Solution to problem 95-2 : Exciton transport

ANDERSEN and LARSEN generalize (1) to all complex n and m and refer to their paper, *Combinatorial summation identities*, in Proceedings from the 21st Nordic Congress in Luleå 1992, (1994), pp. 1-23.

Exciton Transport

Problem 95-2, *by* M. L. GLASSER (Clarkson University).

Evaluate

$$\int_0^1 J_0(t\sqrt{1-x^2})\, e^{tx}\, dx$$

in terms of Bessel and Struve functions. This problem arose in a study of the role of coherence in exciton transport measurements [1]. (There is a misprint in eq. (3.12); under the radical, ϵ should be replaced by $\epsilon + \alpha$.)

REFERENCE

[1] Y. WONG AND V. KERKRE, *Extension of exiton-transport theory for transient grating experiments into the intermediate coherence domain*, Phys. Rev., B22 (1980), pp. 3072–3077.

Solution by J. BOERSMA (Eindhoven University of Technology, Eindhoven, the Netherlands).

The integral, to be denoted by $I(t)$, is evaluated by expansion of the integrand in power series followed by a term-by-term integration. As a first result we find

$$\begin{aligned} I(t) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^m (t/2)^{2m} t^n}{m!m!n!} \int_0^1 (1-x^2)^m x^n\, dx \\ &= \frac{1}{2}\sqrt{\pi} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^m (t/2)^{2m+n}}{m!\Gamma(\frac{1}{2}n+1)\Gamma(m+\frac{1}{2}n+\frac{3}{2})} \\ &= \frac{1}{2}\sqrt{\pi} \sum_{p=0}^{\infty} \frac{(t/2)^p}{\Gamma(\frac{1}{2}p+\frac{3}{2})} \sum_{m=0}^{[p/2]} \frac{(-1)^m}{m!\Gamma(\frac{1}{2}p+1-m)}. \end{aligned}$$

For the inner sum, to be denoted by S_p, we have the closed-form results

$$S_{2p} = \sum_{m=0}^{p} \frac{(-1)^m}{m!(p-m)!} = \begin{cases} 1, & \text{when } p = 0, \\ 0, & \text{when } p = 1, 2, 3, \dots; \end{cases}$$

$$S_{2p+1} = \sum_{m=0}^{p} \frac{(-1)^m}{m!\Gamma(p+\frac{3}{2}-m)} = \frac{F(p-\frac{1}{2}, -p; -p; 1)}{\Gamma(p+\frac{3}{2})} = \frac{2}{\sqrt{\pi}} \frac{(-1)^p}{p!(2p+1)},$$

obtained by use of the known value of the hypergeometric function F with unit argument. On inserting these results for S_p, we are led to

$$I(t) = 1 + \sum_{p=0}^{\infty} \frac{(-1)^p (t/2)^{2p+1}}{p!(p+1)!(2p+1)}.$$

Here the infinite series is recognized as the expansion of the Bessel-function integral

$$\int_0^t \frac{J_1(s)}{s}\, ds = \int_0^t [-J_1'(s) + J_0(s)]\, ds = -J_1(t) + \int_0^t J_0(s)\, ds.$$

The latter integral is expressible in terms of Bessel and Struve functions; see, e.g., [1, form. 11.1.7]. Thus we obtain

$$I(t) = 1 - J_1(t) + tJ_0(t) + \frac{\pi t}{2}[\mathbf{H}_0(t)J_1(t) - \mathbf{H}_1(t)J_0(t)]$$

as our final result.

REFERENCE

[1] M. ABRAMOWITZ AND I. A. STEGUN, EDS., *Handbook of Mathematical Functions*, Dover, New York, 1965.

Also solved by ROBIN CHAPMAN (University of Exeter, Exeter, UK), THOMAS DICKENS (Exxon Production Research, Houston, TX), CARL C. GROSJEAN (University of Ghent, Ghent, Belgium), W. B. JORDAN (Scotia, NY), ALLEN R. MILLER (Washington, DC), NORBERT ORTNER (University of Innsbruck, Austria), *and the proposer.*

Editorial note. MILLER generalizes the integral to

$$\int_0^1 x^{\alpha-1}(1-x^2)^{\nu/2} J_\nu(t\sqrt{1-x^2})e^{tx}\,dx,$$

which he expresses in terms of ${}_1F_2$ hypergeometric functions. ORTNER evaluates the integral

$$\int_0^{\frac{\pi}{2}} J_\mu(t\sin\phi)K_\nu(t\cos\phi)\sin^{\mu+1}(\phi)\cos^{\nu+1}(\phi)\,d\phi$$

in terms of Lommel functions and specializes ($\mu = 0$, $\nu = -1/2$).

Comment. Taking the Laplace transformation gives

$$\int_0^\infty \exp(-st)I(t)\,dt = \int_0^1 \frac{dx}{(s^2+1-2sx)^{1/2}} = (1 - s + (1+s^2)^{1/2})/s,$$

and the result follows from standard tables.

Nonsymmetric Cyclic Pursuit on a Sphere

Problem 95-3*, *by* M. S. KLAMKIN (University of Alberta).

Three bugs, A, B, C, starting from the vertices of an arbitrary spherical triangle, pursue each other cyclically at the same constant speeds, i.e., A always heads directly towards B, while B heads towards C, and C heads towards A, along minor great circular arcs. Prove or disprove that there is simultaneous capture. For the plane case, it is known that there is simultaneous capture, and upper and lower bounds are given for the time to capture [1].

REFERENCE

[1] M. S. KLAMKIN AND D. J. NEWMAN, *Cyclic pursuit or "the three bugs problem,"* Amer. Math. Monthly, 78 (1971), pp. 631-639.

Solution by H. E. DE MEYER *and* C. C. GROSJEAN (University of Ghent, Belgium).

From the six differential equations describing in spherical coordinates the instantaneous motion of the three bugs on the sphere, one easily obtains by elementary trigonometric calculations another set of six first-order differential equations expressing the rate of change of the arcs a, b, c and the angles A, B, C of the spherical triangle with the bugs as vertices. On