Available online at www.sciencedirect.com

ScienceDirect

Journal of Economic Theory 194 (2021) 105254

JOURNAL OF Economic Theory

www.elsevier.com/locate/jet

Common priors under endogenous uncertainty ☆

Pierfrancesco Guarino [a,*], Elias Tsakas [b]

[a] *Alpen-Adria-Universität Klagenfurt (Institut für Volkswirtschaftslehre), Austria*
[b] *Department of Microeconomics and Public Economics (MPE), Maastricht University, Netherlands*

Received 29 July 2020; final version received 3 February 2021; accepted 8 April 2021
Available online 15 April 2021

Abstract

For a fixed game and a type structure that admits a common prior, Action Independence states that the conditional beliefs induced by the common prior do not depend on the players' own strategies. It has been conjectured that Action Independence can be behaviorally characterized by means of a suitable no-betting condition (Dekel and Siniscalchi, 2015), but whether this is indeed the case remains an open problem. In this paper, we prove this conjecture true by focusing on strategy-invariant bets, which are bets that cannot be manipulated by the players. In particular, first, we show that at least one of the common priors satisfies Action Independence if and only if there exists no mutually acceptable strategy-invariant bet among the players. Second, we show that, all common priors satisfy Action Independence if and only if there exists no mutually acceptable strategy-invariant bet among the players and an outside observer. These results give us a deeper understanding of existing foundations of solution concepts using only epistemic conditions that are expressed in terms of type structures and are therefore elicitable.

JEL classification: C70; D82

Keywords: Common prior; No-betting condition; Endogenous uncertainty; Action independence; Strategy-invariant bets

☆ We would like to thank Pierpaolo Battigalli, Giacomo Bonanno, the editor in charge of the manuscript, Marciano Siniscalchi, and two anonymous referees for valuable comments and suggestions that ultimately lead to the present version. Of course, all errors are our own. Pierfrancesco thankfully acknowledges financial support from the Austrian Science Fund (FWF) (P31248-G27).

* Corresponding author.
E-mail addresses: pf.guarino@hotmail.com (P. Guarino), e.tsakas@maastrichtuniversity.nl (E. Tsakas).
URL: http://www.elias-tsakas.com (E. Tsakas).

https://doi.org/10.1016/j.jet.2021.105254

1. Introduction

1.1. Motivation & results

When using game-theoretic solution concepts in order to derive predictions, it is important to have a proper understanding of the underlying assumptions upon which they rely. The epistemic approach to game theory focuses on identifying the players' beliefs that would lead each player to act in accordance with these predictions. Such beliefs are the product of some—arguably reasonable—restrictions on players' belief hierarchies, called *epistemic conditions*. Most such restrictions are related to rationality in some way or form, e.g., *Rationality and Common Belief in Rationality* (Böge and Eisele, 1979; Brandenburger and Dekel, 1987; Tan and da Costa Werlang, 1988), which expresses the idea that it is transparent across all the players that everybody is rational.

One notable exception is the *Common Prior Assumption* (henceforth, CPA), which posits the existence of a common prior on the state space used as a platform to capture the players' interactive beliefs.[1] Intuitively, by means of the CPA it is possible to have beliefs of different players be somehow consistent with each other. Indeed, this consistency is achieved by assuming the existence of a probability measure—the common prior—over the state space representing all parameters of interest (i.e., the strategy profiles and the players' belief hierarchies), which happens to be simultaneously consistent via conditioning with all players' belief hierarchies, i.e., all the information contained in the different belief hierarchies of different players is captured by this one measure.

The interest in the CPA has largely stemmed from the fact that it plays a crucial role in the epistemic characterization of well-known equilibrium solution concepts, such as Nash Equilibrium by Aumann and Brandenburger (1995, Theorem B, Section 4), Objective Correlated Equilibrium by Aumann (1987, Main Theorem, Section 3), and Bayes Correlated Equilibrium by Bergemann and Morris (2016, Theorem 1, Section 2.2, p. 495). This is not surprising, since equilibrium concepts have inherently built-in the idea that players hold correct beliefs about each other, i.e., they have to be somehow consistent with each other. And in fact the consistency imposed by the CPA suffices in this respect.

However, there is a caveat. On the one hand, the CPA is conceptually well-founded when uncertainty is *exogenous*, i.e., when belief hierarchies are about exogenous parameters that the players do not control themselves (e.g., when belief hierarchies are about private values in an auction). On the other hand, Dekel and Siniscalchi (2015) recently pointed out that it may not always be an innocent assumption in presence of *endogenous* uncertainty, i.e., when beliefs are about each other's strategies. In particular, these authors worked with *epistemic type structures*, the—by now benchmark—framework based on product state spaces, where players hold beliefs on other players' strategies and belief hierarchies, but do *not* over their own strategies. There, they noticed that a common prior may postulate that a player's beliefs depend on this same player's own strategy. This observation is clearly at odds with the predominant Bayesian view according to which beliefs are updated only when new information arrives. All this is particularly relevant for the epistemic characterization of equilibrium concepts, which employ the CPA under endogenous uncertainty.

[1] There is an extensive literature, accompanied by a debate, on the conceptual underpinnings and the foundations of the CPA (Aumann, 1976, 1987, 1998; Nau and McCardle, 1990; Gul, 1998; Bonanno and Nehring, 1999). See also Morris (1995) for a methodological discussion of the topic.

In response to this problem, Dekel and Siniscalchi (2015, Definition 12.15) proposed a condition that a common prior must satisfy in order to rule out such phenomena, called *Action Independence* or *Aumann Independence* or simply *AI condition*.[2] According to this condition, the beliefs that a player inherits from the epistemic type structure are the same as the conditional beliefs induced by the common prior, irrespectively of the strategy chosen by the player.[3] Then they went on to emphasize the importance of providing a behavioral foundation for the AI condition and conjectured that this should be possible by means of a suitable no-betting condition. Since then, this has remained an open problem.

In this paper, we address this question and provide an affirmative answer. In particular, we classify each epistemic type structure that admits a (not necessarily unique) common prior into one of three categories: either all common priors satisfy the AI condition, or some common priors satisfy the AI condition and some do not, or none of the common priors satisfies the AI condition. We do so by means of our two main results. First, our Theorem 1 shows that there exists some common prior satisfying the AI condition if and only if there exists no mutually acceptable bet among the players. Second, assuming that there exists at least one common prior satisfying the AI condition, Theorem 2 shows that all common priors satisfy the AI condition if and only if there exists no mutually acceptable bet among the players and an outside observer. Combined, the two results allow us to pin down each epistemic type structure into one of the three categories mentioned above. Crucially, both our results restrict attention to strategy-invariant bets, i.e., bets that pay for each type the same expected payoff irrespectively of the strategy chosen by this type. In this way we avoid providing incentives that may affect the behavior of the players during the upcoming game and—as a consequence—the bets we employ do not indirectly affect the underlying epistemic type structures that aim to characterize.

Besides providing foundations for the AI condition per se, our results provide deeper insights into existing epistemic foundations of solution concepts using epistemic type structures.[4] Focusing on Objective Correlated Equilibrium, a recent result shows that for every epistemic type structure admitting a common prior that satisfies the AI condition, Rationality and Common Belief in Rationality implies an Objective Correlated Equilibrium distribution (Dekel and Siniscalchi, 2015, Theorem 12.4). However, the Objective Correlated Equilibrium distribution is obtained by taking the marginal of the common prior over the strategy space. That is, the underlying epistemic type structure does not contain enough information in order for the analyst to pin down how exactly the players' strategies are correlated. Instead, this last part depends on the common prior that the analyst has chosen, among the possibly *multiple* common priors. But then, if an epistemic type structure belongs to our second category (i.e., it admits some common priors that satisfy the AI condition and some that do not), it can be the case that some common prior induces an Objective Correlated Equilibrium distribution, while some other common prior does not (see the declination of Example 1 in Section 4). This is exactly where our results come in handy: using Theorem 2, we can conclude that, if all common priors admitted by an epistemic

[2] "Action Independence" is the way in which this condition is called in Battigalli et al. (in preparation), while "Aumann Independence" is the name used in Dekel and Siniscalchi (2015).

[3] In a private communication, Giacomo Bonanno directed us to Stalnaker (1998), where—in Footnote 5 at page 25—the author addresses the need to deal with the same issues covered by the AI condition in the framework he employs.

[4] Using epistemic type structures allows us to express epistemic conditions using properties that can be elicited. This is in contrast to alternative epistemic models (e.g., Aumann's partitional model) where belief hierarchies are attached to strategies and are—therefore—unelicitable. For a more detailed discussion of this issue, see Dekel and Siniscalchi (2015, Sections 12.1.1 & 12.2.6).

type structure satisfy the AI condition, it will necessarily be the case that all of them induce an Objective Correlated Equilibrium distribution. Hence, the conditions of Dekel and Siniscalchi (2015, Theorem 12.4) can now be expressed entirely in the language of epistemic type structures without any reference to a specific common prior.

1.2. Related literature

This paper belongs to a rather rich literature on the behavioral foundations of epistemic assumptions. Regarding the CPA, Aumann (1976), Milgrom and Stokey (1982), Sebenius and Geanakoplos (1983), Morris (1994), Samet (1998), Bonanno and Nehring (1999), Feinberg (2000), Heifetz (2006) focused on providing behavioral characterizations of the CPA. All these contributions dealt with exogenous uncertainty, i.e., the epistemic models they employed represented belief hierarchies about exogenous parameters, as opposed to our case where belief hierarchies are defined on players' strategies. A notable exception is the work of Barelli (2009), where there is a behavioral characterization of a weakening of the CPA called "Action Consistency", that explicitly works in a game theoretic setting. However, it has to be pointed out that, to capture interactive reasoning, Barelli (2009) employs a framework where a player's belief hierarchy is inextricably linked to a specific strategy of hers.

1.3. Synopsis

The paper is structured as follows. Section 2 introduces the building blocks of our analysis, i.e., epistemic type structures, common priors, the AI condition, and bets on state spaces. In Section 3, we obtain our characterization results. In Section 4, we analyze the implications of our results in the study of Objective Correlated Equilibrium. Finally, in Section 5, we address various issues related with our analysis. Proofs are relegated to Appendix A.

2. Theoretical framework

2.1. Epistemic type structures

We begin by fixing a game in its strategic form $\Gamma := \langle I, (S_i, u_i)_{i\in I}\rangle$ (henceforth, the *game*). As usual, I is the set of players, which without loss of generality is assumed to contain only two players, viz., $I := \{\text{Ann}(a), \text{Bob}(b)\}$. For every $i \in I$, we let S_i be player i's (finite) set of strategies and $u_i : S_a \times S_b \to \Re$ the payoff function.

Given a game, a standard *epistemic type structure* (henceforth, type structure) attached on it is a tuple

$$\mathscr{T} := \langle I, (T_i, \beta_i)_{i\in I}\rangle$$

where T_i is player i's finite set of *types* and $\beta_i : T_i \to \Delta(S_j \times T_j)$ is her *belief function*. As it is well-known, every type $t_i \in T_i$ of every player $i \in I$ (inductively) encodes an infinite hierarchy of beliefs (Brandenburger and Dekel, 1993, Section 2). We say that the type structure is *non-redundant* if every type encodes a different infinite hierarchy of beliefs. Throughout the paper, without loss of generality, we focus on non-redundant type structures (see Section 5.2).

Let $\Omega := S_a \times S_b \times T_a \times T_b$ denote the state space induced by the type structure. For each $s_i \in S_i$ and $t_i \in T_i$, we define the events $[\![s_i]\!] := \{s_i\} \times T_i \times S_j \times T_j$ and $[\![t_i]\!] := S_i \times \{t_i\} \times S_j \times T_j$ respectively, with $[\![s_i, t_i]\!] := [\![s_i]\!] \cap [\![t_i]\!]$.

Definition 1 *(Common prior).* A type structure $\mathscr{T}$ *admits a common prior* if there exists a $\pi \in \Delta(\Omega)$ such that, for every $i \in I$ and every $t_i \in T_i$,

i) $\pi(\llbracket t_i \rrbracket) > 0$,
ii) $\beta_i(t_i)(s_j, t_j) = \pi(\llbracket s_j, t_j \rrbracket \mid \llbracket t_i \rrbracket)$, for all $(s_j, t_j) \in S_j \times T_j$.

If there exists such π, then π is deemed a *common prior* of $\mathscr{T}$, with $\mathscr{T}$ admitting it. The set of all common priors of $\mathscr{T}$ is denoted by $\Pi_{\mathscr{T}} \subseteq \Delta(\Omega)$.

Obviously, not all type structures admit a common prior. Nevertheless, throughout this paper we are only interested in type structures that admit at least one common prior. It is not difficult to see that there may exist multiple common priors admitted by a given type structure.

Example 1. Let $S_a = \{U, D\}$ and $S_b = \{L, R\}$, and consider the type structure with type spaces $T_a = \{t_a\}$ and $T_b = \{t_b\}$, and belief functions illustrated below.

	(L, t_b)	(R, t_b)
$\beta_a(t_a)$	$1/2$	$1/2$

	(U, t_a)	(D, t_a)
$\beta_b(t_b)$	$1/2$	$1/2$

Note that this type structure admits multiple common priors. Indeed, it can be easily verified that the probability measure in the following table constitutes a common prior for every $\varepsilon \in [0, \frac{1}{2}]$.

	(L, t_b)	(R, t_b)
(U, t_a)	$1/2 - \varepsilon$	ε
(D, t_a)	ε	$1/2 - \varepsilon$

Table 1. A family of common priors.

Notably, even this very simple type structure has uncountably many common priors.

It has to be emphasized that we do not take π as a primitive of our epistemic model, as we do not consider a (fictitious) *ex ante* stage from which the players' beliefs are updated conditioning on the realization of the types (see the references in Footnote 1 for a discussion of this issue). Instead, the only primitives of our model are the belief hierarchies that are encoded in the type structure. In this sense, the common prior enriches our model by inserting additional information to our model in terms of beliefs that are not described within the type structure. Take for instance the previous example with $\varepsilon = 0$ and observe that the common prior essentially introduces new beliefs for Ann, viz., according to this common prior, once Ann has chosen U, she believes that Bob will choose L with probability 1. Clearly, we have to be careful in how we interpret and use these additional beliefs, and—ideally—we want them to be inconsequential for our game-theoretic analysis.

2.2. *The AI condition*

As we have discussed in the previous section, a common prior often contains additional information—beyond what is described by the type structure—due to the fact that players form conditional beliefs given each of their own strategies. As a result, we have to be aware of the fact that, upon having introduced a common prior, inconsistencies may arise. We illustrate this point in the following example.

Example 1 *(cont).* First, recall that Ann's only type puts equal probability to (L, t_b) and (R, t_b) according to her belief function. Take the common prior induced by setting $\varepsilon = 0$. Now, observe that, if Ann chooses to play U, her beliefs induced by the common prior attach probability 1 to L. However, if she chooses to play D, her beliefs induced by the common prior attach probability 1 to R. ◇

The inconsistency in the previous example is due to a conceptually awkward property of the common prior. Namely, according to the chosen common prior, Ann updates her beliefs without having received any new information. She does so after having conditioned with respect to an endogenous variable, viz., her own planned strategy. This is clearly at odds with the standard Bayesian view according to which updating is the result of taking into account new information. This discrepancy was first identified by Dekel and Siniscalchi (2015, Example 12.4, pp. 642–643), who went on to propose the following property that a common prior must satisfy in order to avoid such inconsistencies.

Definition 2 *(AI condition).* The common prior $\pi \in \Pi_{\mathscr{T}}$ satisfies the *AI condition* if, for all $i \in I$ and all $(s_i, t_i), (s_i', t_i) \in S_i \times T_i$ with $\pi([\![s_i, t_i]\!]) > 0$ and $\pi([\![s_i', t_i]\!]) > 0$,

$$\pi([\![s_j, t_j]\!] \mid [\![s_i, t_i]\!]) = \pi([\![s_j, t_j]\!] \mid [\![s_i', t_i]\!]), \tag{2.1}$$

for every $(s_j, t_j) \in S_j \times T_j$, with $\Pi_{\mathscr{T}}^{AI} \subseteq \Pi_{\mathscr{T}}$ denoting the set of common priors that satisfy the AI condition.

The acronym AI can stand for "Action Independence" or "Aumann Independence". It is not difficult to verify that the only common prior in Example 1 that satisfies the AI condition is the one obtained by setting $\varepsilon = \frac{1}{4}$. Indeed, for any other ε, it is the case that $\pi([\![L, t_b]\!] | [\![U, t_a]\!]) = \frac{1}{2} - \varepsilon \neq \varepsilon = \pi([\![L, t_b]\!] | [\![D, t_a]\!])$, implying that Ann's beliefs depend on her own strategy.

An important remark is warranted here: the AI condition has a bite when the type structure represents belief hierarchies about endogenous variables, viz., the players' own strategies. Whenever the type structure represents beliefs about exogenous variables (e.g., about the preferences of the players), it is clearly the case that the choice of a player's own strategy does not affect her own beliefs about said exogenous variables.

Remark 1. A common prior $\pi \in \Pi_{\mathscr{T}}$ satisfies the AI condition if and only if, for all $(s_j, t_j) \in S_j \times T_j$,

$$\pi([\![s_j, t_j]\!] \mid [\![s_i, t_i]\!]) = \beta_i(t_i)(s_j, t_j)$$

for every $i \in I$ and $(s_i, t_i) \in S_i \times T_i$ with $\pi([\![s_i, t_i]\!]) > 0$, i.e., whenever conditional beliefs are derived from a common prior that satisfies the AI condition, said conditional beliefs coincide with those inherited from the type structure.

Following the previous remark—while assuming that the type structure $\mathscr{T}$ admits some (not necessarily unique) common prior—we can classify $\Pi_{\mathscr{T}}$ into one of the following categories:

(Π_1) all common priors satisfy the AI condition, i.e., $\Pi_{\mathscr{T}} = \Pi_{\mathscr{T}}^{AI} \neq \emptyset$;

(Π_2) there are multiple common priors, some satisfying the AI condition and some not, i.e., $\Pi_{\mathscr{T}} \supsetneq \Pi^{AI}_{\mathscr{T}} \neq \emptyset$;
(Π_3) there exists no common prior satisfying the AI condition, i.e., $\Pi^{AI}_{\mathscr{T}} = \emptyset$.

Since the AI condition allows us to epistemically characterize equilibrium concepts such as Objective Correlated Equilibrium (Dekel and Siniscalchi, 2015, Theorem 12.4) or Nash Equilibrium (Dekel and Siniscalchi, 2015, Theorem 12.7), we would like to know in which of the aforementioned categories each type structure is classified. This is actually the main research question of this paper, which we address in the following sections.

2.3. Bets on the state space

It is common practice to characterize (epistemic) properties in terms of appropriate no-betting conditions in light of the fact that a player's willingness to accept a bet is observable. Therefore, these characterizations allow us to build testable hypotheses about the property of interest. This approach dates back to the early contributions of Milgrom and Stokey (1982) and Sebenius and Geanakoplos (1983), and the subsequent work of Samet (1998) and Feinberg (2000) among others, which have led to a full characterization of common priors in models with exogenous uncertainty. In their recent review article, Dekel and Siniscalchi (2015) conjectured that a similar characterization of the AI condition in terms of a suitable no-betting condition should be possible in type structures that represent beliefs about endogenous variables.

A *bet* on the state space Ω is a profile of random variables $g := (g_a, g_b)$, with $g_i \in \Re^{\Omega}$ for each $i \in I$, such that $g_a + g_b = 0$, i.e., simply put, a bet is a zero-sum contingent claim.

Definition 3 *(Willingness to bet).* Player $i \in I$ *accepts* g at some state in $[\![s_i, t_i]\!]$ if

$$\mathbb{E}[g_i|s_i, t_i] := \sum_{s_j \in S_j} \sum_{t_j \in T_j} g_i(s_i, t_i, s_j, t_j) \cdot \beta_i(t_i)(s_j, t_j) \geq 0.$$

Player i *strictly accepts* the bet at some state in $[\![s_i, t_i]\!]$ if the previous inequality is strict.

Crucially, note that a player's willingness to accept (resp., strictly accept) a bet at some state depends on the beliefs she inherits from the type structure. That is, even if we fix a common prior $\pi \in \Pi_{\mathscr{T}}$, the player does not evaluate the bet using the conditional beliefs $\pi(\cdot|[\![s_i, t_i]\!])$, but rather using the conditional beliefs $\pi(\cdot|[\![t_i]\!])$ which always coincide with $\beta_i(t_i)$. This assumption follows naturally from the fact that our primitive concept is a type structure and not a common prior that is admitted by it. We illustrate this point in the context of Example 1.

Example 1 *(cont).* Take the bet that pays Ann the following amounts at each state.

	(L, t_b)	(R, t_b)
(U, t_a)	20	−10
(D, t_a)	−10	10

It is clear that according to the beliefs that Ann inherits from the type structure, she (weakly) accepts the bet at every state in $[\![D, t_a]\!]$, as she attaches probability $\frac{1}{2}$ to each of Bob's strategy-type pair thus yielding zero expected payoff at both states in $[\![D, t_a]\!]$. If we instead used the common prior to evaluate the bet, Ann's willingness to accept it would depend on which—among the multiple common priors—we would fix, viz., for $\varepsilon = \frac{1}{4}$ her willingness to bet would be the same as above, for $\varepsilon < \frac{1}{4}$ she would reject the bet in $[\![D, t_a]\!]$, whereas for $\varepsilon > \frac{1}{4}$ she would strictly accept it in $[\![D, t_a]\!]$. But, again, which common prior is fixed from the set $\Pi_{\mathscr{T}}$ is an arbitrary modeling choice that the analyst makes and it is not based on parameters one can elicit. $\diamond$

Going a step further, even if we use the beliefs that come from the type structure to evaluate the bet at each state, players can influence the outcome of the bet by suitably choosing their own strategy. For instance, in the previous example, it is clear that Ann would prefer to choose U to get access to the good payoff of 20. However, this would have implications for her behavior in the underlying game Γ and—*a fortiori*—on the beliefs in our type structure. This would be clearly undesirable, in the sense that the bet that we would be using to test a property of the type structure (viz., the AI condition) would affect the type structure itself, i.e., we would be having a Heisenberg type of effect. To avoid such a phenomenon, we restrict attention to bets that cannot be manipulated by the players.

Definition 4 *(Strategy-invariant bet).* A bet g is called *strategy-invariant* (henceforth, SI) *for type $t_i \in T_i$* of player $i \in I$ if

$$\mathbb{E}[g_i|s_i, t_i] = \mathbb{E}[g_i|s_i', t_i]$$

for every $s_i, s_i' \in S_i$. A bet is SI if it is SI with respect to every $t_i \in T_i$ and every $i \in I$.

The underlying idea is that, conditional on a type t_i, player i receives the same expected payoff irrespective of the strategy chosen. Hence, the bet does not interfere with the incentives that the player faces in the underlying game and—given that this is the case for every type of every player—the introduction of the bet does not affect the underlying type structure. For instance, in the previous example, to obtain an SI bet we can replace Ann's payoff of 20 with a payoff of 10. Indeed, notice that in such a case, Ann would receive 0 in expectation irrespective of whether she chooses U or D and the same is true for Bob irrespective of whether he chooses L or R. Throughout the rest of the paper, we focus exclusively on SI bets.

Definition 5 *(No-betting condition).* An SI bet g is *mutually acceptable* if, for every $i \in I$ and every $(s_i, t_i) \in S_i \times T_i$,

$$\mathbb{E}[g_i|s_i, t_i] \geq 0,$$

with at least one inequality being strict.

That is, we call an SI bet *mutually acceptable* if every player accepts it at every state and there exists at least one player who strictly accepts it at some state. Our main research question then becomes whether we can classify type structures that admit a common prior into one of the categories (Π_1), (Π_2), (Π_3) from the existence of mutually acceptable bets. The remaining of the paper addresses this question.

3. Characterization results

3.1. Is there a common prior satisfying the AI condition?

We begin by identifying the type structures $\mathscr{T}$ which admit a common prior that satisfies the AI condition. Formally, what we do first is to provide necessary and sufficient conditions in terms of the existence of mutually acceptable SI bets so that $\Pi^{AI}_{\mathscr{T}} \neq \emptyset$. In other words, our first result indicates if the type structure belongs to one of the first two categories (viz., (Π_1) or (Π_2)) or whether it belongs to the third category (viz., (Π_3)). This result already answers affirmatively the conjecture of Dekel and Siniscalchi (2015).

Theorem 1. *Given a type structure $\mathscr{T}$, the following hold.*

i) *If there exists some common prior that satisfies the AI condition (i.e., if $\Pi^{AI}_{\mathscr{T}} \neq \emptyset$), then there exists no mutually acceptable SI bet.*
ii) *If there exists no common prior satisfying the AI condition (i.e., if $\Pi^{AI}_{\mathscr{T}} = \emptyset$), then there exists a mutually acceptable SI bet.*

The full proof of the result is relegated to Appendix A. For the time being, we provide some intuition together with an illustration by means of examples.

The underlying idea behind the proof of part (i) is similar to the one in Sebenius and Geanakoplos (1983). In particular, we begin with the observation (first made in Remark 1) that we can arbitrarily replace the beliefs $\beta_i(t_i)$ inherited from the type structure with the conditional beliefs $\pi(\cdot|[\![s_i, t_i]\!])$ given by a common prior that satisfies the AI condition. Hence, we can construct an auxiliary Aumann structure over the same state space Ω with the common prior π such that, for every SI bet, the willingness to accept the bet using the beliefs $\beta_i(t_i)$ is the same as the willingness to accept it using the beliefs $\pi(\cdot|[\![s_i, t_i]\!])$ at every state and for every player. But then, by Sebenius and Geanakoplos (1983, Proposition 2), no mutually acceptable bet exists in the auxiliary Aumann structure and—*a fortiori*—not in our original type structure either. Let us provide an illustration.

Example 1 *(cont).* Consider the unique common prior that satisfies the AI condition, viz., let $\varepsilon = \frac{1}{4}$, implying that $\pi \in \Pi^{AI}_{\mathscr{T}}$ is uniformly distributed in Ω. Take an SI bet that pays to Ann the following amounts at each state.

	(L, t_b)	(R, t_b)
(U, t_a)	v_1	v_2
(D, t_a)	v_3	v_4

Notice that, in order for this bet to be mutually acceptable, it must be the case that the following two inequalities hold for Ann:

$$v_1 + v_2 \geq 0,$$

$$v_3 + v_4 \geq 0.$$

Likewise, it must be the case that the following two inequalities hold for Bob:

$$-v_1 - v_3 \geq 0,$$
$$-v_2 - v_4 \geq 0.$$

At the same time, one of the previous four inequalities must be strict. Clearly, this cannot occur. If this was the case, by adding the respective sides of the four inequalities, we would obtain $0 > 0$, which is an obvious contradiction. $\diamond$

Now, let us switch to part (ii) of the theorem. Here, the argument is slightly more involved. In the first step we make use of earlier results by Samet (1998) and Feinberg (2000). In particular, once again we construct an auxiliary Aumann structure over Ω with the conditional beliefs given each information set $[\![s_i, t_i]\!]$ being set the same as the ones given by $\beta_i(t_i)$. The key observation is that, since $\Pi^{AI}_{\mathscr{T}} = \emptyset$, there exists no common prior in our auxiliary Aumann structure generating these conditional beliefs. Hence, by Samet (1998, Claim, p. 173) and Feinberg (2000, Theorem 2, p. 146), there exists a mutually acceptable bet in the Aumann structure. However, this bet is not necessarily SI. Thus, with a sequence of transformations, we obtain a new SI bet which keeps the willingness of every player to bet at every state unchanged. As a result, since the original bet was mutually acceptable, so is this new SI bet, thus completing the proof. We illustrate the key ideas with an example.

Example 2. Once again consider the strategy sets $S_a = \{U, D\}$ and $S_b = \{L, R\}$, and take the type structure with type spaces $T_a = \{t_a\}$ and $T_b = \{t_b, t'_b\}$ and belief functions illustrated below.

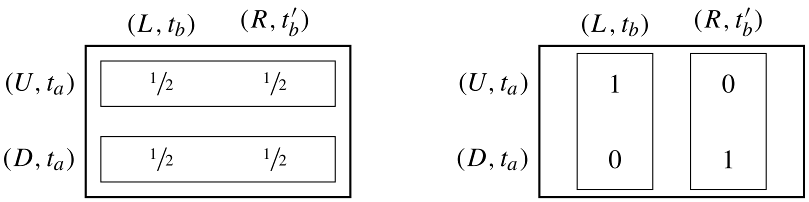
	(L, t_b)	(R, t'_b)
$\beta_a(t_a)$	$1/2$	$1/2$

	(U, t_a)	(D, t_a)
$\beta_b(t_b)$	1	0
$\beta_b(t'_b)$	0	1

Observe that the only common prior that this type structure admits is the one below.

	(L, t_b)	(R, t'_b)
(U, t_a)	$1/2$	0
(D, t_a)	0	$1/2$

Clearly, this common prior does not satisfy the AI condition. Thus, let us begin by taking the following auxiliary Aumann structure over Ω with the following information partitions (with Ann's partition on the left and Bob's on the right) and the corresponding conditional beliefs given each cell of the partition.



Notice that the conditional beliefs given each $[\![s_i, t_i]\!]$ coincide with those given by the type structure. However, these conditional beliefs cannot have been derived from a common prior, because, had such a prior existed, it would have belonged to $\Pi^{AI}_{\mathscr{T}}$, which we know is empty. Therefore, we can find a mutually acceptable bet in this Aumann space. Say this bet is the one that pays Ann the amounts shown below.

	(L, t_b)	(R, t'_b)
(U, t_a)	v_1	v_2
(D, t_a)	v_3	v_4

Of course, if this bet is SI, then we are done. Thus, suppose it is not, implying—without loss of generality—that

$$v_1 + v_2 > v_3 + v_4 \geq 0.$$

Then take the constant

$$c := \frac{1}{2}(v_1 + v_2) - \frac{1}{2}(v_3 + v_4) > 0$$

and subtract it from every payment in the upper information set of Ann. This will yield a new transformed bet that pays Ann the amounts shown below.

	(L, t_b)	(R, t'_b)
(U, t_a)	$v_1 - c$	$v_2 - c$
(D, t_a)	v_3	v_4

First of all, observe that the new bet is SI, as the expected payoff of Ann is the same at all states within $[\![t_a]\!]$. At the same time, Bob has become better off, as our transformation only subtracted payoffs from Ann and—*a fortiori*—added payoffs to Bob. Finally, observe that this transformation does not affect whether the bet is SI for Bob's types, i.e., the transformed bet is SI for any given type of Bob if and only if the original one was SI (which—incidentally—it trivially was). This is because we add the same constant payment to all states that correspond to any given strategy-type pair of Ann. As a result, the new transformed bet that we obtain is both mutually acceptable and SI. For the sake of illustration, such a bet could be one that pays $g_a(U, t_a, L, t_b) = g_a(D, t_a, R, t'_b) = -1$ and $g_a(D, t_a, R, t_b) = g_a(U, t_a, L, t'_b) = 1$.

3.2. *Do all common priors satisfy the AI condition?*

In the previous section we provided necessary and sufficient conditions that identify whether a type structure admits a common prior that satisfies the AI condition. In particular, if the answer is positive, we can conclude that the type structure is classified in category (Π_1) or (Π_2), i.e., either all common priors satisfy the AI condition, or there are additional common priors that do not satisfy the AI condition. In this section, we provide a second result that identifies—again in terms of a no-betting condition—to which of the two it is the case.

We begin with the premise that $\Pi_{\mathscr{T}}^{AI} \neq \emptyset$, i.e., there exists a common prior satisfying the AI condition, implying (by Theorem 1) that there exists no mutually acceptable SI bet between Ann and Bob. Now, we extend our type structure by introducing a dummy player (henceforth, the *outside observer*). The outside observer has a single strategy that is completely inconsequential to anybody and his beliefs are given by a common prior $\pi \in \Pi_{\mathscr{T}}$.

Formally, given a game Γ, the *extended game* is the tuple $\overline{\Gamma} := \langle \overline{I}, (\overline{S}_i, \overline{u}_i)_{i \in \overline{I}} \rangle$, where $\overline{I} := I \cup \{d\}$ is the original set of players augmented by the outsider observed, $\overline{S}_i := S_i$ for each $i \in I$ and $\overline{S}_d := \{s_d\}$ for the outside observer, and finally $\overline{u}_i(s_a, s_b, s_d) := u_i(s_a, s_b)$ for each original player $i \in I$ and $\overline{u}_d(s_a, s_b, s_d) := 0$ for the outside observer, for each extended strategy profile (s_a, s_b, s_d).

Now, take a standard type structure $\mathscr{T}$ (over the original game) admitting common priors and fix an arbitrary $\pi \in \Pi_{\mathscr{T}}$. Extend the type structure to

$$\overline{\mathscr{T}}_\pi := \langle \overline{I}, (\overline{T}_i, \overline{\beta}_i)_{i \in \overline{I}} \rangle,$$

such that $\overline{T}_i := T_i$ for every original player $i \in I$ and $\overline{T}_d := \{t_d\}$ for the outside observer, with $\overline{\beta}_i(t_i)(s_j, t_j, s_d, t_d) := \beta_i(t_i)(s_j, t_j)$ for each $i \in I$ and $\overline{\beta}_d(t_d)(s_i, t_i, s_j, t_j) := \pi(s_i, t_i, s_j, t_j)$ for the outside observer, for each original state $(s_i, t_i, s_j, t_j) \in \Omega$. That is, the original players inherit the beliefs from the original standard type structure, while the outside observer adopts the beliefs that are given by the common prior. We deem this object the *extended type structure*.

It is straightforward to verify that the extended state space $\overline{\Omega} = \overline{S}_a \times \overline{T}_a \times \overline{S}_b \times \overline{T}_b \times \overline{S}_d \times \overline{T}_d$ is homeomorphic to the original state space Ω. Indeed, we simply need to take each $(s_a, t_a, s_b, t_b) \in \Omega$ and augment it with the outside observer's unique strategy-type pair to obtain the corresponding state $(s_a, t_a, s_b, t_b, s_d, t_d) \in \overline{\Omega}$. In this sense, with a slight abuse of notation and without loss of generality, we keep using the original state space to also identify states in the extended state space.

An extended bet on the state space Ω is a profile of random variables $\overline{g} := (\overline{g}_a, \overline{g}_b, \overline{g}_d)$, with $\overline{g}_i \in \Re^\Omega$ for each $i \in \overline{I}$, such that $\overline{g}_a + \overline{g}_b + \overline{g}_d = 0$. We keep restricting attention to SI bets and notice that every extended bet is trivially SI for the outside observer: hence, we only need to make sure that an extended bet is SI for the original players. Willingness (resp., strict willingness) to accept a bet is naturally extended to the present framework, viz., player $i \in \overline{I}$ accepts $\overline{g}$ at some state in $[\![s_i, t_i]\!]$ if $\mathbb{E}[\overline{g}_i | s_i, t_i] \geq 0$ and strictly accepts it if the inequality is strict. Then we can naturally state the no-betting condition appropriate for this framework as follows: an extended SI bet $\overline{g}$ is mutually acceptable if every player $i \in \overline{I}$ accepts it at every state and there exists at least one player who strictly accepts it at some state.

We now state our second result, which allows us to identify whether the original type structure $\mathscr{T}$ is classified in category (Π_1) or in category (Π_2) by means of the existence of a mutually acceptable extended SI bet.

Theorem 2. *Fix a standard type structure that admits a common prior that satisfies the AI condition, i.e.,* $\Pi^{AI}_{\mathscr{T}} \neq \emptyset$. *Then the following hold.*

i) *If all common priors in* $\Pi_{\mathscr{T}}$ *satisfy the AI condition (i.e., if* $\Pi^{AI}_{\mathscr{T}} = \Pi_{\mathscr{T}}$*), then there exists no mutually acceptable extended SI bet.*
ii) *If there exists a common prior in* $\Pi_{\mathscr{T}}$ *that does not satisfy the AI condition (i.e., if* $\Pi^{AI}_{\mathscr{T}} \subsetneq \Pi_{\mathscr{T}}$*), then there exists a mutually acceptable extended SI bet.*

Once again, we relegate the formal proof to Appendix A and here we only provide some intuition.

Proving part (i) of the theorem follows directly from Theorem 1. In particular, pick an arbitrary common prior (which, by hypothesis, satisfies the AI condition) and—once again—construct an auxiliary Aumann model in which the conditional beliefs given each information set are derived from said common prior. Hence, there exists no mutually acceptable extended SI bet. Finally, observe that the willingness to accept any extended SI bet in the auxiliary Aumann model is the same as the willingness to accept the same bet using the beliefs that are derived from the extended type structure, which completes our argument. We now provide an illustration of what above.

Example 3. Once again, let $S_a = \{U, D\}$ and $S_b = \{L, R\}$, and consider the type structure with type spaces $T_a = \{t_a, t'_a\}$ and $T_b = \{t_b, t'_b\}$, and belief functions illustrated below.

	(L, t_b)	(R, t'_b)
$\beta_a(t_a)$	1	0
$\beta_a(t'_a)$	0	1

	(U, t_a)	(D, t'_a)
$\beta_b(t_b)$	1	0
$\beta_b(t'_b)$	0	1

Notice that the distribution below is a common prior admitted by the type structure for every $\delta \in (0, 1)$.

	(L, t_b)	(R, t'_b)
(U, t_a)	δ	0
(D, t'_a)	0	$1-\delta$

In fact, these are all the common priors that our type structure admits. Now, it is clear that they all satisfy the AI condition and it is not difficult to see that there exists no extended SI bet which is mutually acceptable. Indeed, take an arbitrary bet that pays Ann and Bob at each state the following amounts.

	(L, t_b)	(R, t'_b)
(U, t_a)	v_1^a, v_1^b	v_2^a, v_2^b
(D, t_a)	v_3^a, v_3^b	v_4^a, v_4^b

Notice that, in order for Ann to (weakly) accept this bet, it must be the case that $v_1^a \geq 0$ and $v_4^a \geq 0$ and, likewise for Bob, to (weakly) accept this bet we must have $v_1^b \geq 0$ and $v_4^b \geq 0$. But then it is the case that the outside observer's expected payoff is equal to

$$-\delta(v_1^a + v_1^b) - (1-\delta)(v_4^a + v_4^b) \leq 0,$$

with equality holding if and only if $v_1^a = v_4^a = v_1^b = v_4^b = 0$. Hence, there exists no mutually acceptable extended SI bet.

Concerning part (ii) of the theorem, here our proof is constructive and proceeds as follows. First of all, since there exists a common prior that satisfies the AI condition, there exists no mutually acceptable SI bet between Ann and Bob (by Theorem 1). Nevertheless, since there also exists a common prior π that does not satisfy the AI condition, it is necessarily the case that some player (among the original players) evaluates a bet at some state using the beliefs that are inherited from the type structure, which differ from the conditional beliefs given from the common prior π. On the other hand, the outside observer does use π to evaluate a bet. This discrepancy allows us to construct a bet that pays Ann and Bob 0 in expectation at every state, and yields a strictly positive expected payoff to the outside observer. This is clearly a mutually acceptable extended SI bet. We now provide an illustration of this point.

Example 1 *(cont).* Consider a common prior that does not satisfy the AI condition, e.g., let $\varepsilon = 0$. Then consider the bet that pays Bob 0 at every state and pays Ann the amounts depicted in the following table.

	(L, t_b)	(R, t_b)
(U, t_a)	-1	1
(D, t_a)	1	-1

Obviously, Ann receives 0 in expectation at all states, as she uses the beliefs she inherits from the type structure, which distribute probability uniformly across 1 and -1. On the other hand, the outside observer uses the beliefs that come from the common prior, which put probability 1 to the two states where Ann receives -1, and—as a result—the outside observer gets 1 in expectation, i.e., he strictly accepts the bet. Therefore, this is a mutually acceptable extended SI bet. $\diamond$

4. Objective correlated equilibrium & the AI condition

We now focus on how our results allow us a deeper understanding of the role of the AI condition for existing epistemic characterizations of Objective Correlated Equilibrium (henceforth, OCE). According to the original characterization of Aumann (1987), common knowledge of rationality together with a common prior yields an OCE distribution. However, Aumann's model has the feature that each type (viz., each partition cell in Aumann's language) is exogenously endowed with a strategy. As a result, the AI condition is automatically satisfied (which is why we do not need to explicitly postulate it in Aumann's theorem), but—at the same time—bundling each type with a given strategy leads to the shortcoming that it is not always possible to elicit the belief hierarchies of a player.[5]

To deal with these issues, Dekel and Siniscalchi (2015) minimally modify Aumann's conditions to obtain the following result: a type structure expressing *Rationality and Common Belief in Rationality*[6] together with a common prior that satisfies the AI condition leads to an OCE distribution. Formally, take a type structure $\mathscr{T}$ that admits a common prior $\pi \in \Pi_{\mathscr{T}}^{AI}$ such that, for every $i \in I$ and every $(s_i, t_i) \in S_i \times T_i$ with $\pi([\![s_i, t_i]\!]) > 0$, we have

$$\sum_{s_j \in S_j} \pi([\![s_j]\!] | [\![s_i, t_i]\!]) \cdot u_i(s_i', s_j) \geq \sum_{s_j \in S_j} \pi([\![s_j]\!] | [\![s_i', t_i]\!]) \cdot u_i(s_i', s_j),$$

for every $s_i' \in S_i$. Then $\text{marg}_S \pi$ is an OCE distribution (Dekel and Siniscalchi, 2015, Theorem 12.4).[7]

Here we should crucially note that the choice of the common prior (among the multiple common priors that exist) plays a significant role in the previous result. This is easily verified by the fact that the common prior is directly used to obtain the OCE distribution. Thus, it might be the case that the same type structure $\mathscr{T}$ induces an OCE for some common priors in $\Pi_{\mathscr{T}}$ and not for some others. We now illustrate that this is indeed the case.

---

[5] A referee pointed out the following additional problem of the model employed by Aumann: given a player's higher-order beliefs, the model does not rule out the possibility that there could be correlation between this player's beliefs over her own actions and her beliefs about the other players' actions and higher-order beliefs.

[6] See Dekel and Siniscalchi (2015, Section 12.3.1).

[7] To be more precise, Dekel and Siniscalchi (2015) also impose a minimality condition, which is not relevant for our discussion, and therefore it is without loss of generality to omit explicit reference to it. As a side remark, this minimality condition is equivalent to our assumption of working with type structures that are non-redundant.

Example 1 *(cont).* Consider the standard symmetric coordination game below.

		Bob	
		L	R
Ann	U	$1, 1$	$0, 0$
	D	$0, 0$	$1, 1$

We append on it the type structure originally described in this example, which we reproduce below.

	(L, t_b)	(R, t_b)
$\beta_a(t_a)$	$1/2$	$1/2$

	(U, t_a)	(D, t_a)
$\beta_b(t_b)$	$1/2$	$1/2$

First of all, given that we let RCBR denote the event in Ω that captures *Rationality and Common Belief in Rationality*, it has to be observed that in this type structure we have $\mathsf{RCBR} = \{(U, t_a, L, t_b), (U, t_a, R, t_b), (D, t_a, L, t_b), (D, t_a, R, t_b)\}$, i.e., RCBR does not rule out anything. Now, given the family of common priors in Table 1, consider the common prior induced by $\varepsilon = \frac{1}{2}$ represented below.

	(L, t_b)	(R, t_b)
(U, t_a)	0	$1/2$
(D, t_a)	$1/2$	0

Clearly, this common prior does not satisfy the AI condition. Moreover, it does not induce an OCE distribution, as it leads the players to miscoordinate with probability 1. If instead we had chosen from Table 1 another common prior that satisfied the AI condition (i.e., if we had chosen $\varepsilon = \frac{1}{4}$), we would have had a type structure expressing *Rationality and Common Belief in Rationality* along with a common prior satisfying the AI condition. Thus, from Dekel and Siniscalchi (2015, Theorem 12.4), the resulting distribution of strategy profiles would have constituted an OCE. $\diamond$

The conclusion is that the epistemic conditions for OCE cannot be expressed entirely within a type structure, but a reference to the choice of the common prior has to be made. This is exactly where our results come in handy: following Theorem 2, if the type structure under scrutiny does not allow for side-betting with an outside observer, then all common priors satisfy the AI condition and—as a result—all common priors that the type structure $\mathscr{T}$ admits do induce an OCE distribution. We emphasize this point in the following corollary.

Corollary 1. *Fix a standard type structure $\mathscr{T}$ such that there exists no mutually acceptable extended SI bet. Then, for every common prior $\pi \in \Pi_{\mathscr{T}}$ such that* $\operatorname{supp}\pi \subseteq \mathsf{RCBR}$*, it is the case that* $\operatorname{marg}_S \pi$ *is an OCE distribution.*

Therefore, the result of Dekel and Siniscalchi (2015) would only make reference to the type structure, implying that we obtain sufficient conditions for OCE using only properties that can be elicited.

5. Discussion

5.1. Rationality and common belief in rationality

As we have already mentioned, the reason behind our choice of focusing on SI bets is that they allow us to isolate the betting element that could potentially arise in a game from the actual play in the game. It is essentially due to this point that in this paper there is no actual reference to specific games: by focusing on SI bets, we make irrelevant the specific game upon which the betting can potentially take place, thus ending up to be in position to focus on the specific property of the type structures (and infinite hierarchies of beliefs) we are actually interested in, namely, the AI condition.

The importance of our results lies in the fact that side-betting on games is strictly related to the presence of endogenous uncertainty, i.e., the players' behavior has an impact on the realization of the state of nature (i.e., the outcome). Focusing on SI bets prevents us from the need to create a larger game to incorporate potential side bets in the actual game under scrutiny. Indeed, SI bets on a game do not alter the players' incentives to choose a strategy instead of another one in the actual play of the game. That is, SI bets make players indifferent between different strategies.

However, it is possible to argue that in an actual game, not all strategies are 'equal', in the sense that a rational player should not choose some strategies. As a matter of fact, we feel this would betray the exercise, since the rationality of a player in a game should not have any bite in the analysis of her infinite hierarchies of beliefs by means of side bets. Nevertheless, the question stands if our results are true when the focus is on strategies that are consistent with the rationality of the players and their mutual beliefs on their rationality. It is clear that all our results naturally extend to a setting where the focus is on those strategies that correspond to the strategies chosen in a type structure where the event *Rationality and Common Belief of Rationality* is nonempty.

5.2. Non-redundancy and elicitability

In Section 2.1, we explicitly assumed that we work with type structures that are non-redundant. First let us stress that our results do not formally rely on the non-redundancy assumption. Thus, why do we impose it? The reason is conceptual. In particular, bets are contingent claims that allocate payments to each player conditional on strategy profiles and belief hierarchies. This is a natural implicit assumption, as belief hierarchies can be elicited and strategies are directly observed. Therefore, allowing for redundancies would lead to multiple types representing the same hierarchy and—as a result—whenever we elicit said hierarchy, it would be ambiguous which type should be used to determine the payments from a bet. Hence, in such a case, we would need to also require bets to be measurable with respect to the belief hierarchies, i.e., in presence of redundancies two different types inducing the same belief hierarchy should originate the exact same payments. However, this would make our model unnecessarily more complex, which is why we rule out redundancies in the first place.

5.3. The n-player case

The extension of our results from the 2-player to the n-player case is straightforward. First of all, the respective parts (i) of both Theorem 1 and Theorem 2 are direct applications of the early contributions (Aumann, 1976; Milgrom and Stokey, 1982; Sebenius and Geanakoplos, 1983) where things extend verbatim from 2 to n players. Thus, we now focus on the respective parts (ii) of the two theorems. In Theorem 1, the proof proceeds in exactly the same way, except for how we transform the mutually acceptable bet $(f_i)_{i\in I}$ employed in the proof (that we have obtained from Samet (1998) and Feinberg (2000)) into an SI mutually acceptable bet. In particular, the idea is to distribute the positive payoff $c_i(s_i, t_i)$ that we subtract from player i at every state in $[\![s_i, t_i]\!]$ to all other players uniformly. This modification in the proof does not affect the validity of our argument. Finally, regarding Theorem 2, the proof again proceeds as in the 2-player case, i.e., we construct a mutually acceptable bet between the player who violates the AI condition and the outside observer, with *all* the remaining actual players getting 0 everywhere.

5.4. Ex ante trade & extended bets

It is possible to state Theorem 2 in terms of *ex ante* trade.[8] To do so, take the conditions of Theorem 1 and replace $g_a + g_b = 0$ with the expectation $\mathbb{E}_\pi[g_a + g_b] \leq 0$ for a given common prior π. Incidentally, this is the same common prior we have assumed to induce the outside observer's beliefs in our construction of extended bets. That is, instead of g being a bet, it is now an investment plan consisting of a state-contingent act for each player such that the sum of expected payoffs is *ex ante* non-positive. Since we do not require the respective payoffs to statewise add up to 0, we can think of the outside observer (in the context of Theorem 2) as an auxiliary *residual claimant* who receives the $-(g_a + g_b)$. Then, obviously such an SI investment plan is mutually acceptable (by the two players) if and only if the corresponding SI extended bet is mutually acceptable (by the two players and the residual claimant).

5.5. Adding exogenous uncertainty

Throughout the paper we have focused explicitly on type structures that represent belief hierarchies about strategies, i.e., models with *only* endogenous uncertainty. As a matter of fact, we can directly generalize our framework so that players have beliefs over *both* strategies-type pairs of the opponents and an exogenous parameter space (Dekel and Siniscalchi, 2015, Section 12.6.2). These models are useful in the study of games with incomplete information and—for instance—they can be used to provide an epistemic foundation to Bayes Correlated Equilibrium of Bergemann and Morris (2016). As it turns out, all our results and proofs can be naturally extended to characterize the AI condition in these models.

Appendix A. Proofs

Proof of Theorem 1. *Part (i):* Assume that there exists a common prior $\pi \in \Pi^{AI}_{\mathscr{T}}$. By the fact that $\pi \in \Pi^{AI}_{\mathscr{T}}$, it is the case that $\beta_i(t_i)(s_j, t_j) = \pi([\![s_j, t_j]\!] | [\![s_i, t_i]\!])$ for every $(s_i, t_i) \in S_i \times T_i$ and every $(s_j, t_j) \in S_j \times T_j$ (see Remark 1). Hence, for every SI bet $\widetilde{g}$, we can rewrite i's expected payoff at each state in $[\![s_i, t_i]\!]$ as

$$\mathbb{E}[\widetilde{g}_i | s_i, t_i] = \sum_{s_j \in S_j} \sum_{t_j \in T_j} \widetilde{g}_i(s_i, t_i, s_j, t_j) \cdot \pi([\![s_j, t_j]\!] | [\![s_i, t_i]\!]).$$

Now, we proceed by contradiction and assume that there exists a mutually acceptable SI bet g, implying that $\mathbb{E}[g_i | s_i, t_i] \geq 0$ for every $(s_i, t_i) \in S_i \times T_i$ and every $i \in I$, with at least one

[8] We are grateful to a referee for having pointed this out.

inequality being strict. Hence, by the fact that $\{[\![s_i, t_i]\!] | (s_i, t_i) \in S_i \times T_i\}$ is a partition of Ω, we obtain

$$\begin{aligned}0 &\leq \sum_{s_j \in S_j} \sum_{t_j \in T_j} \pi([\![s_i, t_i]\!]) \cdot \mathbb{E}[g_i | s_i, t_i] \\ &= \sum_{s_i \in S_i} \sum_{t_i \in T_i} \pi([\![s_i, t_i]\!]) \sum_{s_j \in S_j} \sum_{t_j \in T_j} g_i(s_i, t_i, s_j, t_j) \cdot \pi([\![s_j, t_j]\!] | [\![s_i, t_i]\!]) \\ &= \sum_{s_i \in S_i} \sum_{t_i \in T_i} \sum_{s_j \in S_j} \sum_{t_j \in T_j} g_i(s_i, t_i, s_j, t_j) \cdot \pi(s_i, t_i, s_j, t_j),\end{aligned}$$

with the inequality being strict for at least one of the two players. Thus, we finally add the respective sides of the inequalities for the two players to obtain

$$0 < \sum_{s_a \in S_a} \sum_{t_a \in T_a} \sum_{s_b \in S_b} \sum_{t_b \in T_b} \Big(g_a(s_a, t_a, s_b, t_b) + g_b(s_a, t_a, s_b, t_b) \Big) \cdot \pi(s_a, t_a, s_b, t_b) = 0,$$

which is the desired contradiction. Hence, there exists no mutually acceptable SI bet.

Part (ii): We begin by defining the following auxiliary Aumann structure $\langle \Omega, (\mathcal{P}_i, \pi_i)_{i \in I} \rangle$, where $\mathcal{P}_i := \{[\![s_i, t_i]\!] | (s_i, t_i) \in S_i \times T_i\}$ is the information partition of $S_i \times T_i$ cylinders and, for each player $i \in I$, $\pi_i \in \Delta(\Omega)$ is a probability measure such that the conditional beliefs given each information set $[\![s_i, t_i]\!]$ agree with the beliefs obtained from the type structure, i.e., we set

$$\pi_i([\![s_j, t_j]\!] | [\![s_i, t_i]\!]) := \beta_i(t_i)(s_j, t_j),$$

for every $(s_j, t_j) \in S_j \times T_j$. Note that it is necessarily the case that $\pi_a \neq \pi_b$, otherwise there would exist a common prior that satisfies the AI condition, which cannot be (by hypothesis). Therefore, our auxiliary Aumann structure does not admit a common prior. Hence, by Samet (1998, Claim, p. 173) and Feinberg (2000, Theorem 2, p. 146), there exists a mutually acceptable bet $f := (f_a, f_b)$, with $f_i \in \Re^\Omega$ for every $i \in I$, such that $f_a + f_b = 0$. In particular, for every $i \in I$ and every $[\![s_i, t_i]\!]$, it is the case that

$$\sum_{s_j \in S_j} \sum_{t_j \in T_j} f_i(s_i, t_i, s_j, t_j) \cdot \pi_i([\![s_j, t_j]\!] | [\![s_i, t_i]\!]) \geq 0,$$

with at least one inequality being strict. By definition of π_i, it follows that, for every $i \in I$ and every $[\![s_i, t_i]\!]$, it is the case that

$$\mathbb{E}[f_i | s_i, t_i] \geq 0,$$

with at least one inequality being strict. Now, if f is an SI bet, we are done. Thus, we assume that it is not and we define a new bet $g := (g_a, g_b)$ as follows. First, for each $i \in I$ and each $(s_i, t_i) \in S_i \times T_i$, we let

$$c_i(s_i, t_i) := \mathbb{E}[f_i | s_i, t_i] - \min_{s'_i \in S_i} \mathbb{E}[f_i | s'_i, t_i] \geq 0.$$

Then, for each state $(s_i, t_i, s_j, t_j) \in \Omega$ and each player $i \in I$, we define

$$g_i(s_i, t_i, s_j, t_j) := f_i(s_i, t_i, s_j, t_j) - c_i(s_i, t_i) + c_j(s_j, t_j).$$

First of all, it is trivially verified that g is indeed a bet. Now, notice that, for any fixed $t_i \in T_i$, we obtain

$$\mathbb{E}[g_i|s_i,t_i] = \min_{s_i' \in S_i} \mathbb{E}[f_i|s_i',t_i] + c_j(s_j,t_j),$$

for all $s_i \in S_i$. Observe that $\mathbb{E}[g_i|s_i,t_i] = \mathbb{E}[g_i|s_i',t_i]$ for all $s_i, s_i' \in S_i$, i.e., g is an SI bet. Moreover, $\mathbb{E}[g_i|s_i,t_i] \geq 0$ for all $(s_i,t_i) \in S_i \times T_i$. Finally, observe that, since f is not SI in the first place, we obtain $c_j(s_j,t_j) > 0$ for at least one player $j \in I$ and at least one pair $(s_j,t_j) \in S_j \times T_j$. Therefore, there exists at least one pair (s_i,t_i) of j's opponent such that $\mathbb{E}[g_i|s_i,t_i] > 0$. This implies that g is mutually acceptable, which completes the proof. □

Proof of Theorem 2. *Part (i):* The proof is almost identical to the one of the first part of Theorem 1. The only caveat is that there are three players now: Ann, Bob, and the outside observer. Nevertheless, the outside observer can only have one belief, viz., it is necessarily the case that $\overline{\beta}_d(t_d) = \pi$ for some $\pi \in \Pi^{AI}_{\mathscr{T}}$. Now, suppose that there exists a mutually acceptable extended SI bet. By following the exact same steps in the proof of Theorem 1, we obtain

$$0 < \sum_{s_a \in S_a} \sum_{t_a \in T_a} \sum_{s_b \in S_b} \sum_{t_b \in T_b} \Big(g_a(s_a,t_a,s_b,t_b) + g_b(s_a,t_a,s_b,t_b) + g_d(s_a,t_a,s_b,t_b) \Big) \cdot \pi(s_a,t_a,s_b,t_b) = 0,$$

which is an obvious contradiction.

Part (ii): The proof is constructive. Fix a common prior $\pi \in \Pi_{\mathscr{T}} \setminus \Pi^{AI}_{\mathscr{T}}$, and take the extended type structure such that the outside observer's beliefs agree with this common prior, i.e., $\overline{\beta}_d(t_d) = \pi$. Since π is a common prior that does not satisfy the AI condition, there exists a $E_j \subseteq S_j \times T_j$ and two strategy-type pairs $(s_i,t_i), (s_i',t_i) \in S_i \times T_i$ such that

$$\pi(E_j|[\![s_i,t_i]\!]) > \beta_i(t_i)(E_j) > \pi(E_j|[\![s_i',t_i]\!]).$$

Obviously, this directly implies that

$$\pi(E_j^c|[\![s_i,t_i]\!]) < \beta_i(t_i)(E_j^c) < \pi(E_j^c|[\![s_i',t_i]\!]),$$

where $E_j^c = (S_j \times T_j) \setminus E_j$. Now, we define the extended bet $\overline{g}$ as follows.

- At all states $(s_i'',t_i'',s_j'',t_j'') \notin [\![s_i,t_i]\!] \cup [\![s_i',t_i]\!]$, set $\overline{g}_k(s_i'',t_i'',s_j'',t_j'') := 0$ for every player $k \in \overline{I}$.
- At all states $(s_i'',t_i'',s_j'',t_j'') \in [\![s_i,t_i]\!] \cup [\![s_i',t_i]\!]$, set $\overline{g}_j(s_i'',t_i'',s_j'',t_j'') := 0$. Hence, for player i and the outside observer we have $\overline{g}_i(s_i'',t_i'',s_j'',t_j'') = -\overline{g}_d(s_i'',t_i'',s_j'',t_j'')$, where i's payments are shown in the table below.

	E_j	E_j^c
(s_i,t_i)	v_1	v_2
(s_i',t_i)	v_3	v_4

That is, payments are measurable with the respect to the events in $\{\{(s_i,t_i)\},\{(s_i',t_i)\}\} \times \{E_j, E_j^c\}$. Then we set i's payments to be such that

$$\beta_i(t_i)(E_j) \cdot v_1 + \beta_i(t_i)(E_j^c) \cdot v_2 = \beta_i(t_i)(E_j) \cdot v_3 + \beta_i(t_i)(E_j^c) \cdot v_4 = 0,$$

with $v_1 < 0$ and $v_4 < 0$, and *a fortiori* $v_2 > 0$ and $v_3 > 0$.

By construction, both i's and j's expected payoffs from the bet are equal to 0 at all states, i.e.,

$$\mathbb{E}[\overline{g}_i|s_i'', t_i''] = \mathbb{E}[\overline{g}_j|s_j'', t_j''] = 0,$$

at all $(s_i'', t_i'', s_j'', t_j'') \in \Omega$. Hence, the bet $\overline{g}$ is SI for players i and j and moreover it is (weakly) acceptable for both of them. Now, observe that

$$\begin{aligned}
\mathbb{E}[\overline{g}_d|s_d, t_d] = & -\pi([\![s_i, t_i]\!]) \cdot \Big(\pi(E_j|[\![s_i, t_i]\!]) \cdot v_1 + \pi(E_j^c|[\![s_i, t_i]\!]) \cdot v_2\Big) \\
& - \pi([\![s_i', t_i]\!]) \cdot \Big(\pi(E_j|[\![s_i', t_i]\!]) \cdot v_3 + \pi(E_j^c|[\![s_i', t_i]\!]) \cdot v_4\Big) \\
> & -\pi([\![s_i, t_i]\!]) \cdot \Big(\beta_i(t_i)(E_j) \cdot v_1 + \beta_i(t_i)(E_j^c) \cdot v_2\Big) \\
& - \pi([\![s_i', t_i]\!]) \cdot \Big(\beta_i(t_i)(E_j) \cdot v_3 + \beta_i(t_i)(E_j^c) \cdot v_4\Big) \\
= & \ 0,
\end{aligned}$$

which implies that the outside observer is strictly willing to accept the extended bet $\overline{g}$. Thus, overall $\overline{g}$ is a mutually acceptable extended SI bet, which completes the proof. □

References

Aumann, R.J., 1976. Agreeing to disagree. Ann. Stat. 4, 1236–1239.
Aumann, R.J., 1987. Correlated equilibrium as an expression of Bayesian rationality. Econometrica 55, 1–18.
Aumann, R.J., 1998. Common priors: a reply to gul. Econometrica 66, 929–938.
Aumann, R.J., Brandenburger, A., 1995. Epistemic conditions for Nash equilibrium. Econometrica 63, 1161–1180.
Barelli, P., 2009. Consistency of beliefs and epistemic conditions for Nash and correlated equilibria. Games Econ. Behav. 67, 363–375.
Battigalli, P., Friedenberg, A., Siniscalchi, M., in preparation. Epistemic Game Theory: Reasoning about Strategic Uncertainty.
Bergemann, D., Morris, S., 2016. Bayes correlated equilibrium and the comparison of information structures in games. Theor. Econ. 11, 487–522.
Böge, W., Eisele, T., 1979. On solutions of Bayesian games. Int. J. Game Theory 8, 193–215.
Bonanno, G., Nehring, K., 1999. How to make sense of the common prior assumption under incomplete information. Int. J. Game Theory 28, 409–434.
Brandenburger, A., Dekel, E., 1987. Rationalizability and correlated equilibria. Econometrica 55, 1391–1402.
Brandenburger, A., Dekel, E., 1993. Hierarchies of beliefs and common knowledge. J. Econ. Theory 59, 189–198.
Dekel, E., Siniscalchi, M., 2015. Epistemic game theory. In: Young, H.P., Zamir, S. (Eds.), Handbook of Game Theory, vol. IV. North-Holland, Amsterdam.
Feinberg, Y., 2000. Characterizing common priors in the form of posteriors. J. Econ. Theory 91, 127–179.
Gul, F., 1998. A comment on Aumann's Bayesian view. Econometrica 66, 923–927.
Heifetz, A., 2006. The positive foundation of the common prior assumption. Games Econ. Behav. 56, 105–120.
Milgrom, P., Stokey, N., 1982. Information, trade and common knowledge. J. Econ. Theory 26, 17–27.
Morris, S., 1994. Trade with heterogeneous prior beliefs and asymmetric information. Econometrica 62, 1327–1347.
Morris, S., 1995. The common prior assumption in economic theory. Econ. Philos. 13, 227–253.
Nau, R.F., McCardle, K.F., 1990. Coherent behavior in noncooperative games. J. Econ. Theory 50, 424–444.
Samet, D., 1998. Common priors and separation of convex sets. Games Econ. Behav. 24, 172–174.
Sebenius, J.K., Geanakoplos, J., 1983. Don't bet on it: contingent agreements with asymmetric information. J. Am. Stat. Assoc. 78, 424–426.
Stalnaker, R., 1998. Belief revision in games: forward and backward induction. Math. Soc. Sci. 36, 31–56.
Tan, T.C.C., da Costa Werlang, S.R., 1988. The Bayesian foundations of solution concepts of games. J. Econ. Theory 45, 370–391.