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On Analytical Expressions for the Distribution of the Filtered Output of Square Envelope Receivers with Signal and Colored Gaussian Noise Input

Idelfonso Tafur Monroy

Abstract—Closed-form expressions for the moment generating function (MGF) of the filtered output of square envelope receivers with signal and colored Gaussian noise input are derived. The informative signal is a binary sequence of rectangular pulses. The considered Gaussian processes are the Wiener process, a Gaussian process with linear covariance (moving average), and the Ornstein–Uhlenbeck process. The derived MGFs are then applied to the problem of finding the quantum limit for optically preamplified, direct detection receivers.

Index Terms—Communication theory, envelope receivers, error analysis, optical communications, preamplified receivers.

I. INTRODUCTION

DETERMINING the distribution of the output of square envelope receivers with colored Gaussian input constitutes a classic problem in communication theory; see, e.g., [1]–[5]. Although the mathematical formalism for determining the statistics of the output of such receivers is well known, e.g., [1], [3], [5], and [6], deriving closed-form expressions for the distribution is a complex task. Moreover, if both signal and noise are present at the input, this task become even more formidable. For the case of noise only input expressions for the moment generating function (MGF) and analytical approximations to the probability density function have been reported for several covariance kernels, e.g., [3], [5], and [7]. For both signal and noise being present at the input closed-form expressions for the distribution are scarcely documented in the literature. In this letter, closed-form expressions for the MGF are derived for the case of an input informative signal composed of a binary sequence of rectangular pulses. The considered Gaussian processes are the Wiener process, a Gaussian process with linear covariance (moving average), and the Ornstein–Uhlenbeck process.

The remain of this letter is structured as follows. In Section II the system model to be discussed is presented. The mathematical formalism to obtain closed-form expressions for the MGF of the receiver's output is also described. Closed-form expressions for the MGF for the considered Gaussian processes are derived in Section III. Section IV is devoted to the application of the derived MGFs to find the quantum limit for optically preamplified, direct detection receivers. Finally, summarizing conclusions are presented in Section V.

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II. SYSTEM MODEL

In this section, we present the reference model for the system under investigation. The receiver schematic diagram is presented in Fig. 1. This receiver has the classic configuration of pre-detection filter $r(t)$, square envelope detector, and post-detection filter. The input is an informative signal corrupted by an additive white Gaussian noise (AWGN) $n(t)$ with spectral density parameter N_0 . In the sequel, equivalent bandpass representation of signal and noise is assumed. The incoming signal is a binary sequence of rectangular pulses $S(t)$. For a given bit pattern, $\mathcal{B} = (\dots, b_{-1}, b_0, b_1, \dots)$, at the output of the filter $r(t)$ the signal is given by

$$Y(t) = S(t) \star r(t)r = \sqrt{\frac{m}{T}} \left[b_0 l(t) + \sum_{k \neq 0} b_k l(t - kT) \right] \quad (1)$$

where $r(t)$ is the pre-detection filter impulse response, $l(t) = g(t) \star r(t)$ in which \star denotes convolution, $g(t)$ is unit rectangular pulse of duration T , $b_k \in \{0, 1\}$ are statistically binary symbols representing a data “zero” and a “one,” respectively. m is the energy content of the signal $S(t)$ in a bit-duration time interval $m = (1/2) \int_0^T |S(t)|^2 dt$. At the pre-detection filter output, the resultant colored Gaussian noise is denoted by $X(t)$, whose in-phase and quadrature components have zero mean and autocovariance

$$K(\tau) = \frac{N_0}{2} R(\tau) \quad R(\tau) = \int_{-\infty}^{\infty} r(t)r^*(t + \tau) dt \quad (2)$$

where $*$ means complex conjugate. With the above notations, the input of the square envelope detector becomes $B(t) = Y(t) + X(t)$. If we consider and integrate-and-dump post-detection filter, then the receiver's output is given by

$$\Lambda = \int_0^T |Y(t) + X(t)|^2 dt. \quad (3)$$

The general mathematical form for the MGF of Λ , $M_\Lambda(s) = E\{e^{s\Lambda}\}$, is well known, e.g., [1], [7]

$$M_\Lambda(s) = \prod_{n=1}^{\infty} \frac{1}{(1 - \lambda_n s)} \exp\left(\sum_{n=1}^{\infty} \frac{|y_n|^2 s}{1 - \lambda_n s}\right) \quad (4)$$

where $y_n = \int_0^T Y(t)f_n^*(t) dt$. The set of orthonormal functions $\{f_n\}$ are the eigenfunctions and λ_n are the eigenvalues of the integral equation

$$\int_0^T K(t, u)f_n(u) du = \lambda_n f_n(t), \quad 0 \leq t \leq T \quad (5)$$

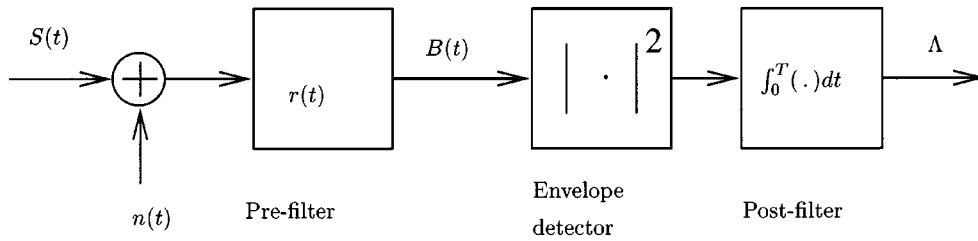


Fig. 1. Square envelope receiver.

in which $K(t, u)$ is the covariance kernel of the process $X(t)$. Let $h(t, u; s; \tau)$ stand for the resolvent kernel associated with the integral equation (5). The MGF in (4) can be represented in terms of the resolvent kernel as [7]

$$M_{\Lambda}(s) = [D(s)]^{-1} \exp[F(s)] \quad (6)$$

where

$$F(s) = s \int_0^T |Y(t)|^2 dt + s^2 \int_0^T \int_0^T Y^*(t) h(t, u; s; T) Y(u) dt du \quad (7)$$

and $D(s)$, also called the Fredholm determinant, is given by

$$D(s) = \exp \left[\int_0^s \int_0^T h(t, t; v; T) dt dv \right]. \quad (8)$$

III. MGF

In this section, closed-form expressions for $M_{\Lambda}(s)$ are presented for three different Gaussian processes.

Case 1: The Wiener Process: Suppose that $r(t)$ is the (normalized) impulse response of an integrate-and-dump filter, then

$$X(t) = \int_0^{\infty} r(t-s)n(s) ds$$

$$r(t-s) = \begin{cases} \frac{1}{T}, & s < t \\ 0, & s > t. \end{cases} \quad (9)$$

The process $X(t)$ is the Wiener process which has covariance given by

$$K(t, u) = \frac{N_0}{T^2} \int_0^t \int_0^u \delta(t-u) dt du = \frac{N_0}{T^2} \min(t, u). \quad (10)$$

As one can observe from the character of $r(t)$, the integrate-and-dump filter does not introduce intersymbol interference (ISI). The resulting expression for the MGF is (see Appendix A for a derivation)

$$M_Z(s) = \cos(\sqrt{N_0 s})^{-1} \exp \left[-\frac{m}{N_0} + \frac{m \tan(\sqrt{N_0 s})}{\sqrt{N_0 s}} \right]. \quad (11)$$

Case 2: Gaussian Process with Linear Covariance: Let $r(t)$ be a finite-time bandpass integrator (moving average) whose impulse response is given by

$$r(t) = \begin{cases} \frac{1}{T}, & 0 \leq t \leq T \\ 0, & \text{otherwise.} \end{cases} \quad (12)$$

The covariance kernel of $X(t)$ is the triangular function (linear covariance [5])

$$K(\tau) = \begin{cases} \frac{N_0}{T} \left(1 - \frac{|\tau|}{T} \right), & 0 \leq \tau \leq T \\ 0, & \text{otherwise.} \end{cases} \quad (13)$$

The analysis for the observation time $[0, T]$ shows that the MGF for Λ is of the same character as (11). As already reported in [5], it can be derived from (11) by substitution of \sqrt{s} by $\sqrt{2s}$. The result is

$$M_Z(s) = \cos(\sqrt{2N_0 s})^{-1} \exp \left[-\frac{m}{N_0} + \frac{m \tan(\sqrt{2N_0 s})}{\sqrt{2N_0 s}} \right].$$

In contradistinction to the integrate-and-dump filter the finite-duration integrator introduces ISI. Communication is only possible if the observation time is shifted from $[0, T]$ to $[(T/2), (3T/2)]$. For this type of filter, only a single past and one succeeding bit produce ISI on the present transmitted bit. Hence the bit sequence of interest is $\mathcal{B} = (b_{-1}, b_0, b_1)$. The expression for $F(s)$ is given by (14), shown at the bottom of the page, with

$$H_1(s) = \text{sinc}(\beta)(5N_0 s + 4) - 2\text{sinc}(\beta/2)(N_0 s + 1) - 2\cos(\beta/2)$$

$$H_2(s) = \text{sinc}(\beta)(N_0 s - 4) - \text{sinc}(\beta/2)(N_0 s - 2) + 2\cos(\beta/2)$$

$$H_3(s) = \text{sinc}(\beta)(N_0 s/2 + 2) - \text{sinc}(\beta/2) - \cos(\beta/2)$$

where $\beta = \sqrt{2N_0 s}$, and $\text{sinc}(x) = \frac{\sin(x)}{x}$. The Fredholm determinant is given by $D(s) = \cos(\sqrt{2N_0 s})$. Appendix B explains the derivation of (14).

Case 3: The Ornstein-Uhlenbeck Process: If $r(t)$ is the impulse response of the Lorentzian filter

$$r(t) = \sqrt{2}\mu e^{-\mu t}, \quad t \geq 0 \quad (15)$$

$$F(s) = \frac{m[b_0 H_1(s) + (\sqrt{b_{-1} b_0} + \sqrt{b_0 b_1}) H_2(s) + (b_{-1} + b_1) H_3(s)]}{4N_0 \cos(\beta)} \quad (14)$$

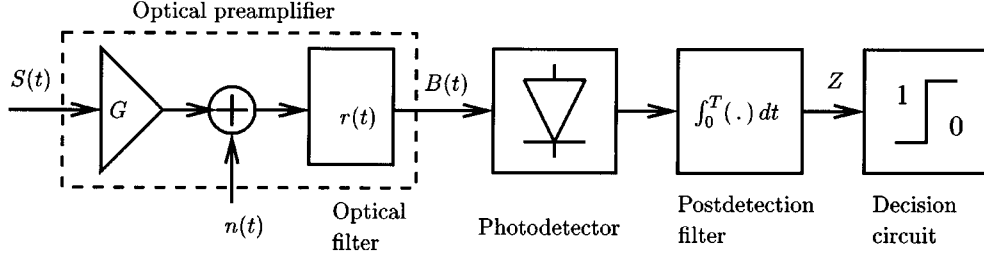


Fig. 2. Optically preamplified OOK receiver.

then the process $X(t)$ is the so called Ornstein–Uhlenbeck process with covariance given by

$$K(\tau) = \mu e^{-\mu|\tau|}, \quad \tau \geq 0. \quad (16)$$

The information signal for $t \in [0, T]$ is given by

$$Y(t) = \sqrt{\frac{2m}{T}} [b_0 + \rho e^{-\mu t}]$$

$$\rho = (e^{\mu T} - 1) \sum_{k=-\infty}^{-1} b_k e^{\mu k T} - b_0.$$

The Lorentzian filter is a causal filter; hence when studying the effect of ISI only a sequence of previous bits with respect to the present transmitted bit b_0 is treated, $\mathcal{B} = (\dots, b_{-2}, b_{-1}, b_0)$. In practice only a small number of previous bit is considered [8] For this case the resolvent kernel is well known, e.g., [5] and [7]. The closed-form expression for $M_\Lambda(s)$ and its detailed derivation has been reported in an earlier paper by the author [8]. The resulting expressions for $D(s)$ and $F(s)$ are presented here

$$F(s) = \frac{2mG}{\mu T} [b_0^2 F_1(s) + b_0 \rho F_2(s) + \rho^2 F_3(s)] \quad (17)$$

where

$$F_1(s) = \mu T s + \frac{2\mu T s^2 \sigma^2}{v^2}$$

$$+ \frac{4s^2 \sigma^2 (2 - [(v+1)e^{\beta T} - (v-1)e^{-\beta T}])}{v^3 [(v+1)^2 e^{\beta T} - (v-1)^2 e^{-\beta T}]}$$

$$F_2(s) = 2s \left[1 - \frac{4 - 2s\sigma^2 [(v+2)e^{\beta T} - (v-2)e^{-\beta T}]}{v [(v+1)^2 e^{\beta T} - (v-1)^2 e^{-\beta T}]} \right]$$

$$F_3(s) = s \left[\frac{1}{2} + \frac{\sigma^2 s (e^{\beta T} - e^{-\beta T}) - [(v-1)e^{\beta T} + (v+1)e^{-\beta T}]}{[(v+1)^2 e^{\beta T} - (v-1)^2 e^{-\beta T}]} \right]$$

in which $v = \sqrt{1 - 2\sigma^2}$, $\beta = v\mu$.

$$D(s) = \frac{(v+1)^2 e^{\beta T} - (v-1)^2 e^{-\beta T}}{4v e^{\mu T}}. \quad (18)$$

The mean and the variance of Z can be found from the properties of the MGF. Namely, from the first and second derivative of the MGF evaluated at $s = 0$ [9]. The mean and the variance of Z can also be expressed in terms of the covariance kernel without the knowledge of the MGF [8]. The validity of the previous derived MGF has been tested by confirming that the mean and the variance obtained by both methods are identical. Moreover,

if only noise is present, then the MGF is given only in terms of the Fredholm determinant $D(s)$, and their expressions are in agreement with those already known in the literature, e.g., [5] and [7].

IV. APPLICATIONS

In this section, we applied the derived MGF to determine the quantum limit for optically preamplified, on–off keying (OOK) direct detection receivers. The schematic diagram of such a receiver is illustrated in Fig. 2. The preamplifier is an EDFA (erbium-doped fiber amplifier) which is modeled as linear optical field amplifier with gain G and AWGN noise $n(t)$ representing the ASE (amplified spontaneous emission) noise. The spectral parameter of $n(t)$ is given by $N_0 = n_{\text{sp}}(G-1)h\nu$, where n_{sp} is the amplifier spontaneous emission factor, h is the Planck's constant, and ν optical frequency. An optical filter $r(t)$ is used to limit the effect of ASE on the system performance, and in the case of WDM (wavelength-division multiplexing) systems to select the desired channel.

By introducing a proper normalization (see [8]), m represents the average number of photons contained in an optical signal $S(t)$ for a transmitted binary “one.” The spectral parameter of $n(t)$ is then given by $N_0 = n_{\text{sp}}(G-1)$. At the output of the photodetector the photocurrent is directly proportional to the square magnitude of the received optical field. Further, the photocurrent is filtered and sampled to form the decision variable Z . Thus, the analysis of optically preamplified, OOK direct detection receivers (Fig. 2) is an example of the classic communication situation of square envelope detectors followed by filtering with colored Gaussian input. Assume that the postdetection filter is an integrate-and-dump filter. The MGF for the receiver decision variable Z is then given by $M_Z(s) = M_\Lambda(e^s - 1)$, where Λ is the so-called *Poisson parameter* [cf. (3)], e.g., [10] and [8].

Assuming independent, equally likely binary symbols the average error probability is given by

$$P_e = \frac{1}{2} [E_{\mathcal{B}} \{P_r(Z < \alpha |_{\mathcal{B}|b_0=1}) + P_r(Z > \alpha |_{\mathcal{B}|b_0=0})\}]. \quad (19)$$

Based on the MGF for the decision variable Z , error probabilities are expeditiously computed by the so-called saddlepoint approximation. For further details on the saddlepoint approximation see [7], [11], and [8] for an application to performance analysis of optically preamplified receivers.

In optical communications, the (standard), *quantum limit* is defined as the average number of photons per bit in the optical signal $S(t)$ needed to achieve a bit-error probability of 10^{-9}

TABLE I
QUANTUM LIMIT FOR OPTICALLY PREAMPLIFIED OOK DIRECT DETECTION,
 $G = 100$, AND $n_{sp} = 1$

Optical filter type	Quantum limit
Ideal bandpass	38.4 [11]
Integrate-and-dump	46.3
Finite-time Bandpass Integrator	59.6
Lorentzian, Optimal $\mu T = 7$	49.9

assuming ideal detection conditions, which for a preamplified receiver means that a large G is assumed.

Suppose we have optical filters described by the impulse responses (equivalent baseband representation) of Cases 1–3 [see (9), (12), and (15)]. The above derived MGFs [(11), (14), and (17)] can then be used [substituting $m \mapsto mG$ to account for amplification, and with $N_0 = n_{sp}(G - 1)$] to find the corresponding quantum limits. In Table I, the obtained results are presented. For comparison, the quantum limit for the situation when the optical signal is assumed to pass the optical filter undistorted and that $X(t)$ is Gaussian bandlimited (ideal bandpass filter) is also included. When the effect of optical filtering is taken into account penalties are observed compared to the case assuming ideal bandpass filtering. For the analyzed optical filters, this penalty in the quantum limit is at least of eight photons per bit. It should be noted that of the considered optical filters only the Lorentzian filter (Case 3) represents practical interest. Widely used in optical transmission systems Fabry–Perot filters are well described by the Lorentzian impulse response of (15). We observe also that there exists an optimum bandwidth bit-time product $BT = \mu T/\pi$, the reason being a tradeoff between ISI and ASE noise (see Table I).

V. CONCLUSIONS

Closed-form expressions, believed to be new, for the MGF of the filtered output of square envelope receivers with signal and colored Gaussian noise have been derived. The Wiener process, a Gaussian process with linear covariance, and the Ornstein–Uhlenbeck process are considered. The informative signal is binary sequence of rectangular pulses. We present an application of the derived MGFs in the performance analysis of optically preamplified, direct detection receivers.

APPENDIX A DERIVATION OF THE MGF: CASE 1

For the covariance function given by (10), the resolvent kernel is given by [5]

$$\begin{aligned}
 h(t, u; s; \tau) &= \frac{\sigma^2}{\beta} \underbrace{[\sin(\beta t) \cos(\beta u) - \theta(t - u) \sin(\beta t) \cos(\beta u)]}_{h_1} \\
 &+ q \underbrace{\frac{\sigma^2}{\beta} \sin(\beta t) \sin(\beta u)}_{h_2} + \theta(t - u) \underbrace{\frac{\sigma^2}{\beta} \cos(\beta t) \sin(\beta u)}_{h_3}
 \end{aligned} \quad (20)$$

where $\sigma^2 = N_0/T^2$, $\beta = \sqrt{\sigma^2 s}$

$$q = \tan(\beta T) \quad \text{and} \quad \theta(t - u) = \begin{cases} 0, & t < u \\ 1, & t > u. \end{cases}$$

We perform integration in (7) first with respect to t and with respect to u .

$$I_1 = \int_0^T Y^*(u) \underbrace{\int_0^T h(t, u; s; T) Y(t) dt}_{I_2} du$$

$$I_2 = \int_0^T \underbrace{Y(t)h_1}_a + \underbrace{Y(t)h_2}_b + \underbrace{Y(t)h_3}_c dt$$

in which $Y(t) = \sqrt{(2m/T)t/T} \equiv At$. Solving the integrals a, b, c , we get

$$I_2 = -A \frac{\sigma^2 u}{\beta^2} + A \frac{\sigma^2 \sin(\beta u)}{\beta^3 \cos(\beta T)}.$$

Subsequently

$$\begin{aligned}
 I_1 &= \int_0^T Au I_2 du = -A^2 T^3 \frac{\sigma^2}{3\beta^2} + \frac{A^2 \sigma^2}{\beta^5} (\tan(\beta T) - \beta T). \\
 F(s) &= s \int_0^T |Y(t)|^2 dt + s^2 I_1 \quad \text{resulting in} \\
 F(s) &= -\frac{m}{N_0} + \frac{m \tan(\sqrt{N_0 s})}{N_0 \sqrt{N_0 s}}. \quad (21)
 \end{aligned}$$

The Fredholm determinant is given by [see (8)]

$$D(s) = \exp \left\{ \int_0^s \frac{\sigma T}{\sqrt{v} 2} \tan(\sigma T \sqrt{v}) dv \right\} = \cos(\sqrt{N_0 s}). \quad (22)$$

A result already obtained in [5] and references therein.

APPENDIX B DERIVATION OF THE MGF: CASE 2

For the covariance kernel given in (13) and an observation interval $[-T/2, T/2]$, the resolvent kernel is presented in [5]. If the observation time is extended to $[0, 2T]$, then the resultant resolvent kernels is given by

$$\begin{aligned}
 h(t, u; s; T) &= \frac{N_0}{\beta T} \left[\tan \beta \cos \left[\frac{\beta(t - u)}{T} \right] - \sin \left[\frac{\beta|t - u|}{T} \right] \right] \quad (23)
 \end{aligned}$$

with $\beta = \sqrt{2N_0 s}$. The MGF is found by performing integration in (7) and (8) with the proper integration limits and the corresponding expression for $Y(t)$. The algebraic procedure is similar to that presented in Appendix A.

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