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Normalized Solutions to the Fractional Schrödinger Equation with Potential

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Abstract. This paper is concerned with the existence of normalized solutions to a class of Schrödinger equations driven by a fractional operator with a parametric potential term. We obtain minimization of energy functional associated with that equations assuming basic conditions for the potential. Our work offers a partial extension of previous results to the non-local case.

Mathematics Subject Classification. 35R11, 35J20, 35J60.

Keywords. Schrödinger equation, fractional operator, normalized solution, potential function.

1. Introduction and Main Result

In this paper, we investigate the attainability of the following constraint minimization problem:

$$\mathcal{I}_m = \inf_{u \in \mathcal{M}_m} E(u), \tag{1.1}$$

where m is a positive constant and the energy functional $E : W \to \mathbb{R}$ is defined by

$$E(u) = \frac{1}{2}[u]^2_{H^s} + \frac{1}{2}\int_{\mathbb{R}^N} \lambda V(x)|u|^2 \mathrm{d}x - \frac{1}{p}\|u\|_p^p,$$
$$\mathcal{M}_m = \left\{u \in W : \|u\|_2^2 = m^2\right\},$$

here, $0 < s < 1$ and $2s < N < 4s$, λ is a positive parameter, $p \in (2, \min\{N/(N-2s), \overline{p}\})$ with L^2-critical exponent $\overline{p} = 2 + 4\,s/N$, the potential function $V(x) : \mathbb{R}^N \to [0, +\infty)$ is continuous and bounded. Moreover, the weighted fractional Sobolev space W is given by

$$W = \left\{u \in H^s(\mathbb{R}^N) \ : \int_{\mathbb{R}^N} \lambda V(x)|u|^2 \mathrm{d}x < +\infty\right\},$$

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with norm

$$\|u\|_W = \left(\int_{\mathbb{R}^N} (\lambda V(x)+1)|u|^2 \mathrm{d}x + [u]_{H^s}^2\right)^{\frac{1}{2}}.$$

In addition, $\|\cdot\|_p$ represents the norm of Lebesgue space L^p and $H^s(\mathbb{R}^N)$ denotes the fractional Sobolev space:

$$H^s(\mathbb{R}^N) = \left\{u \in L^2(\mathbb{R}^N) \ : \ [u]_{H^s}^2 = \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y < \infty\right\},$$

where $[u]_{H^s}^2$ is the Gagliardo semi-norm, The norm of $H^s(\mathbb{R}^N)$ is defined by

$$\|u\|_{H^s} = (\|u\|_2^2 + [u]_{H^s}^2)^{\frac{1}{2}}.$$

According to the definition of W, it is clear that the embedding $W \hookrightarrow H^s(\mathbb{R}^N)$ is continuous, furthermore, we know that $W \hookrightarrow L^h(\mathbb{R}^N)$ is also continuous for any $h \in [2, 2_s^*]$ (see Di Nezza et al. [12, Theorem 6.5]), where $2_s^* = 2N/(N-2s)$ is the fractional critical exponent. It is well known that problem (1.1) plays an important role in studying the standing wave of the following fractional Schrödinger equation

$$i\phi_t = (-\Delta)^s\phi + \lambda V(x)\phi - |\phi|^{p-2}\phi, \text{ for all } (x,t) \in (\mathbb{R}^N, \mathbb{R}), \tag{1.2}$$

where i is the imaginary unit and $(-\Delta)^s$ is the fractional Laplacian, which is defined by

$$(-\Delta)^s\phi(x) = C(N,s) \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^N\setminus B_\varepsilon(x)} \frac{\phi(x)-\phi(y)}{|x-y|^{N+2s}} \mathrm{d}y,$$

where $C(N,s)$ is a constant that depends on N and s (for more information we refer again to [12]). Putting the standing wave $\phi(x,t) = e^{i\mu t}u(x)$ into (1.2), we obtain the equation

$$(-\Delta)^s u + \mu u + \lambda V(x)u - |u|^{p-2}u = 0 \text{ in } \mathbb{R}^N. \tag{1.3}$$

This is a classical fractional Schrödinger equation, which is originated by Laskin' work in [19]. In recent decades a great deal of attention has been paid to this kind of problem because of its important applications in many disciplines such as physics. In general, the role of parameter μ in Eq. (1.3) is reflected in two aspects. First fixed the number $\mu \in \mathbb{R}$, one can look to the solution of (1.3), that is the so-called *fixed frequency problem*. A large number of researchers have carried out in-depth studies using variational and topological methods, here we just mention the works by Berestycki and Lions [6,7] (minimization problems in unbounded domains), Liang et al. [21] (critical Choquard-Kirchhoff equations). Then, taking μ as unknown, which appears as a Lagrange multiplier, one can look to prescribed L^2-norm solutions. From a physical point of view, the study seems particularly interesting because of the conservation of mass. To the best of our knowledge, there have been some works that have considered this aspect but without involving a potential term. For example, Luo and Zhang [18] studied a class of fractional Schrödinger equations and established several results concerning the existence of normalized solutions. For more details on normalized solutions of fractional problems we refer to Appolloni e Secchi [4] (mass supercritical nonlinearity), Li and Zou [20] and Zhen and Zhang [27] (L^2-subcritical (or L^2-critical or L^2-supercritical) perturbation). When $V(x) \equiv 0$ and $s = 1$ in Eq. (1.3), namely the fractional Laplace operator $(-\Delta)^s$ reduces to the classical Laplace operator, then a large body of literatures studied the following problem:

$$-\Delta u + \mu u - f(u) = 0 \text{ in } \mathbb{R}^N, \quad \|u\|_2^2 = m^2. \tag{1.4}$$

We mention the works of Alves et al. [1,2] (critical growth case), Jeanjean [17] (semilinear elliptic equations), Stefanov [23] and Soave [24,25] (ground states and combined power nonlinearities). In a recent paper, Cingolani et al. [11] studied the fractional counterpart of Eq. (1.4) in the case when the function f satisfies suitable Berestycki–Lions type conditions. They show the existence of solutions by means of a variant of the Palais-Smale condition together with some deformation arguments. This variational approach is based on previous results and ideas introduced by Ikoma and Tanaka [16], to obtain existence and multiplicity results in the local case. We note that when $V(x) \not\equiv 0$, the scaled function $u(\xi x)$ may not be useful to approach the problem (see Ikoma et al. [14] and refer to the well-known scaling function method to solve optimization problems). This makes it more difficult to deal with the constraint minimization problem. For this reason, the papers concerned with this topic are very few, we point out that Ikoma et al. in [14] studied a constraint minimization problem and proved the existence and nonexistence of the minimizer of this problem for integer-order cases (for some differences between the cases $V(x) \equiv 0$ and $V(x) \not\equiv 0$ see [14, Remark 1.2]). Subsequently, the ideas of paper [14] are applied to a nonlinear Schrödinger system with potentials in Ikoma et al. [15]. For more information on normalized (hence prescribed L^2-norm) solutions of the Schrödinger equation with different types of potentials, interested readers can refer to Alves and Ji [3] (L^2-subcritical growth), Bartsch et al. [9] (mass supercritical case), Bellazzini et al. [10] and Zhong and Zou [28] (ground state solutions). However, as far as we know, the above works only consider the local case, and so it is natural to investigate whether some of the results still hold for the non-local case. For this, Peng and Xia [22] proved that a fractional Schrödinger equation admits normalized solutions in the mass supercritical case, using technical assumptions on the potential function. They consider a potential vanishing at zero and note that the energy functional associated with the principal equation possesses a mountain pass geometry with the same mountain pass value of the case when $V(x) \equiv 0$ but it is not achieved. Thus, it is very worth paying attention to the mass subcritical case with respect to this problem as well. When we go on this way, we will face new math challenge such as how to properly overcome the difficulties caused by the non-local operator and the potential function.

To state our main result, we need the following assumptions about the potential function $V(x) : \mathbb{R}^N \to [0, +\infty)$:

(V_1) there exists a positive constant $C_0 > 0$ such that the measure of the set $\Omega = \{x \in \mathbb{R}^N : V(x) < C_0\}$ is finite.

(V_2) $V \in L^\infty(\mathbb{R}^N)$.

Remark 1.1. The conditions (V_1)–(V_2) originate from the finding in [8], but we get suitable modifications (namely, we consider weaker assumptions) because of the different needs of Lemmas 2.3 and 2.4 in Sect. 2. In particular, we do not impose conditions about the nonemptiness of the interior of $V^{-1}(0)$ and the positivity of the measure of Ω (which are involved in [8] to get the Palais-Smale condition for the energy functional associated with the semilinear elliptic partial differential equation considered therein).

Then, our main result is as follows.

Theorem 1.2. *Let $2s < N < 4s$, $p \in (2, \min\{N/(N-2s), 2+4s/N\})$ and suppose that conditions (V_1)-(V_2) hold. Then, for each $m > 0$, there exist $\lambda^* > 0$ and $\zeta = \zeta(m) > 0$ such that $\mathcal{I}_m < 0$ for all $\lambda \in [\lambda^*, +\infty)$ satisfying $\lambda \|V\|_\infty < \zeta$. Furthermore, the infimum $\mathcal{I}_m$ is attained at a point $u \in \mathcal{M}_m$, which is a solution of (1.3) with $\mu = \mu_m$ as a Lagrange multiplier.*

Remark 1.3. Although [3, Theorem 1.3] shows the existence of solutions to the local version of our problem (1.3) under analogous assumptions (see condition (V_4) of [3]), we point out that the hypothesis of [3, Lemma 2.2] (the key lemma to prove [3, Theorem 1.3]) makes $V(x) = 0$ a.e. in $\mathbb{R}^N$. Differently, here we do not need a similar behavior (see Lemma 2.3) and we consider a condition (see assumption (V_2)), which is weaker than condition (V_3) in [3]. We note that $\lambda^* > 0$ in Theorem 1.2 is independent of m, as can be seen following the proof of Lemma 2.8 (see Sect. 2).

2. Proofs of Auxiliary Lemmas and Theorem 1.2

In this section, we first note an important fractional Gagliardo-Nirenberg inequality (see Frank et al. [13]), which will be used in the proofs of some auxiliary lemmas. Hence, we denote $\gamma_p = \frac{N}{s}(\frac{1}{2} - \frac{1}{p})$ and, if it is not said differently in the statement, we assume $p \in (2, 2+4\,s/N)$ in all lemmas.

Lemma 2.1. *Let $u \in H^s(\mathbb{R}^N)$ and $p \in (2, 2_s^*)$, then there exists a positive constant $C(N,s,p)$ such that the following is the case*

$$\|u\|_p^p \leq C(N,s,p)[u]_{H^s}^{\gamma_p p} \|u\|_2^{(1-\gamma_p)p}.$$

As a consequence of Lemma 2.1, we obtain the following result.

Lemma 2.2. *The energy functional $E(\cdot)$ is bounded from below on $\mathcal{M}_m$ for all $m > 0$.*

Proof. It follows from Lemma 2.1 that

$$\begin{aligned} E(u) &= \frac{1}{2}[u]_{H^s}^2 + \frac{1}{2}\int_{\mathbb{R}^N} \lambda V(x)|u|^2 \mathrm{d}x - \frac{1}{p}\|u\|_p^p \\ &\geq \frac{1}{2}[u]_{H^s}^2 - \frac{C(N,s,p)}{p} m^{(1-\gamma_p)p}[u]_{H^s}^{\gamma_p p}. \end{aligned}$$

Since $p \in (2, 2+4s/N)$, thus $\gamma_p p < 2$. So we deduce that the result is true and the definition of $\mathcal{I}_m$ is reasonable. □

Next, we establish the following key lemma (scaling result).

Lemma 2.3. *If condition* (V_2) *holds, then for any* $m > 0$ *there exists* $\zeta = \zeta(m) > 0$ *such that* $\mathcal{I}_m < 0$ *if* $\lambda\|V\|_\infty < \zeta$.

Proof. For any $m > 0$, we set $u_0 \in \mathcal{M}_m$ and make a scaling that, keeping L^2-norm invariant, gives us:

$$(\eta \diamond u_0)(x) = e^{\frac{\eta N}{2}} u_0(e^\eta x) \text{ for any } \eta \in \mathbb{R} \text{ and } x \in \mathbb{R}^N,$$

which is borrowed from the ideas of Jeanjean [17]. Through simple calculations, we get that

$$\begin{aligned}
[\eta \diamond u_0]_{H^s}^2 &= e^{2\eta s} \int_{\mathbb{R}^{2N}} \frac{|u_0(x) - u_0(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y = e^{2\eta s}[u_0]_{H^s}^2, \\
\|\eta \diamond u_0\|_2^2 &= \|u_0\|_2^2 = m^2, \\
\|\eta \diamond u_0\|_\alpha^\alpha &= e^{\frac{(\alpha-2)N\eta}{2}} \|u_0\|_\alpha^\alpha \quad \text{for any } \alpha > 2.
\end{aligned}$$

Thus, we have

$$E(\eta \diamond u_0) \leq \frac{1}{2} e^{2\eta s}[u_0]_{H^s}^2 + \frac{\lambda}{2}\|V\|_\infty m^2 - \frac{e^{\frac{(p-2)N\eta}{2}}}{p}\|u_0\|_p^p.$$

Since $p \in (2, 2 + 4s/N)$, then we get that $(p-2)N < 4s$ and so there exists $\eta < 0$ such that

$$\frac{1}{2} e^{2\eta s}[u_0]_{H^s}^2 - \frac{e^{\frac{(p-2)N\eta}{2}}}{p}\|u_0\|_p^p = Z_\eta < 0.$$

Hence, fixed the number $\zeta = \frac{-Z_\eta}{m^2}$ and in view of the assumption $\lambda\|V\|_\infty < \zeta$, we deduce that $E(\eta \diamond u_0) < 0$, which implies that $\mathcal{I}_m < 0$. □

Even if not explicitly stated, in the following series of lemmas we always assume that

$$(u_n)_{n\in\mathbb{N}} \subset \mathcal{M}_m \text{ is a minimizing sequence for } \mathcal{I}_m.$$

The first lemma gives us a positive bound-from-below condition.

Lemma 2.4. *If condition* (V_2) *holds, then for any* $m > 0$, *there exist* $\zeta(m) > 0$ *and* $M > 0$ *such that*

$$\liminf_{n\to+\infty} \int_{\mathbb{R}^N} |u_n|^p \, dx \geq M \ \textit{if} \ \lambda\|V\|_\infty < \zeta. \tag{2.1}$$

Proof. From Lemma 2.3 we know that we can find $\zeta(m) > 0$ and $M > 0$ such that $\mathcal{I}_m < -M$ if $\lambda\|V\|_\infty < \zeta$. Now, since $(u_n) \subset \mathcal{M}_m$ is a minimizing sequence for $\mathcal{I}_m$, then we have

$$\begin{aligned}
\mathcal{I}_m + o_n(1) &= E(u_n) \\
&= \frac{1}{2}[u_n]_{H^s}^2 + \frac{1}{2}\int_{\mathbb{R}^N} \lambda V(x)|u_n|^2 \mathrm{d}x - \frac{1}{p}\|u_n\|_p^p,
\end{aligned}$$

furthermore, we obtain that

$$-M + o_n(1) \geq -\frac{1}{p}\|u_n\|_p^p,$$

hence (2.1) holds. □

Next result establishes a quantitative relationship between the two ordered values of m and the corresponding values of $\mathcal{I}_m$.

Lemma 2.5. *If condition* (V_2) *holds and if* $0 < m_1 < m_2$*, then* $\frac{\mathcal{I}_{m_2}}{m_2^2} < \frac{\mathcal{I}_{m_1}}{m_1^2}$.

Proof. The proof of this lemma is almost the same as that of [3, Lemma 2.3], so we omit the details. □

Now, we prove a (strong) convergence result in the weighted fractional Sobolev space W.

Lemma 2.6. *If condition* (V_2) *holds and if* $u_n \rightharpoonup u$ *in* W*,* $u_n(x) \to u(x)$ *a.e. in* $\mathbb{R}^N$ *and* $u \neq 0$*, then* $u \in \mathcal{M}_m$*,* $E(u) = \mathcal{I}_m$ *and* $u_n \to u$ *in* W*.*

Proof. We note that if $\|u\|_2 = k \neq m$, according to the Fatou lemma and the assumption $u \neq 0$, we get that $k \in (0, m)$. From the continuity of embedding $W \hookrightarrow L^h(\mathbb{R}^N)$ for any $h \in [2, 2_s^*]$ and two kinds of Brézis–Lieb lemmas in [26] and [5], we have that

$$\begin{aligned} [u_n]_{H^s}^2 &= [u_n - u]_{H^s}^2 + [u]_{H^s}^2 + o_n(1), \\ \|u_n\|_2^2 &= \|u_n - u\|_2^2 + \|u\|_2^2 + o_n(1), \\ \|u_n\|_p^p &= \|u_n - u\|_p^p + \|u\|_p^p + o_n(1). \end{aligned} \tag{2.2}$$

Let $v_n = u_n - u$, $\|v_n\|_2 = l_n$ and assume that $\|v_n\|_2 \to l$, by (2.2) we infer that $m^2 = k^2 + l^2$ and $l_n \in (0, m)$ for sufficiently big n, which implies that

$$\begin{aligned} \mathcal{I}_m + o_n(1) &= E(u_n) \\ &= E(v_n) + E(u) + o_n(1) \\ &\geq \mathcal{I}_{l_n} + \mathcal{I}_k + o_n(1) \\ &\geq \frac{l_n^2}{m^2}\mathcal{I}_m + \mathcal{I}_k + o_n(1), \end{aligned}$$

thanks to Lemma 2.5. Letting $n \to +\infty$, we obtain the inequality

$$\mathcal{I}_m \geq \frac{l^2}{m^2}\mathcal{I}_m + \mathcal{I}_k.$$

Using the fact that $k \in (0, m)$ and Lemma 2.5 in the above inequality (this time for $\mathcal{I}_k$), we deduce that

$$\mathcal{I}_m > \frac{l^2}{m^2}\mathcal{I}_m + \frac{k^2}{m^2}\mathcal{I}_m = \mathcal{I}_m,$$

a contradiction. Therefore $\|u\|_2 = m$, that is, $u \in \mathcal{M}_m$. It follows from $\|u_n\|_2 = \|u\|_2 = m$ and $u_n \rightharpoonup u$ in $L^2(\mathbb{R}^N)$ (since $W \hookrightarrow L^2(\mathbb{R}^N)$ is a continuous embedding) that $u_n \to u$ in $L^2(\mathbb{R}^N)$, which together with interpolation inequality gives us that $u_n \to u$ in $L^p(\mathbb{R}^N)$. In addition, since $\int_{\mathbb{R}^N} \lambda V(x)|u|^2 \mathrm{d}x + [u]_{H^s}^2$ is convex and continuous in W, we know that it is a weak lower semicontinuous, namely

$$\liminf_{n\to+\infty} \left(\int_{\mathbb{R}^N} \lambda V(x)|u_n|^2 \mathrm{d}x + [u_n]_{H^s}^2 \right) \geq \int_{\mathbb{R}^N} \lambda V(x)|u|^2 \mathrm{d}x + [u]_{H^s}^2.$$

Keeping in mind that $\mathcal{I}_m = \lim_{n\to+\infty} E(u_n)$, we infer that $\mathcal{I}_m \geq E(u)$, again by the definition of $\mathcal{I}_m$ and $u \in \mathcal{M}_m$ we get that $\mathcal{I}_m = E(u)$, and hence

$$\lim_{n\to+\infty} E(u_n) = E(u).$$

Finally, in view of the convergences $u_n \to u$ in $L^2(\mathbb{R}^N)$ and $u_n \to u$ in $L^p(\mathbb{R}^N)$, we conclude that $u_n \to u$ in W. □

We give the following short proof of the boundedness of a minimizing sequence for $\mathcal{I}_m$, in the weighted fractional Sobolev space W.

Lemma 2.7. *The sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in W.*

Proof. It follows from the definition of minimizing sequence that

$$\mathcal{I}_m = \lim_{n\to+\infty} E(u_n),$$

which combined with Lemma 2.2 gives us that $([u_n]_{H^s})_{n\in\mathbb{N}}$ is a bounded sequence. Consequently, the sequence $(\int_{\mathbb{R}^N} \lambda V(x)|u_n|^2 \mathrm{d}x)_{n\in\mathbb{N}}$ is also bounded. As a result, $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence in W. □

We now establish a technical result, which will be successfully involved in the proof (by contradiction) of a subsequent lemma.

Lemma 2.8. *Let $p \in [1, N/(N-2s)]$. If conditions (V_1)-(V_2) hold, then there exist $R > 0$ and $\lambda^* > 0$ such that for any $\lambda > \lambda^*$, we have*

$$\limsup_{n\to+\infty} \int_{B_R^c(0)} |u_n|^p \, dx \leq \frac{M}{2}, \tag{2.3}$$

where $B_R^c(0) = \{x \in \mathbb{R}^N : |x| > R\}$, $M > 0$ as given in Lemma 2.4.

Proof. Following the methods of Bartsch and Wang [8], for $R > 0$, we consider two sets

$$X(R) := \{x \in \mathbb{R}^N : |x| > R, V(x) \geq C_0\}$$

and

$$Y(R) := \{x \in \mathbb{R}^N : |x| > R, V(x) < C_0\}.$$

Let $C > 0$ be a constant whose value may change from line to line. According to Lemma 2.7 we know that $\|u_n\|_W^2 \leq C$ for all $n \in \mathbb{N}$, hence we have

$$\begin{aligned}
\int_{X(R)} u_n^2 \mathrm{d}x &\leq \frac{1}{\lambda C_0 + 1} \int_{\mathbb{R}^N} (\lambda V(x) + 1) u_n^2 \mathrm{d}x \\
&\leq \frac{1}{\lambda C_0 + 1} \left(\int_{\mathbb{R}^N} (\lambda V(x) + 1) u_n^2 \mathrm{d}x + [u_n]_{H^s}^2 \right) \\
&= \frac{1}{\lambda C_0 + 1} \|u_n\|_W^2 \\
&\leq \frac{C}{\lambda C_0 + 1}.
\end{aligned}$$

Again by the Hölder inequality, the continuous embedding $H^s(\mathbb{R}^N) \hookrightarrow L^{2p}(\mathbb{R}^N)$ for any $p \in [1, N/(N-2s)]$ (see [12, Theorem 6.5]) and condition (V_1), we obtain

$$\begin{aligned}\int_{Y(R)} u_n^2 \mathrm{d}x &\leq \left(\int_{Y(R)} |u_n|^{2p} \mathrm{d}x\right)^{\frac{1}{p}} \left(\int_{Y(R)} \mathrm{d}x\right)^{\frac{1}{q}} \\ &\leq \left(\int_{\mathbb{R}^N} |u_n|^{2p} \mathrm{d}x\right)^{\frac{1}{p}} \left(\int_{Y(R)} \mathrm{d}x\right)^{\frac{1}{q}} \\ &\leq C\|u_n\|_{H^s}^2 |Y(R)|^{\frac{1}{q}} \\ &= C|Y(R)|^{\frac{1}{q}},\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. It follows from Lemma 2.1 that

$$\begin{aligned}\int_{B_R^c(0)} |u_n|^p \mathrm{d}x &\leq C(N,s,p)[u_n|_{B_R^c(0)}]_{H^s}^{\gamma_p p} \|u_n|_{B_R^c(0)}\|_2^{(1-\gamma_p)p} \\ &\leq C(N,s,p)C^{\gamma_p p}\|u_n\|_W^{\gamma_p p} \left(\int_{X(R)} u_n^2 \mathrm{d}x + \int_{Y(R)} u_n^2 \mathrm{d}x\right)^{\frac{(1-\gamma_p)p}{2}} \\ &\leq C \left(\int_{X(R)} u_n^2 \mathrm{d}x + \int_{Y(R)} u_n^2 \mathrm{d}x\right)^{\frac{(1-\gamma_p)p}{2}}.\end{aligned}$$

The first term on the right-hand side of the above inequality can be arbitrarily small when $\lambda > \lambda^*$ (large enough). The second term on the right-hand side of the above inequality can also be arbitrarily small if R is big enough since $|Y(R)| \to 0$ as $R \to +\infty$, thanks to condition (V_1). This concludes the proof, that is, (2.3) holds. □

The last auxiliary lemma concerns the existence of a suitable weak limit, always in the setting of weighted fractional Sobolev space W.

Lemma 2.9. *If conditions (V_1)–(V_2) hold, then there exists $\lambda^{**} > 0$ such that for any $\lambda > \lambda^{**}$ the sequence $(u_n)_{n\in\mathbb{N}}$ admits a nontrivial weak limit u in W.*

Proof. According to Lemma 2.7, we know that there exists $u \in W$ and a subsequence of $(u_n)_{n\in\mathbb{N}}$, still denoted as itself, such that

$$u_n \rightharpoonup u \text{ in } W,\ u_n(x) \to u(x) \text{ a.e. in } \mathbb{R}^N.$$

Now, arguing by contradiction, we assume $u = 0$ for some $\lambda > \lambda^*$ as given in Lemma 2.8. Based on the compactness of the embedding over the bounded domain, we get that $u_n \to 0$ in $L^p(B_R(0))$ for any $R > 0$. In view of Lemmas

2.4 and 2.8, we have that

$$\begin{aligned} M &\le \liminf_{n\to+\infty} \int_{\mathbb{R}^N} |u_n|^p \mathrm{d}x \\ &= \liminf_{n\to+\infty} \int_{B^c_R(0)} |u_n|^p \mathrm{d}x \\ &\le \limsup_{n\to+\infty} \int_{B^c_R(0)} |u_n|^p \mathrm{d}x \\ &\le \frac{M}{2}, \end{aligned}$$

a contradiction. Thus we conclude that there exists $\lambda^{**} \le \lambda^*$ such that u is nontrivial for any $\lambda > \lambda^{**}$. □

Finally, we can give the proof of our main result.

Proof of Theorem 1.2. From Lemmas 2.7 and 2.9 it follows that there exists a minimizing sequence $(u_n)_{n\in\mathbb{N}} \subset \mathcal{M}_m$ for $\mathcal{I}_m$, which is bounded in W and its weak limit u is nontrivial. In view of Lemma 2.6, we have that $u \in \mathcal{M}_m$, $E(u) = \mathcal{I}_m$ and $u_n \to u$ in W. Hence, employing the Lagrange multiplier method, there exists $\mu_m \in \mathbb{R}$ solving the equation

$$E^{'}(u) + \mu_m J^{'}(u) = 0 \text{ in } W^*, \tag{2.4}$$

where W^* is the dual space of W and $J : W \to \mathbb{R}$ is defined by

$$J(u) = \|u\|_2, \ \ u \in W.$$

By (2.4), we deduce that

$$(-\Delta)^s u + \mu_m u + \lambda V(x)u - |u|^{p-2}u = 0 \text{ in } \mathbb{R}^N.$$

The proof of Theorem 1.2 is now complete. □

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