

# Perturbation and approximation properties for abstract evolution equations of fractional order

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**PERTURBATION AND APPROXIMATION PROPERTIES  
FOR ABSTRACT EVOLUTION EQUATIONS  
OF FRACTIONAL ORDER**

**Emilia Bazhlekova**

**Abstract**

We investigate the abstract evolution equation of fractional order

$$\mathbf{D}^\alpha u = Au, \quad \alpha > 0,$$

where  $\mathbf{D}^\alpha$  is the Caputo fractional derivative of order  $\alpha$  and  $A$  is an unbounded closed operator in a Banach space  $X$ . Some perturbation properties are presented. Using a numerical approximation of  $\mathbf{D}^\alpha$  by fractional differences, a representation formula for the solution operator  $S_\alpha$  is obtained and applied for studying of the convergence of the corresponding numerical method. The results generalize known facts about  $C_0$ -semigroups and cosine operator functions.

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*Key Words and Phrases:* fractional calculus,  $C_0$ -semigroup, cosine operator function, Mittag-Leffler function

**1. Introduction**

Consider a linear closed operator  $A$  densely defined in a Banach space  $X$ . Let  $\alpha > 0$  and  $n \in \mathbf{N}$ . Given  $x \in X$ , we investigate the following Cauchy problem:

$$\begin{aligned} \mathbf{D}_t^\alpha u(t) &= Au(t), \quad t > 0, \quad n - 1 < \alpha \leq n, \\ u(0) &= x, \quad u^{(k)}(0) = 0, \quad k = 1, 2, \dots, n - 1. \end{aligned} \tag{1.1}$$

Here  $\mathbf{D}_t^\alpha$  is the *Caputo fractional derivative* of order  $\alpha$ :

$$\mathbf{D}_t^\alpha u(t) = J_t^{n-\alpha} D_t^n u(t), \quad n - 1 < \alpha \leq n, \tag{1.2}$$

where  $D_t^n = \left(\frac{d}{dt}\right)^n$  and  $J_t^\beta$  is the *Riemann-Liouville fractional integral*:

$$J_t^\beta u(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} u(s) ds, \quad \beta > 0; \quad J_t^0 u(t) = u(t). \tag{1.3}$$

The connection between  $\mathbf{D}_t^\alpha$  and the *Riemann-Liouville fractional derivative*

$$D_t^\alpha u(t) = D_t^n J_t^{n-\alpha} u(t), \quad n - 1 < \alpha \leq n, \tag{1.4}$$

is given by

$$\mathbf{D}_t^\alpha u(t) = D_t^\alpha \left( u(t) - \sum_{k=0}^{n-1} u^{(k)}(0) \frac{t^k}{k!} \right), \quad n-1 < \alpha \leq n. \quad (1.5)$$

For more details on fractional calculus and applications see [6] and [9]. For results on the abstract problem (1.1) (also with nonzero initial conditions) see [1], [2], [5] and references there.

Section 2. contains preliminaries. In Section 3. we study perturbation properties of problem (1.1). In Section 4. we derive a representation formula for the solution operator  $S_\alpha(t; A)$ , in terms of the resolvent of its generator  $A$ . The starting point is a numerical approximation of  $\mathbf{D}_t^\alpha$  by fractional differences. The convergence rate of the formula when  $\alpha \geq 1$  is estimated. The results generalize some facts concerning  $C_0$ -semigroups and cosine operator functions (COF) and also exhibit some new features.

## 2. Preliminaries

Throughout this paper  $D(A)$  is the domain,  $\varrho(A)$  is the resolvent set,  $R(\lambda, A) = (\lambda I - A)^{-1}$  is the resolvent operator of  $A$ ;  $\mathcal{B}(X)$  is the space of all bounded operators from  $X$  into itself;  $z^\alpha$  denotes the principal branch of  $z^\alpha$  in  $\mathbf{C}$  cut along the negative real axis.

DEFINITION 2.1. A family  $\{S_\alpha(t; A)\}_{t \geq 0} \subset \mathcal{B}(X)$  is called a *solution operator* for (1.1), if the following conditions are satisfied:

- a)  $S_\alpha(t; A)$  is strongly continuous for  $t \geq 0$  and  $S_\alpha(0; A) = I$ ;
- b)  $S_\alpha(t; A)D(A) \subset D(A)$  and  $AS_\alpha(t; A)x = S_\alpha(t; A)Ax$  for all  $x \in D(A)$ ;
- c)  $S_\alpha(t; A)x$  is a solution of (1.1) for all  $x \in D(A)$ ,  $t \geq 0$ .

DEFINITION 2.2. The solution operator  $S_\alpha(t; A)$  is called *exponentially bounded*, if there are constants  $M \geq 1$  and  $\omega \geq 0$  such that

$$\|S_\alpha(t; A)\| \leq M e^{\omega t}, \quad t \geq 0. \quad (2.1)$$

An operator  $A$  is said to belong to  $\mathcal{C}^\alpha(M, \omega)$ , if the problem (1.1) has a solution operator  $S_\alpha(t; A)$  satisfying (2.1). Denote  $\mathcal{C}^\alpha = \bigcup \{\mathcal{C}^\alpha(M, \omega); M \geq 1, \omega \geq 0\}$ .

Let us note that if  $A \in \mathcal{C}^\alpha(M, \omega)$  and  $\lambda \in \mathbf{C}$  such that  $\operatorname{Re} \lambda > \omega$  then  $\lambda^\alpha$  belongs to the resolvent set  $\varrho(A)$  of  $A$ ,  $R(\lambda^\alpha, A)$  is analytic in  $\lambda$  and

$$\lambda^{\alpha-1} R(\lambda^\alpha, A)x = \int_0^\infty e^{-\lambda s} S_\alpha(s)x ds, \quad x \in X, \quad (2.2)$$

(see e.g. [2, eqs (2.2), (2.3)]).

A characterization of  $\mathcal{C}^\alpha$  is given in the following generation theorem, a particular case of [13, Theorem 1.3].

THEOREM 2.1.  $A \in \mathcal{C}^\alpha(M, \omega)$  iff  $(\omega^\alpha, \infty) \subset \varrho(A)$  and

$$\|(\lambda^{\alpha-1}R(\lambda^\alpha, A))^{(n)}\| \leq Mn!(\lambda - \omega)^{-(n+1)}, \lambda > \omega, n \in \mathbf{N}_0. \quad (2.3)$$

It is known (see [2, Theorem 2.1]) that if  $\alpha > 2$ , then  $A \in \mathcal{C}^\alpha$  iff  $A \in \mathcal{B}(X)$ . That is why we consider only  $\alpha \in (0, 2]$ . In general, it is difficult to prove that  $A \in \mathcal{C}^\alpha$  verifying conditions (2.2) directly. Therefore it is useful to develop a perturbation theory for problem (1.1).

In this direction, the following *subordination principle* [2] could be also helpful.

THEOREM 2.2. Let  $0 < \alpha < \beta \leq 2$ ,  $\gamma = \alpha/\beta$ . If  $A \in \mathcal{C}^\beta$  then  $A \in \mathcal{C}^\alpha$ ,  $S_\alpha(t; A)$  is analytic in  $\{t \in \mathbf{C} \setminus \{0\}; |\arg t| < \min\{(1/\gamma - 1)\pi/2, \pi/2\}\}$  and the following representation holds

$$S_\alpha(t; A)x = \int_0^\infty \varphi_{t,\gamma}(s)S_\beta(s; A)x ds, \quad t > 0, x \in X, \quad (2.4)$$

where  $\varphi_{t,\gamma}(s) = t^{-\gamma}\Phi_\gamma(st^{-\gamma})$ ,  $\Phi_\gamma(z) = \sum_{n=0}^\infty \frac{(-z)^n}{n!(-\gamma n + 1 - \gamma)}$ .

### 3. Perturbation properties

A classical result (see [11] and [10]) is: if  $A$  is the generator of a  $C_0$ -semigroup (COF) and  $B \in \mathcal{B}(X)$ , then  $A+B$  is again a generator of a  $C_0$ -semigroup (COF). This is not true in general for solution operators of (1.1) with  $0 < \alpha < 1$ , as the following example shows.

EXAMPLE 3.1. Let  $0 < \alpha < 1$  be fixed. Assume  $X = l^1$  - the Banach space of all sequences  $x = \{x_n\}_{n=1}^\infty$ ,  $x_n \in \mathbf{C}$ , with norm  $\|x\| = \sum_{n=1}^\infty |x_n| < \infty$ . Let  $A_\alpha$  be an operator defined by  $A_\alpha x = \{\exp(i\alpha\pi/2)nx_n\}_{n=1}^\infty$  with domain  $D(A_\alpha) = \{x \in l^1 : \sum_{n=1}^\infty n|x_n| < \infty\}$ . We are going to prove that  $A_\alpha \in \mathcal{C}^\alpha$ , but  $A_\alpha + I \notin \mathcal{C}^\alpha$ .

Applying (A.2), it follows that if  $S_\alpha(t; A)$  exists, it is given by the formula

$$S_\alpha(t; A)x = \{E_\alpha(\exp(i\alpha\pi/2)nt^\alpha)x_n\}_{n=1}^\infty.$$

Since  $\arg(\exp(i\alpha\pi/2)nt^\alpha) = \alpha\pi/2$  for  $t \geq 0$ , the asymptotic property of Mittag-Leffler functions (A.3) implies  $|E_\alpha(\exp(i\alpha\pi/2)nt^\alpha)| \simeq 1/\alpha$ ,  $n \rightarrow \infty$ . Hence  $\|S_\alpha(t; A)x\| < (1/\alpha + \varepsilon)\|x\|$  for some  $\varepsilon > 0$  and any  $x \in l^1$ , hence  $A_\alpha \in \mathcal{C}^\alpha$ .

Similarly,

$$S_\alpha(t; A_\alpha + I)x = \{E_\alpha((\exp(i\alpha\pi/2)n + 1)t^\alpha)x_n\}_{n=1}^\infty.$$

Using again the asymptotic relation (A.3),

$$|E_\alpha((\exp(i\alpha\pi/2)n + 1)t^\alpha)| \simeq (1/\alpha) \exp(\operatorname{Re}(\exp(i\alpha\pi/2)n + 1)^{1/\alpha}t), \quad n \rightarrow \infty, \quad (3.1)$$

we shall show that, given  $t > 0$ , there is no constant  $C$  such that  $\|S_\alpha(t; A_\alpha + I)x\| \leq C\|x\|$ . Indeed, let us use the representation

$$\operatorname{Re}((\exp(i\alpha\pi/2)n + 1)^{1/\alpha}) = \operatorname{Re}(r_n \exp(i\theta_n))^{1/\alpha} = r_n^{1/\alpha} \cos(\theta_n/\alpha), \quad (3.2)$$

where

$$r_n = (n^2 + 1 + 2n \cos(\alpha\pi/2))^{1/2}, \quad \theta_n = \arctan \frac{n \sin(\alpha\pi/2)}{1 + n \cos(\alpha\pi/2)}. \quad (3.3)$$

Now we shall find the asymptotic behaviour of  $\cos(\theta_n/\alpha)$  as  $n \rightarrow \infty$ . Using a well-known school formula and (3.3) we obtain

$$\tan(\alpha\pi/2 - \theta_n) = \frac{\tan(\alpha\pi/2) - \tan \theta_n}{1 + \tan(\alpha\pi/2) \tan \theta_n} = \frac{\sin(\alpha\pi/2)}{n + \cos(\alpha\pi/2)} = O\left(\frac{1}{n}\right), \quad n \rightarrow \infty,$$

that is  $\pi/2 - \theta_n/\alpha = O(1/n)$ ,  $n \rightarrow \infty$ . Hence  $\cos(\theta_n/\alpha) = \sin(\pi/2 - \theta_n/\alpha) = O(1/n)$ ,  $n \rightarrow \infty$ . Together with  $r_n^{1/\alpha} = O(n^{1/\alpha})$ ,  $n \rightarrow \infty$ , we obtain

$$r_n^{1/\alpha} \cos(\theta_n/\alpha) = O(n^{1/\alpha-1}), \quad n \rightarrow \infty.$$

Since  $1/\alpha - 1 > 0$ , using (3.1) and (3.2), we have the desired result.

In contrast to the case  $\alpha \in (0, 1)$ , in the case  $\alpha \in (1, 2)$  perturbations by bounded operators are always possible. In the next theorem we prove this even in the case of bounded time-dependent perturbations. For  $\alpha = 2$  an analogous theorem is presented in [7].

**THEOREM 3.1.** *Let  $\alpha \in (1, 2)$ ,  $A \in \mathcal{C}^\alpha(M, \omega)$  and for every  $t \in \mathbf{R}_+$   $B(t) \in \mathcal{B}(X)$ . If the function  $t \rightarrow B(t)$  is continuous in the uniform operator topology then for every  $x \in D(A)$  the Cauchy problem*

$$\mathbf{D}_t^\alpha u(t) = (A + B(t))u(t), \quad t > 0, \quad (3.4)$$

$$u(0) = x, \quad u'(0) = 0, \quad (3.5)$$

admits an uniquely determined solution  $u(t)$  given by the formula

$$u(t) = S_\alpha(t; A + B)x = \sum_{n=0}^{\infty} S_{\alpha, n}(t; A)x, \quad (3.6)$$

where

$$S_{\alpha, 0}(t; A) = S_\alpha(t; A);$$

$$S_{\alpha, n}(t; A) = \int_0^t R_\alpha(t-s; A)B(s)S_{\alpha, n-1}(s; A) ds, \quad n \in \mathbf{N}; \quad (3.7)$$

$$R_\alpha(t; A) = \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} S_\alpha(s; A) ds. \quad (3.8)$$

Moreover, if  $K_T = \max_{t \in [0, T]} \|B(t)\|$ , we have for all  $t \in [0, T]$  the bounds

$$\begin{aligned} \|u(t)\| &\leq M e^{\omega t} E_\alpha(M K_T t^\alpha) \|x\|, \\ \|u(t) - S_\alpha(t; A)x\| &\leq M e^{\omega t} (E_\alpha(M K_T t^\alpha) - 1) \|x\|. \end{aligned} \quad (3.9)$$

**P r o o f.** From (2.1) and (3.8) it follows

$$\|R_\alpha(t; A)\| \leq M e^{\omega t} t^{\alpha-1} / \Gamma(\alpha). \quad (3.10)$$

Then, applying the identity  $\int_0^t (t-s)^{a-1} s^{b-1} ds = t^{a+b-1} B(a, b)$  to (3.7), we obtain by induction

$$\|S_{\alpha, n}(t; A)\| \leq M^{n+1} e^{\omega t} K_t^n t^{\alpha n} / \Gamma(\alpha n + 1), \quad n \in \mathbf{N}_0. \quad (3.11)$$

From these bounds it follows that the series representing  $S_\alpha(t; A + B)$  in (3.6) are uniformly convergent on compact subsets of  $\mathbf{R}_+$  with respect to the operator norm topology. Hence,  $S_\alpha(t; A + B)$  is a strongly continuous function on  $\mathbf{R}_+$  with values in  $\mathcal{B}(X)$ . Furthermore, the bounds (3.9) follow directly from (3.11).

Next we prove that  $u(t)$  satisfies (3.4) and (3.5). Since  $S_\alpha(0; A) = I$ ,  $S_{\alpha, n}(0; A) = 0$ ,  $n \in \mathbf{N}$ ,  $S'_{\alpha, n}(0; A) = 0$ ,  $n \in \mathbf{N}_0$ , we have  $S_\alpha(0; A + B) = I$ ,  $S'_\alpha(0; A + B) = 0$ , i.e. the initial conditions are satisfied. Applying (3.6) and (3.7), it follows

$$\begin{aligned} u(t) &= S_\alpha(t; A)x + \sum_{n=0}^{\infty} \int_0^t R_\alpha(t-s; A) B(s) S_{\alpha, n}(s; A) x ds \\ &= S_\alpha(t; A)x + \int_0^t R_\alpha(t-s; A) B(s) u(s) ds, \end{aligned} \quad (3.12)$$

where the interchanging of the summation and integration is justified by the uniform convergence of the series. Integrodifferentiating (3.12), we obtain

$$\mathbf{D}_t^\alpha u(t) = A S_\alpha(t; A)x + \mathbf{D}_t^\alpha \int_0^t R_\alpha(t-s; A) B(s) u(s) ds. \quad (3.13)$$

Let for shortness,  $(f * g)(t) = \int_0^t f(t-s)g(s) ds$ . Then, setting  $h(t) = B(t)u(t)$  and using (1.5), (1.4),  $(R_\alpha * h)(0) = (R_\alpha * h)'(0) = 0$ , the property  $J_t^\beta(f * g) = (J_t^\beta f) * g$  and the semigroup property for operators of fractional integration, we obtain

$$\begin{aligned} \mathbf{D}_t^\alpha (R_\alpha(t; A) * h(t)) &= D_t^\alpha (R_\alpha(t; A) * h(t)) = D_t^2 J_t^{2-\alpha} (J_t^{\alpha-1} S_\alpha(t; A) * h(t)) \\ &= D_t^1 (S_\alpha(t; A) * h(t)) = (D_t^1 S_\alpha(t; A)) * h(t) + S_\alpha(0; A)h(t). \end{aligned} \quad (3.14)$$

Since  $S'_{\alpha, n}(0; A) = 0$ ,  $n \in \mathbf{N}_0$ , it follows

$$D_t^1 S_\alpha(t; A)x = J_t^1 D_t^2 S_\alpha(t; A)x = J_t^{\alpha-1} J_t^{2-\alpha} D_t^2 S_\alpha(t; A)x = J_t^{\alpha-1} \mathbf{D}_t^\alpha S_\alpha(t; A)x$$

$$= J_t^{\alpha-1} S_\alpha(t; A) Ax = R_\alpha(t; A) Ax = AR_\alpha(t; A)x. \quad (3.15)$$

Combining (3.13), (3.14) and (3.15) and using the closedness of  $A$ , we obtain that  $u(t)$  satisfies (3.4).

To prove the uniqueness, let  $v : \mathbf{R}_+ \rightarrow D(A)$  be a solution of (3.4) with  $v(0) = v'(0) = 0$ . Then, using the property  $J_t^\alpha \mathbf{D}_t^\alpha u(t) = u(t) - u(0) - tu'(0)$  we have  $v(t) = J_t^\alpha Av(t) + J_t^\alpha B(t)v(t)$  and applying the variation of parameters formula (see [13, Prop. 1.2]),  $v(t)$  satisfies the integral equation

$$v(t) = \int_0^t R_\alpha(t-s; A)B(s)v(s) ds.$$

Setting  $m_t = \max_{s \in [0, t]} \|v(s)\|$ , we see that for  $m_t > 0$

$$m_t \leq \frac{MK_t m_t}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{\omega(t-s)} ds \leq \frac{MK_t m_t}{\Gamma(\alpha+1)} t^\alpha e^{\omega t} < m_t,$$

if  $t > 0$  is chosen sufficiently small. Thus,  $v(t) = 0$  on  $[0, t_0]$  with  $t_0 > 0$ . Iteration of this argument leads to  $v(t) = 0$  on  $\mathbf{R}_+$ .  $\square$

Next we present an additional perturbation result.

**THEOREM 3.2.** *Let  $\alpha \in (0, 2)$ ,  $A \in \mathcal{C}^\alpha$  and  $B$  be a linear operator with domain  $D(B)$  satisfying  $D(B) \supset D(A)$ . Assume that there exist constants  $\omega \geq 0$ ,  $M \geq 1$  such that  $(\omega^\alpha, \infty) \subset \varrho(A)$ ;  $BR(\lambda^\alpha, A)$  is strongly infinitely differentiable when  $\lambda > \omega$  and satisfies*

$$\|(BR(\lambda^\alpha, A))^{(n)}x\| \leq Mn! \|x\| / (\lambda - \omega)^{n+1}, \quad x \in X, \lambda > \omega, n \in \mathbf{N}_0. \quad (3.16)$$

Then  $A + B \in \mathcal{C}^\alpha$ .

**P r o o f.** The proof imitates the proof of an analogous theorem for COF (see [14]). Here we take  $\lambda^{\alpha-1}R(\lambda^\alpha, A)$  instead of  $\lambda R(\lambda^2, A)$  and use Theorem 2.1.  $\square$

We conclude this section with two open problems:

1) Let  $\alpha \in (1, 2)$ . If  $A \in \mathcal{C}^\alpha$  does there exist  $\omega \in \mathbf{R}$  such that the solution operator  $S_\alpha(t; A - \omega I)$  is uniformly bounded ( $\|S_\alpha(t; A - \omega I)\| \leq M, t \geq 0$ )?

For  $\alpha = 1$  the existence of such a constant is trivial and it is of great help in dealing with  $C_0$ -semigroups. For  $\alpha = 2$  it is shown by an example (see [4]) that the answer is negative. Applying the subordination principle (Theorem 2.2), we can give an answer to the problem for noninteger  $\alpha$  only in some particular cases.

2) Let  $\alpha \in (1, 2)$ . If  $A_1, A_2 \in \mathcal{C}^\alpha$  and they commute, is it true that  $\overline{A_1 + A_2} \in \mathcal{C}^\alpha$ ?

Again for  $\alpha = 1$  the answer is positive (see [15]) and for  $\alpha = 2$  – negative (see [4]). Here we notice only a weaker property of  $\overline{A_1 + A_2}$ , as follows.

**PROPOSITION 3.1.** *Let  $\alpha \in (1, 2)$ ,  $A_1, A_2 \in \mathcal{C}^\alpha$  and they commute. Then  $\overline{A_1 + A_2}$  generates a semigroup analytic in  $\Delta_\alpha = \{t \in \mathbf{C} \setminus \{0\}; |\arg t| < (\alpha - 1)\pi/2\}$ .*

Let us note that the analyticity of  $S_1(t; \overline{A_1 + A_2})$  in  $\Delta_\alpha$  does not imply in general  $\overline{A_1 + A_2} \in \mathcal{C}^\alpha$  (see [2, Example 3.2]).

**P r o o f.** According to Theorem 2.2,  $A_1$  and  $A_2$  generate  $C_0$ -semigroups analytic in  $\Delta_\alpha$  and given by the formulas

$$S_1(t; A_j)x = \int_0^\infty \varphi_{t,1/\alpha}(s)S_\alpha(t; A_j)x ds, \quad j = 1, 2.$$

Since  $S_\alpha(t; A_1)$  and  $S_\alpha(t; A_2)$  commute for all  $t, s > 0$ , then  $S_1(t; A_1)$  and  $S_2(t; A_2)$  commute for  $t, s > 0$ . Hence (see [15])  $\overline{A_1 + A_2} \in \mathcal{C}^1$  and  $S_1(t; \overline{A_1 + A_2}) = S_1(t; A_1)S_2(t; A_2)$  is a strongly continuous semigroup analytic in  $\Delta_\alpha$ .  $\square$

#### 4. Approximate solutions

The exponential representation for  $C_0$ -semigroups

$$S_1(t; A)x = \lim_{n \rightarrow \infty} (I - \frac{t}{n}A)^{-n}x, \quad x \in X, \quad t \geq 0, \quad (4.1)$$

where the convergence is uniform in bounded  $t$ -intervals for each fixed  $x$ , is very well known. This formula has important implication for the numerical approximation of the trajectories of  $S_1(t; A)$  especially for implicit approximation schemes. Results in this direction for cosine operator functions are given e.g. in [16] and [8].

First we recall why the exponential formula (4.1) is important for the numerical approximation of  $S_1(t; A)$ . Usually, to find an approximation of the value of the solution of the problem

$$D_t^1 u(t) = Au(t), \quad u(0) = x, \quad (4.2)$$

at a fixed time  $t > 0$  we divide the interval  $[0, t]$  into  $n$  equal parts and approximate the derivative by a difference. If we take the two-point backward difference we obtain the following implicit difference scheme

$$\frac{1}{h}[u_n(jh) - u_n((j-1)h)] = Au_n(jh), \quad j = 1, \dots, n, \quad u_n(0) = x, \quad (4.3)$$

with  $h = t/n$ . The equations (4.3) can be solved explicitly and their solution  $u_n(t)$  given by  $u_n(t) = (I - \frac{t}{n}A)^{-n}x$  for  $n$  sufficiently large is an approximation of the solution  $u(t)$  of (4.2). Then the exponential formula (4.1) implies that  $u_n(t) \rightarrow u(t)$ , as  $n \rightarrow \infty$ . So, the solution of the difference scheme (4.3) converges



to the solution of the differential equation (4.2). Some estimates of the range of convergence could be also done.

We generalize this classical result to the case of the fractional order problem (1.1). Consider the numerical method for solving fractional differential equations using the following approximation of  $D_t^\alpha$ , by the backward fractional difference (see [12, Ch.7,8]):

$$D_\tau^\alpha f(\tau) \approx h^{-\alpha} \sum_{i=0}^{[\tau/h]} (-1)^i \binom{\alpha}{i} f(\tau - ih), \quad (4.4)$$

where

$$\binom{\alpha}{i} = \frac{\alpha(\alpha-1)\dots(\alpha-i+1)}{i!}, \quad i \in \mathbf{N}.$$

Applying (1.5) to the solution  $u(t)$  of (1.1) we obtain  $\mathbf{D}_t^\alpha u(t) = D_t^\alpha(u(t) - x)$ . Then using (4.4) to approximate the equation in (1.1) and forward differences to approximate the initial conditions in (1.1)

$$u^{(k)}(0) \approx h^{-k} \sum_{i=0}^k (-1)^i \binom{k}{i} u((k-i)h), \quad k = 0, 1, \dots, m-1,$$

we obtain the following difference scheme

$$h^{-\alpha} \sum_{i=0}^j (-1)^i \binom{\alpha}{i} (u_n((j-i)h) - x) = Au_n(jh), \quad j = m, m+1, \dots; \quad (4.5)$$

$$u_n(jh) = x, \quad j = 0, 1, \dots, m-1, \quad (4.6)$$

where  $h = t/n$ . It can be solved explicitly and the result is presented in the next

**THEOREM 4.1.** *Let  $A \in \mathcal{C}^\alpha(M, \omega)$  and  $u_n(t)$  be the approximation defined by (4.5-6) to the solution  $u(t)$  of (1.1). Then*

$$u_n(t) = \frac{1}{(n-m)!} \sum_{k=1}^{n-m+1} b_{k,n-m+1}^\alpha (I - (t/n)^\alpha A)^{-k} x, \quad x \in X, \quad (4.7)$$

where  $b_{k,n}^\alpha$  are given by the recurrence relations

$$\begin{aligned} b_{1,1}^\alpha &= 1, \quad b_{k,n}^\alpha = (n-1-k\alpha)b_{k,n-1}^\alpha + \alpha(k-1)b_{k-1,n-1}^\alpha, \quad 1 \leq k \leq n, \quad n = 2, 3, \dots, \\ b_{k,n}^\alpha &= 0, \quad k > n, \quad n = 1, 2, \dots; \end{aligned} \quad (4.8)$$

and  $u_n(t)$  converges to  $u(t)$  as  $n \rightarrow \infty$  uniformly on bounded subsets of  $t \geq 0$ .

To prove this theorem we need two lemmas.

LEMMA 4.1. Let  $\alpha > 0$ . If  $\lambda \in \mathbf{C}$  is such that  $R(\lambda^\alpha, A)$  is infinitely differentiable then

$$(\lambda^{\alpha-1}R(\lambda^\alpha, A))^{(n)}x = (-1)^n \lambda^{-(n+1)} \sum_{k=1}^{n+1} b_{k,n+1}^\alpha (\lambda^\alpha R(\lambda^\alpha, A))^k, \quad (4.9)$$

for  $n = 0, 1, \dots$  where  $b_{k,n}^\alpha$  are given by the recurrence relations (4.8).

LEMMA 4.2. Let  $f(\cdot) : [0, \infty) \rightarrow X$  be a continuous function such that  $\|f(t)\| \leq M e^{\omega t}$ ,  $t \geq 0$ , for some  $M \geq 1$  and  $\omega \geq 0$ . Then for any integer  $k \geq 0$  we have

$$\lim_{n \rightarrow \infty} \frac{n^{n-k+1}}{(n-k)!} \int_0^\infty \sigma^{n-k} e^{-n\sigma} [f(t) - f(t\sigma)] d\sigma = 0$$

uniformly on bounded subsets of  $t \geq 0$ .

Lemma 4.1 follows by a simple inductive argument on  $n$ , Lemma 4.2 can be proven by a method similar to that in [11, p.34], so we omit the proofs.

PROOF. (of Theorem 4.1) The case  $t = 0$  is trivial, applying  $\sum_{k=1}^{n+1} b_{k,n+1}^\alpha = n!$ , which can be obtained from (4.9) with  $A \equiv 0$ . Consider  $t > 0$ . Then, using (4.6) and the identity

$$\sum_{i=0}^j (-1)^i \binom{\alpha}{i} = \binom{j-\alpha}{j},$$

(4.5) is equivalent to

$$(I - h^\alpha A) u_n((m+j)h) = \sum_{i=1}^j (-1)^{i+1} \binom{\alpha}{i} u_n((m+j-i)h) + \binom{j-\alpha}{j} x, \quad j = 0, 1, \dots, \quad (4.10)$$

(here we let  $\sum_{i=1}^0 = 0$ ).

If we choose  $n$  such that  $n/t > \omega$  and apply (2.2) then  $(I - h^\alpha A)^{-1} = h^{-\alpha} R(h^{-\alpha}, A)$  exists and we obtain from (4.10)

$$u_n((m+j)h) = h^{-\alpha} R(h^{-\alpha}, A) \left[ \sum_{i=1}^j (-1)^{i+1} \binom{\alpha}{i} u_n((m+j-i)h) + \binom{j-\alpha}{j} x \right]. \quad (4.11)$$

Denote for shortness  $F(\lambda) = \lambda^{\alpha-1} R(\lambda^\alpha, A)$ . Next we shall prove by induction on  $j$  that  $u_n((m+j)h) = v_n(j, h)$ ,  $j = 0, 1, \dots$ , where

$$v_n(j, h) = \left[ \frac{(-1)^j \lambda^{j+1}}{j!} F^{(j)}(\lambda) x \right]_{\lambda=1/h}, \quad j = 0, 1, \dots \quad (4.12)$$

For  $j = 0$  this is trivial. Suppose that  $u_n((m+l)h) = v_n(l, h)$  is true for all  $l \leq j-1$ . By (4.11) and the induction hypothesis it follows

$$u_n((m+j)h) = \left[ \lambda F(\lambda) \left( (-1)^{j+1} \sum_{i=1}^j \binom{\alpha}{i} \frac{\lambda^{j+1-i}}{(j-i)!} F^{(j-i)}(\lambda)x + \binom{j-\alpha}{j} x \right) \right]_{\lambda=1/h}. \quad (4.13)$$

Further, we use the identity

$$AF^{(j)}(\lambda)x = (\lambda^\alpha F(\lambda))^{(j)}x + (-1)^{j+1} j! \lambda^{-j+\alpha-1} \binom{j-\alpha}{j} x, \quad j = 0, 1, \dots, \quad (4.14)$$

which can be easily proven by induction. (Note that  $AF^{(j)}(\lambda)$  is a bounded operator by (4.9) and the fact that  $AR(\lambda^\alpha, A)$  is bounded and it is sufficient to prove (4.14) for  $x \in D(A)$ .) Applying the operator  $-R(\lambda^\alpha, A)$  to both sides of (4.14) and using that  $AR(\lambda^\alpha, A)x = R(\lambda^\alpha, A)Ax = \lambda F(\lambda)x - x$ ,  $x \in D(A)$ ,  $R(\lambda^\alpha, A) = \lambda^{1-\alpha} F(\lambda)$ , we obtain

$$F^{(j)}(\lambda)x = F(\lambda) \left[ -\lambda^{1-\alpha} (\lambda^\alpha F(\lambda))^{(j)}x + \lambda F^{(j)}(\lambda)x + (-1)^j j! \lambda^{-j} \binom{j-\alpha}{j} x \right]. \quad (4.15)$$

With the aid of the Leibniz rule

$$(\lambda^\alpha F(\lambda))^{(j)} = \sum_{i=0}^j \binom{j}{i} \alpha(\alpha-1) \dots (\alpha-i+1) \lambda^{\alpha-i} F^{(j-i)}(\lambda)$$

and (4.15) can be written in the form

$$F^{(j)}(\lambda)x = \frac{(-1)^j j!}{\lambda^{j+1}} \lambda F(\lambda) \left[ (-1)^{j+1} \sum_{i=1}^j \binom{\alpha}{i} \frac{\lambda^{j+1-i}}{(j-i)!} F^{(j-i)}(\lambda)x + \binom{j-\alpha}{j} x \right]. \quad (4.16)$$

Now (4.13) and (4.16) imply  $u_n((m+j)h) = v_n(j, h)$ . Taking  $j = n-m$  we obtain

$$u_n(t) = \left[ \frac{(-1)^{n-m} \lambda^{n-m+1}}{(n-m)!} F^{(n-m)}(\lambda)x \right]_{\lambda=n/t}, \quad (4.17)$$

and this representation together with (4.9) implies (4.7).

Differentiating (2.2)  $n-m$  times with respect to  $\lambda$  and inserting the result into (4.17) we find

$$u_n(t) = \left[ \frac{\lambda^{n-m+1}}{(n-m)!} \int_0^\infty s^{n-m} e^{-\lambda s} S_\alpha(s; A)x ds \right]_{\lambda=n/t}. \quad (4.18)$$

Noting that  $\lambda^{n-m+1} \int_0^\infty s^{n-m} e^{-\lambda s} ds = (n-m)!$ , it follows

$$u(t) - u_n(t) = \left[ \frac{\lambda^{n-m+1}}{(n-m)!} \int_0^\infty s^{n-m} e^{-\lambda s} (S_\alpha(t; A)x - S_\alpha(s; A)x) ds \right]_{\lambda=n/t}, \quad (4.19)$$

or, after the change of variables  $s = t\sigma$

$$u(t) - u_n(t) = \frac{n^{n-m+1}}{(n-m)!} \int_0^\infty \sigma^{n-m} e^{-n\sigma} (S_\alpha(t; A)x - S_\alpha(t\sigma; A)x) d\sigma. \quad (4.20)$$

It remains to apply Lemma 4.2 and the proof of Theorem 4.1 is completed.  $\square$

REMARK 4.1. A similar representation for  $u(t)$  has been obtained in [1, Corollary 3.2], applying the Post-Widder inversion formula. In our notations it can be written as  $u(t) = \lim_{n \rightarrow \infty} v_n(n, t/n)$ . In fact, (4.20) together with Lemma 4.2 shows that  $u(t) = \lim_{n \rightarrow \infty} v_n(n-k, t/n)$  for any integer  $k \geq 0$ .

REMARK 4.2. Note that if  $0 < \alpha < 1$  then  $b_{k,n}^\alpha > 0$ . This fact can be applied for studying the positivity properties of the solution operator.

Next we estimate the rate of convergence of representation (4.7) when  $\alpha \geq 1$ .

THEOREM 4.2. *Let  $\alpha \geq 1$ ,  $A \in \mathcal{C}^\alpha(M, \omega)$  and  $x \in D(A)$ . If  $u_n(t)$  is the approximation defined by (4.5-6) to the solution  $u(t)$  of (1.1) then*

$$\|u(t) - u_n(t)\| = O(n^{-1/2}), \quad (4.21)$$

uniformly on  $t$  in compacts of  $[0, \infty)$ . If  $A \in \mathcal{C}(M, 0)$  then the more precise estimate

$$\|u(t) - u_n(t)\| \leq C_\alpha M n^{-1/2} t^\alpha \|Ax\|, \quad x \in D(A), \quad (4.22)$$

holds, where  $C_\alpha$  depends only on  $\alpha$ .

P r o o f. We start from (4.19). Let us find a bound for  $S_\alpha(t; A)x - S_\alpha(s; A)x$ . Applying  $D_t^1 J_t^\alpha$  to both sides of the equation in (1.1) and using the property

$$J_t^\alpha \mathbf{D}_t^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0) \frac{t^k}{k!}$$

we obtain

$$S'_\alpha(t)x = D_t^1 J_t^\alpha A S_\alpha(t)x = D_t^1 J_t^1 J_t^{\alpha-1} S_\alpha(t)Ax = J_t^{\alpha-1} S_\alpha(t)Ax. \quad (4.23)$$

Therefore  $S_\alpha(t; A)x - S_\alpha(s; A)x = \int_s^t R_\alpha(\tau; A)Ax d\tau$  with  $R_\alpha(t; A)$  defined as in (3.8). Applying (3.10) it follows

$$\|S_\alpha(t; A)x - S_\alpha(s; A)x\| \leq M\Gamma(\alpha)^{-1} |t-s| \max_{\tau \in [s,t]} \tau^{\alpha-1} e^{\omega\tau} \|Ax\| =$$

$$\frac{M\|Ax\|}{\Gamma(\alpha)} \begin{cases} (t-s)t^{\alpha-1}e^{\omega t}, & t \geq s; \\ (s-t)s^{\alpha-1}e^{\omega s}, & s \geq t. \end{cases}$$

Inserting these bounds in (4.19) we get

$$\begin{aligned} & \|u(t) - u_n(t)\| \leq \\ & \frac{M\|Ax\|}{\Gamma(\alpha)(n-m)!} \left(\frac{n}{t}\right)^{n-m+1} \left[ \int_0^t s^{n-m} e^{-ns/t} (t-s)t^{\alpha-1} e^{\omega t} ds + \right. \\ & \quad \left. \int_t^\infty s^{n-m} e^{-ns/t} (s-t)s^{\alpha-1} e^{\omega s} ds \right] = \\ & \frac{M\|Ax\|t^\alpha e^{\omega t}}{\Gamma(\alpha)(n-m)!} \left[ \gamma(n-m+1, n) - \frac{1}{n}\gamma(n-m+2, n) + \right. \\ & \quad \frac{n^{n-m+1} e^{-\omega t}}{(n-\omega t)^{n-m+\alpha+1}} \Gamma(n-m+\alpha+1, n-\omega t) - \\ & \quad \left. \frac{n^{n-m+1} e^{-\omega t}}{(n-\omega t)^{n-m+\alpha}} \Gamma(n-m+\alpha, n-\omega t) \right], \end{aligned} \quad (4.24)$$

where  $\gamma(a, b) = \int_0^b e^{-t} t^{a-1} dt$ ,  $\Gamma(a, b) = \int_b^\infty e^{-t} t^{a-1} dt$  are the incomplete Gamma functions (see [3], vol.1). Using the identities  $\gamma(a+1, b) = a\gamma(a, b) - b^a e^{-b}$ ,  $\Gamma(a+1, b) = a\Gamma(a, b) + b^a e^{-b}$ , we simplify the last expression and obtain

$$\begin{aligned} \|u(t) - u_n(t)\| & \leq \frac{M\|Ax\|t^\alpha e^{\omega t}}{\Gamma(\alpha)(n-m)!} \left[ \frac{m-1}{n}\gamma(n-m+1, n) + \right. \\ & \quad \left. e^{-\omega t} \left(1 - \frac{\omega t}{n}\right)^{-n} \frac{n^{1-m}(\alpha + \omega t - m)}{(n-\omega t)^{\alpha-m+1}} \Gamma(n-m+\alpha, n-\omega t) + \frac{2n-\omega t}{n-\omega t} n^{n-m} e^{-n} \right]. \end{aligned} \quad (4.25)$$

Applying the inequalities  $\gamma(n+1, n) < \Gamma(n+1) = n!$ ,  $\Gamma(n+\alpha, n-\omega t) < \Gamma(n+\alpha)$ ,  $2(n!)^{-1}n^n e^{-n} \leq (2/\pi n)^{1/2}$  and the asymptotic property of the Gamma functions  $\Gamma(n+\alpha)/n! \simeq n^{\alpha-1}(1+O(n^{-1}))$ , ([3], vol. 1), we see that the last term in the brackets is dominating as  $n \rightarrow \infty$ , that implies (4.21). In case  $\omega = 0$  the estimate (4.25) reduces to

$$\|u(t) - u_n(t)\| \leq \frac{M\|Ax\|t^\alpha}{\Gamma(\alpha)(n-m)!} \left[ \frac{m-1}{n}\gamma(n+1, n) + \frac{\alpha-m}{n^\alpha}\Gamma(n+\alpha, n) + 2n^{n-m} e^{-n} \right]$$

that by remarks above implies (4.22)  $\square$

## Appendix

The *Mittag-Leffler function* (see [3, vol. 3]), defined as follows

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad \alpha > 0, z \in \mathbf{C}, \quad (\text{A.1})$$

is an entire function which satisfies the fractional order differential relation

$$\mathbf{D}_t^\alpha E_\alpha(\lambda t^\alpha) = \lambda E_\alpha(\lambda t^\alpha). \quad (\text{A.2})$$

Its asymptotic expansion as  $z \rightarrow \infty$  for  $0 < \alpha < 2$  is:

$$E_\alpha(z) = \frac{1}{\alpha} \exp(z^{1/\alpha}) + \varepsilon_\alpha(z), \quad |\arg z| \leq \frac{1}{2}\alpha\pi, \quad (\text{A.3})$$

$$E_\alpha(z) = \varepsilon_\alpha(z), \quad |\arg(-z)| < (1 - \frac{1}{2}\alpha)\pi, \quad (\text{A.4})$$

where

$$\varepsilon_\alpha(z) = - \sum_{n=1}^{N-1} \frac{z^{-n}}{\Gamma(1 - \alpha n)} + O(|z|^{-N}), \quad z \rightarrow \infty.$$

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