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Eindhoven University
of Technology

Department of Mathematics and Computing Sciences

Memorandum COSOR 98-12

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Eindhoven, July 1998
The Netherlands

Precedence Probability, Prediction Interval and A Combinatorial Identity

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SUMMARY

Precedence tests are simple yet useful nonparametric tests based on two specified order statistics from independent random samples or, equivalently, on the count of the number of observations from one of the samples preceding some order statistic of the other sample. The probability that an order statistic from the second sample exceeds an order statistic from the first sample is termed the precedence probability. When the distributions are the same, this probability can be calculated exactly, without any specific knowledge of the underlying common continuous distribution. This fact can be utilized to set up nonparametric prediction intervals in a number of situations. In this paper, prediction intervals are considered for the number of second sample observations that exceed a particular order statistic of the first sample. To aid the user, tables are provided for small sample sizes, where exact calculations are most necessary. The same tables can be used to implement a precedence test for small sample sizes. Finally, a combinatorial identity is proved.

Keywords: *Distribution-free; Extremes; Exceedance and Precedence; Nonparametric; Order statistics.*

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Support provided in part by NATO Collaborative Research Grant No. CRG920287.

1. Introduction

Let $X_{(1)} < X_{(2)} < .... < X_{(m)}$ be the order statistics of a random sample of size m from a continuous c.d.f. F and let $Y_{(1)} < Y_{(2)} < ... < Y_{(n)}$ be the order statistics of a second, independent, random sample of size n from a continuous c.d.f. G. Consider the probability that the jth Y-order statistic exceeds the ith X-order statistic, $\theta=\theta_{ij}(F,G)=P(Y_{(j)}>X_{(i)})$. The parameter θ can be interpreted in several ways. Two such interpretations are: (i) it is the probability that the number of Y observations that precede $X_{(i)}$ is at most equal to j-1 and (ii) it is the probability that the number of Y observations that exceed $X_{(i)}$ is at least equal to n-j+1. According to the first interpretation, θ is termed a "precedence" probability whereas according to the second interpretation θ is referred to as an "exceedance" probability. Both interpretations can be found in the literature as the quantity θ arises in various applications. The fields of applications include quality control and reliability where θ can be associated with the so-called "warranty time" of a product. In these problems, the underlying probability distributions are often not completely known and frequently can not be assumed to be normal. Thus, a study of the precedence probability, from a distribution-free point of view is useful.

The study of precedence and exceedances goes back to at least the early 40's. Some of this literature will be referred to later on. Nelson (1963) proposed a simple nonparametric test, called the precedence test, for the usual two sample problem H_0: $F(t) = G(t)$. Against the one-sided alternative H_1:$G(t) < F(t)$, that the Y's are stochastically larger than the X's, the precedence test rejects H_0 iff, say, $Y_{(j)} > X_{(i)}$. Thus the precedence probability θ is simply the power of the precedence test.

The concept of precedence (or exceedance) is easy to grasp and is intuitively appealing (one just needs to compare two ordered values from the two samples) to practitioners in many statistical inference problems. Since a precedence test is based on ordered values, in situations (such as life-testing) where data are collected sequentially, such a test can lead to savings in time and resources by allowing an early decision (rejecting H_0 or not) before all the data are collected.

Recently, there has been a resurgence of interest in precedence tests. Nelson (1993) revisited the precedence test. Lin and Sukhatme (1992) studied "best" precedence tests under Lehmann alternatives. Liu (1992) investigated some properties of precedence probabilities, and obatined some results, mainly for the equal sample size case. Chakraborti and van der Laan (1996,1997) provided comprehensive surveys of the area of precedence and precedence-type tests for two- and multi-sample problems, for the complete and the right-censored data, respectively. Further, van der Laan and Chakraborti (1998) studied "best" precedence tests, based on power, for several types of Lehmann and proportional-hazards alternatives. In this paper the focus is mainly on the precedence probability and in this context the problem of some nonparametric prediction intervals is considered based on exceedance statistics. Necessary formulas and tables are presented so that these can be implemented in practice. In the sequel, an interesting combinatorial identity is obtained.

2. Precedence Probability and Prediction Intervals

First note that in general an expression for the precedence probability θ can be easily obtained from the distributions of the order statistics $X_{(i)}$ and $Y_{(j)}$. It can be shown (see for example Chakraborti and van der Laan, 1996; hereafter referred to as CV) that θ depends on the unknown c.d.f.'s only through the so-called "conversion" function $C(u)=FG^{-1}(u)$, $0<u<1$. Thus θ can be calculated explicitly when the conversion function is completely specified. This includes common situations where parametric model assumptions (such as normal or exponential) are made about F and G.

However, when $F \equiv G$ (for example, under H_0), $C(u)=u$, and the expression for θ reduces to an incomplete beta integral that can be calculated using tables of the incomplete beta function or via the c.d.f. of a binomial distribution. Note that in practice there are situations where the precedence probability θ, when $F \equiv G$, is important. This is particularly true in problems of prediction. Suppose that a random sample of observations is available from some (continuous) population and based on this sample one wishes to estimate some characteristics of a future sample drawn from the same population. For example, the interest might be to estimate the number of observations in the second random sample that will exceed (or precede) some ordered (say the median or the largest) value of the first sample. This type of problem is important, for example, in studies of the extremes (in environmental monitoring; hydrology, etc.) and in quality control. For instance, in a production process producing a certain type of light fuses, it might be of interest to estimate, with some degree of confidence, the number of fuses in a future sample that would last longer than say the longest working fuse from the current sample. Such a number or the proportion could be interpreted as one measure of the "quality" of this type of fuses.

Since E is a random variable, an answer to the above problem might be given by a prediction interval. For various problems in the context of prediction intervals the reader is referred to the recent book by Hahn and Meeker (1991). For a brief introduction, one can also refer to Vardeman (1992). For our problem, let V_i (or E_i) denote the number of Y observations that precede (or exceed) $X_{(i)}$. The statistic V_i is called a “precedence” statistic and a test based on V_i is called a “precedence” test (on the other hand one could just as easily use the "exceedance" statistic E_i, and could call the resulting test an "exceedance" test). Recall that according to the first interpretation of the precedence probability, θ is simply the c.d.f. of V_i at j-1. Also, since $P(V_i \leq j-$

1) = $P(E_i \geq n-j+1)$, one can consider either the exceedances or the precedances in testing hypotheses or in constructing prediction intervals.

When $F \equiv G$, it has been shown that (see for example, CV)

$$P(V_i = v) = \frac{\binom{i+v-1}{v}\binom{m+n-i-v}{n-v}}{\binom{m+n}{n}}, \qquad v = 0,1,\ldots, n;\ i = 1,2,\ldots, m. \tag{1}$$

Remark 1 For $1 \leq a < b \leq m$, it can be seen that $P(X_{(a)} \leq Y_{(j)} \leq X_{(b)}) = P(a \leq W_j \leq b-1)$, where W_j is the number of X's preceding $Y_{(j)}$. The distribution of W_j can be obtained from (1) by writing m for n and j for i. This result gives a nonparametric prediction interval for $Y_{(j)}$ based on two X-order statistics. See Fligner and Wolfe (1976,1979) for further details.

Remark 2 The probability distribution of E_i, the number of Y "exceedances" (the number of Y's that exceed $X_{(i)}$) follows from (1) and is given for completeness

$$P(E_i = e) = \frac{\binom{i+n-e-1}{n-e}\binom{m-i+e}{e}}{\binom{m+n}{n}}, \qquad e = 0,1,\ldots, n;\ i = 1,2,\ldots, m. \tag{2}$$

Remark 3 In some applications (such as in the analysis of extremes) the probability distribution of F_i, the number of Y observations that exceed $X_{(m-i+1)}$, the ith largest (note that $X_{(i)}$ is the ith smallest) of the X's, is needed. This exceedance probability is easily obtained from (2) by substituting m-i+1 for i. After some simplification the result can be expressed as

$$P(F_i = f) = \frac{i}{m+n}\frac{\binom{m}{i}\binom{n}{f}}{\binom{m+n-1}{i+f-1}}, \qquad f = 0,1,\ldots, n;\ i = 1,2,\ldots, m. \tag{3}$$

These and other related expressions have been obtained by several authors, particularly in the 50's and the 60's, using a variety of mathematical-statistical as well as combinatorial techniques. The starting point for many of these works appears to be the classic paper by Wilks

(1942). Some rather old but still useful references on this topic are: Gumbel and von Schelling (1950), Epstein (1954) and Rosenbaum (1954).

The distribution of E_i is computed and presented in Tables 3 and 4 for selected 'small' values of m and n: m,n=3(2)15 and i=(m+1)/2 and i=m. Thus, the tables cover exceedances above the median and the largest, respectively. Note that for values of m, n and i not covered by the tables, it is not hard to use the explicit formulas given above. First, these tables can be used to implement a precedence test as proposed in Nelson (1963). To illustrate this, for example, suppose m=9, n=9 and a size α=.05 precedence test is desired at the X-median, so that i=5. Using Table 4, the rejection region can be found as follows. Since V_5=n-E_5, where E_5 is the number of Y's exceeding $X_{(5)}$, from Table 4, we first find the smallest integer r so that $P(E_5 \leq r) \geq .95$. This yields r=9 so that n-r=2 and the precedence test has rejection region $V_5 \leq 2$, with an exact size equal to .0379. Also, using either of the two interpretations for θ, this corresponds to j=3 and the precedence rejection region can be equivalently expressed in terms of two order statistics: $Y_{(3)} > X_{(5)}$. Secondly, tables 3 and 4 are useful in the calculation of prediction intervals. This is discussed in the following section.

2.1 Prediction intervals

The exact distribution of the exceedance statistic when $F \equiv G$, can be used to set up a prediction interval on the number (or the proportion) of future observations that exceed a current order statistic. For example, suppose m=9, n=7 and i=5, so the interest is in the number of future exceedances in sample of size 7 over the median of a current sample of size 9. The distribution of Y-exceedances over the X-median, E_5, in this case is found from Table 3 and is given in Table 1 for quick reference.

Table 1: Distribution of number of exceedances E with m=9,n=7 and i=5

	0	1	2	3	4	5	6	7
prob	0.02885	0.09178	0.16521	0.21416	0.21416	0.16521	0.09178	0.02885
cuprob	0.02885	0.12063	0.28584	0.50000	0.71412	0.87937	0.97115	1.00000

From Table 1 it is seen that the distribution of E_5 in this case is symmetric and bimodal. The number of future observations that are expected to exceed the current sample median is 3.5. Now, suppose we want a 90% prediction interval on E_5, the number of Y-observations exceeding the X-median. From Table 1, using the cumulative probabilities (cuprob), it is found that the required prediction interval is between 1 and 6, with both endpoints included. This interval is conservative in the sense that the exact confidence coefficient is 0.9423, which is higher than the nominal 0.90. Equivalently, the proportion of future observations that The distribution given in Table 1 also demonstrates the well-known fact not all typical confidence coefficients might be available for all m, n and i, owing to the discreteness of the E_i statistic. In general, a two-sided prediction interval [a,b] for E_i, with confidence coefficient $1-\alpha$, can be calculated by solving for two integers a and b so that

$$\sum_{e=a}^{b} \mathrm{P}(\mathrm{E}_i = \mathrm{e} \mid \mathrm{F} \equiv \mathrm{G}) = 1-\alpha, \tag{4}$$

where $\mathrm{P}(\mathrm{E}_i=\mathrm{e}|\mathrm{F} \equiv \mathrm{G})$ is given by (2).

When i corresponds to the median of the X-sample, the distribution of E_i is symmetric. In this case one can set a to be the largest integer such that $\sum_{e=0}^{a-1} \mathrm{P}(\mathrm{E}_i = \mathrm{e} \mid \mathrm{F} \equiv \mathrm{G}) \leq \alpha/2$ and take b=n-a.

In some problems only a one-sided prediction interval (or a prediction bound), say of the form [0,c] is needed. In this case (4) can be easily modified and tables 3-4 can be used to find the interval.

Now suppose that for the same m and n, i=9, so that the interest is in the number of future exceedances over the largest value of the current sample. For this case, the distribution of E is found from Table 4 and is given in Table 2.

Table 2: Distribution of number of exceedances E with m=9, n=7 and i=9

	0	1	2	3	4	5	6	7
prob	0.56250	0.26250	0.11250	0.04327	0.01442	0.00393	0.00079	0.00009
cuprob	0.56250	0.82500	0.93750	0.98077	0.99519	0.99913	0.99991	1.00000

Thus there is a 11.25% probability that in a future (Y-) sample of 7, 2 will exceed the largest of the current (X-) sample of 9 observations from the same continuous population. As it might be expected, this distribution is highly skewed to the right. The probability is over 56% that none of the Y-sample values will exceed the maximum. It may be noted that for these small values of m, n there are no prediction intervals for E_9 (exceedances over the current maximum), given typical confidence coefficients such as 0.95 or 0.90.

<<Tables 3 and 4 Here>>

Normal Approximation

Though exact formulas are given, when m and n are large, the practitioner may find it more convenient to employ a normal approximation to calculate a and b. To this end, note that it has been shown (see for example, CV, equation (8)) that V_i has, approximately, a normal distribution. Specifically, when $F \equiv G$, and m and n are large, the precedence statistic V_i is approximately normally distributed with mean $\mu = n(1-\frac{i}{m})$ and variance $\sigma^2 = n(\frac{m+n}{m})(\frac{i}{m})(1-\frac{i}{m})$. It follows that a normal approximation to and b are

$$a = n - [\frac{n}{m}\{i - z_{\alpha/2}\sqrt{i(m-i)(\frac{1}{m}+\frac{1}{n})}\}]$$

and (5)

$$b = n - [\frac{n}{m}\{i + z_{\alpha/2}\sqrt{i(m-i)(\frac{1}{m}+\frac{1}{n})}\}] - 1$$

respectively, where $z_{\alpha/2}$ is the upper $100\alpha/2$th standard normal percentile and [x] denotes the greatest integer not exceeding x. Using these formulas for our example, with m=9, n=7, i=5 and α=.10, we get a=1 and b=6, the same solutions that were found using the exact distribution.

2.2 A combinatorial identity

Recall that the precedence probability θ is simply the value of the c.d.f. of V_i at j-1. Thus, when $F \equiv G$, we have

$$\begin{aligned} \theta &= P(V_i \le j-1) \\ &= \sum_{r=0}^{j-1} P(V_i = v) \\ &= \sum_{v=0}^{j-1} \frac{\binom{i+v-1}{v}\binom{m+n-i-v}{n-v}}{\binom{m+n}{n}}. \end{aligned} \tag{6}$$

On the other hand, when $F \equiv G$, the probability θ can also be viewed (see for example, van der Laan, 1970; Liu, 1992) as the probability that at least i of the X's are in the first i+j-1 positions out of the total m+n ordered X's and Y's, where m X's and n Y's are drawn from the same distribution. It follows that

$$\theta = \sum_{r=i}^{i+j-1} \frac{\binom{m}{r}\binom{n}{i+j-r-1}}{\binom{m+n}{i+j-1}} \tag{7}$$

Thus (6) should be equal to (7). The result is stated as a combinatorial identity in the following lemma. An alternative algebraic proof is also provided below.

Lemma

$$\sum_{r=0}^{j-1} \frac{\binom{i+r-1}{r}\binom{m+n-j-r}{n-r}}{\binom{m+n}{n}} = \sum_{r=i}^{i+j-1} \frac{\binom{m}{r}\binom{n}{i+j-r-1}}{\binom{m+n}{i+j-1}} \qquad (8)$$

Proof

Note that we can write

$$\sum_{r=i}^{i+j-1} \frac{\binom{m}{r}\binom{n}{i+j-r-1}}{\binom{m+n}{i+j-1}} = \sum_{r=0}^{j-1} \frac{\binom{m}{i+r}\binom{n}{j-1-r}}{\binom{m+n}{i+j-1}},$$

so we need to show that

$$\sum_{r=0}^{j-1} \frac{\binom{i+r-1}{r}\binom{m+n-i-r}{n-r}}{\binom{m+n}{n}} = \sum_{r=0}^{j-1} \frac{\binom{m}{i+r}\binom{n}{j-1-r}}{\binom{m+n}{i+j-1}}.$$

Canceling the common factorials on both sides, it can be seen that the above amounts to showing that

$$\sum_{r=0}^{j-1}\binom{i+j-1}{i+r}\binom{m+n-i-j+1}{m-i-r} - \sum_{r=0}^{j-1}\binom{i+r-1}{i-1}\binom{m+n-i-r}{m-i} = 0. \qquad (9)$$

Proof of (9) is given by induction on j.

For j=1, (9) is true since

$$\binom{i}{i}\binom{m+n-i}{m-i} - \binom{i-1}{i-1}\binom{m+n-i}{m-i} = 0.$$

To illustrate the general approach, note that for j=2 we have to show that

$$\binom{i+1}{i}\binom{m+n-i-1}{m-i} + \binom{i+1}{i+1}\binom{m+n-i-1}{m-i-1} - \binom{i-1}{i-1}\binom{m+n-i}{m-i} - \binom{i}{i-1}\binom{m+n-i-1}{m-i} = 0. \quad (10)$$

For this, using the well-known identity

$$\binom{M}{s}+\binom{M}{s-1}=\binom{M+1}{s}, \text{ so that } \binom{M}{s}-\binom{M+1}{s}=-\binom{M}{s-1},$$

for any positive integer M and any non-negative integer s=0,1,...,M, we get,

$$\binom{i+1}{i}\binom{m+n-i-1}{m-i}-\binom{i}{i-1}\binom{m+n-i-1}{m-i}=\binom{m+n-i-1}{m-i}\binom{i}{i},$$

$$\binom{i}{i}\binom{m+n-i-1}{m-i}-\binom{i-1}{i-1}\binom{m+n-i}{m-i}=-\binom{m+n-i-1}{m-i-1},$$

and therefore the L.H.S. of (8) equals

$$-\binom{m+n-i-1}{m-i-1}+\binom{i+1}{i+1}\binom{m+n-i-1}{m-i-1}=0.$$

Now we suppose the identity (1) is true for fixed j, then we have to prove that the identity is also true for j+1. Thus we have to prove

$$\sum_{r=0}^{j}\binom{i+j}{i+r}\binom{m+n-i-j}{m-i-r}-\sum_{r=0}^{j}\binom{i+r-1}{i-1}\binom{m+n-i-r}{m-i}=0 \tag{11}$$

Note that the second sum can be rewritten as

$$\sum_{r=0}^{j-1}\binom{i+r-1}{i-1}\binom{m+n-i-r}{m-i}+\binom{i+j-1}{i-1}\binom{m+n-i-j}{m-i}. \tag{12}$$

By the induction hypothesis, the first sum in (12) is equal to

$$\sum_{r=0}^{j-1}\binom{i+j-1}{i+r}\binom{m+n-i-j+1}{m-i-r},$$

so that the left hand side of (11) equals

$$\sum_{r=0}^{j}\binom{i+j}{i+r}\binom{m+n-i-j}{m-i-r}-\sum_{r=0}^{j-1}\binom{i+j-1}{i+r}\binom{m+n-i-j+1}{m-i-r}-\binom{j+i-1}{i-1}\binom{m+n-i-j}{m-i},$$

which reduces to

$$\sum_{r=1}^{j}\binom{i+j}{i+r}\binom{m+n-i-j}{m-i-r}-\sum_{r=0}^{j-1}\binom{i+j-1}{i+r}\binom{m+n-i-j+1}{m-i-r}+\binom{j+i-1}{i-1}\binom{m+n-i-j}{m-i}$$

The last expression can be further simplified as follows

$$=\sum_{r=1}^{j}\binom{i+j}{i+r}\binom{m+n-i-j}{m-i-r}-\sum_{r=1}^{j-1}\binom{i+j-1}{i+r}\binom{m+n-i-j+1}{m-i-r}-\binom{i+j-1}{i}\binom{m+n-i-j}{m-i-1}$$

$$=\sum_{r=2}^{j}\binom{i+j}{i+r}\binom{m+n-i-j}{m-i-r}-\sum_{r=1}^{j-1}\binom{i+j-1}{i+r}\binom{m+n-i-j+1}{m-i-r}+\binom{i+j-1}{i+1}\binom{m+n-i-j}{m-i-1}$$

$$=\sum_{r=2}^{j}\binom{i+j}{i+r}\binom{m+n-i-j}{m-i-r}-\sum_{r=2}^{j-1}\binom{i+j-1}{i+r}\binom{m+n-i-j+1}{m-i-r}-\binom{i+j-1}{i+1}\binom{m+n-i-j}{m-i-2}$$

....

$$=\sum_{r=j-1}^{j}\binom{i+j}{i+r}\binom{m+n-i-j}{m-i-r}-\sum_{r=j-1}^{j-1}\binom{i+j-1}{i+r}\binom{m+n-i-j+1}{m-i-r}-\binom{i+j-1}{i+j-2}\binom{m+n-i-j}{m-i-j+1}$$

$$=\sum_{r=j}^{j}\binom{i+j}{i+r}\binom{m+n-i-j}{m-i-r}-\sum_{r=j-1}^{j-1}\binom{i+j-1}{i+r}\binom{m+n-i-j+1}{m-i-r}+\binom{i+j-1}{i+j-1}\binom{m+n-i-j}{m-i-j+1}$$

$$=\binom{i+j}{i+j}\binom{m+n-i-j}{m-i-j}-\binom{i+j-1}{i+j-1}\binom{m+n-i-j+1}{m-i-j+1}+\binom{i+j-1}{i+j-1}\binom{m+n-i-j}{m-i-j+1}$$

$$=\binom{m+n-i-j}{m-i-j}-\binom{m+n-i-j}{m-i-j}=0,$$

and the proof is complete.

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Table 3. Distribution of E for exceedances over the median

m=3,i=2													
n=3		n=5		n=7		n=9		n=11		n=13		n=15	
0	.2	0	.107	0	.067	0	0.045455	0	0.032967	0	0.02500000	0	0.01960784
1	.3	1	.179	1	.117	1	0.081818	1	0.060440	1	0.04642857	1	0.03676471
2	.3	2	.214	2	.150	2	0.109091	2	0.082418	2	0.06428571	2	0.05147059
3	.2	3	.214	3	.167	3	0.127273	3	0.098901	3	0.07857143	3	0.06372549
		4	.179	4	.167	4	0.136364	4	0.109890	4	0.08928571	4	0.07352941
		5	.107	5	.150	5	0.136364	5	0.115385	5	0.09642857	5	0.08088235
				6	.117	6	0.127273	6	0.115385	6	0.10000000	6	0.08578431
				7	.067	7	0.109091	7	0.109890	7	0.10000000	7	0.08823529
						8	0.081818	8	0.098901	8	0.09642857	8	0.08823529
						9	0.045455	9	0.082418	9	0.08928571	9	0.08578431
								10	0.060440	10	0.07857143	10	0.08088235
								11	0.032967	11	0.06428571	11	0.07352941
										12	0.04642857	12	0.06372549
										13	0.02500000	13	0.05147059
												14	0.03676471
												15	0.01960784
m=5,i=3													
n=3		n=5		n=7		n=9		n=11		n=13		n=15	
0	.179	0	.083	0	.046	0	.028	0	0.017857	0	0.012255	0	0.00877193
1	.321	1	.179	1	.106	1	.067	1	0.04533	1	0.031863	1	0.02321981
2	.321	2	.238	2	.159	2	.108	2	0.075549	2	0.054622	2	0.04063467
3	.179	3	.238	3	.189	3	.140	3	0.103022	3	0.077031	3	0.05869453
		4	.179	4	.189	4	.157	4	0.123626	4	0.096289	4	0.0754644
		5	.083	5	.159	5	.157	5	0.134615	5	0.110294	5	0.08939628
				6	.106	6	.140	6	0.134615	6	0.117647	6	0.09932921
				7	.046	7	.107	7	0.123626	7	0.117647	7	0.10448916
						8	.067	8	0.103022	8	0.110294	8	0.10448916
						9	.028	9	0.075549	9	0.096289	9	0.09932921
								10	0.04533	10	0.077031	10	0.08939628
								11	0.017857	11	0.054622	11	0.0754644
										12	0.031863	12	0.05869453
										13	0.012255	13	0.04063467
												14	0.02321981
												15	0.00877193

m=7,i=4													
n=3		n=5		n=7		n=9		n=11		n=13		n=15	
0	.167	0	.071	0	.035	0	0.019231	0	0.011438	0	0.007224	0	0.004785
1	.333	1	.178	1	.098	1	0.057692	1	0.035948	1	0.023478	1	0.015949
2	.333	2	.253	2	.163	2	0.104895	2	0.069130	2	0.046956	2	0.032836
3	.167	3	.253	3	.204	3	0.146853	3	0.103695	3	0.073787	3	0.053359
		4	.178	4	.204	4	0.171329	4	0.131976	4	0.099329	4	0.074702
		5	.071	5	.163	5	0.171329	5	0.147813	5	0.119195	5	0.093911
				6	.098	6	0.146853	6	0.147813	6	0.130031	6	0.108359
				7	.035	7	0.104895	7	0.131976	7	0.130031	7	0.116099
						8	0.057692	8	0.103695	8	0.119195	8	0.116099
						9	0.019231	9	0.069130	9	0.099329	9	0.108359
								10	0.035948	10	0.073787	10	0.093911
								11	0.011438	11	0.046956	11	0.074702
										12	0.023478	12	0.053359
										13	0.007224	13	0.032836
												14	0.015949
												15	0.004785

m=9,i=5													
n=3		n=5		n=7		n=9		n=11		n=13		n=15	
0	0.159091	0	0.062937	0	0.028846	0	0.014706	0	0.008127	0	0.004785	0	0.002964
1	0.340909	1	0.174825	1	0.091783	1	0.050905	1	0.029799	1	0.018294	1	0.011702
2	0.340909	2	0.262238	2	0.165210	2	0.101810	2	0.063854	2	0.041162	2	0.027304
3	0.159091	3	0.262238	3	0.214161	3	0.151172	3	0.103150	3	0.070433	3	0.048719
		4	0.174825	4	0.214161	4	0.181407	4	0.137533	4	0.100619	4	0.073078
		5	0.062937	5	0.165210	5	0.181407	5	0.157538	5	0.125387	5	0.096463
				6	0.091783	6	0.151172	6	0.157538	6	0.139319	6	0.114837
				7	0.028846	7	0.101810	7	0.137533	7	0.139319	7	0.124933
						8	0.050905	8	0.103150	8	0.125387	8	0.124933
						9	0.014706	9	0.063854	9	0.100619	9	0.114837
								10	0.029799	10	0.070433	10	0.096463
								11	0.008127	11	0.041162	11	0.073078
										12	0.018294	12	0.048719
										13	0.004785	13	0.027304
												14	0.011702
												15	0.002964

m=11,i=6													
n=3		n=5		n=7		n=9		n=11		n=13		n=15	
0	0.153846	0	0.057692	0	0.024887	0	0.011920	0	0.006192	0	0.003432	0	0.002007
1	0.346154	1	0.173077	1	0.087104	1	0.045975	1	0.025542	1	0.014874	1	0.009030
2	0.346154	2	0.269231	2	0.166290	2	0.099024	2	0.059598	2	0.036748	2	0.023288
3	0.153846	3	0.269231	3	0.221719	3	0.154037	3	0.102167	3	0.067371	3	0.044851
		4	0.173077	4	0.221719	4	0.189045	4	0.141462	4	0.101057	4	0.071234
		5	0.057692	5	0.166290	5	0.189045	5	0.165039	5	0.129930	5	0.097947
				6	0.087104	6	0.154037	6	0.165039	6	0.146588	6	0.119713
				7	0.024887	7	0.099024	7	0.141462	7	0.146588	7	0.131929
						8	0.045975	8	0.102167	8	0.129930	8	0.131929
						9	0.011920	9	0.059598	9	0.101057	9	0.119713
								10	0.025542	10	0.067371	10	0.097947
								11	0.006192	11	0.036748	11	0.071234
										12	0.014874	12	0.044851
										13	0.003432	13	0.023288
												14	0.009030
												15	0.002007

m=13,i=7													
n=3		n=5		n=7		n=9		n=11		n=13		n=15	
0	0.15	0	0.053922	0	0.022136	0	0.010062	0	0.004958	0	0.002609	0	0.001449
1	0.35	1	0.171569	1	0.083437	1	0.042260	1	0.022457	1	0.012494	1	0.007246
2	0.35	2	0.274510	2	0.166873	2	0.096594	2	0.056143	2	0.033318	2	0.020290
3	0.15	3	0.274510	3	0.227554	3	0.156037	3	0.101057	3	0.064676	3	0.041648
		4	0.171569	4	0.227554	4	0.195046	4	0.144367	4	0.101057	4	0.069413
		5	0.053922	5	0.166873	5	0.195046	5	0.171019	5	0.133395	5	0.098811
				6	0.083437	6	0.156037	6	0.171019	6	0.152451	6	0.123514
				7	0.022136	7	0.096594	7	0.144367	7	0.152451	7	0.137630
						8	0.042260	8	0.101057	8	0.133395	8	0.137630
						9	0.010062	9	0.056143	9	0.101057	9	0.123514
								10	0.022457	10	0.064676	10	0.098811
								11	0.004958	11	0.033318	11	0.069413
										12	0.012494	12	0.041648
										13	0.002609	13	0.020290
												14	0.007246
												15	0.001449

m=15,i=8													
n=3		n=5		n=7		n=9		n=11		n=13		n=15	
0	0.147059	0	0.051084	0	0.020124	0	0.008749	0	0.004119	0	0.002070	0	0.001099
1	0.352941	1	0.170279	1	0.080495	1	0.039373	1	0.020137	1	0.010766	1	0.005997
2	0.352941	2	0.278638	2	0.167183	2	0.094495	2	0.053305	2	0.030598	2	0.017991
3	0.147059	3	0.278638	3	0.232198	3	0.157491	3	0.099946	3	0.062330	3	0.038981
		4	0.170279	4	0.232198	4	0.199892	4	0.146588	4	0.100828	4	0.067703
		5	0.051084	5	0.167183	5	0.199892	5	0.175905	5	0.136117	5	0.099298
				6	0.080495	6	0.157491	6	0.175905	6	0.157291	6	0.126556
				7	0.020124	7	0.094495	7	0.146588	7	0.157291	7	0.142375
						8	0.039373	8	0.099946	8	0.136117	8	0.142375
						9	0.008749	9	0.053305	9	0.100828	9	0.126556
								10	0.020137	10	0.062330	10	0.099298
								11	0.004119	11	0.030598	11	0.067703
										12	0.010766	12	0.038981
										13	0.002070	13	0.017991
												14	0.005997
												15	0.001099

Table 4: Distribution of E for exceedances above the largest

m=3,i=3													
n=3		n=5		n=7		n=9		n=11		n=13		n=15	
0	0.50	0	0.37500	0	0.300000	0	0.250000	0	0.214286	0	0.187500	0	0.166667
1	0.30	1	0.267857	1	0.233333	1	0.204545	1	0.181319	1	0.162500	1	0.147059
2	0.15	2	0.178571	2	0.175000	2	0.163636	2	0.151099	2	0.139286	2	0.128676
3	0.05	3	0.107143	3	0.125000	3	0.127273	3	0.123626	3	0.117857	3	0.111520
		4	0.053571	4	0.083333	4	0.095455	4	0.098901	4	0.098214	4	0.095588
		5	0.017857	5	0.050000	5	0.068182	5	0.076923	5	0.080357	5	0.080882
				6	0.025000	6	0.045455	6	0.057692	6	0.064286	6	0.067402
				7	0.008333	7	0.027273	7	0.041209	7	0.050000	7	0.055147
						8	0.013636	8	0.027473	8	0.037500	8	0.044118
						9	0.004545	9	0.016484	9	0.026786	9	0.034314
								10	0.008242	10	0.017857	10	0.025735
								11	0.002747	11	0.010714	11	0.018382
										12	0.005357	12	0.012255
										13	0.001786	13	0.007353
												14	0.003676
												15	0.001225

m=5,i=5													
n=3		n=5		n=7		n=9		n=11		n=13		n=15	
0	0.625000	0	0.500000	0	0.416667	0	0.357143	0	0.3125	0	0.277778	0	0.25
1	0.267857	1	0.277778	1	0.265152	1	0.247253	1	0.229167	1	0.212418	1	0.197368
2	0.089286	2	0.138889	2	0.159091	2	0.164835	2	0.16369	2	0.159314	2	0.153509
3	0.017857	3	0.059524	3	0.088384	3	0.104895	3	0.113324	3	0.11683	3	0.117389
		4	0.019841	4	0.044192	4	0.062937	4	0.075549	4	0.08345	4	0.088042
		5	0.003968	5	0.018939	5	0.034965	5	0.048077	5	0.057773	5	0.064564
				6	0.006313	6	0.017483	6	0.028846	6	0.038515	6	0.046117
				7	0.001263	7	0.007493	7	0.016026	7	0.02451	7	0.031927
						8	0.002498	8	0.008013	8	0.014706	8	0.021285
						9	0.000500	9	0.003434	9	0.00817	9	0.013545
								10	0.001145	10	0.004085	10	0.008127
								11	0.000229	11	0.001751	11	0.004515
										12	0.000584	12	0.002257
										13	0.000117	13	0.000967
												14	0.000322
												15	6.45E-05

m=7,i=7													
n=3		n=5		n=7		n=9		n=11		n=13		n=15	
0	0.700000	0	0.583333	0	0.500000	0	0.437500	0	0.388889	0	0.35	0	0.318182
1	0.233333	1	0.265152	1	0.269231	1	0.262500	1	0.251634	1	0.239474	1	0.227273
2	0.058333	2	0.106061	2	0.134615	2	0.150000	2	0.157271	2	0.159649	2	0.159091
3	0.008333	3	0.035354	3	0.061189	3	0.080769	3	0.094363	3	0.103302	3	0.108852
		4	0.008838	4	0.024476	4	0.040385	4	0.053922	4	0.064564	4	0.072568
		5	0.001263	5	0.008159	5	0.018357	5	0.029035	5	0.038738	5	0.046956
				6	0.002040	6	0.007343	6	0.014517	6	0.022136	6	0.029347
				7	0.000291	7	0.002448	7	0.006599	7	0.01192	7	0.017608
						8	0.000612	8	0.00264	8	0.00596	8	0.010062
						9	8.74E-05	9	0.00088	9	0.002709	9	0.005418
								10	0.00022	10	0.001084	10	0.002709
								11	3.14E-05	11	0.000361	11	0.001231
										12	9.03E-05	12	0.000493
										13	1.29E-05	13	0.000164
												14	4.1E-05
												15	5.86E-06

m=9,i=9													
n=3		n=5		n=7		n=9		n=11		n=13		n=15	
0	0.750000	0	0.642857	0	0.562500	0	0.5	0	0.45	0	0.409091	0	0.375
1	0.204545	1	0.247253	1	0.262500	1	0.264706	1	0.260526	1	0.253247	1	0.244565
2	0.040909	2	0.082418	2	0.112500	2	0.132353	2	0.144737	2	0.151948	2	0.155632
3	0.004545	3	0.022478	3	0.043269	3	0.061765	3	0.076625	3	0.08797	3	0.096344
		4	0.004496	4	0.014423	4	0.026471	4	0.038313	4	0.048872	4	0.057806
		5	0.000500	5	0.003934	5	0.010181	5	0.017879	5	0.025874	5	0.033467
				6	0.000787	6	0.003394	6	0.007663	6	0.012937	6	0.018593
				7	8.74E-05	7	0.000926	7	0.002947	7	0.006037	7	0.009843
						8	0.000185	8	0.000982	8	0.002587	8	0.004922
						9	2.06E-05	9	0.000268	9	0.000995	9	0.002297
								10	5.36E-05	10	0.000332	10	0.000984
								11	5.95E-06	11	9.05E-05	11	0.000379
										12	1.81E-05	12	0.000126
										13	2.01E-06	13	3.44E-05
												14	6.88E-06
												15	7.65E-07
m=11,i=11													
n=3		n=5		n=7		n=9		n=11		n=13		n=15	
0	0.785714	0	0.687500	0	0.611111	0	0.55	0	0.5	0	0.458333	0	0.423077
1	0.181319	1	0.229167	1	0.251634	1	0.260526	1	0.261905	1	0.259058	1	0.253846
2	0.03022	2	0.065476	2	0.094363	2	0.115789	2	0.130952	2	0.141304	2	0.148077
3	0.002747	3	0.015110	3	0.031454	3	0.047678	3	0.06203	3	0.074017	3	0.083696
		4	0.002518	4	0.008987	4	0.017879	4	0.027569	4	0.037008	4	0.045652
		5	0.000229	5	0.002074	5	0.00596	5	0.011352	5	0.01753	5	0.023913
				6	0.000346	6	0.001703	6	0.004257	6	0.007791	6	0.011957
				7	3.14E-05	7	0.000393	7	0.001419	7	0.003208	7	0.005664
						8	6.55E-05	8	0.000405	8	0.001203	8	0.002517
						9	5.95E-06	9	9.36E-05	9	0.000401	9	0.001036
								10	1.56E-05	10	0.000115	10	0.000389
								11	1.42E-06	11	2.64E-05	11	0.00013
										12	4.41E-06	12	3.7E-05
										13	4.01E-07	13	8.54E-06
												14	1.42E-06
												15	1.29E-07

m=13,i=13													
n=3		n=5		n=7		n=9		n=11		n=13		n=15	
0	0.8125	0	0.722222	0	0.65	0	0.590909	0	0.541667	0	0.5	0	0.464286
1	0.1625	1	0.212418	1	0.239474	1	0.253247	1	0.259058	1	0.26	1	0.257937
2	0.023214	2	0.053105	2	0.079825	2	0.101299	2	0.117754	2	0.13	2	0.138889
3	0.001786	3	0.010621	3	0.023478	3	0.037321	3	0.050466	3	0.062174	3	0.072222
		4	0.001517	4	0.005869	4	0.01244	4	0.020186	4	0.028261	4	0.036111
		5	0.000117	5	0.001174	5	0.003659	5	0.007437	5	0.012112	5	0.017271
				6	0.000168	6	0.000915	6	0.002479	6	0.004845	6	0.00785
				7	1.29E-05	7	0.000183	7	0.000729	7	0.001785	7	0.003364
						8	2.61E-05	8	0.000182	8	0.000595	8	0.001346
						9	2.01E-06	9	3.65E-05	9	0.000175	9	0.000496
								10	5.21E-06	10	4.37E-05	10	0.000165
								11	4.01E-07	11	8.75E-06	11	4.86E-05
										12	1.25E-06	12	1.22E-05
										13	9.61E-08	13	2.43E-06
												14	3.47E-07
												15	2.67E-08

m=15,i=15													
n=3		n=5		n=7		n=9		n=11		n=13		n=15	
0	0.833333	0	0.75	0	0.681818	0	0.625	0	0.576923	0	0.535714	0	0.500000
1	0.147059	1	0.197368	1	0.227273	1	0.244565	1	0.253846	1	0.257937	1	0.258621
2	0.018382	2	0.04386	2	0.068182	2	0.088933	2	0.105769	2	0.119048	2	0.129310
3	0.001225	3	0.00774	3	0.017943	3	0.029644	3	0.041388	3	0.052381	3	0.062261
		4	0.000967	4	0.003987	4	0.008893	4	0.01505	4	0.021825	4	0.028736
		5	6.45E-05	5	0.000704	5	0.00234	5	0.005017	5	0.00854	5	0.012644
				6	8.8E-05	6	0.00052	6	0.001505	6	0.003106	6	0.005268
				7	5.86E-06	7	9.18E-05	7	0.000396	7	0.001035	7	0.002061
						8	1.15E-05	8	8.8E-05	8	0.000311	8	0.000750
						9	7.65E-07	9	1.55E-05	9	8.17E-05	9	0.000250
								10	1.94E-06	10	1.82E-05	10	7.5E-05
								11	1.29E-07	11	3.2E-06	11	1.97E-05
										12	4.01E-07	12	4.38E-06
										13	2.67E-08	13	7.74E-07
												14	9.67E-08
												15	6.45E-09