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An Accelerated Poincaré-map method for finding the PSS of Autonomous Oscillators

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Abstract

A novel time-domain method for finding the periodic steady-state of a free-running electrical oscillator is introduced. The method is based on the extrapolation technique MPE. This method is applied to the well-known Colpitt's Oscillator, for which it turns out to have super-linear convergence.

1 Introduction

The prediction of the behaviour of a particular electrical circuit, is of importance for the construction and design of efficient electronic devices. This paper focuses on determining the periodic steady-state (PSS) of a radio frequent (RF) oscillator circuit, which is a typical component of for instance microprocessors. One may distinguish between *non-autonomous* and *autonomous* oscillators. Non-autonomous (or *driven*) oscillators have a time-dependent input signal. A common situation is that the input signal is periodic, and the output signal is periodic with the same period as the input signal. In this case, the period T of the output signal is known a priori. On the other hand, autonomous (or *free-running*) oscillators have no time-dependent input signal, which means that it is in general not possible to predict the period T a priori.

For non-autonomous oscillators many efficient solution methods exist. For an overview, see [4], for a more recent overview see [6] or [3]. However, for *autonomous* or *free-running* oscillators, the situation is less satisfactory. Here, the period T is an additional degree of freedom, which makes the resulting system under-determined. Most methods proposed for the autonomous case have been based on methods for non-autonomous oscillators. Typically, this is done by considering the period T as an additional degree of freedom. Unfortunately, many of these methods are very sensitive with respect to the initial guess T_0 for the circuit's period T. They converge only for T_0 in a small neighbourhood of T.

This paper presents a novel method for the solution of autonomous oscillators. The paper is composed as follows. Section 2 introduces the concept of a periodic steady state. Section 3 presents a straightforward Poincaré-map method. Although this method is robust, it converges linearly for most real-world circuits. In particular, it converges linearly for Colpitt's oscillator, which

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is a well-known benchmark problem in the literature. For that reason, section 4 introduces an accelerated variant of the Poincaré-map method, based on minimal polynomial extrapolation (MPE). This method gives super-linear convergence for Colpitt's oscillator. Conclusions are presented in section 5. For a more detailed discussion of the method, see [2].

2 Periodic steady-states

In this paper, we will concentrate on methods for finding a stable periodic steady-state (see def. 1 and 3 below). Periodic steady-states that are not stable are not interesting for the IC designer, since they do not correspond to any physical behaviour of the modelled circuit. In fact, we want to actively avoid non-stable periodic steady-states for this reason.

Definition 1. *Consider an autonomous differential-algebraic equation (DAE) of the form:*

$$\frac{d}{dt}\mathbf{q}(\mathbf{x}) + \mathbf{j}(\mathbf{x}) = \mathbf{0}. \tag{1}$$

A function $\mathbf{x} : \mathbf{R} \to \mathbf{R}^n$ *is called a* periodic steady-state *(PSS) of* (1) *if:*

1. $\mathbf{x}$ *is a solution to* (1).
2. $\mathbf{x}$ *is periodic, i.e. there is a* $T > 0$ *such that for all* $t \in \mathbf{R}$, $\mathbf{x}(t) = \mathbf{x}(t+T)$.

Note that according to this definition, a stationary solution, i.e. a solution of the form $\mathbf{x}(t) \equiv \mathbf{x}_0$, is also a PSS.

Definition 2. *The* limit cycle $\mathcal{C}(\mathbf{x})$ *of a PSS* $\mathbf{x}$ *is the range of the function* $\mathbf{x}(t)$, *i.e.*

$$\mathcal{C}(\mathbf{x}) = \{\mathbf{x}(t) \mid t \in \mathbf{R}\}. \tag{2}$$

A set C *is called a limit cycle of* (1) *if there is a PSS* $\mathbf{x}$ *of* (1) *so that* $C = \mathcal{C}(\mathbf{x})$.

Definition 3. *A periodic steady-state* $\mathbf{x}$ *is called* stable[1] *if there is a* $\delta > 0$ *so that the following holds: For every solution* $\mathbf{x}^*$ *to* (1) *which has the property that*

$$\exists_{\tau_1 > 0} ||\mathbf{x}^*(0) - \mathbf{x}(\tau_1)|| < \delta, \tag{3}$$

there exists a $\tau_2 > 0$ *so that*

$$\lim_{t \to \infty} ||\mathbf{x}^*(t)(0) - \mathbf{x}(t + \tau_2)|| = 0 \tag{4}$$

A limit cycle is called stable when all of its periodic steady-states is stable.

A well-known example of a free-running oscillator is Colpitt's Oscillator. Its network schematics are shown in figure 1. Colpitt's oscillator converges to a PSS after some time.

[1] some authors prefer the term *strongly stable*

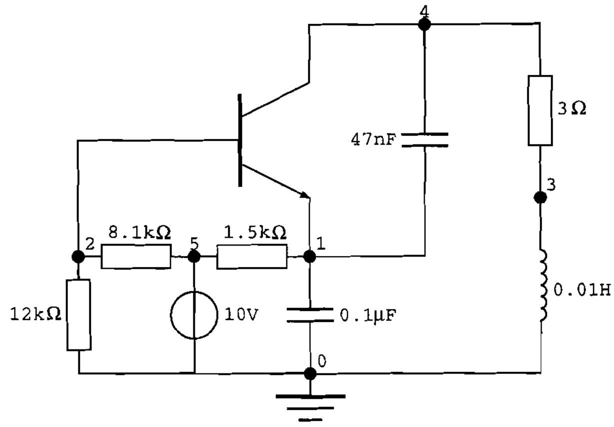


Figure 1: Colpitt's Oscillator

3 The Poincaré-map method

The Poincaré-map method for solving (1) is based on the following observation. Provided we start sufficiently close to a stable limit cycle $\mathcal{C}$, a transient simulation will eventually converge towards $\mathcal{C}$. After all, this is implied in the definition of a stable limit cycle. Therefore, we can simply approximate the PSS by starting at some point $\mathbf{x}_0$ and then performing a transient simulation until the solution $\mathbf{x}(t)$ has approached the stable limit cycle sufficiently close. There are, however, two problems with this approach

- We have to find a way to detect when we have approached the stable limit cycle close enough, so that we know when to stop.

- Convergence will be linear at best, which means that excessive computing time is needed to arrive at a solution.

This section addresses the first problem. The proposed solution method is still hampered by the second problem; therefore, it will be rather slow. However, in the next section we shall show how we can accelerate the method.

First we note that the length of the period can be estimated by looking for periodic recurring features in the computed circuit behaviour. A possible recurring feature is the point at which a specific condition is satisfied. This is equivalent to carrying out a Poincaré-map iteration, see [1], section I.16. The idea is to cut the transient solution $\mathbf{x}(t)$ by a hyperplane. The hyperplane is defined by an affine equation of the form $(\mathbf{x}(t), \mathbf{n}) = \alpha$, for some vector $\mathbf{n}$ and scalar α. This equation is called the *switch equation.* The situation is visualised in Figure 2. The unaccelerated Poincaré-map method can now be described as follows.

Algorithm 1. *Let an approximate solution* $\mathbf{x}_0$ *and a required accuracy tolerance* $\varepsilon > 0$ *be given. The approximated solution* $\tilde{\mathbf{x}}$ *and period* $\tilde{T}$ *is computed by:*

$i := 0$, $t_0 := 0$, $\mathbf{x}_0 :=$ *some initial guess for* $\mathbf{x}$
repeat

Starting with $t = t_i$, $\mathbf{x}(t_i) = \mathbf{x}_i$, *integrate* (1) *until* $(\mathbf{x}(t), \mathbf{n}) = \alpha$ *and* $d(\mathbf{x}(t), \mathbf{n})/dt > 0$.
$\mathbf{x}_{i+1} := \mathbf{x}(t)$, $t_{i+1} := t$
$\delta := \|\mathbf{x}_{i+1} - \mathbf{x}_i\|$
$i := i + 1$
until $\delta \leq \varepsilon$
$\tilde{T} := t_i - t_{i-1}$, $\tilde{\mathbf{x}} := \mathbf{x}_i$

This method has been tested on Colpitt's Oscillator. The errors after each iteration have been plotted in Figure 3 a.

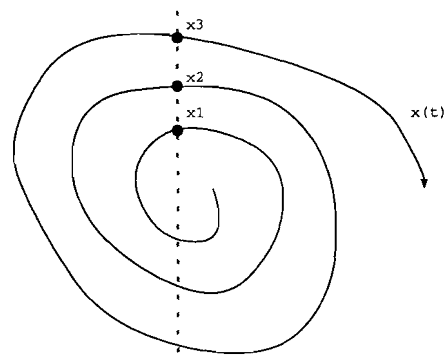


Figure 2: The trajectory of a solution, cut with a hyperplane

4 The MPE accelerated Poincaré-map method

Summarised, the Poincaré-map method consists of finding a fixed point of a function $F : \mathbf{R}^n \to \mathbf{R}^n$, which is defined as

$$F(\mathbf{x}_0) := \mathbf{x}(T_0). \tag{5}$$

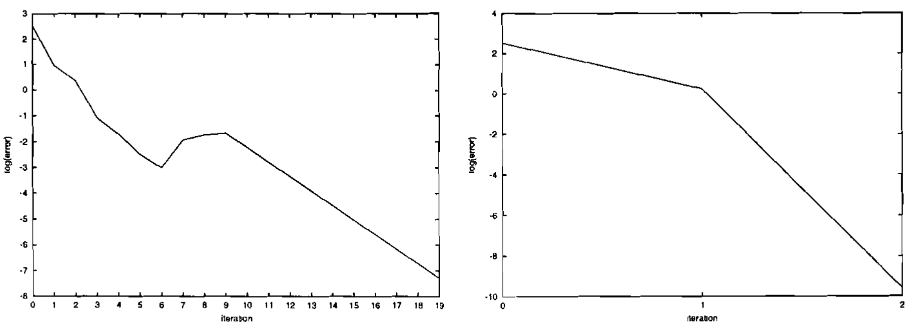
Here, $\mathbf{x}(t)$ is the solution of (1) with $\mathbf{x}(0) = \mathbf{x}_0$, and T_0 is the smallest $t > 0$ such that $(\mathbf{x}(t), \mathbf{n}) = \alpha$ and $d(\mathbf{x}(t), \mathbf{n})/dt > 0$.

For given $\mathbf{x}_0$, $F(\mathbf{x})$ is computed by use of Algorithm 1, i.e. by applying the ordinary Poincaré-map method. The successive approximations of the Poincaré-map method satisfy the recursion relation

$$\mathbf{x}_{n+1} = F(\mathbf{x}_n) \tag{6}$$

Suppose that the sequence (6) converges linearly to some fixed point $\tilde{\mathbf{x}}$ of F. We look for a way to accelerate this to super-linear convergence. An acceleration method operates on the first k vectors of a sequence $\{\mathbf{x}_n\}$, and produces an approximation $\mathbf{y}$ to the limit of $\{\mathbf{x}_n\}$. This approximation is then used to restart (6) and generate the beginning of a new sequence $\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2, \ldots$. Again, the acceleration method can be applied to this new sequence, resulting in a new approximation $\mathbf{z}$ of the limit. The idea is that the sequence $\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots$ converges much faster to the limit of $\{\mathbf{x}_n\}$ than the sequence $\{\mathbf{x}_n\}$ itself. Typically, if $\{\mathbf{x}_n\}$

converges linearly, then $\{\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots\}$ converges super-linearly. A well-known acceleration method is minimal polynomial extrapolation (MPE). Rather than describing MPE here in detail, the reader is referred to [5]. The MPE-accelerated Poincaré-map method has also been tested on Colpitt's Oscillator. The errors after each iteration have been plotted in Figure 3 b.



(a) Poincaré-map method

(b) MPE-accel. Poincaré-map method

Figure 3: $\log(\|x_i - x_{i-1}\|)$ after each iteration for Colpitt's Osc.

5 Conclusions

The Poincaré-map method and the MPE-accelerated Poincaré-map method have been tested on Colpitt's Oscillator. For the Poincaré-map method, convergence becomes linear after several iterations. The MPE-accelerated method leads to much faster convergence than the unaccelerated method.

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