalty costs equals p_i for each backlogged product at end-stockpoint i. For a product at stockpoint i or in transfer to one of its successors the holding costs equals $h_i + \sum_{k \in U_i} h_k$, where U_i represents all stockpoints on the path from the supplier to i. Notice that h_i can be regarded as an additional holding cost due to value added in stockpoint i. No fixed ordering costs are assumed. Note that because all excess customer demand is backordered, linear variable ordering costs do not influence any control policy and can therefore be omitted. The objective of the analysis is to determine a cost-optimal replenishment policy, i.e., minimizing the expected total costs per period on the long run.

For clarity in the remainder of this paper we refer to the length of a review period as one period ($R := 1$). Furthermore, we introduce the following notation:

$ech(i)$	:=	Set of stockpoints that constitute the echelon of stockpoint i (e.g. $ech(5) = \{5, 8, 9\}$),
$pre(i)$	:=	Preceding stockpoint of stockpoint i (e.g. $pre(8) = 5$),
U_i	:=	Set of stockpoints on path from supplier to stockpoint i (e.g. $U_1 = \varnothing$ and $U_6 = \{1, 3\}$),
V_i	:=	All stockpoints which are supplied by i (e.g. $V_1 = \{2, 3, 4\}$),
E	:=	Set of all end-stockpoints (e.g. $E = \{2, 6, 8, 9, 10\}$),
I	:=	Set of all intermediate stockpoints (e.g. $I = \{1, 3, 4, 5, 7\}$),
N	:=	Number of stages in inventory system (e.g. $N = 4$).

The examples between the brackets refer to the situation of Figure 1.

3 Analysis

Diks & De Kok [1996] performed an average cost analysis of the above model. They proved that decomposition is exact. In order to explain this decomposition properly we first elaborate on the system dynamics, and introduce some additional notation. Suppose at the beginning of a review period a stockpoint i has an echelon stock of y products. All stockpoints $j \in V_i$ want to raise their echelon inventory position to y_j. If $y \geq \sum_{j \in V_i} y_j$ then the echelon inventory position of stockpoint j just after rationing yields y_j, and the remainder $y - \sum_{j \in V_i} y_j$ is retained at stockpoint i. However, if $y < \sum_{j \in V_i} y_j$ we have to deal with one of the main difficulties of divergent multi-echelon systems: How should stockpoint i ration the available stock over its successors? To overcome this problem we define a rationing function $z_i^j[y]$. This means that the rationing policy allocates $z_i^j[y]$ to echelon j, and no products are retained at stockpoint i. Decomposition of the network yields that the order-up-to-level at stockpoint i and the rationing functions to its successors are determined so as to minimize the cost-function $D_i(x, \Psi_i)$. This cost-function represents the expected total costs in $ech(i)$ at the end of an arbitrary period given that the order-up-to-level at stockpoint i equals x and its downstream control parameters are given by Ψ_i. By definition $\Psi_i := \bigcup_{j \in V_i} (z_i^j, y_j, \Psi_j)$ for an intermediate stockpoint i, and $\Psi_i := \varnothing$ for an end-stockpoint i. In Diks & De Kok [1996] it is shown that

Theorem 3.1. *If $i \in E$,*

$$D_i(x) = h_i(x - (L_i + 1)\mu_{ech(i)}) + \int_x^\infty (h_i + \sum_{j \in U_i} h_j + p_i)(u - x) dF_{L_i+1}^{ech(i)}(u), \tag{1a}$$

if $i \in I$,

$$D_i(x, \Psi_i) = h_i(x - (L_i + 1)\mu_{ech(i)}) + \sum_{j \in V_i} \Big[D_j(y_j, \Psi_j) + \tag{1b}$$

$$\int_{x - \sum_{j \in V_i} y_j}^\infty D_j(z_i^j[x - u], \Psi_j) - D_j(y_j, \Psi_j) dF_{L_i}^{ech(i)}(u) \Big],$$

where $F_L^{ech(i)}$ represents the cdf of demand at all end-stockpoints in $ech(i)$ during L periods (if $L = 1$ we suppress the index). Notice that if $\Psi_i = \varnothing$ then Ψ_i is omitted. □

In Section 3.1 we briefly overview the most important results of Diks & De Kok [1996]. These results are needed for the determination of appropriate linear rationing functions in Section 3.2.

3.1 Optimal rationing functions

In Diks & De Kok [1996] necessary conditions and properties of an optimal set of rationing functions are derived.

Theorem 3.2. *Necessary conditions for an optimal set of rationing functions $\{\hat{z}_i^j\}_{j \in V_i}$ are*
(i)

$$\sum_{j \in V_i} \hat{z}_i^j[x] = x.$$

(ii) For every successor j of stockpoint i:

$$\left. \frac{\partial D_j(y, \Psi_j)}{\partial y} \right|_{y = \hat{z}_i^j[x]} = \lambda_i[x].$$

□

As a result of these necessary conditions some interesting properties can be derived.

Corollary 3.1. *For every optimal rationing-function* $\hat{z}_i^j[x]$ *holds*

(i)

$$\frac{d\hat{z}_i^j[x]}{dx} \geq 0.$$

(ii)

$$\hat{z}_i^j[\sum_{j \in V_i} \hat{y}_j] = \hat{y}_j \quad \textit{with} \quad \hat{y}_j := \text{argmin}\left\{y | \partial D_j(y, \Psi_j)/\partial y = 0\right\}$$

□

Finally, we state Theorem 3.3 which enables us to simplify the expressions derived in the next section considerably.

Theorem 3.3. *Let* $\alpha_k^i(y)$ *denotes the non-stock out probability of an end-stockpoint k in a divergent echelon system, in which the most upstream stockpoint i uses an order-up-to-policy with order-up-to-level y. If for every rationing function in stockpoint i holds* $z_i^j[\sum_{j \in V_i} y_j] = y_j$, *then for* $j \in V_i$ *and* $k \in E \cap ech(j)$:

$$\alpha_k^i(y) = \begin{cases} F_{L_k+1}^{ech(k)}(y) & i \in E, \\ \int_0^\infty \alpha_k^j(z_i^j[y-u])\, dF_{L_i}^{ech(i)}(u) & i \in I,\ y < \sum_{j \in V_i} y_j \\ \int_0^\infty \alpha_k^j(z_i^j[\sum_{j \in V_i} y_j - u])\, d\left(F_{L_i}^{ech(i)}\right)^{y - \sum_{j \in V_i} y_j}(u) & i \in I,\ y \geq \sum_{j \in V_i} y_j, \end{cases} \tag{2}$$

where $\left(F_L^{ech(i)}\right)^\Delta(x)$ *equals 0 for* $x < 0$, *and equals* $F_L^{ech(i)}(x + \Delta)$ *for* $x \geq 0$. □

3.2 Linear rationing functions

Diks & De Kok [1996] proved that the decomposition approach yields the optimal replenishment policy, given the balance assumption. For practical purposes, however, it is rather cumbersome to determine the optimal rationing functions. In order to keep the analysis tractable we restrict to linear rationing functions in this paper, i.e.,

$$z_i^j[x] = q_i^j x + c_i^j. \tag{3}$$

We like to emphasize that many of the rationing policies used in the literature are linear. For instance, the well-known Fair-Share rationing policy of Eppen & Schrage [1981] for a two-echelon system defines

$$q_{pre(j)}^j = \frac{\sigma_j}{\sum_{k \in E} \sigma_k}, \qquad c_{pre(j)}^j = 0 \qquad \text{for} \quad j \in E.$$

And, the more general Consistent Appropriate Share (CAS) rationing policy of De Kok, Lagodimos & Seidel [1994] for a two-echelon system:

$$q_{pre(j)}^j = \frac{y_j - (L_j+1)\mu_j}{\sum_{k \in E}(y_k - (L_k+1)\mu_k)}, \qquad c_{pre(j)}^j = (L_j+1)\mu_j - q_{pre(j)}^j \sum_{k \in E}(L_k+1)\mu_k \qquad \text{for} \quad j \in E.$$

In the remainder of this paper we refer to q_i^j as the so-called *allocation-fractions.*

In this paper $\{q_i^j\}$ and $\{c_i^j\}$ are defined such that the linear rationing functions $\{z_i^j\}$ have as many similar properties as the optimal rationing functions. Hence, from Theorem 3.2*(i)* we have $\sum_{j \in V_i} q_i^j = 1$. From Corollary 3.1*(i)* it follows $q_i^j \geq 0$. If $q_i^j = 0$ from eq. (3) it follows that $z_i^j[x]$ equals a fixed value, *independent* of x. This does not at all coincide with the behavior of an optimal rationing function, therefore we require $q_i^j > 0$. Finally, from Corollary 3.1*(ii)* we define $c_i^j := \hat{y}_j - q_i^j(\sum_{k\in V_i}\hat{y}_k - x)$. The aforementioned definitions implies

$$z_i^j[x] = \hat{y}_j - q_i^j\left(\sum_{k\in V_i}\hat{y}_k - x\right) \quad \text{with} \quad \sum_{j\in V_i} q_i^j = 1, \quad q_i^j > 0. \tag{4}$$

This rationing function has already been used in several papers (cf. Van Houtum [1990] and Van der Heijden [1996]). Recall that $\hat{y}_i$ denotes the optimal order-up-to-level at stockpoint i **given** all its downstream control parameters, denoted by Ψ_i. Furthermore, since the considered rationing functions are linear:

$$\Psi_i = \bigcup_{j\in V_i}(z_i^j, y_j, \Psi_j) \overset{(4)}{\equiv} \bigcup_{j\in V_i}(q_i^j, y_j, \Psi_j).$$

We refer to $\hat{\Psi}_i$ as *quasi-optimal* if for every stockpoint j in $ech(i)$ holds $\partial D_j(x, \hat{\Psi}_j)/\partial x = 0$.

In order to determine a quasi-optimal replenishment policy we need a tractable expression to evaluate $\partial D_j(x, \hat{\Psi}_j)/\partial x$. From Theorem 3.1, 3.3 and the definition of quasi-optimality we prove the next theorem.

Theorem 3.4. *For every end-stockpoint i:*

$$\frac{\partial D_i(y)}{\partial y} = h_i - (h_i + \sum_{j\in U_i} h_j + p_i)(1 - \alpha_i^i(y)). \tag{5a}$$

For an intermediate stockpoint $i \in W_n$ with quasi-optimal $\hat{\Psi}_i$:

$$\begin{aligned}\frac{\partial D_i(y, \hat{\Psi}_i)}{\partial y} =& h_i + \sum_{i_{n-1}\in V_i} q_i^{i_{n-1}}[h_{i_{n-1}} + \sum_{i_{n-2}\in V_{i_{n-1}}} q_{i_{n-1}}^{i_{n-2}}[\ldots + \\ &\sum_{i_1\in V_{i_2}} q_{i_2}^{i_1}[h_{i_1} - (h_{i_1} + \sum_{j\in U_{i_1}} h_j + p_{i_1})(1 - \alpha_{i_1}^i(y))]\ldots]].\end{aligned} \tag{5b}$$

Proof. The proof is by induction on i. If i is an end-stockpoint equation $(5a)$ immediately follows after differentiating $(1a)$ to x. From $(1b)$ it follows

$$\begin{aligned}\frac{\partial D_i(x, \hat{\Psi}_i)}{\partial x} &= \frac{\partial}{\partial x}\Bigg\{h_i(x - (L_i + 1)\mu_{ech(i)}) + \sum_{j\in V_i} D_j(y_j, \hat{\Psi}_j) + \\ &\qquad \int_{x-\sum_{j\in V_i}\hat{y}_j}^{\infty} D_j(z_i^j[x-u], \hat{\Psi}_j) - D_j(y_j, \hat{\Psi}_j)\, dF_{L_i}^{ech(i)}(u)\Bigg\} \\ &= h_i + \sum_{j\in V_i}\int_{x-\sum_{j\in V_i}\hat{y}_j}^{\infty} \left.\frac{dz_i^j[y]}{dy}\right|_{y=x-u} \left.\frac{\partial D_j(y, \hat{\Psi}_j)}{\partial y}\right|_{y=z_i^j[x-u]} dF_{L_i}^{ech(i)}(u).\end{aligned}$$

Differentiation of $z_i^j[y]$ to y yields q_i^j. Hence,

$$\frac{\partial D_i(x, \hat{\Psi}_i)}{\partial x} = h_i + \sum_{j\in V_i} q_i^j \int_{x-\sum_{j\in V_i}\hat{y}_j}^{\infty} \left.\frac{\partial D_j(y, \hat{\Psi}_j)}{\partial y}\right|_{y=z_i^j[x-u]} dF_{L_i}^{ech(i)}(u).$$

We distinguish between $x < \sum_{j\in V_i}\hat{y}_j$ and $x \geq \sum_{j\in V_i}\hat{y}_j$. In the former case we obtain

$$\frac{\partial D_i(x, \hat{\Psi}_i)}{\partial x} = h_i + \sum_{j\in V_i} q_i^j \int_0^{\infty} \left.\frac{\partial D_j(y, \hat{\Psi}_j)}{\partial y}\right|_{y=z_i^j[x-u]} dF_{L_i}^{ech(i)}(u).$$

In the latter case we obtain

$$\frac{\partial D_i(x,\hat{\Psi}_i)}{\partial x} = h_i + \sum_{j\in V_i} q_i^j \int_0^\infty \left.\frac{\partial D_j(y,\hat{\Psi}_j)}{\partial y}\right|_{y=z_i^j[\sum_{j\in V_i}\hat{y}_j - u]} d\left(F_{L_i}^{ech(i)}\right)^{x-\sum_{j\in V_i}\hat{y}_j}(u),$$

where $\left(F_L^{ech(i)}\right)^\Delta(x)$ equals 0 for $x<0$, and equals $F_L^{ech(i)}(x+\Delta)$ for $x\geq 0$.
In order to prove the above result we notice that

$$\left.\frac{\partial D_j(y,\hat{\Psi}_j)}{\partial y}\right|_{y=z_i^j[\sum_{j\in V_i}\hat{y}_j]} = \left.\frac{\partial D_j(y,\hat{\Psi}_j)}{\partial y}\right|_{y=\hat{y}_j} = 0.$$

To complete this proof we use induction and Theorem 3.3. □

From (5) we can show the following corollary.

Corollary 3.2. *The cost-function $D_i(y,\hat{\Psi}_i)$ with quasi-optimal policy $\hat{\Psi}_i$ is convex in y. Specifically, if $F^{ech(i)}(x)$ is strictly increasing for $x\geq 0$ then $D_i(y,\hat{\Psi}_i)$ is strict convex in y.*

Proof. Taking the derivative of (5) to y_i yields

$$\frac{\partial^2 D_i(y,\hat{\Psi}_i)}{\partial y^2} = \begin{cases} (h_i + \sum_{j\in U_i} h_j + p_i)\dfrac{d\alpha_i^i(y)}{dy} & i\in E \\ \sum_{i_{n-1}\in V_i} q_i^{i_{n-1}} \sum_{i_{n-2}\in V_{i_{n-1}}} q_{i_{n-1}}^{i_{n-2}} \cdots \sum_{i_1\in V_{i_2}} q_{i_2}^{i_1}(h_{i_1} + \sum_{j\in U_{i_1}} h_j + p_{i_1})\dfrac{d\alpha_{i_1}^i(y)}{dy} & i\in I. \end{cases}$$

Using the monotonicity of the non-stock out probability completes the proof. □

Consider a divergent multi-echelon system with positive penalty costs at the end-stockpoints. We conclude that if h_i is positive then there exist a finite y such that $D_i(y,\hat{\Psi}_i)$ with quasi-optimal $\hat{\Psi}_i$ is minimized for y. Specifically, if $F^{ech(i)}(x)$ is strictly increasing for $x\geq 0$ the unicity of this minimum is guaranteed. These results follows from Corollary 3.2 and the observation that when y tends to minus infinity $D_i(y,\hat{\Psi}_i)$ converges to $-\sum_{j\in U_i} h_j - \sum_{i_{n-1}\in V_i} q_i^{i_{n-1}} \sum_{i_{n-2}\in V_{i_{n-1}}} q_{i_{n-1}}^{i_{n-2}} \cdots \sum_{i_1\in V_{i_2}} q_{i_2}^{i_1} p_{i_1} < 0$, and when y tends to infinity $D_i(y,\hat{\Psi}_i)$ converges to h_i. Hence, if h_i equals 0 the minimum is attained in infinity.

From (5) we derive the the following **optimality conditions** for a cost-optimal replenishment policy (given linear rationing functions (4)):
The order-up-to-level of an end-stockpoint i, say y_i, satisfies

$$h_i - (h_i + \sum_{j\in U_i} h_j + p_i)(1-\alpha_i^i(y_i)) = 0. \tag{6a}$$

The order-up-to-level of an intermediate stockpoint i, say y_i, and its downstream allocation-fractions satisfy

$$h_i + \sum_{i_{n-1}\in V_i} q_i^{i_{n-1}}[h_{i_{n-1}} + \sum_{i_{n-2}\in V_{i_{n-1}}} q_{i_{n-1}}^{i_{n-2}}[\ldots + \sum_{i_1\in V_{i_2}} q_{i_2}^{i_1}[h_{i_1} - (h_{i_1} + \sum_{j\in U_{i_1}} h_j + p_{i_1})(1-\alpha_{i_1}^i(y_i))]\ldots]] = 0, \tag{6b}$$

where the allocation-fractions of every stockpoint in $ech(i)$ sum up to one, and are positive.

Diks & De Kok [1996] proved that for the optimal replenishment policy (using optimal rationing functions) the order-up-to-level in a stockpoint i satisfies a newsboy-style expression:

$$\alpha_k^i(\hat{y}_i) = \frac{\sum_{j\in U_i} h_j + p_k}{h_k + \sum_{j\in U_k} h_j + p_k} \quad \text{for every} \quad k\in ech(i)\cap E. \tag{7}$$

In this section we have approximated the optimal rationing functions by linear rationing functions. Therefore (7) does not necessarily have to hold any longer. However, if the order-up-to-level at stockpoint i is determined such that (7) holds, it also satisfies the optimality conditions of (6).

4 Algorithm

Consider a divergent N-echelon distribution system in which the holding costs at every stockpoint are identical. Hence, for every stockpoint i (except the most upstream stockpoint) the minimum of $D_i(x, \hat{\Psi}_i)$ is attained in infinity, i.e., the optimal order-up-to-level for every stockpoint equals infinity. This implies that no stock is retained at intermediate stockpoints. In Verrijdt & De Kok [1995] this divergent N-echelon system without intermediate stocks is addressed. They assume CAS rationing functions and developed an algorithm to determine the allocation-fractions of the rationing policies at each stockpoint and the order-up-to-level of the most upstream stockpoint so as to meet the predetermined target fill rates at the end-stockpoints. In this paper, however, these parameters are determined so as to minimize the expected total costs per period.

In the remainder of this section we focus on divergent N-echelon systems for which in every stockpoint value is added to the product (e.g. production systems). The system has a cost-structure as defined in Section 2. It is rather cumbersome to determine the optimal replenishment policy. Therefore we formulate an algorithm based on the linear rationing functions developed in Section 3.2. This algorithm determines the control parameters of a near-optimal policy. It distincts itself from most algorithms treated in the literature by the wide applicability. There already exists several algorithms for the 2-echelon case, however, for the N-echelon case (with $N \geq 3$) it appears to be far more difficult to determine the optimal control parameters.

Before addressing the algorithm, some attention should be given to the ordering at which the control parameters need to be determined. A low level code (LLC) is assigned to every stockpoint. By definition the low level code of an end-stockpoint i equals 1, i.e., LLC(i):=1. For an intermediate stockpoint i we have LLC(i):=1+$\max_{j \in V_i}$ LLC(j). Let W_n denote the set of stockpoints with low level code n.

Algorithm based on decomposition:

(i). $n := 1$.

(ii). Consider a stockpoint $i \in W_n$. *Define* for every end-stockpoint $k \in ech(i)$ a target non-stock out probability α_k^i:

$$\alpha_k^i := \frac{\sum_{j \in U_i} h_j + p_k}{h_k + \sum_{j \in U_k} h_j + p_k}.$$

(iii). (a) Initialize the order-up-to-level at stockpoint i, denoted by y_i.

(b) For every stockpoint $j \in V_i$:
Determine for every end-stockpoint $k \in ech(j)$ the allocation-fraction, $q_i^{j[k]}$ say, such that $\alpha_k^i(y_i) = \alpha_k^i$. In Appendix A we give an algorithm to fastly compute a good approximation of $\alpha_k^i(y_i)$.
Define the allocation-fraction q_i^j by

$$q_i^j := \frac{\sum_{k \in ech(j) \cap E} q_i^{j[k]}}{|ech(j) \cap E|}.$$

(c) If $\sum_{j \in V_i} q_i^j < 1$ we increase y_i and return to step *(b)*.
If $\sum_{j \in V_i} q_i^j > 1$ we decrease y_i and return to step *(b)*.
Repeat this adaptation of y_i until $\sum_{j \in V_i} q_i^j$ is sufficiently close to 1, and then define $\hat{y}_i := y_i$.

(iv). Repeat step *(ii)* and *(iii)* till for all stockpoints $i \in W_n$ the order-up-to-levels $\{\hat{y}_i\}$ and the allocation-fractions $\{q_i^j\}$ are determined.

(v). (a) $m := n$.

(b) If for a stockpoint $i \in W_m$ holds $\hat{y}_i < \sum_{j \in V_i} \hat{y}_j$ we use the following adaptation procedure: For all stockpoints $j \in V_i$ we redefine the order-up-to-levels

$$\hat{y}_j := z_i^j[\hat{y}_i], \quad j \in V_i.$$

(c) $m := m - 1$.

(d) If $m = 1$ we are finished, otherwise return to step *(b)*.

(vi). If $n < N$ then n:=n+1 and proceed with step *(ii)*.

In step *(iii)* of the algorithm the order-up-to-level $\hat{y}_i$ of an end-stockpoint i can easily be determined from (2):

$$F^i_{L_i+1}(\hat{y}_i) = \alpha_i^i.$$

For an intermediate stockpoint i we need the existence of an order-up-to-level $\hat{y}_i$ such that for every end-stockpoint $k \in ech(i)$ holds $\alpha_k^i(\hat{y}_i) = \alpha_k^i$ in order to satisfy (6). If $i \in W_2$ we use a similar heuristic as developed by De Kok, Lagodimos & Seidel [1994] to determine this optimal order-up-to-level $\hat{y}_i$ and the allocation-fractions from stockpoint i to the end-stockpoints such that these target service-levels are attained. However, if $i \in W_n$ with $n > 2$ we have too little degrees of freedom to guarantee this service level α_k^i for every end-stockpoint $k \in ech(i)$. To illustrate this we consider the subsystem $ech(3)$ of Figure 1. Suppose the decomposition algorithm already computed the allocation-fractions q_5^8, q_5^9, and the order-up-to-levels $\hat{y}_5$, $\hat{y}_8$ and $\hat{y}_9$ (recall that we do not alter these already determined allocation-fractions and order-up-to-levels). In order to obtain an optimal replenishment policy we have to choose q_3^5, q_3^6 and $\hat{y}_3$ such that the service requirement for every end-stockpoint is met, and $q_3^5 + q_3^6 = 1$. It is clear that in general the existence of such q_3^5, q_3^6 and $\hat{y}_3$ satisfying these constraints is not guaranteed, since the number of constraints exceeds the number of variables. In order to still use the decomposition approach we suggest to define q_i^j as in step *(iii)b*. This method is only justifiable when the differences between the values of $q_i^{j[k]}$ for the different end-stockpoints are small. Because otherwise averaging these allocation-fractions $q_i^{j[k]}$ implies that for some end-stockpoints the defined value q_i^j is too large and consequently the resulting service performance is too low, or the defined q_i^j is too small and consequently the resulting service performance is too large. This probably results in a bad performance of the replenishment policy, since (6) does no longer hold.

The adaptation procedure in step *(v)* is based on the following theorem.

Theorem 4.1. *Consider a multi-echelon system, where the order-up-to-level of the most upstream stockpoint, i say, equals y_i. The rationing functions are defined as in (4). If y_i is less than the sum of the order-up-to-levels of its successors we redefine these order-up-to-levels $\hat{y}_j := z_i^j[y_i]$. For this adapted replenishment policy holds*
(i) The total expected costs in the multi-echelon system do not alter.
(ii) The non-stock out probabilities at the end-stockpoints do not alter. □

5 Numerical Results

In this section the performance of the algorithm of Section 4 is tested by considering 500 instances of the divergent 3-echelon system as depicted in Figure 2. The lead time of each intermediate stock-

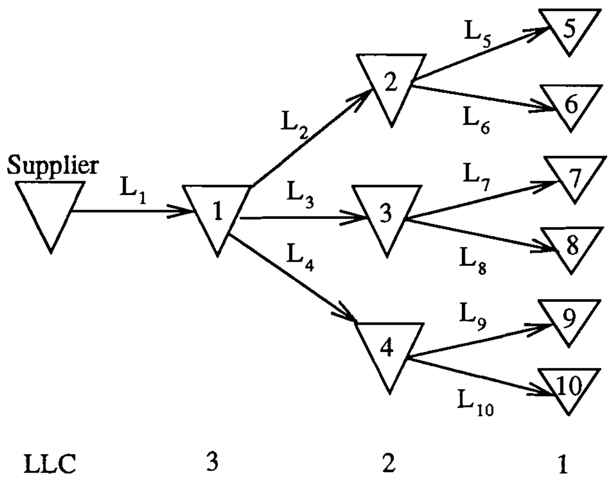
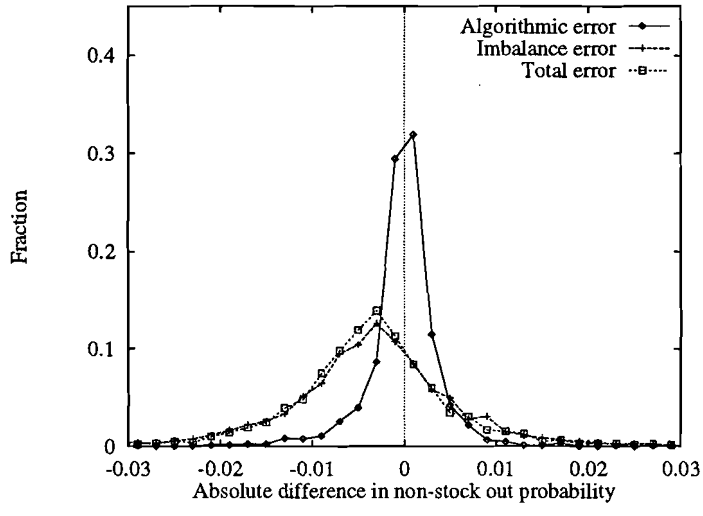


Figure 2: Divergent 3-echelon inventory system.

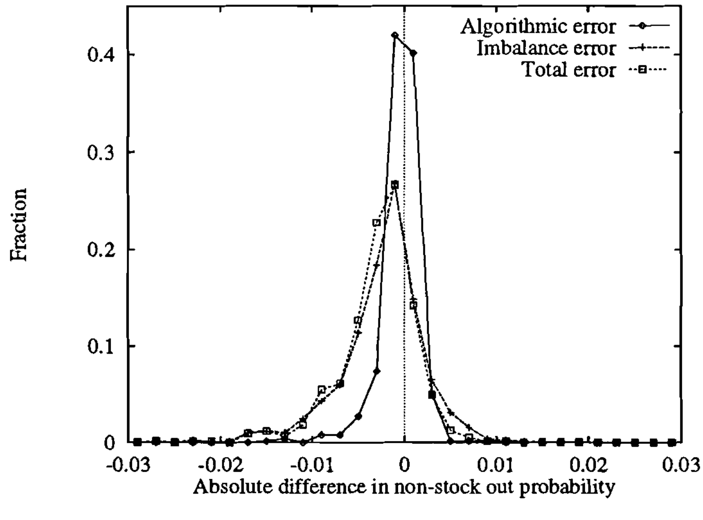
point is drawn from an uniform distribution on $\{1, \ldots, 8\}$, and the lead time of an end-stockpoint is drawn from an uniform distribution on $\{1, \ldots, 5\}$. The penalty costs at each end-stockpoint are chosen such that the non-stock out probability under optimal control is equal to an uniformly distributed value on $[0.85, 0.99]$. The mean demand and squared coefficient of variation per review period at an end-stockpoint is drawn from a uniform distribution on [10,25] and [0.5, 1.5], respectively. Every stock-point adds some value to the product. The amount added is uniformly distributed on $[0.1, 3]$.

Consider an end-stockpoint k. When the system is optimally controlled (assuming the balance assumption) this stockpoint attains a non-stockout probability, α_k say. However, in Section 3.2 we approximate the optimal rationing functions by linear ones. Furthermore, in Section 4 the algorithm uses an approximate step in *(iii)*. The non-stock out probability in stockpoint k resulting from the algorithm (still assuming the balance assumption) is denoted by α_k^A. Generally α_k is not equal too α_k^A. Hence the optimality constraints (6) generally do not hold any longer. Every instance is simulated with the control parameters obtained by applying the algorithm. In case the imbalance would not affect the attained non-stock out probability at stockpoint k, it would be equal to α_k^A. However, usually the phenomenon of imbalance does have some effect on the stock out probability. Hence, stockpoint k attains a non-stock out probability of α_k^S.

Figure 3*a* and *b* depict the absolute differences $\alpha_k^A - \alpha_k$, $\alpha_k^S - \alpha_k^A$ and $\alpha_k^S - \alpha_k$ for an end-stockpoint k with $\alpha_k \leq 0.95$ and $\alpha_k > 0.95$, respectively. Notice that $\alpha_k^A - \alpha_k$ represents the 'Algorithmic error' due to linearizing the rationing functions and averaging $q_i^{j[k]}$ in step *(iii)* of the algorithm, $\alpha_k^S - \alpha_k^A$ represents the 'Imbalance error' due to the violation of the balance assumption, and $\alpha_k^S - \alpha_k$ represents the 'Total error'. We distinguish between these two cases $\alpha_k \leq 0.95$ and $\alpha_k > 0.95$, since an absolute error of 0.01 is acceptable in case α_k is not to large (e.g. 0.85), although, when α_k is large (e.g. 0.99) such an error is intolerable. Comparing these two figures indicates that when α_k is large the absolute errors diminish. Furthermore, we conclude that the algorithm works very well. Most of the difference between α_k and α_k^S is caused by imbalance.



(a) $\alpha_k \leq 0.95$.



(b) $\alpha_k > 0.95$

Figure 3: Performance of algorithm.

6 Conclusions

In this paper we addressed the problem of determining the control parameters of a divergent multi-echelon inventory system such that the expected holding and penalty costs per period are minimized. Diks & De Kok [1996] proved that given the balance assumption a decomposition of the system is exact. Hence the complex multi-dimensional problem of determining these parameters reduces to the problem of determining (for every stockpoint): (1) the optimal order-up-to-policy, and (2) the optimal rationing functions to its successors. It is rather cumbersome and time-consuming to determine these optimal rationing functions. Therefore we restrict ourselves to a special class of linear functions. An algorithm is developed to compute the order-up-to-level and its allocation-fractions so as to minimize the expected total costs as much as possible. The algorithm is tested by a simulation study on a 3-echelon system. It turns out that the algorithm performs very well. When comparing the non-stock out probability of an end-stockpoint controlled by the optimal policy (given the balance assumption) and the policy obtained by the algorithm the difference is very small. Specifically, this difference is mainly caused by imbalance and not by linearization of the rationing functions and an approximate step in the algorithm.

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A Algorithm to compute the non-stock out probability

In this appendix we show how the non-stock out probability $\alpha_k^i(y_i)$ can be approximated. For sake of clarity we restrict ourselves to the case where the most upstream stockpoint i supplies a stockpoint j, which supplies the end-stockpoint k. The generalization to the case where more stockpoints are on the path from i to k is straightforward. It can be shown that

$$\alpha_k^i(y_i) = Pr\{Q_{L_k+1}^k - \Delta_k + q_j^k \left(Q_{L_j}^j - \Delta_j + q_i^j \left(Q_{L_i}^i - \Delta_i\right)^+\right)^+ < 0\}, \tag{8}$$

with $\Delta_m := \begin{cases} y_m & m \in E \\ y_m - \sum_{n \in V_m} y_n & m \in I \end{cases}$

$Q_L^m :=$ Demand at the end-stockpoints in $ech(m)$ during L periods.

It is cumbersome to determine (8) exact, therefore we propose an approximate procedure as in Van Houtum & Zijm [1991a]. This procedure works as follows: First, we determine the first two moments of $X := Q_{L_i}^i - \Delta_i$ and fit a mixture of Erlang distributions on these moments. For details of the fit procedure we refer to Van Houtum & Zijm [1991a] and Tijms [1994]. Next, determine the first two moments of $Y := Q_{L_j}^j - \Delta_j + q_i^j X^+$ and again fit a mixture of Erlang distributions on its moments. Finally, we determine the first two moments of $Q_{L_k+1}^k - \Delta_k + q_j^k Y^+$ and fit a mixture of Erlang distributions on its moments. The probability mass of the resulting distribution on the negative halfspace approximates $\alpha_k^i(y_i)$.