

Stochastic analysis of a dependent parallel system

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Theory and Methodology

Stochastic analysis of a dependent parallel system

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Abstract: This article discusses the stochastic behaviour of a two-unit parallel redundant repairable system with statistically dependent units. Important performance measures for the system, namely reliability, mean time to system failure, availability, stationary availability, joint availability and interval reliability are obtained in an explicit form. The transient behaviour of the system is characterised for a wide class of repair time distributions. The lifetimes of the units are modelled as bivariate exponential to capture the statistical dependence of the units. The article concludes with a detailed investigation of the stochastic point process induced by entries to various states, which correspond to the number of failed components in the system.

Keywords: Stochastic process; Parallel redundancy; Reliability (interval); Availability (joint); Intensity function; Product density

Motivation

The analysis of a parallel redundant system where the units can be repaired on failure has extensive literature. The fundamental and original contribution is due to Gaver (1963, 1964), who considered a two-unit parallel redundant system with constant hazard rate (for the individual independent units) and arbitrarily distributed repair times. Gaver used supplementary variables (Cox, 1955) to derive the mean time to system failure and stationary availability. Ever since this reported research there have been many attempts to derive the mean time to system failure (MTSF) and stationary availability under relaxed assumptions on the lifetime and repair time distributions of the units in the system. Some of the notable contributions are Kodama et al. (1974), Linton (1976), Subramanian et al. (1979), Takeda et al. (1979), Ravichandran (1981), Osaki (1985) and Liebowitz (1986). Ravichandran (1991) reviewed the state of the art for this system. An explicit derivation of the system's operating characteristics, when the failure and repair rates are non-constant, is not easy and regenerative simulation (Rubenstein, 1981) is a meaningful alternative.

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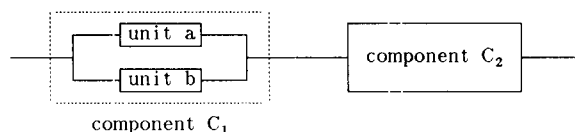


Figure 1. Parallel redundancy with dependent units represented by a two-component series system

There are several directions in which the fundamental system can be extended to meet the reality of a practical situation. In this article the situation is analysed where the units are not statistically independent: it is assumed that common cause failures may occur, which destroy both units simultaneously. Harris (1968) used a bivariate exponential (BVE) distribution to model the lifetimes of the units and derived the MTSF by using the supplementary variable technique for an arbitrary repair time distribution. Osaki (1970, 1980) extended the analysis to obtain the availability of the system by using a variant of a semi-Markov process with some non-regeneration points.

Here it is shown that the analysis of a two-unit dependent parallel system can be carried out by using an appropriate imbedded renewal process. Using the imbedded renewal process, explicit expressions for several operating characteristics, such as reliability, MTSF, availability, stationary availability, joint availability and interval reliability are obtained. The transient behaviour of the system for a wide class of repair time distributions is also reported. Finally, the intensity functions associated with the counting process of various stochastic point events are investigated; the expected value of the counting measure of the number of events in $(0, t)$ and its variance are derived.

There is yet another interesting way to approach the case of parallel redundancy with dependent units. Using a BVE to represent the lifetimes of the units, it is easy to see that the hazard rate in various states (which are defined by the number of failed units) is expressed as a sum of two terms, one corresponding to the 'baseline' hazard rate for the system and the other due to common cause failure. In other words: the system's hazard rate equals the hazard rate of a two components *series* system, consisting of components C_1 and C_2 , where:

1. C_1 represents a two unit parallel redundant system which units have independently (exponentially) distributed lifetimes.
2. C_2 represents an artificial component, used (in this particular case) to model dependence between the units of component C_1 (see Figure 1).

From this point of view, the system is a specific, but interesting example of the additive hazards model, proposed and reviewed in Pijenburg (1990) in contrast to the proportional hazards model of Cox (1972).

1. Model description

A representation for the joint survival function of the lifetimes of the components 1 and 2 is given by the BVE introduced by Marshall et al. (1967). Hence

$$\Pr\{X_1 > t_1, X_2 > t_2\} = \exp(-\lambda_1 t_1 - \lambda_2 t_2 - \lambda_{12} \max(t_1, t_2)) \quad (1.1)$$

where $t_1, t_2 \geq 0$ and $\lambda_1, \lambda_2, \lambda_{12}$ are non-negative constants. It is useful to note that:

1. The joint lifetime of both components is characterised by (1.1), hence the instantaneous failure rate, when both units are operable, is $\lambda_1 + \lambda_2 + \lambda_{12}$.
2. The lifetime of unit 1 (2), when the other unit is not operating, is negative exponentially distributed with parameter $\lambda_1 + \lambda_{12}$ ($\lambda_2 + \lambda_{12}$).
3. The time until both units fail simultaneously, due to common cause, is negative exponentially distributed with parameter λ_{12} .

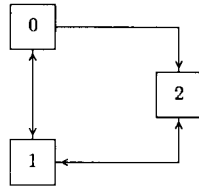


Figure 2. One-step transition diagram, similar units

Observations 1–3 are crucial for the subsequent analysis and bring out the additive hazard nature (from various sources) of the system. Some of the mathematical properties and results of the BVE can be found in Barlow and Proschan (1975, pp. 128, 129).

The following model assumptions are made:

1. The system consists of two units in a parallel configuration. The system requires only one unit for operation.
2. The units are repairable. On failure they are repaired by a single server repair facility with FIFO repair policy. Repairs are assumed to restore the normal operational efficiency of the units perfectly.
3. The identification of the operable and non-operable status of a unit is perfect. A unit is switched from the operating position to the repair facility and vice versa instantaneously.
4. The lifetimes of the units are statistically dependent as they are subjected to a common cause failure which occurs with rate λ_{12} . The joint lifetime distribution of the units is represented by the BVE (1.1).
5. The repair time durations of the units are identically distributed random variables with pdf $g(\cdot)$.

2. Stochastic behaviour of the system

For the present analysis the units are assumed to be physically identical but statistically dependent. A state description of the system can be given as the number of units operating (or under failure). Let $X(t)$ be the state of the system at time t , representing the number of failed units in the system. Then Figure 2 gives the one-step transition diagram of the system. State 2 is said to be the down state and the states 0 and 1 are called up states.

Questions about the operating characteristics can be translated into equivalent questions about the process $\{X(t), t \geq 0\}$. At time $t = 0$ we assume the system is operable with both units working. The various operating characteristics given $X(0) = 0$ are expressed in Table 1 in terms of the $X(t)$ process.

To study the behaviour of the induced stochastic process it is convenient to introduce the events E_i as E_i : Entrance into state i , $i = 0, 1, 2$.

Table 2 summarises the various possibilities for the occurrence of E_i -events and the properties associated with them.

In our analysis the occurrence of E_1 -events plays a crucial role as they are not only regenerative, but also form a renewal process (Cox, 1962). We exploit this structure of the E_1 -events in obtaining the system's operating characteristics.

Table 1

Measure	Symbol	Expression in terms of $\{X(t), t \geq 0\}$
Reliability	$R(t)$	$R(t) = \Pr\{X(u) \neq 2, 0 \leq u \leq t \mid X(0) = 0\}$
Availability	$A(t)$	$A(t) = \Pr\{X(t) \neq 2 \mid X(0) = 0\}$
Interval reliability	$R(t, \tau)$	$R(t, \tau) = \Pr\{X(u) \neq 2, t \leq u \leq t + \tau \mid X(0) = 0\}$
Joint availability	$A(t_1, t_2)$	$A(t_1, t_2) = \Pr\{X(t_1) \neq 2, X(t_2) \neq 2 \mid X(0) = 0\}$

Table 2

Event	Possible occurrences	Description	Nature of the event
E_0	Initial occurrence	Both the units are operating	Regenerative
E_0	From state 1	By repair completion	Regenerative
E_1	From state 0	By failure of one of the units	Regenerative
E_1	From state 2	By repair completion	Regenerative
E_2	From state 0	By failure of both the units	Regenerative
E_2	From state 1	By failure of operating unit before repair completion of the other unit	Non-regenerative

Notation

We conclude this section with a brief list of the notation that is used in this article:

- $f(t)$ = pdf of a random variable.
- $F(t) = \int_0^t f(u) du$.
- $\bar{F}(t) = 1 - F(t)$.
- $f^{(n)}(t)$ = n -fold convolution of $f(t)$ with itself.
- $f^*(s)$ = Laplace transform of $f(t)$.
- $*$ = Convolution symbol: $f(t) * g(t) = \int_0^t f(u)g(t-u) du$.
- $\bar{F}_0(t) = \exp(-(2\lambda + \lambda_{12})t)$.
- $\bar{F}_1(t) = \exp(-(\lambda + \lambda_{12})t)$.
- $f_1(t) = (\lambda + \lambda_{12}) \exp(-(\lambda + \lambda_{12})t)$.

3. The imbedded renewal process

Let X_{11} be a random variable representing the time interval between successive visits to state 1. At every new visit to state 1 an E_1 -event occurs. Let $\{t_i\}$ be the epochs at which an E_1 -event occurs. Then the durations $\{t_{i+1} - t_i\}$ are realisations of the random variable X_{11} . By the observed property of the E_1 -events, the durations $\{t_{i+1} - t_i\}$ correspond to intervals in a renewal process. The pdf of the random variable X_{11} characterises this renewal process. The random variable X_{11} is composed of three distinct parts (as detailed below) corresponding to the possible paths taken by the process $\{X(t), t \geq 0\}$:

- a) The process starts from state 1, visits state 2 by the failure of the operating unit before the repair completion of the other unit, and recovers subsequently, resulting in an occurrence of E_1 .
- b) The process starts from state 1, visits state 0 after repair completion of one of the units before failure of the other unit, and enters state 1 by failure of one of the units, inducing an E_1 -event.
- c) The process starts from state 1, visits state 0 after repair completion of one of the units before failure of the other unit, enters state 2 (causing a system failure) and recovers through a repair completion, causing the occurrence of an E_1 -event.

Before writing a formal expression for the pdf of the random variable X_{11} , we need to obtain the pdf of the random variables characterising an entry to the states 1 and 2 from state 0. Using $f_{0i}(t)$ to denote the pdf of the random variables representing the length of a stay in state 0, followed by a transition to state i we have

$$f_{01}(t) = 2\lambda \exp(-(2\lambda + \lambda_{12})t), \tag{3.1}$$

$$f_{02}(t) = \lambda_{12} \exp(-(2\lambda + \lambda_{12})t). \tag{3.2}$$

From (3.1)–(3.2), the earlier description of the process, the independence of the failure and repair times and the repair policy of the units, $f_{11}(t)$ is

$$f_{11}(t) = g(t)F_1(t) + [g(t)\bar{F}_1(t)] * f_{01}(t) + [g(t)\bar{F}_1(t)] * f_{02}(t) * g(t). \tag{3.3}$$

Relation (3.3) completely characterises the renewal process induced by E_1 -events. It is evident that the random variable X_{11} is proper. Its Laplace transform $f_{11}^*(s)$ is

$$f_{11}^*(s) = g^*(s) + g^*(\lambda + \lambda_{12} + s) \left(\frac{2\lambda + \lambda_{12}g^*(s)}{2\lambda + \lambda_{12} + s} - 1 \right). \quad (3.4)$$

From (3.4) the expected value $\mathbb{E}[X_{11}]$ of X_{11} is

$$\mathbb{E}[X_{11}] = \mu + \frac{g^*(\lambda + \lambda_{12})}{2\lambda + \lambda_{12}} (1 + \lambda_{12}\mu) \quad (3.5)$$

where μ is the mean repair time.

To go further, the pdf of a modified version of the random variable X_{11} is needed. Denote by \tilde{X}_{11} the time interval between two successive visits to state 1 (i.e. two occurrences of E_1 -events), with the provision that in between these visits the process $X(t)$ does *not* visit state 2, the system's down state. This pdf is denoted as $\tilde{f}_{11}(t)$ and is obtained by dropping the first term in (3.3) (transitions 1–2–1) and the third term (transitions 1–0–2–1) and retaining the second term (transitions 1–0–1). Hence,

$$\tilde{f}_{11}(t) = [g(t)\bar{F}_1(t)] * f_{01}(t). \quad (3.6)$$

The density given by (3.6) for obvious reasons is defective. Its Laplace transform is given by

$$\tilde{f}_{11}^*(s) = \frac{2\lambda g^*(\lambda + \lambda_{12} + s)}{2\lambda + \lambda_{12} + s}. \quad (3.7)$$

4. Reliability and availability

It is now possible to obtain the operating characteristics of the system under investigation. Let $R_1(t)$ and $A_1(t)$ be the reliability and the availability of the system, conditioned on an E_1 -event at the time origin. The next result specifies these measures.

Result 1. Let

$$\alpha(t) = \bar{G}(t)\bar{F}_1(t) + [g(t)\bar{F}_1(t)] * \bar{F}_0(t). \quad (4.1)$$

then

$$R_1(t) = \alpha(t) + \sum_{n=1}^{\infty} \tilde{f}_{11}^{(n)}(t) * \alpha(t), \quad (4.2)$$

$$A_1(t) = \alpha(t) + \sum_{n=1}^{\infty} f_{11}^{(n)}(t) * \alpha(t). \quad (4.3)$$

The derivation of expressions (4.1)–(4.3) is achieved by observing the stochastic behaviour of the system. The function $\alpha(t)$ is used both in the reliability and availability function. It represents the probability that in an interval initiated by an E_1 -event, the system *neither* fails *nor* induces the occurrence of an E_1 -event. The derivation is obtained by considering the mutually exclusive cases that the repair, which commenced at the time origin, is completed or not before time t . Further, expressions (4.2) and (4.3) are obtained by classifying the time interval under consideration as:

- a) No occurrence of E_1 -event in $(0, t)$.
- b) Several occurrences of E_1 -events in $(0, t)$ and the last one occurs in $(u, u + du)$, $u < t$.

In (4.3) the system can visit state 2 in between two successive E_1 -events, whereas in (4.2) such visits are not possible. These requirements are met by using the functions $f_{11}(\cdot)$ and $\tilde{f}_{11}(\cdot)$.

It should be remarked that (4.1)–(4.3) are fundamental for the analysis of the system and the reasoning used in their derivations is typical and standard for the subsequent results in this work.

Expressions (4.2) and (4.3) give the two key operating characteristics of the system. More details regarding their behaviour and computational feasibility are postponed to the second part of this article. Two important summary measures associated with the performance of the system, viz. the mean time to system failure (MTSF) and the stationary availability (β) of the system, are derived. They are well known in reliability literature (Birolini, 1985) in terms of their direct physical interpretation.

The best starting point for the MTSF and β is directly from the Laplace transforms of (4.1)–(4.3). From (4.1),

$$\alpha^*(s) = \frac{1}{\lambda + \lambda_{12} + s} + g^*(\lambda + \lambda_{12} + s) \left(\frac{1}{2\lambda + \lambda_{12} + s} - \frac{1}{\lambda + \lambda_{12} + s} \right).$$

Using X to represent the lifetime of the system starting with an E_1 -event at the origin, the reliability $R_1(t)$ of the system is equivalent to $R_1(t) = \Pr\{X > t\}$ and $\text{MTSF} = \mathbb{E}[X] = R_1^*(s)|_{s=0}$.

From (4.2),

$$R_1^*(s) = \frac{\alpha^*(s)}{1 - \tilde{f}_{11}^*(s)} = \frac{2\lambda + \lambda_{12} + s - \lambda g^*(\lambda + \lambda_{12} + s)}{(2\lambda + \lambda_{12} + s - 2\lambda g^*(\lambda + \lambda_{12} + s))(\lambda + \lambda_{12} + s)} \quad (4.4)$$

and

$$\text{MTSF} = \frac{2\lambda + \lambda_{12} - \lambda g^*(\lambda + \lambda_{12})}{(2\lambda + \lambda_{12} - 2\lambda g^*(\lambda + \lambda_{12}))(\lambda + \lambda_{12})}. \quad (4.5)$$

Using (4.3), the Laplace transform of the availability of the system conditioned on an event E_1 is

$$\begin{aligned} A_1^*(s) &= \frac{\alpha^*(s)}{1 - f_{11}^*(s)} \\ &= \frac{2\lambda + \lambda_{12} + s - \lambda g^*(\lambda + \lambda_{12} + s)}{((2\lambda + \lambda_{12} + s + \lambda_{12} g^*(\lambda + \lambda_{12} + s))(1 - g^*(s)) + s g^*(\lambda + \lambda_{12} + s))(\lambda + \lambda_{12} + s)}. \end{aligned} \quad (4.6)$$

The stationary availability of the system conditioned by an event E_1 at the origin (in any case, the stationary availability is independent of the initial event at time $t = 0$) is obtained as the limiting value of $A_1(t)$ as $t \rightarrow \infty$. By the key renewal theorem,

$$\beta = \frac{1}{\mathbb{E}[X_{11}]} \int_0^\infty \alpha(u) \, du = \frac{2\lambda + \lambda_{12} - \lambda g^*(\lambda + \lambda_{12})}{(\lambda + \lambda_{12})(\mu(2\lambda + \lambda_{12}) + g^*(\lambda + \lambda_{12})(1 + \lambda_{12}\mu))}. \quad (4.7)$$

An alternative way to compute the stationary availability is by using Tauberian theorems, which state that β may be computed as $\beta = \lim_{s \rightarrow 0} s A_1^*(s)$.

The stationary measure β can also be obtained directly from the stochastic process $X(t)$, the number

of failed units at time t . Using the imbedded renewal process corresponding to the entries of the E_1 -events and arguments similar to the derivation of (4.2) and (4.3), $X(t)$ is characterised. Define

$$P_i(t) = \Pr\{X(t) = i \mid E_1 \text{ at } t = 0\}, \quad i = 0, 1, 2.$$

By the stated behaviour of the process, it follows that

$$P_i(t) = \gamma_i(t) + \sum_{n=1}^{\infty} f_{11}^{(n)}(t) * \gamma_i(t) \quad (4.8)$$

where

$$\gamma_0(t) = [g(t)\bar{F}_1(t)] * \bar{F}_0(t), \quad (4.9a)$$

$$\gamma_1(t) = \bar{G}(t)\bar{F}_1(t), \quad (4.9b)$$

$$\gamma_2(t) = \bar{G}(t)F_1(t) + [g(t)\bar{F}_1(t)] * f_{02}(t) * \bar{G}(t). \quad (4.9c)$$

The key renewal theorem (Smith, 1958) applied in (4.8) gives the stationary distribution $\{\pi_i\}$ of the process $X(t)$ as

$$\pi_0 = \frac{g^*(\lambda + \lambda_{12})}{\mu(2\lambda + \lambda_{12}) + g^*(\lambda + \lambda_{12})(1 + \lambda_{12}\mu)}, \quad (4.10a)$$

$$\pi_1 = \frac{(2\lambda + \lambda_{12})(1 - g^*(\lambda + \lambda_{12}))}{(\lambda + \lambda_{12})(\mu(2\lambda + \lambda_{12}) + g^*(\lambda + \lambda_{12})(1 + \lambda_{12}\mu))}, \quad (4.10b)$$

$$\pi_2 = \frac{\mu(\lambda + \lambda_{12})(2\lambda + \lambda_{12}) - (2\lambda + \lambda_{12})(1 - g^*(\lambda + \lambda_{12})) + \lambda_{12}\mu(\lambda + \lambda_{12})g^*(\lambda + \lambda_{12})}{(\lambda + \lambda_{12})(\mu(2\lambda + \lambda_{12}) + g^*(\lambda + \lambda_{12})(1 + \lambda_{12}\mu))}. \quad (4.10c)$$

It is easily verified that $\pi_0 + \pi_1$, determined by (4.10a–c), agrees with the stationary availability β obtained in (4.7).

Expressions (4.8)–(4.10) capture the stationary distribution of the process $\{X(t), t \geq 0\}$. In Pijenburg et al. (1991) the stationary distribution of the process $\{X(t), t \geq 0\}$ is investigated, under the additional condition that the process has not visited state 2. Further, the limiting residual lifetime distribution is subject of research. Such limiting distributions conditioned on an event whose probability tends to zero in the long range, are known as quasi-stationary distributions in stochastic process literature (Cavender, 1978) and they are useful when the system rarely enters the failed state during its life. This aspect has not been widely used in the reliability literature: in fact, the only known reported contribution in this context is that of Kalpakam et al. (1983).

5. More general performance measures

Expressions (4.2) and (4.3) give the reliability and the availability of the system, conditioned by an E_1 -event at the time origin. It is quite easy and useful to extend these measures for an event E_0 at the time origin. Using $R_0(t)$ and $A_0(t)$ to represent the reliability and availability under the changed initial condition, we readily obtain

$$R_0(t) = \bar{F}_0(t) + f_{01}(t) * R_1(t) \quad (5.1)$$

and

$$A_0(t) = \bar{F}_0(t) + [f_{01}(t) + f_{02}(t) * g(t)] * A_1(t) \quad (5.2)$$

where $R_1(t)$ and $A_1(t)$ are determined by expressions (4.2) and (4.3).

The derivation of (5.1) and (5.2) is based on considering whether there is a failure or not in the time interval under consideration. When there is no failure in $(0, t)$, with probability $\bar{F}_0(t)$, the system is obviously reliable in $(0, t)$ (and hence available at t). When there is a failure, an E_1 -event is induced and hence the required probability is related to $R_1(t)$ and $A_1(t)$ as in the second terms of (5.1) and (5.2).

The analysis is now extended to some of the more general operating characteristics namely interval reliability and joint availability. As expected, the imbedded renewal process described by (3.3) will play a dominant role in the derivations. Define

$$R_1(t, \tau) = \Pr\{X(t) = 0 \vee X(t) = 1, t \leq u \leq t + \tau \mid E_1 \text{ at } t = 0\},$$

$$A_1(t_1, t_2) = \Pr\{X(t) = 0 \vee X(t) = 1, t = t_1, t_2 \mid E_1 \text{ at } t = 0\}.$$

The function $R_1(t, \tau)$ represents the probability that the system is available for a duration τ , beginning at time t , conditioned by an E_1 -event at the origin. This function is a combined measure of availability and reliability introduced earlier. The measures reliability ($R_1(t)$) and availability ($A_1(t)$) are recovered from $R_1(t, \tau)$ by setting t and τ , respectively, to zero. The function $A_1(t_1, t_2)$ represents the joint probability that the system is available at the time epochs t_1 and t_2 , given an E_1 -event at $t = 0$. Explicit expressions for the interval reliability and joint availability are obtained below.

Result 2. The interval reliability of the system, conditioned by an E_1 -event at $t = 0$, is given by

$$R_1(t, \tau) = \varphi(t, \tau) + \sum_{n=1}^{\infty} \int_0^t f_{11}^{(n)}(u) \varphi(t - u, \tau) \, du \quad (5.3)$$

where

$$\varphi(t, \tau) = \bar{G}(t + \tau) \bar{F}_1(t + \tau) + \int_0^{t+\tau} g(u) \bar{F}_1(u) \bar{F}_0(t + \tau - u) \, du$$

$$+ \int_t^{t+\tau} \tilde{f}_{11}(u) R_1(t + \tau - u) \, du.$$

Result 3. The joint availability of the system, conditioned by an E_1 -event at $t = 0$, is given by

$$A_1(t_1, t_2) = \psi(t_1, t_2) + \sum_{n=1}^{\infty} \int_0^{t_1} f_{11}^{(n)}(u) \psi(t_1 - u, t_2 - u) \, du \quad (5.4)$$

where

$$\psi(t_1, t_2) = \bar{G}(t_2) \bar{F}_1(t_2) + \int_{t_1}^{t_2} g(u) \bar{F}_1(u) A_0(t_2 - u) \, du$$

$$+ \int_{t_1}^{t_2} f_{11}(u) \int_0^{t_2-u} g(u + v) A_1(t_2 - u - v) \, dv \, du$$

$$+ \int_0^{t_1} g(u) \bar{F}_1(u) \bar{F}_0(t_2 - u) \, du + \int_0^{t_1} g(u) \bar{F}_1(u) \int_{t_1-u}^{t_2-u} f_{01}(v) A_1(t_2 - u - v) \, dv \, du$$

$$+ \int_0^{t_1} g(u) \bar{F}_1(u) \int_{t_1-u}^{t_2-u} f_{02}(v) \int_0^{t_2-u-v} g(w) A_1(t_2 - u - v - w) \, dw \, dv \, du.$$

Expression (5.3) is obtained by classifying the events according to the number of E_1 -events which occurs in the intervals $(0, t)$ and $(t, t + \tau)$. In $\varphi(t, \tau)$ the first two terms follow from the non-occurrence of an E_1 -event in both $(0, t)$ and $(t, t + \tau)$ and the third term follows from a non-occurrence in $(0, t)$ and one or more occurrences in $(t, t + \tau)$. Finally, the integral in (5.3) represents the probability of exactly n occurrences, $n \geq 1$, in $(0, t)$ and any number in $(t, t + \tau)$. The derivation of (5.4) is rather delicate and involves a careful follow-up of the process until time t_2 . The major classification is the number of occurrences of E_1 -events in $(0, t_1)$. The further subclassification is based on whether the process is in state 0 or state 1 at time t_1 and whether the process remains in this state or not during (t_1, t_2) .

6. Intensity of the event E_i

In this section we study the point events generated by the process $\{X(t), t \geq 0\}$. The objective is to obtain expressions for the expected value, the variance and covariance of the counting measures associated with the E_i -events ($i = 0, 1, 2$). The next observations follow from the discussion of the stochastic process $\{X(t), t \geq 0\}$.

- The process $\{X(t), t \geq 0\}$ induces the event E_i corresponding to the entry to state i ($i = 0, 1, 2$).
- The entries to the states 0 and 1 are regenerative. Also, the events E_0 and E_1 induce a sequence of renewal events.
- The entry to state 2 is non-regenerative if it occurs from state 1 (hereafter called a β -event) and is regenerative if it occurs from state 0 (hereafter called an α -event).

Let $N_i(t)$ be the counting measure associated with event i , $i = 0, 1, \alpha, \beta$. The objective is to obtain $\mathbb{E}[N_i(t)]$, $\text{Var}[N_i(t)]$ and $\text{Cov}(N_\alpha(t), N_\beta(t))$. The basic approach is to use the product densities associated with the events. The product densities are the counter parts of renewal density in the context of non-renewal stochastic point processes and are systematically discussed and illustrated in Srinivasan (1974) and also in Cox et al. (1980).

The analysis begins with E_1 -events. By the assumed stationarity of the $X(t)$ -process the counting process $N_i(0, t)$ would be represented by $N_i(t)$. The renewal density of E_1 -events is used to obtain the expected value and the second order characteristics. Further, define

$$h_1(t) = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \Pr\{N_1(t + \Delta t) - N_1(t) = 1 \mid E_1 \text{ at } t = 0\},$$

and, for $t_1 \neq t_2$,

$$h_1(t_1, t_2) = \lim_{\Delta t_1, \Delta t_2 \rightarrow 0} \frac{1}{\Delta t_1 \Delta t_2} \Pr\{N_1(t_1 + \Delta t_1) - N_1(t_1) = 1, i = 1, 2 \mid E_1 \text{ at } t = 0\}.$$

The functions $h_1(t)$ and $h_1(t_1, t_2)$ can be interpreted as follows: $h_1(t)\Delta t$ represents the probability of an occurrence of an E_1 -event in $(t, t + \Delta t)$, and $h_1(t_1, t_2)\Delta t_1\Delta t_2$ represents the joint occurrence of an E_1 -event in $(t_1, t_1 + \Delta t_1)$ and another E_1 -event in $(t_2, t_2 + \Delta t_2)$. From the renewal nature of E_1 -events and the possibility that the required event in $(t, t + \Delta t)$ may be the first or any subsequent event,

$$h_1(t) = \sum_{n=1}^{\infty} f_{11}^{(n)}(t) \quad (6.1)$$

and, for $t_1 < t_2$,

$$h_1(t_1, t_2) = h_1(t_1)h_1(t_2 - t_1). \quad (6.2)$$

In (6.2) and later product densities the assumption $t_1 < t_2$ is made; for obvious reasons the roles of t_1 and t_2 are interchanged if $t_2 < t_1$.

The expected number of E_1 -events in $(0, t)$ is obtained by integrating the function $h_1(u)$ over $(0, t)$. The second factorial moment is (Cox et al., 1980)

$$\mathbb{E}\{N_1(t)[N_1(t) - 1]\} = \int_{\substack{0 < t_1, t_2 \leq t \\ t_1 \neq t_2}} h_1(t_1, t_2) dt_1 dt_2, \quad (6.3)$$

and hence

$$\text{Var}[N_1(t)] = \int_{\substack{0 < t_1, t_2 \leq t \\ t_1 \neq t_2}} h_1(t_1, t_2) dt_1 dt_2 + \mathbb{E}[N_1(t)] - \{\mathbb{E}[N_1(t)]\}^2. \quad (6.4)$$

The stationary frequency Λ_1 of E_1 -events is obtained from the limiting behaviour of $h_1(t)$:

$$\Lambda_1 = \lim_{t \rightarrow \infty} h_1(t) = \frac{1}{\mathbb{E}[X_{11}]} = \frac{2\lambda + \lambda_{12}}{\mu(2\lambda + \lambda_{12}) + g^*(\lambda + \lambda_{12})(1 + \lambda_{12}\mu)}. \quad (6.5)$$

Subsequently, the asymptotic behaviour of $h_1(t_1, t_2)$ for $t_1 \rightarrow \infty, t_2 \rightarrow \infty, t_2 - t_1 = \tau$ is

$$\lim_{\substack{t_1, t_2 \rightarrow \infty \\ t_2 - t_1 = \tau}} h_1(t_1, t_2) = \Lambda_1 h_1(\tau). \quad (6.6)$$

Finally, in the stationary case the variance of E_1 -events in an interval of length τ , denoted by $V_1(\tau)$ is obtained by using (6.6):

$$V_1(\tau) = \lim_{\substack{t_1, t_2 \rightarrow \infty \\ t_2 - t_1 = \tau}} \text{Var}[N_1(t_1, t_2)] = \Lambda_1 \tau (1 - \Lambda_1 \tau) + 2\Lambda_1 \int_0^\tau \int_0^u h_1(v - u) du dv. \quad (6.7)$$

The point events E_0 can be examined in the same way.

A more interesting sequence of point events is generated by E_2 -events. It is clear that in the stationary case the frequency Λ_2 of E_2 -events is $\Lambda_2 = \Lambda_\alpha + \Lambda_\beta$, where Λ_α and Λ_β are the stationary frequencies corresponding to α and β events.

The stationary intensity associated with E_2 events is

$$h_2(t) = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \Pr\{N_2(t + \Delta t) - N_2(t) = 1 \mid E_1 \text{ at } t = 0\}.$$

It is immediate that

$$h_2(t) = a(t) + \sum_{n=1}^{\infty} f_{11}^{(n)}(t) * a(t) \quad (6.8)$$

where

$$a(t) = [g(t)\bar{F}_1(t)] * f_{02}(t) + \bar{G}(t)f_1(t). \quad (6.9)$$

Alternatively,

$$h_2(t) = h_\alpha(t) + h_\beta(t) \quad (6.10)$$

where

$$h_\alpha(t) = a_\alpha(t) + \sum_{n=1}^{\infty} f_{11}^{(n)} * a_\alpha(t), \quad h_\beta(t) = a_\beta(t) + \sum_{n=1}^{\infty} f_{11}^{(n)}(t) * a_\beta(t),$$

and

$$a_\alpha(t) = [g(t)\bar{F}_1(t)] * f_{02}(t), \quad a_\beta(t) = \bar{G}(t)f_1(t).$$

Hence,

$$\Lambda_2 = \frac{1}{\mathbb{E}[X_{11}]} \int_0^\infty a(u) \, du = \frac{2\lambda + \lambda_{12} - 2\lambda g^*(\lambda + \lambda_{12})}{\mu(2\lambda + \lambda_{12}) + g^*(\lambda + \lambda_{12})(1 + \lambda_{12}\mu)}, \quad (6.11a)$$

and

$$\Lambda_\alpha = \frac{1}{\mathbb{E}[X_{11}]} \int_0^\infty a_\alpha(u) \, du = \frac{\lambda_{12} g^*(\lambda + \lambda_{12})}{\mu(2\lambda + \lambda_{12}) + g^*(\lambda + \lambda_{12})(1 + \lambda_{12}\mu)}, \quad (6.11b)$$

$$\Lambda_\beta = \frac{1}{\mathbb{E}[X_{11}]} \int_0^\infty a_\beta(u) \, du = \frac{(2\lambda + \lambda_{12})(1 - g^*(\lambda + \lambda_{12}))}{\mu(2\lambda + \lambda_{12}) + g^*(\lambda + \lambda_{12})(1 + \lambda_{12}\mu)}. \quad (6.11c)$$

The α -events, generated as a special sequence of E_2 -events, are regenerative (more specifically, they are renewal as well). The second-order properties are similar to E_1 -event or E_0 -event, and hence are not repeated here. However, the β -events are non-regenerative and it is useful to obtain the second-order properties of the counting process $N_\beta(t)$. This is obtained by using the appropriate product densities.

Define, for $t_1 \neq t_2$,

$$h_{\beta\beta}(t_1, t_2) = \lim_{\Delta t_1, \Delta t_2 \rightarrow 0} \frac{1}{\Delta t_1 \Delta t_2} \Pr\{N_\beta(t_i + \Delta t_i) - N_\beta(t_i) = 1, i = 1, 2 \mid E_1 \text{ at } t = 0\}.$$

Then

$$\begin{aligned} h_{\beta\beta}(t_1, t_2) &= f_1(t_1) \bar{G}(t_1) \int_0^{t_2-t_1} \frac{g(t_1+u)}{\bar{G}(t_1)} h_\beta(t_2-t_1-u) \, du \\ &\quad + \sum_{n=1}^\infty \int_0^{t_1} f_{11}^{(n)}(t_1-v) f_1(v) \bar{G}(v) \int_0^{t_2-t_1} \frac{g(v+u)}{\bar{G}(v)} h_\beta(t_2-t_1-u) \, du \, dv. \end{aligned} \quad (6.12)$$

Expression (6.12) is derived by considering the following specifications. It is required to obtain a β -event (a transition of state 1 to state 2) at t_1 and another β -event at t_2 ($t_1 \neq t_2$) conditioned by an E_1 -event at the origin:

- a) The β -event realised at t_1 may be caused by the repair not being completed until t_1 and a failure of the operating unit at t_1 . This gives the term $f_1(t_1) \bar{G}(t_1)$.
- b) The β -event realised at t_1 is non-regenerative, and the (elapsed) repair duration at t_1 is precisely the time until the last E_1 -event. For the occurrence of a β -event at t_2 , a repair completion is necessary between t_1 and t_2 : this gives the term $\{g(t_1+u)/\bar{G}(t_1)\} \, du$.
- c) When the repair (which began at the last E_1 -event before t_1) is completed, a fresh E_1 -event is generated and using this, the required β -event at t_2 is identified as the first-order product density in the appropriate interval.
- d) Alternatively, the β -event at t_1 may be preceded by several E_1 -events before t_1 and the last one occurred at $t_1 - v$, so that the elapsed repair duration of a unit at t_1 is v . Then the rest of the analysis is followed by using reasoning similar to a)-c).

The stationary behaviour $h_{\beta\beta}(\tau)$ of $h_{\beta\beta}(t_1, t_2)$, with $t_2 - t_1 = \tau$, is equivalent to

$$\begin{aligned} h_{\beta\beta}(\tau) &= \lim_{\substack{t_1, t_2 \rightarrow \infty \\ t_2 - t_1 = \tau}} h_{\beta\beta}(t_1, t_2) = \lim_{t_1 \rightarrow \infty} \sum_{n=1}^\infty \int_0^{t_1} f_{11}^{(n)}(t_1-v) f_1(v) \int_0^\tau g(u+v) h_\beta(\tau-u) \, du \, dv \\ &= \lim_{t_1 \rightarrow \infty} \int_0^{t_1} \hat{f}_\beta(\tau, t_1-v) \, dm(v) \end{aligned}$$

where

$$\hat{f}_\beta(\tau, v) = f_1(v) \int_0^\tau g(u+v) h_\beta(\tau-u) \, du$$

and $m(v)$ is the renewal function for X_{11} :

$$m(v) = \sum_{n=1}^{\infty} F_{11}^{(n)}(v).$$

From the key renewal theorem,

$$h_{\beta\beta}(\tau) = \frac{1}{\mathbb{E}[X_{11}]} \int_0^{\infty} \hat{f}_{\beta}(\tau, v) \, dv. \tag{6.13}$$

To complete the analysis of the stochastic point events associated with event E_2 , an expression for the second-order cross product densities is obtained.

For $t_1 < t_2$,

$$\begin{aligned} h_{\alpha\beta}(t_1, t_2) &= \lim_{\Delta_1, \Delta_2 \rightarrow 0} \frac{1}{\Delta_1 \Delta_2} \Pr\{N_{\alpha}(t_1 + \Delta_1) - N_{\alpha}(t_1) = 1, N_{\beta}(t_2 + \Delta_2) - N_{\beta}(t_2) = 1 \mid E_1 \text{ at } t = 0\}. \end{aligned}$$

The function $h_{\beta\alpha}(t_1, t_2)$ is defined similarly by interchanging the role of α and β .

From the renewal nature of the α -events it immediately follows that

$$h_{\alpha\beta}(t_1, t_2) = h_{\alpha}(t_1) \int_0^{t_2-t_1} g(u) h_{\beta}(t_2 - t_1 - u) \, du, \tag{6.14}$$

$$h_{\alpha\beta}(\tau) = \lim_{\substack{t_1, t_2 \rightarrow \infty \\ t_2 - t_1 = \tau}} h_{\alpha\beta}(t_1, t_2) = \Lambda_{\alpha} \int_0^{\tau} g(u) h_{\beta}(\tau - u) \, du. \tag{6.15}$$

With the observation that the β -events are non-regenerative and every β -event is followed by an E_1 -event, for $t_1 < t_2$,

$$\begin{aligned} h_{\beta\alpha}(t_1, t_2) &= f_1(t_1) \bar{G}(t_1) \int_0^{t_2-t_1} \frac{g(t_1+v)}{\bar{G}(t_1)} h_{\alpha}(t_2 - t_1 - v) \, dv \\ &\quad + \sum_{n=1}^{\infty} \int_0^{t_1} f_{11}^{(n)}(t_1 - v) f_1(v) \bar{G}(v) \int_0^{t_2-t_1} \frac{g(u+v)}{\bar{G}(v)} h_{\alpha}(t_2 - t_1 - u) \, du \, dv. \end{aligned} \tag{6.16}$$

The limiting behaviour of $h_{\beta\alpha}(t_1, t_2)$ with $t_1, t_2 \rightarrow \infty$ and $t_2 - t_1 = \tau$ is

$$h_{\beta\alpha}(\tau) = \frac{1}{\mathbb{E}[X_{11}]} \int_0^{\infty} \hat{f}_{\alpha}(\tau, v) \, dv \tag{6.17}$$

where

$$\hat{f}_{\alpha}(\tau, v) = f_1(v) \int_0^{\tau} g(u+v) h_{\alpha}(\tau - u) \, du.$$

Relations (6.1)–(6.17) summarise the properties of the counting measures associated with the E_i -events. Among the relations connecting the counting measures is the covariance function of the counting process associated with α - and β -events in $(0, t)$:

$$\text{Cov}[N_{\alpha}(t), N_{\beta}(t)] = \int_0^t \int_0^v \{h_{\alpha\beta}(u, v) + h_{\beta\alpha}(u, v)\} \, du \, dv - \mathbb{E}[N_{\alpha}(t)] \mathbb{E}[N_{\beta}(t)].$$

paper advances the state of art of a two-unit parallel redundant system, a basic model in reliability modelling, to an important dimension of dependent units. Several variations of models, like intermittently used systems and imperfect switch-over, may be handled similarly (Srinivasan et al., 1980; Ravichandran, 1991). In principle the analysis can be extended to more than two dependent components by using a multivariate exponential distribution. However, it seems more interesting to explore the use of multivariate distributions other than BVE to handle dependency structures, even in the context of two-unit systems. Some of the issues are under investigation and will be reported subsequently.

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