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$\mathfrak{P}$-ADIC CONTINUED FRACTIONS FOR NUMBER FIELDS

Abstract. A classical problem posed by Rosen was, given a number field K, to devise a continued fraction algorithm having the property of providing a finite expansion exactly when applied to an entry in K. We investigate this problem in a non archimedean setting. Starting from the p-adic case studied by Browkin and Ruban, we give a general definition of $\mathfrak{P}$-adic continued fraction, when $\mathfrak{P}$ is an integral ideal of K. We find some necessary and sufficient conditions for the finiteness property, studying in details the case of norm Euclidean quadratic fields

1. Introduction

The classical continued fraction algorithm provides an integer sequence $[a_0, a_1, \ldots]$ that represents a real number α_0 by means of the following recursive algorithm:

$$\begin{cases} a_n = \lfloor \alpha_n \rfloor \\ \alpha_{n+1} = \dfrac{1}{\alpha_n - a_n} \quad \text{if } \alpha_n - a_n \neq 0, \end{cases}$$

for all $n \geq 0$, where $\lfloor \cdot \rfloor$ denotes the integral part of a real number. The Euclidean algorithm ensures that, for classical continued fractions, the procedure eventually stops if and only if α_0 is a rational number.

Motivated by this property, Rosen [18] posed the problem of finding more general definitions of continued fraction expansions characterizing all the elements of an algebraic number field K by means of having a finite expansion and providing approximations for those elements not belonging to K, in terms of elements in K. See [7] for a overview of the results on this topic.

The problem of Rosen can be naturally translated into the context of p–adic numbers. In this context, however, there is no natural definition of a p–adic continued fraction, since there is no canonical definition for a "p-adic floor function". The two main definitions of a p–adic continued fraction algorithm are due to Browkin [4] and Ruban [19]; they are both based on the definition of a p–adic floor function

$$s(\alpha) = \sum_{n=k}^{0} x_n p^n \in \mathbb{Q}, \quad \text{where } \alpha = \sum_{n=k}^{\infty} x_n p^n \in \mathbb{Q}_p,$$

where k is the p-adic valuation of α, and the coefficients x_n's are the representatives modulo p in the interval $(-p/2, p/2)$ in Browkin's definition, while Ruban's algorithm considers them in the interval $[0, p-1]$. It was proved that rational numbers always have finite Browkin continued fraction expansion [5], and finite or eventually periodic in Ruban's case [12].

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We are interested here in the p-adic analogue of Rosen question, which was extensively addressed in [7]: given a number field K and a prime ideal $\mathfrak{P}$ in its ring of integers $\mathscr{O}_K$, we firstly recall the general definition of $\mathfrak{P}$-adic continued fractions, and the main convergence and approximation properties. Then we present the results concerning the finiteness of the expansion. Finally, we consider some explicit examples in the case of quadratic fields.

2. $\mathfrak{P}$-adic continued fractions

For every rational prime p, let $|\cdot|_p$ denote the p-adic absolute value, defined as $|x|_p = p^{-v_p(x)}$, where $v_p(x)$ is the standard p-adic valuation function. The archimedean absolute value on $\mathbb{R}$ or $\mathbb{C}$ will be denoted by $|\cdot|$ or by $|\cdot|_\infty$.

Let K be a number field of degree d over $\mathbb{Q}$, and let $\mathscr{O}_K$ be its ring of integers. We fix a prime ideal $\mathfrak{P}$ of $\mathscr{O}_K$ lying over an odd prime p. Let $\mathscr{M}_K$ be a set of representatives for the places of K. For every rational prime q and every $v \in \mathscr{M}_K$ above q let K_v be the completion of K w.r.t. the v-adic valuation and $\mathscr{O}_v$ be its valuation ring; we put $d_v = [K_v : \mathbb{Q}_q]$. Let $|\cdot|_v = |N_{K_v/\mathbb{Q}_q}(\cdot)|_q^{\frac{1}{d_v}}$ be the unique extension of $|\cdot|_q$ to K_v.
Let $v_0 \in \mathscr{M}_K$ be the place corresponding to $\mathfrak{P}$. We define

$$\mathscr{O}_{K,\{v_0\}} = \{\alpha \in K \mid |\alpha|_v \leq 1 \text{ for every non archimedean } v \neq v_0 \text{ in } \mathscr{M}_K\}.$$

2.1. $\mathfrak{P}$-adic floor functions and types

We recall the main definitions presented in [7, §3.1].

DEFINITION 1. *A* $\mathfrak{P}$-adic floor function *for K is a function $s : K_{v_0} \to K$ such that*

a) $|\alpha - s(\alpha)|_{v_0} < 1$ *for every* $\alpha \in K_{v_0}$*;*

b) $|s(\alpha)|_v \leq 1$ *for every non archimedean* $v \in \mathscr{M}_K \setminus \{v_0\}$*;*

c) $s(0) = 0$*;*

d) $s(\alpha) = s(\beta)$ *if* $|\alpha - \beta|_{v_0} < 1$.

The choice of a $\mathfrak{P}$-adic floor function is equivalent to choose a set $\mathscr{Y}$ of representatives of the cosets of $\mathfrak{P}\mathscr{O}_{v_0}$ in K_{v_0} containing 0 and contained in $\mathscr{O}_{K,\{v_0\}}$.
We shall call the triplet $\tau = (K, \mathfrak{P}, s)$ (or $(K, \mathfrak{P}, \mathscr{Y})$) a *type*.
In the case where $\mathfrak{P}$ is principal, there is a more natural way of defining a floor function associated to $\mathfrak{P}$. Indeed, let $\pi \in \mathscr{O}_K$ be generator and let $\mathscr{R}$ be a complete set of representatives of $\mathscr{O}_K/\mathfrak{P}$ containing 0. Then, every $\alpha \in K_{v_0}$ can be expressed uniquely as a Laurent series $\alpha = \sum_{j=-n}^{\infty} c_j \pi^j$, where $c_j \in \mathscr{R}$ for every j. It is possible to define

a $\mathfrak{P}$-adic floor function by

$$s(\alpha) = \sum_{j=-n}^{0} c_j \pi^j \in K.$$

In this case, we shall denote the types $\tau = (K, \mathfrak{P}, s)$ obtained as by $\tau = (K, \pi, \mathscr{R})$, and we call them *special types*.

EXAMPLE 1 (Browkin and Ruban types over $\mathbb{Q}$). When $K = \mathbb{Q}$ and $\pi = p$ is an odd prime, two main special types have been studied in the literature:

- the *Browkin type* $\tau_B = (\mathbb{Q}, p, \mathscr{R}_B)$ where $\mathscr{R}_B = \{-\frac{p-1}{2}, \dots, \frac{p-1}{2}\}$ (see [1–6]);
- the *Ruban type* $\tau_R = (\mathbb{Q}, p, \mathscr{R}_R)$ where $\mathscr{R}_R = \{0, \dots, p-1\}$ (see [8, 12, 19, 20]).

2.2. $\mathfrak{P}$-adic continued fractions associated to types

Let $\tau = (K, \mathfrak{P}, s)$ be a type and put

$$\mathscr{Y}_s = s(K_{v_0}), \quad \mathscr{Y}_s^1 = \{a \in \mathscr{Y}_s \mid |a|_{v_0} > 1\}.$$

Then, $\mathscr{Y}_s$ is a discrete subset of K_{v_0}.

DEFINITION 2. *Let $\tau = (K, \mathfrak{P}, s)$ be a type. A* continued fraction *of type τ is a (possibly infinite) expression of the form*

$$[a_0, a_1, \dots] = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \dots}}$$

of elements of $\mathscr{Y}_s$ such that $a_n \in \mathscr{Y}_s^1$ for $n \geq 1$.

We define the sequences $(A_n)_{n=-1}^{\infty}$, $(B_n)_{n=-1}^{\infty}$ by putting

$$A_{-1} = 1, A_0 = a_0, A_n = a_n A_{n-1} + A_{n-2},$$
$$B_{-1} = 0, B_0 = 1, \ B_n = a_n B_{n-1} + B_{n-2},$$

for $n \geq 1$. By using matrices we can write

(1)
$$\mathscr{A}_n = \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \quad \text{for } n \geq 0,$$
$$\mathscr{B}_n = \begin{pmatrix} A_n & A_{n-1} \\ B_n & B_{n-1} \end{pmatrix} \quad \text{for } n \geq 0;$$

then,

$$\mathscr{B}_n = \mathscr{B}_{n-1} \mathscr{A}_n = \mathscr{A}_0 \mathscr{A}_1 \dots \mathscr{A}_n.$$

Notice that

$$\det(\mathscr{A}_n) = -1, \quad \det(\mathscr{B}_n) = (-1)^{n-1}.$$

We define the $n^{th}-$*convergent* to be

$$Q_n = \frac{A_n}{B_n} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}} \quad \text{for } n \geq 0.$$

We notice that the sequence of the convergents $\{Q_n\}_{n\in\mathbb{N}}$ $\mathfrak{P}$-adically converges; indeed an easy induction shows that $|B_n|_{v_0} = \prod_{j=1}^n |a_j|_{v_0}$, and $|Q_n - Q_{n-1}|_{v_0} = \frac{1}{|B_n|_{v_0}|B_{n-1}|_{v_0}}$. Then the claim follows by the hypothesis that $|a_n|_{v_0} > 1$ for every $n \geq 1$.

Conversely, every $\alpha \in K_{v_0}$ is the limit of a (unique) continued fraction of type τ obtained applying the following algorithm:

$$(2) \qquad \begin{cases} \alpha_0 & = \alpha, \\ \alpha_{n+1} & = \frac{1}{\alpha_n - a_n}, \\ a_n & = s(\alpha_n). \end{cases}$$

The sequence $[a_0, a_1, \ldots]$ obtained by algorithm (2) is called the *continued fraction expansion of type* τ for α.

3. Finiteness properties

Let $\tau = (K, \mathfrak{P}, s)$ be a type. We are interested in giving necessary and sufficient conditions on τ in order to ensure that every element of K has finite continued fraction expansion of type τ.

Following [16], where the authors address this problem in the Archimedean case, we introduce the following definitions.

DEFINITION 3.

a) We say that τ *satisfies the* Continued Fraction Finiteness property (CFF) *(resp. the* Continued Fraction Periodicity property (CFP)*) if every* $\alpha \in K$ *has a finite (resp. finite or periodic)* τ*-expansion.*

b) We say that the field K satisfies the $\mathfrak{P}$-adic Continued Fraction Finiteness property (CFF) *(resp. the* $\mathfrak{P}$-adic Continued Fraction Periodicity property (CFP)*) if there exists a type* $\tau = (K, \mathfrak{P}, s)$ *satisfying the* CFF *(resp.* CFP*) property.*

It was proven in [4, §3] that the Browkin types τ_B satisfy the CFF property, for every odd prime p; see also [7, Proposition 4.3] for a more effective result. On the other hand it is easy to see that the Ruban type cannot satisfy CFF, since negative rational numbers cannot have a terminating Ruban continued fraction. Laohakosol [12] and, independently, Wang [20] proved that τ_R satisfies CFP, proving in particular that, if a rational number as non-terminating τ_R-expansion, then the tail is equal to $\left[\overline{1 - \frac{1}{p}}\right]$.

However, none of these arguments were effective; more recently, in [8] the authors gave a quantitative estimation for the length of the expansion when this is finite, and an estimation on the length of the pre-periodic part in terms of the height of the rational number.

4. General results

Firstly, it is possible to prove a strong necessary condition for CFF:

PROPOSITION 1. *[7, Proposition 7.1 and Corollary 7.2] Assume that the field K satisfies the $\mathfrak{P}$-adic* CFF *property. Then the ideal class group of K is cyclic and generated by $[\mathfrak{P}]$.*

Moreover, $\mathcal{O}_K$ is a PID *when either $\mathfrak{P}$ is principal, or if K satisfies the $\mathfrak{P}$-adic* CFF *property for all but finitely many prime ideals $\mathfrak{P}$.*

4.1. A criterion for CFF

For $x \in \mathbb{C}$, we define

$$\theta(x) = \frac{1}{2}(|x|_\infty + \sqrt{|x|_\infty^2 + 4}).$$

In [7] the authors prove that, given a type $\tau = (K, \mathfrak{P}, s)$, a suitable bound involving the $\mathfrak{P}$-adic absolute values and the values of θ on the elements in the image of s and on their conjugates will guarantee the CFF (resp. CFP) property for τ:

THEOREM 1. *[7, Theorem 4.5] Let $\tau = (K, \mathfrak{P}, s)$ be a type. Let Σ be the set of embeddings of K in $\mathbb{C}$, and let us denote by*

$$\nu_\tau = \sup \left\{ \frac{\prod_{\sigma \in \Sigma} \theta(a^\sigma)}{|a|_{v_0}^{d_{v_0}}} \mid a \in \mathscr{Y}_s^1 \right\}.$$

Then,

a) if $\nu_\tau \leq 1$, then τ satisfies CFP;

b) if $\nu_\tau < 1$, then τ satisfies CFF.

The above criterion becomes more explicit in the case of special types:

THEOREM 2. *[7, Theorem 4.6] Let $\tau = (K, \pi, \mathscr{R})$ be a special type, and let Σ be the set of embeddings of K in $\mathbb{C}$. For every $\sigma \in \Sigma$, let $L_\sigma = \max\{|c^\sigma|_\infty \mid c \in \mathscr{R}\}$, and $\lambda_\sigma = |\pi^\sigma|_\infty$. Assume that, for every $\sigma \in \Sigma$,*

$$\lambda_\sigma > 1 \text{ and } L_\sigma \leq (\lambda_\sigma - 1)\left(1 - \frac{1}{\lambda_\sigma^2}\right);$$

then,

a) τ *satisfies the* CFP *property;*

b) *if moreover* $L_\sigma < (\lambda_\sigma - 1)\left(1 - \frac{1}{\lambda_\sigma^2}\right)$ *for at least one* $\sigma \in \Sigma$*, then* τ *satifies the* CFF *property.*

4.2. The CFF property for norm Euclidean number fields

As an application of Theorem 1, by using basic notions in geometry of numbers and the fundamental properties of the norm Euclidean minimum in [9], the following result was proven in [7]:

THEOREM 3. *[7, Theorem 5.6] Assume that* K *is a norm Euclidean number field having Euclidean minimum* $M(K) < 1$*. Then* K *satisfies the* $\mathfrak{P}$*-adic* CFF*-property for all but finitely many prime ideals* $\mathfrak{P}$ *of* $\mathscr{O}_K$*.*

As pointed out in [7, Remark 5.7] the condition $M(K) < 1$ in Theorem 3 is verified for "almost all" norm Euclidean number fields.

Theorem 3 admits an interesting generalisation to prime ideals lying in a *norm Euclidean ideal class* in the sense of [14]; indeed, the paper [17] extends the results on the Euclidean minimum presented in [9] to this setting.

THEOREM 4. *[7, Theorem 7.4] Let* K *be a number field and assume that* K *has a norm Euclidean ideal class* $\mathscr{C}$ *having Euclidean minimum* $M_{\mathscr{C}} < 1$ *and rank of units* $r > 1$*. Then,* K *satisfies the* $\mathfrak{P}$*-adic* CFF *property for all but finitely many* $\mathfrak{P} \in \mathscr{C}$*.*

We notice that non principal Euclidean classes exist for example for fields like $\mathbb{Q}(\sqrt{-15})$ and $\mathbb{Q}(\sqrt{-20})$ (see [14, Prop. 2.1]), and $\mathbb{Q}(\sqrt{10})$, $\mathbb{Q}(\sqrt{15})$, $\mathbb{Q}(\sqrt{85})$ (see [14, 2.5]); other examples can be found in [15].

5. CFF for norm Euclidean quadratic fields

In the case of norm Euclidean quadratic fields, we obtained more explicit results.

5.1. Imaginary norm Euclidean quadratic fields

Let $K = \mathbb{Q}(\sqrt{-D})$ with D a square free integer > 0. It is known that the Euclidean minimum $M(K)$ is given by

$$M(K) = \begin{cases} \frac{D+1}{4} & \text{if } D \equiv 1,2 \pmod 4 \\ \frac{(D+1)^2}{16D} & \text{if } D \equiv 3 \pmod 4 \end{cases}$$

(see for example [13, Prop. 4.2]). It follows that the only norm Euclidean quadratic imaginary fields $K = \mathbb{Q}[\sqrt{-D}]$ with $M(K) < 1$ are those having $D = 1,2,3,7,11$.

PROPOSITION 2. *[7, Proposition 6.1] Let $K = \mathbb{Q}(\sqrt{-D})$ be a imaginary quadratic norm Euclidean field. Let $\mathfrak{P}$ be a prime ideal of $\mathcal{O}_K$ with odd residual characteristics. Put $\lambda = \sqrt{N_{K/\mathbb{Q}}(\mathfrak{P})}$. Then*

a) if $\sqrt{M(K)} < 1 - \frac{1}{\lambda^2}$, then K satisfies the $\mathfrak{P}$-adic CFF *property.*

b) if $\sqrt{M(K)} < \left(1 - \frac{1}{\lambda}\right)^2 \left(1 + \frac{1}{\lambda}\right)$, then there exists a special type $\tau = (K, \pi, \mathscr{R})$ satisfying the CFF *property.*

The following list summarises the behaviour of the $\mathfrak{P}$-adic CFF property for imaginary norm Euclidean fields $K = \mathbb{Q}(\sqrt{-D})$; p denotes the rational prime $\mathfrak{P} \cap \mathbb{Z}$.

	D	1	2	3	7	11
CFF property	for $p \geq$	3	5	2	3	7
CFF special type	for $p \geq$	7	23	11	13	127

5.2. Real norm Euclidean quadratic fields: some explicit constructions

It is well known that a real quadratic field $\mathbb{Q}(\sqrt{D})$ is norm Euclidean if and only if $D = 2, 3, 11, 13, 17, 19, 21, 29, 33, 37, 41, 57, 73$ (see for example [11]). In [10], the authors give a constructive proof of this fact by showing that, in each of these cases, the fundamental region is covered by (finitely many) unit neighborhoods of the plane, explicitely exhibiting a family of such neighborhoods for each field. Using this in combination with the construction of the proof of Theorem 3, it is shown in [7, §6.2] how to construct explicitly a $\mathfrak{P}$-adic floor function for a prime ideal $\mathfrak{P}$ of $\mathcal{O}_K$. We give a sketch of this construction: let $K = \mathbb{Q}(\sqrt{D})$ be a real norm Euclidean field; we consider the plane embedding given by

$$\begin{aligned} j : K &\longrightarrow \mathbb{R}^2 \\ a + b\sqrt{D} &\longmapsto (a, b); \end{aligned}$$

this gives a representation of the elements of K as the points of the plane with rational coordinates.

Under this plane embedding, the algebraic integers correspond to the lattice points $\mathbb{Z}^2$, if $D \equiv 2, 3 \pmod 4$, and to the mid-lattice points $\frac{1}{2}\mathbb{Z}^2$ if $D \equiv 1 \pmod 4$.

For any $\lambda \in \mathcal{O}_K$, we define the neighborhood of λ in K of radius ε to be the set

$$V_\varepsilon(\lambda) = \{\beta \in \mathbb{Q}(\sqrt{D}) \mid |N_{K/\mathbb{Q}}(\beta - \lambda)| < \varepsilon\};$$

using the plane embedding, this maps to

$$V_\varepsilon(x, y) = \{(r, s) \in \mathbb{Q}^2 \mid |(r - x)^2 - D(s - y)^2| < \varepsilon\},$$

where $(x,y) = j(\lambda)$. Notice that these are infinite "X-shaped" regions in the plane bounded by conjugate hyperbolas.

It is then clear that, since we are assuming that K is norm Euclidean, each $\beta \in \mathbb{Q}(\sqrt{D})$ lies in the neighborhood $V_\varepsilon(\lambda)$ for some $\lambda \in \mathscr{O}_K$, i.e. each point $(r,s) \in \mathbb{Q}^2$ lies in some neighborhood $V_\varepsilon(x,y)$ in the plane, where $(x,y) = j(\lambda)$ for some $\lambda \in \mathscr{O}_K$.

Let $\mathfrak{P}$ be a prime ideal in $\mathscr{O}_K$; we can associate to every generator $\pi \in \mathfrak{P}$ a type $\tau_\pi = (\mathbb{Q}, \mathfrak{P}, s_\pi)$, where the floor function s_π is defined by the following algorithm: given a coset $\alpha + \mathfrak{P}\mathscr{O}_{v_0}$ in K_{v_0}, we can find, by strong approximation, an element $\alpha' \in K$ belonging to this coset such that $|\alpha'|_v < 1$ for every non-archimedean $v \in \mathscr{M}_K \setminus \{v_0\}$; in particular, $\alpha' \in \mathscr{O}_K[\frac{1}{\pi}]$. We can now translate $\frac{\alpha'}{\pi}$ by a suitable element $\mu \in \mathscr{O}_K$ so that $j(\beta) := j(\alpha' - \mu)$ belongs to the region

$$F(D) := \left\{ (r,s) \in \mathbb{Q}^2 \;\mid\; -\frac{1}{2} < r \leq \frac{1}{2},\ -\frac{1}{2} < s \leq \frac{1}{2} \right\},$$

and such β is unique. We call $F(D)$ fundamental region. Notice that $\alpha' \equiv \pi\beta \pmod{\mathfrak{P}}$. By [10], we have that $F(D)$ is covered by a finite number of neighborhoods or radius $\varepsilon < 1$ (depending of D) $V_\varepsilon(x_k, y_k)$; hence, $j(\beta)$ lies in (almost) one of these neighbourhood. We choose a neighbourhood $V_\varepsilon(x', y')$ such that $j(\beta)$ lies in it and, for every $\gamma \in \alpha + \mathfrak{P}\mathscr{O}_{v_0}$, we put

$$s_\pi(\gamma) := \pi(\beta - j^{-1}(x', y')).$$

EXAMPLE 2. Let us consider the case $D = 17$; since $D \equiv 1 \pmod 4$, then $\mathscr{O}_K = \mathbb{Z}\left[\frac{1+\sqrt{D}}{2}\right]$. Let us divide the fundamental region $F(17)$ into six subsets, namely:

- $F_1 = \{(x,y) \in \mathbb{Q}^2 \mid 0 < r \leq 1/2,\ -1/4 < s \leq 1/4\}$;
- $F_2 = \{(x,y) \in \mathbb{Q}^2 \mid -1/2 < r \leq 0,\ -1/4 < s \leq 1/4\}$;
- $F_3 = \{(x,y) \in \mathbb{Q}^2 \mid 0 < r \leq 1/2,\ 1/4 < s \leq 1/2\}$;
- $F_4 = \{(x,y) \in \mathbb{Q}^2 \mid 0 < r \leq 1/2,\ -1/2 < s \leq -1/4\}$;
- $F_5 = \{(x,y) \in \mathbb{Q}^2 \mid -1/2 < r \leq 0,\ 1/4 < s \leq 1/2\}$;
- $F_6 = \{(x,y) \in \mathbb{Q}^2 \mid -1/2 < r \leq 0,\ -1/2 < s \leq -1/4\}$.

Then, $F(17)$ is equal to the union of these regions, and the union is disjoint, hence every $\beta \in F(17)$ belongs to one F_k. We have now to associate to every F_k a unit neighborhood $V(x,y)$ that covers the corresponding region; this can be of course done in many ways.

We use an argument analogous to [10]. By easy calculations, we have that the point $(1/2, 1/4) \in F_1$ lies of the top boundary of the neighborhood $V_{13/16}(1,0)$, hence the preimage of every point in F_1 satisfies $N_{K/\mathbb{Q}}(\beta - 1) \leq 13/16$. Similarly, the point $(1/2, 1/4)$ lies on the bottom boundary of the neighborhood $V_{13/16}(1, 1/2)$, hence F_3 is contained in its closure. Using the symmetry properties of $F(17)$, it is easy to see that $F_2 \subset \overline{V_{13/16}(-1,0)}$, $F_4 \subset \overline{V_{13/16}(1,-1/2)}$, $F_5 \subset \overline{V_{13/16}(-1,1/2)}$ and $F_6 \subset \overline{V_{13/16}(-1,-1/2)}$. For every $k = 1,\dots,6$, let us denote by δ_k the preimage in $\mathscr{O}_K$ of the center of the corresponding neighborhood, i.e. $\delta_k := j^{-1}(x_k, y_k)$. Using this, we can perform the algorithm described above.

Given a prime ideal $\mathfrak{P} \subset \mathscr{O}_{v_0}$, choose a suitable generator π of $\mathfrak{P}\mathscr{O}_{v_0}$. Then, for every coset $\alpha + \mathfrak{P}\mathscr{O}_{v_0}$, choose $\alpha' \in \alpha + \mathfrak{P}\mathscr{O}_{v_0} \cap \mathscr{O}_K[1/\pi]$, and translate it by an element $\mu \in \mathscr{O}_K$ so that the image of $\beta := \alpha - \mu$ lies in the fundamental region; then $j(\beta) \in F_k$ for some $k = 1,\dots,6$.

Take any $\gamma \in \alpha + \mathfrak{P}\mathscr{O}_{v_0}$; then, we denote by

$$s_\pi(\gamma) := \pi(\beta - \delta_i).$$

Let us show for example that, if p is an odd prime which is inert in $\mathscr{O}_K$, then this choice of the floor function gives rise to a type satisfying CFF property.

If p is inert, then we can take $\pi = p$, hence $N_{K/\mathbb{Q}}(\mathfrak{P}) = p^2$. To apply Theorem 1, we have to estimate $\theta(a^\sigma)$ for every $a \in \mathscr{Y}_s^1$ and every embedding σ of K into $\mathbb{R}$, which are exactly the identity and the one sending $\sqrt{17}$ to $-\sqrt{17}$. By the above choice of the neighborhoods covering the fundamental region and the corresponding construction of the floor function, we have that, for every $\beta \in F_k$ and for every centre of the corresponding neighborhood δ_k, $|\beta^\sigma - \delta_k^\sigma| \leq \sqrt{5}/4$, and $N_{K/\mathbb{Q}}(\beta^\sigma - \delta_k^\sigma) \leq 1/4$, hence for every $a \in \mathscr{Y}_s^1$, we have $N_{K/\mathbb{Q}}(a) \leq \frac{13}{16}p^2$.

It follows that

$$\prod_{\sigma\in\Sigma} \theta(a^\sigma) < \prod_{\sigma\in\Sigma} (1 + |a^\sigma|) \leq 1 + \frac{\sqrt{5}}{2}p + \frac{13}{16}p^2.$$

Since $|a|_p^{d_p} \geq p^2$, we have that $\nu_{\tau_\pi} < 1$ if

$$1 + \frac{\sqrt{5}}{2}p + \frac{13}{16}p^2 < p^2,$$

which holds for every prime $p \geq 3$. We finally point out that a similar argument involving another choice of the generator of the prime ideal $\mathfrak{P}$ can be used in the case where p split.

Section 6.3 of [7] is devoted to study in details the case $K = \mathbb{Q}(\sqrt{2})$:

THEOREM 5. *[7, Theorem 6.3] The field $\mathbb{Q}(\sqrt{2})$ has the $\mathfrak{P}$-adic* CFF *property, for every prime ideal $\mathfrak{P}$ of odd residual characteristics.*

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