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Ideals generated by the inner 2-minors of collections of cells

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Author:
Francesco NAVARRA

Supervisor:
Prof. Marilena CRUPI

Co-Supervisor:
Prof. Rosanna UTANO

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in

Mathematics and Computational Sciences

Declaration of Authorship

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Signed: Francesco Navarra

Date: 08/06/2023

"Mathematics is the most beautiful and most powerful creation of the human spirit."

Stefan Banach

UNIVERSITY OF PALERMO

Abstract

Faculty Name
Department of Mathematics and Computer Sciences

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Ideals generated by the inner 2-minors of collections of cells

by Francesco NAVARRA

In 2012 Ayesha Asloob Qureshi connected collections of cells to Commutative Algebra assigning to every collection $\mathcal{P}$ of cells the ideal of inner 2-minors, denoted by $I_{\mathcal{P}}$, in the polynomial ring $S_{\mathcal{P}} = K[x_v : v \text{ is a vertex of } \mathcal{P}]$ ([37]). Investigating the main algebraic properties of $K[\mathcal{P}] = S_{\mathcal{P}}/I_{\mathcal{P}}$ depending on the shape of $\mathcal{P}$ is the purpose of this research. Many problems are still open and they seem to be fascinating and exciting challenges.
In this thesis we prove several results about the primality of $I_{\mathcal{P}}$ and the algebraic properties of $K[\mathcal{P}]$ like Cohen-Macaulyness, normality and Gorensteiness, for some classes of non-simple polyominoes. The study of the Hilbert-Poincaré series and the related invariants as Krull dimension and Castelnuovo-Mumford regularity are given. Finally we provide the code of the package `PolyominoIdeals` developed for `Macaulay2`.

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Dedicated to my parents

Introduction

The work [45] of R. Stanley, in which he proves the upper bound conjecture for the spheres, is the starting point of a new trend which combines Commutative Algebra and Combinatorics. This area of research has obtained a huge interest during the years and has become gradually fashionable. Today the most known references are [3], [15], [22], [27], [43] and [46].

Since 1990s the study of the ideals of t-minors of an $m \times n$ matrix of indeterminates has become a central topic in Combinatorial Commutative Algebra. The determinantal ideals are studied in [10], [11] and [12], the ideals of adjacent 2-minors in [21], [29] and [35] as well as the ideals generated by an arbitrary set of 2-minors of a $2 \times n$ matrix in [26]. This work is devoted to the study of some binomial ideals arising from 2-minors, which are the *polyomino ideals*. The class of polyomino ideals or, more in general, of the inner 2-minors ideals of collections of cells generalizes the class of the ideals generated by 2-minors of $m \times n$ matrices of indeterminates.

Polyominoes are plane figures, made up of squares of the same size joined edge by edge. They appeared for the first time in recreational mathematics and combinatorics and they are studied especially in tiling problems of the plane. Even though some problems like the enumeration of pentominoes have their origins in antiquity, polyominoes were formally defined by Golomb first in 1953 and later, in 1996, in his monograph [18]. The study of polyominoes reveals many connections to different subjects. For instance, there seems to be a nice relation between polyominoes and Dyck words and Motzkin words [13] as well as with statistical physics: polyominoes and their higher-dimensional analogues appear as models of branched polymers and of percolation clusters [49].

In [37] A.A. Qureshi connects polyominoes or more in general collections of cells to Commutative Algebra. She defines a binomial ideal attached to a collection $\mathcal{P}$ of cells, as the ideal generated by all inner 2-minors of $\mathcal{P}$ in the ring $S_{\mathcal{P}}$, that is the polynomial ring over a field K in the variables x_v, where v is a vertex of a cell of $\mathcal{P}$. Such an ideal is denoted by $I_{\mathcal{P}}$ and the quotient ring $K[\mathcal{P}] = S_{\mathcal{P}}/I_{\mathcal{P}}$ is called *coordinate ring* of $\mathcal{P}$. The study of the main algebraic properties of $K[\mathcal{P}]$, depending on the shape of $\mathcal{P}$, provides an exciting line of research.

Recently, the literature on this topic has significantly expanded and contains plenty of interesting and exciting challenges. One of the most fascinating topic in this context is the study of the primality of $I_{\mathcal{P}}$. To shorten the notation, we say that a polyomino is prime if its associated polyomino ideal is prime. In [24] and in [39] it is proved that simple polyominoes, which are roughly speaking the polyominoes without holes, are prime. Therefore the study is applied to multiply connected polyominoes, which are polyominoes with one or more holes. In [28] and [41], the authors prove that the polyominoes obtained by removing a convex polyomino from a rectangle in $\mathbb{N}^2$ are prime, using two different demonstrative techniques. In [31] the authors introduce a very useful tool to provide a characterization of prime polyominoes. They define a particular sequence of inner intervals of $\mathcal{P}$, called a *zig-zag walk*, and they prove that if $I_{\mathcal{P}}$ is prime then $\mathcal{P}$ does not contain zig-zag walks. Using a computational method, they show that for polyominoes consisting of at most

fourteen cells the non-existence of zig-zag walks in $\mathcal{P}$ is a sufficient condition in order to $I_\mathcal{P}$ is prime. Therefore, they conjecture that non-existence of zig-zag walks in a polyomino characterizes its primality. Not only the primality of $I_\mathcal{P}$ is still an open problem but also the radicality of $I_\mathcal{P}$. In fact the study of the Gröbner basis of $I_\mathcal{P}$ is quite elaborated in general and, in particular, to prove that the initial ideal of $I_\mathcal{P}$ is squarefree for a suitable monomial order is very difficult for all polyominoes. In [25] the authors prove that for a simple polyomino the polyomino ideal has a squarefree Gröbner basis with respect to any monomial order, so the initial ideal is always squarefree. In [32] the authors give some conditions so that the set of generators of $I_\mathcal{P}$ forms a reduced Gröbner basis with respect to some suitable degree reverse lexicographic monomial orders and they prove that in these cases the polyomino is prime. Using this method, they prove the primality of two classes of thin polyominoes, which are polyominoes not containing the square tetromino. The experience seems to show that every polyomino ideal admits squarefree initial ideal with respect to suitable monomial orders and hence it is radical. Also the Cohen-Macaulay property and the normality of $K[\mathcal{P}]$ are still completely unknown, except for some classes of polyominoes, like simple polyominoes ([24], [25] and [39]), rectangle minus a convex polyomino ([28]) and grid polyominoes ([32]). There is no known example of a non-simple polyomino $\mathcal{P}$ whose $K[\mathcal{P}]$ is not Cohen-Macaulay or not normal, as well as for the radicality of $I_\mathcal{P}$.
A particular attention is devoted recently to the study of the Hilbert-Poincaré series and the Castelnuovo-Mumford regularity of $K[\mathcal{P}]$ in terms of the rook polynomial of $\mathcal{P}$ and, in particular, to the Gorenstein property. By using different approaches, in [37] Qureshi gives a characterization of the Gorenstein stack polyominoes and, later, in [1] the author classifies all Gorenstein convex polyominoes. In [17] the authors give a new combinatorial interpretation of the regularity of the coordinate ring attached to an L-convex polyomino, as the rook number of $\mathcal{P}$, that is the maximum number of rooks which can be arranged in $\mathcal{P}$ in non-attacking positions. In [40] the Hilbert-Poincaré series of simple thin polyominoes is studied relating them to the rook polynomial of $\mathcal{P}$. More precisely, it is showed that if $\mathcal{P}$ is a simple thin polyomino then the h-polynomial $h(t)$ of $K[\mathcal{P}]$ is the rook polynomial $r_\mathcal{P}(t) = \sum_{i=0}^{n} r_i t^i$ of $\mathcal{P}$, whose coefficient r_i represents the number of distinct possibilities of arranging i rooks on cells of $\mathcal{P}$ in non attacking positions (with the convention $r_0 = 1$). Gorenstein simple thin polyominoes are also characterized using the S-property and finally it is conjectured that a polyomino is thin if and only if $h(t) = r_\mathcal{P}(t)$. In [30] it is also discussed this conjecture for a certain class of polyominoes. In a recent paper [38] the authors introduce a particular equivalence relation on the rook complex of a simple polyomino and they conjecture that the number of equivalence classes of i non-attacking rooks arrangements is exactly the i-th coefficient of the h-polynomial in the reduced Hilbert-Poincaré series. Moreover they prove it for the class of parallelogram polyominoes and by a computational method also for all simple polyominoes with rank at most eleven.

Starting from this background, we investigate the main algebraic properties of the coordinate ring associated to new classes of collections of cells. Motivated by the results already mentioned about the primality, we start our research defining a new family of non-simple polyominoes, called *closed paths*, and studying the primality of their polyomino ideal. Inspired by the conjecture that states that the polyomino ideal is prime if and only if the polyomino contains no zig-zag walks, we classify all closed paths having no zig-zag walks introducing the L-configurations and the ladders of at least three steps. In Sub-Section 2.2.1 we define an L-configuration, that consists of a path of five cells $A_1, \dots, A_5$ such that the two blocks A_1, A_2, A_3 and

A_3, A_4, A_5 go in orthogonal directions, and we prove that if a closed path has an L-configuration, then it does not contain zig-zag walks. Besides that, we define a more elaborated structure of cells in a polyomino, called a ladder (see Definition 2.2.10), and we show that having a ladder of at least three steps is a sufficient condition for a closed path to have no zig-zag walks. In Sub-Section 2.2.2 we give a toric representation of a closed path with an L-configuration or a ladder of at least three steps, using an appropriate argument inspired by a strategy provided by Shikama in [41]. Finally, in Sub-Section 2.2.3 we show that having an L-configuration or a ladder of at least three steps is a necessary and sufficient condition in order to have no zig-zag walks for a closed path, providing a complete characterization of the structure of closed paths containing no zig-zag walks. This result gives an important geometric condition, which allows to characterize the primality of the polyomino ideal attached to a closed path by the non-existence of zig-zag walks. Hence the class of closed answers positively to the conjecture mentioned before.
The L-configuration and the ladder as well as the technique used to deal the primality of closed paths suggest some classes of prime polyominoes, presented in Section 3.1, which can be viewed as generalizations of closed paths. In particular we give some sufficient conditions for the primality of these non-simple polyominoes but to find out the necessary ones, and so proving the conjecture of [31] for such classes, seems to be a non-easy task, that we leave as an open question. Motivated by the wish to examine in deep the primality of the polyomino ideal by zig-zag walks, we introduce the class of *weakly closed path* polyominoes. While reading Section 3.3, we invite the reader to pay attention to the difference between closed path and weakly closed path because they seem to be very similar but actually they are deeply different. One of the crucial point in the study of the primality of a closed path is that, if we remove certain cells from the L-configuration or a ladder in a closed path, then we obtain a simple polyomino (see Proposition 2.2.5, Theorems 2.2.15 and 2.2.18). Unfortunately this fact does not hold in general for a weakly closed path, because we could get a simple collection of cells with two connected components by removing cells from the polyomino. Therefore the study of the primality of a weakly closed path requires to examine firstly the primality of simple and weakly connected collections of cells (see Section 3.2). In particular we show that the binomial ideal associated to a simple and weakly connected collection of cells coincides with the toric ideal of the edge ring of a weakly chordal bipartite graph, generalizing [39, Theorem 3.10]. As an application of this result and following the strategy used to characterize the primality of closed paths, in Section 3.3 we give a characterization of the primality of the polyomino ideal of a weakly closed path.
Inspired by two well known theorems of Sturmfels ([46]) and of Hochster ([3]), which state that a toric ring whose defining ideal admits a squarefree initial ideal is a normal Cohen-Macaulay domain, we start to study the Gröbner basis of polyomino ideals. Firstly, in Section 4.1 we give several results that provide necessary and sufficient conditions in order to the S-polynomial of two generators of a polyomino ideal reduces to zero with respect to a lexicographic order induced by any total order on the set of the variables. Later in Section 4.2 we deal the Gröbner basis for polyomino ideals attached to closed paths. In particular we define four specific sub-polyominoes of a closed path, which are the W-pentominoes, the LD-horizontal and vertical skew tetrominoes and hexominoes and the RW-heptominoes. For each case in which one of the previous configurations is not in the closed path, we provide a set of suitable vertices (see Algorithm 4.2.7 for the general case) which allow us to define some nice monomial orders in order to prove that the set of the generators of the polyomino ideal of a closed path forms the reduced Gröbner basis with respect

to these monomial orders. As a consequence we get the radicality of the associated polyomino ideal and, moreover, that the coordinate ring of a closed path having no zig-zag walks is a normal Cohen-Macaulay domain. Finally, stimulated by the fact that the polyomino ideal of a simple polyomino has a squarefree universal Gröbner basis (see [25]), we show that the smallest closed path with respect to the set inclusion provides a significant example of a non-simple polyomino which does not admit squarefree universal Gröbner basis.
Among the Cohen-Macaulay rings, it is interesting to determine the ones that are Gorenstein. The nice result of Stanley [44, Theorem 4.4] suggests to study the Hilbert-Poincaré series to investigate the Gorenstein property for Cohen-Macaulay domains. Motivated also by the well known results on the rook polynomial of a simple thin polyomino, we study the Hilbert-Poincaré series of some non-simple polyominoes and we interpret the h-polynomial of $K[\mathcal{P}]$, where $\mathcal{P}$ is a prime closed path, like the rook polynomial of $\mathcal{P}$. In Section 5.1 we introduce a particular polyomino $\mathcal{L}$ and we define the class of $(\mathcal{L},\mathcal{C})$-polyominoes, where $\mathcal{C}$ is a generic polyomino. Firstly we provide some results on the primality of some sub-polyominoes of an $(\mathcal{L},\mathcal{C})$-polyomino and, in particular, on the isomorphism of the polyomino ring of an $(\mathcal{L},\mathcal{C})$-polyomino modulo a certain colon ideal and the tensor product of K-algebras between the coordinate ring of a suitable sub-polyomino of the $(\mathcal{L},\mathcal{C})$-polyomino and a suitable polynomial ring. Thanks to these results we give an explicit formula for the Hilbert-Poincaré series of the coordinate rings of a prime $(\mathcal{L},\mathcal{C})$-polyomino, depending on the Hilbert-Poincaré series of some polyominoes obtained by eliminating specific cells from the $(\mathcal{L},\mathcal{C})$-polyomino. Moreover, when $\mathcal{C}$ is a simple polyomino, we improve this formula and we compute also the Krull dimension. In Section 5.2 we assume that $\mathcal{P}$ is a closed path polyomino and we deal the case in which $\mathcal{P}$ has no L-configuration but it contains a ladder of at least three steps. Here we examine several cases depending on the shape of the ladder and we need to describe also the initial ideals of some sub-polyominoes of $\mathcal{P}$ with respect to some monomial orders, in order to obtain the formulas of the Hilbert-Poincaré series of the coordinate rings of such polyominoes in terms of suitable sub-polyominoes. The case in which $\mathcal{P}$ has an L-configuration is a particular case of $(\mathcal{L},\mathcal{C})$-polyomino, where $\mathcal{C}$ is a path, so we fulfil the class of closed paths without zig-zag walks for what concerns the study of the Hilbert-Poincaré series. In Section 5.3 we prove that if $\mathcal{P}$ is a closed path without zig-zag walks then the h-polynomial of $K[\mathcal{P}]$ is the rook polynomial of $\mathcal{P}$, obtaining as a consequence the regularity and the Krull dimension of $K[\mathcal{P}]$. Finally we characterize all Gorenstein prime closed paths, proving that $K[\mathcal{P}]$ is Gorenstein if and only if $\mathcal{P}$ consists of maximal blocks of length three.
All the examples which inspired the results described in this dissertation have been obtained using the Algebra Software `Macaulay2`. Motived by the will to make several examples for our researches, we implement a package for the Algebra Software `Macaulay2`, called `PolyominoIdeals` ([5]). The aim of this package is to provide some tools to help mathematicians in the study of the polyomino ideals. Encoding a fixed collection of cells by a list of lists containing the diagonal corners of each cell, the package provides three functions and some related options. Among them, the function `polyoIdeal` allows to define the inner 2-minor ideal of $\mathcal{P}$. For this function, we also give three options. The option `RingChoice` allows one to choose between two rings having two different monomial orders: one is the lexicographic order induced by the natural partial order on the set $V(\mathcal{P})$ of the vertices of $\mathcal{P}$; the other one, based on a function called `polyoMatrix` which gives the matrix attached to $\mathcal{P}$, is the monomial order $<$ given in [34] for the weakly chordal bipartite graphs. This one can be induced in a *convex* polyomino as shown in [37], so one can deduce that

for a convex polyomino $\mathcal{P}$ the generators of the ideal $I_{\mathcal{P}}$ form the reduced Gröbner basis with respect to $<$. The latter can be useful to compute the quadratic squarefree initial ideal of a convex polyomino with respect to $<$ and later the related simplicial complex, for example. For the other two options, the first one is `Field` which allows to change the base field K in the base polynomial ring of $I_{\mathcal{P}}$, and the second is `TermOder`, which allows to replace the lexicographic order with other term orders when `RingChoice` is defined in the first way as described above. Finally we have the function `polyoToric`, which allows to generate the toric ideal defined in [31] and which can be useful to study the primality of the polyomino ideal.

This thesis is organized as follows. Chapter 1 is devoted to introduce some well known algebraic notations and concepts in Commutative Algebra.

In Chapter 2 we define collections of cells and we give some basic combinatorial definitions and properties. We define the ideal $I_{\mathcal{P}}$ of the inner 2-minors of $\mathcal{P}$, where $\mathcal{P}$ is a collection of cells, and the coordinate ring $K[\mathcal{P}]$ of $\mathcal{P}$. In the second part, we introduce the class of closed paths and we study the primality of their polyomino ideal.

In Chapter 3 we present firstly some classes of prime polyominoes viewed as generalizations of closed paths, whose primality can be provided using some techniques similar to those applied in Chapter 2. Later, we focus on the primality of simple and weakly connected collections of cells and finally we give a characterization of the primality of the polyomino ideal of the weakly closed paths.

Chapter 4 is devoted to study the Gröbner basis of polyomino ideals, in particular we give some monomial orders in order to the reduced Gröbner basis of the polyomino ideal of a closed path with respect to these monomial orders is the set of the generators of the ideal.

In Chapter 5 we study the Hilbert-Poincaré series of some non-simple polyominoes and we prove especially that the h-polynomial of the coordinate ring of a closed path without zig-zag walks is the rook polynomial of $\mathcal{P}$. Finally a characterization of Gorenstein prime closed paths is given in terms of the length of the blocks of the polyomino.

In Chapter 6 we describe the functions which we implemented in the package `PolyominoIdeals` for `Macaulay2`. In Section 6.1 we explain the use of the three functions `polyoIdeals`, `polyoMatrix` and `polyoToric`, and the related options `Field`, `TermOrder` and `RingChoice`. The whole code is provided in Section 6.2. Every contribution to improve it is very welcome.

We conclude the dissertation giving some hints for future possible researches on this topic.

Chapter 1

Basics on Commutative Algebra

In this chapter we recall some basic notions of Commutative Algebra and some useful results that will be useful along the dissertation. Two nice books to approach and to examine in deep some of these topics are [2] and [3].

Let us start providing for the sake of the reader the elementary definitions of primality and radicality of an ideal, which will be investigated for some classes of ideals in this work. We assume that R is a unitary commutative ring throughout this chapter, unless specified otherwise.

Definition. Let I be a proper ideal of R. We say that I is *prime* if for all $x, y \in R$ such that $xy \in I$ it follows that $x \in I$ or $y \in I$.
The ideal $\sqrt{I} := \{x \in R : x^n \in I, \text{ for some } n \in \mathbb{N}\}$ is called the *radical of* I. Moreover, I is said to be *radical* if $I = \sqrt{I}$.

In the next sections we discuss briefly the Krull dimension of Noetherian commutative rings, the Castelnuovo-Mumford regularity of a graded finitely generated module and the Hilbert-Poincaré series of a graded K-algebra. Later we present Cohen-Macaulay and Gorenstein properties for unitary commutative rings and we provide some basics on Gröbner bases of an ideal in a polynomial ring. Finally, we conclude the chapter giving a brief review on normal domains and toric ideals with some related well known results.

1.1 Krull dimension, Castelnuovo-Mumford regularity and Hilbert series

At the beginning of the twentieth century Emmy Noether introduced the condition of ascending chains for ideals of R, which is equivalent to state that every ideal of R is finitely generated. Today such rings are known as *Noetherian* ring. It is known by Hilbert's basis Theorem that every polynomial ring over a Noetherian ring in a finite number of indeterminates is Noetherian.
In the theory of Noetherian commutative rings one of the most important notions is the Krull dimension which measures the longest possible chain of prime ideals in the ring. We start providing the definition of height and dimension of a prime ideal.

Definition 1.1.1. An ascending chain of prime ideals is a family $\{\mathfrak{p}_j\}_{j\in J}$ of prime ideals, where J is a countable set and $\mathfrak{p}_j \subset \mathfrak{p}_{j+1}$ for $j \in J$.

Definition 1.1.2. Let $\mathfrak{C} : \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$ be a finite ascending chain of prime ideals of R. The integer n is called *length of* $\mathfrak{C}$.

Definition 1.1.3. Let $\mathfrak{p}$ be a prime ideal. We call *height of* $\mathfrak{p}$ the supremum of the lengths of the ascending chains of prime ideals contained in $\mathfrak{p}$. We denote it by $\mathrm{ht}(\mathfrak{p})$.

Example 1.1.4. In $K[X_1, \dots, X_n]$ the prime ideal $(X_1, \dots, X_n)$ has height equal to n, in fact a chain of prime ideals of maximum length is given by $(0) \subset (X_1) \subset (X_1, X_2) \cdots \subset (X_1, \dots, X_{n-1}) \subset (X_1, \dots, X_n)$.

Denote by $\mathrm{Spec}(R)$ the set of all prime ideals of R.

Definition 1.1.5. The *Krull dimension of* R, denoted by $\dim(R)$, is defined as

$$\sup\{\mathrm{ht}(\mathfrak{p}) : \mathfrak{p} \in \mathrm{Spec}\, R\}.$$

Example 1.1.6. The Krull dimension of $K[X_1, \dots, X_n]$ is n, since $(0) \subset (X_1) \subset (X_1, X_2) \cdots \subset (X_1, \dots, X_{n-1}) \subset (X_1, \dots, X_n)$ is one of the longest chains of prime ideals.

More in general, the *Krull dimension* of an R-module M is defined as the maximal length of the chains of prime ideals $\mathfrak{p}$ such that $M_{\mathfrak{p}}$ is non null, that is $\dim(R/Ann(M))$. We recall that a ring is said to be *local* if there exists a unique maximal ideal.

Proposition 1.1.7. *Let* $(R, \mathfrak{m})$ *be a local ring, where* $\mathfrak{m}$ *is the maximal ideal of* R*. Then* $\dim R = \mathrm{ht}(\mathfrak{m})$.

Definition 1.1.8. Let I be an ideal of R. We define the *height* of I as

$$\min\{\mathrm{ht}(\mathfrak{p}) : \mathfrak{p} \in \mathrm{Spec}\, R, I \subseteq \mathfrak{p}\}.$$

We denote it by $\mathrm{ht}(I)$.

Example 1.1.9. Let $\mathfrak{p}$ be a prime ideal of R and consider the local ring $R_{\mathfrak{p}}$ whose maximal ideal is

$$\mathfrak{p}R_{\mathfrak{p}} := \left\{ \frac{x}{y} : x \in \mathfrak{p}, y \notin \mathfrak{p} \right\}$$

In a certain sense the ideals in $R_{\mathfrak{p}}$ are given by just the ideals in R contained in $\mathfrak{p}$. Hence from the known bijection between the prime ideals of R and of the ring of fractions, we have that

$$\dim R_{\mathfrak{p}} = \mathrm{ht}(\mathfrak{p}R_{\mathfrak{p}}) = \mathrm{ht}(\mathfrak{p}).$$

We state a famous theorem concerning the height of an ideal in a Noetherian ring.

Theorem 1.1.10. *Let* R *be a Noetherian ring and* I *be an ideal of* R*. The height of* I *is finite. In addition, if* R *is also local then* R *has a finite Krull dimension.*

Now, we recall briefly some results on the Krull dimension of polynomial rings.

Theorem 1.1.11. *Let* R *be a Noetherian ring. Then*

$$\dim R[X] = \dim R + 1.$$

Corollary 1.1.12. *The following hold:*

1. *if* R *is a Noetherian ring, then* $\dim R[X_1, \dots, X_n] = \dim R + n$*;*
2. *if* K *is a field, then* $\dim K[X_1, \dots, X_n] = n$*.*

Example 1.1.13. The Krull dimensions of the rings $\mathbb{R}[X]$, $\mathbb{R}[X,Y]$ and $\mathbb{Z}[X,Y]$ are respectively $1,2$ and 3.

Let $S = K[X_1,\dots,X_n]$. We recall that $S = \oplus_{i\in\mathbb{Z}} S_i$ is a *graded* K-algebra, setting S_j as the K-vector space spanned by all monomials of degree j, $S_0 = K$ and $S_j = (0)$ for all $j < 0$. The K-vector spaces S_i are called *homogeneous components* of S and the elements in S_i are defined as *homogeneous* elements of degree i.
A *complex* $\mathbb{F}$ *over* S is a sequence of S-modules and homomorphisms of S-modules as

$$\mathbb{F} : \cdots \to F_{j+1} \xrightarrow{d_{j+1}} F_j \xrightarrow{d_j} F_{j-1} \to \cdots \to F_0 \to \cdots$$

such that $d_{i-1}d_i = 0$ for all $i \in \mathbb{Z}$. The *i-th homology* of $\mathbb{F}$ is defined by $H_i(\mathbb{F}) = \ker(d_i)/\operatorname{Im}(d_{i+1})$, for all $i \in \mathbb{Z}$. We say that $\mathbb{F}$ is *exact* if $H_i(\mathbb{F}) = 0$ for all $i \in \mathbb{Z}$.

Let M be a graded finitely generated S-module. A *graded free resolution* of M is a complex of finitely generated graded free S-modules as

$$\mathbb{F}(M) : \cdots \to F_{j+1} \xrightarrow{d_{j+1}} F_j \xrightarrow{d_j} F_{j-1} \to \cdots \to F_0 \xrightarrow{d_0} M \to 0$$

such that $H_i(\mathbb{F}) = 0$ for all $i \neq 0$, $M \cong F_0/\operatorname{Im}(d_1)$ and d_i is an homogeneous map of degree 0. Recall that in general such a resolution is not unique.
The length of $\mathbb{F}(M)$ is defined by $\sup\{i | F_i \neq 0\}$. We say that $\mathbb{F}(M)$ is *finite* if its length is finite, otherwise it is *infinite*. We say that $\mathbb{F}(M)$ is *minimal* if $d_{i+1}(F_{i+1}) \subseteq (x_1,\dots,x_n)F_i$ for all $i \geq 0$.
It is well known from the famous Hilbert's Syzygy Theorem that a minimal graded free resolution of a graded finitely generated S-module is finite and its length is at most n. Moreover, it is proved that a minimal graded free resolution of a graded finitely generated S-module M always exists and it is unique up to isomorphisms. In such a case we can write $\mathbb{F}(M)$ as

$$0 \to \bigoplus_{j\in\mathbb{Z}} S(-j)^{\beta_{t,j}} \xrightarrow{d_t} \cdots \to \bigoplus_{j\in\mathbb{Z}} S(-j)^{\beta_{i,j}} \xrightarrow{d_i} \cdots \to \bigoplus_{j\in\mathbb{Z}} S(-j)^{\beta_{1,j}} \xrightarrow{d_1} \bigoplus_{j\in\mathbb{Z}} S(-j)^{\beta_{0,j}} \xrightarrow{d_0} M \to 0$$

The numbers $\beta_{i,j}$ are called the *graded Betti numbers* of M. Moreover, the *i-th Betti number* $\beta_i(M)$ is defined by $\sum_{j\in\mathbb{Z}} \beta_{i,j}$, which is the rank of F_i.
Let M be a graded finitely generated S-module. The *projective dimension* of M is $\operatorname{pd}(M) = \max\{i : \beta_i(M) \neq 0\}$, which is the length of the minimal graded free resolution of M.
The *Castelnuovo-Mumford regularity* (or simply *regularity*) of M is

$$\max\{j : \beta_{i,i+j} \neq 0 \; for \; some \; i\}$$

and it is denoted by $\operatorname{reg}(M)$.
The graded Betti numbers of M can be displayed in a table called *Betti diagram*. Table 1.1 is that one provided in several computer programs, in particular in `Macaulay2`.

Example 1.1.14. Let $S = K[x_1,x_2,x_3,x_4]$ and I be the ideal generated by $f_1 = x_1x_3 - x_2^2$, $f_2 = x_3^2 - x_1x_2$ and $f_3 = x_4^2$. Using Macaulay2, we find the minimal

$$
\begin{array}{c|ccccc}
 & \beta_0 & \beta_1 & \beta_2 & \dots \\
\hline
0 & \beta_{0,0} & \beta_{1,1} & \beta_{2,2} & \dots \\
1 & \beta_{0,1} & \beta_{1,2} & \beta_{2,3} & \dots \\
2 & \beta_{0,2} & \beta_{1,3} & \beta_{2,4} & \dots \\
3 & \beta_{0,3} & \beta_{1,4} & \beta_{2,5} & \dots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

TABLE 1.1

free resolution of S/I

$$0 \to S \xrightarrow{\begin{pmatrix} -x_4^2 \\ -x_2^2+x_1\, x_3 \\ x_1\, x_2-x_3^2 \end{pmatrix}} S^3 \xrightarrow{\begin{pmatrix} x_2^2-x_1\, x_3 & -x_4^2 & 0 \\ -x_1\, x_2+x_3^2 & 0 & -x_4^2 \\ 0 & x_1\, x_2-x_3^2 & x_2^2-x_1\, x_3 \end{pmatrix}} S^3 \xrightarrow{\begin{pmatrix} x_1\, x_2-x_3^2 & x_2^2-x_1\, x_3 & x_4^2 \end{pmatrix}} S \to S/I \to 0$$

and the Betti diagram

$$
\begin{array}{rcccc}
\text{total:} & 1 & 3 & 3 & 1 \\
0: & 1 & . & . & . \\
1: & . & 3 & . & . \\
2: & . & . & 3 & . \\
3: & . & . & . & 1
\end{array}
$$

We conclude this section recalling some notions on the Hilbert-Poincaré series of a graded K-algebra R/I.
Let R be a standard graded ring and I be an homogeneous ideal of R. Then R/I has a natural structure of graded K-algebra as $\bigoplus_{k\in\mathbb{N}}(R/I)_k$. The numerical function $\mathrm{H}_{R/I} : \mathbb{N} \to \mathbb{N}$ with $\mathrm{H}_{R/I}(k) = \dim_K(R/I)_k$ is called the *Hilbert function* of R/I.
The formal series

$$\mathrm{HP}_{R/I}(t) = \sum_{k\in\mathbb{N}} \mathrm{H}_{R/I}(k)t^k$$

is called the *Hilbert-Poincaré series* of R/I.
It is known by Hilbert-Serre theorem that there exists a polynomial $h(t) \in \mathbb{Z}[t]$ with $h(1) \neq 0$ such that

$$\mathrm{HP}_{R/I}(t) = \frac{h(t)}{(1-t)^d}$$

where d is the Krull dimension of R/I. Such an expression of $\mathrm{HP}_{R/I}$ is known as the *reduced* Hilbert-Poincaré series of R/I and the polynomial $h(t)$ is called the *h-polynomial* of R/I. For instance, if $S = K[x_1, \dots, x_n]$ then $\mathrm{HP}_S(t) = \frac{1}{(1-t)^n}$. Using Macaulay2 we find that the Hilbert-Poincaré series of S/I in the Example 1.1.14 is given by

$$\frac{1+3\,t+3\,t^2+t^3}{1-t}.$$

Proposition 1.1.15. *[48, Chapter 5] Let R be a graded K-algebra and I be a graded ideal of R. Let q be an homogeneous element of R of degree m and let*

$$0 \longrightarrow R/(I:q) \longrightarrow R/I \longrightarrow R/(I,q) \longrightarrow 0$$

be a short exact sequence. Then $\mathrm{HP}_{R/I}(t) = \mathrm{HP}_{R/(I,q)}(t) + t^m \mathrm{HP}_{R/(I:q)}(t)$.

Proposition 1.1.16. *Let A and B be standard graded K-algebras over a field K. Then* $\mathrm{HP}_{A\otimes_K B}(t) = \mathrm{HP}_A(t) \cdot \mathrm{HP}_B(t)$.

We conclude mentioning [48, Proposition 3.1.33], which will be useful in several results of Chapter 5 of this work.

Proposition 1.1.17. *If $\mathbf{X}$ is a set of indeterminates, $\mathbf{X}_1, \mathbf{X}_2 \subset \mathbf{X}$ form a partition of $\mathbf{X}$ into disjoint non-empty subsets and I is an ideal of $K[\mathbf{X}]$, K a field, such that each generator of I belongs to $K[\mathbf{X}_j]$ for $j \in \{1,2\}$, then $K[\mathbf{X}]/I \cong K[\mathbf{X}_1]/I_1 \otimes_K K[\mathbf{X}_2]/I_2$, where $I_j = I \cap K[\mathbf{X}_j]$ for $j \in \{1,2\}$.*

1.2 Cohen-Macaulay rings, injective dimension and Gorenstein rings

We introduce the concept of regular sequence and later the class of Cohen-Macaulay rings. Keep in mind that R is a unitary commutative ring.

Definition 1.2.1. A sequence $x_1, \ldots, x_n$ of elements of R is said to be *regular* in R if it satisfies the following conditions:

1. $(x_1, \ldots, x_n) \neq R$;
2. x_1 is not a zero-divisor in R;
3. $\overline{x_i}$ is not a zero-divisor in $R/(x_1, \ldots, x_{i-1})$ for all $i = 1, \ldots, n$, where $\overline{x_i}$ is the image of x_i in the canonical surjective morphism in $R/(x_1, \ldots, x_{i-1})$.

Example 1.2.2. In $K[X_1, \ldots, X_n]$ the sequence $X_1, \ldots, X_n$ is regular.

Definition 1.2.3. We define the *depth* of an ideal I of R by

$$\sup\{n \in \mathbb{N} : (x_1, \ldots, x_n) \text{ is a regular sequence in } I\}.$$

We denote it by $\mathrm{depth}(I)$.

Definition 1.2.4. We define the *homological codimension* of R by

$$\sup\{n \in \mathbb{N} : (x_1, \ldots, x_n) \text{ is a regular sequence in } R\}.$$

We denote it by $\mathrm{codh}(R)$.

If (R, m) is a local ring, then $\mathrm{depth}(\mathfrak{m})$ is the maximum length of a regular sequence in R and $\mathrm{codh}(R) = \mathrm{depth}(\mathfrak{m})$. In general $\mathrm{depth}(\mathfrak{m}) \leqslant \mathrm{ht}(\mathfrak{m})$, so $\mathrm{codh}(R) \leqslant \dim(R)$.

Definition 1.2.5. Let R be a local ring. If $\mathrm{codh}(R) = \dim(R)$ then we say that R is a *Cohen-Macaulay* ring.

Definition 1.2.6. Let R be a ring. We say that R is *Cohen-Macaulay* if $R_\mathfrak{m}$ is a local Cohen-Macaulay ring for all maximal ideals $\mathfrak{m}$ of R.

Example 1.2.7. The polynomial ring $K[X_1, \ldots, , X_n]$ or the ring $K[[X_1, \ldots, X_n]]$ of the formal power series, with K a field, are examples of Cohen-Macaulay rings.

In general, it holds that $\dim(R/I) + \mathrm{ht}(I) \leq \dim(R)$ for all ideals I of R.

Proposition 1.2.8. *Let R be a Cohen-Macaulay ring. Then* $\dim(R/I) + \mathrm{ht}(I) = \dim(R)$ *for all ideals I of R.*

Now we introduce the notion of injective dimension for R-modules and later the class of Gorenstein rings.

Definition 1.2.9. An R-module J is called *injective* if for every R-homomorphism $g : N \longrightarrow J$ and any injective R-homomorphism $f : N \longrightarrow M$ there exists an R-homomorphism $h : M \longrightarrow J$ such that $h \circ f = g$.

$$\begin{tikzcd} N \arrow[r, hook, "f"] \arrow[d, "g"] & M \arrow[dl, "h"] \\ J & \end{tikzcd}$$

Definition 1.2.10. Let M be an R-module. An *injective resolution* of M is an exact complex as

$$\mathcal{I} :\ 0 \to M \to I_0 \to I_1 \to I_2 \to \dots$$

where I_j is an injective R-module, for all $j \geq 0$.

It is known that every R-module can be embedded in an injective R-module, so as a consequence every R-module has an injective resolution.
The *injective dimension* of an R-module M, denoted $\mathrm{injdim}(M)$, is the smallest integer n, if there exists, such that there is an injective resolution of M as

$$I :\ 0 \to M \to I_0 \to I_1 \to \cdots \to \cdots \to I_n \to 0$$

If such an integer does not exist, we say that the injective dimension of M is infinite.

Proposition 1.2.11. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M be an R-module with finite injective dimension. Then* $\mathrm{injdim}(M) = \mathrm{depth}(R)$.

We are ready to give the definition of Gorenstein rings.

Definition 1.2.12. A Noetherian local ring R is said to be *Gorenstein* if the injective dimension of R as a R-module is finite.

Definition 1.2.13. A Noetherian ring R is called *Gorenstein* if $R_{\mathfrak{m}}$ is Gorenstein for all maximal ideals $\mathfrak{m}$ of R.

It is known that every Gorenstein ring is Cohen-Macaulay.

For the graded K-algebras we can characterize the Gorenstein property and the regularity using the invariants which appear in the related reduced Hilbert-Poincaré series. For a reference see [44].

Theorem 1.2.14. *Let $R = K[x_1, \dots, x_n]$ be a standard graded polynomial ring and I be a homogeneous ideal of R such that R/I is a Cohen-Macaulay domain. Consider the reduced Hilbert series of R/I, that is*

$$\mathrm{HP}_{R/I}(t) = \frac{\sum_{i=0}^{s} h_i t^i}{(1-t)^{\dim(R/I)}}.$$

Then the following hold:

1. $\mathrm{reg}(R/I) = s$;
2. *R/I is Gorenstein if and only if $h_i = h_{s-i}$, for all $i \in [s] = \{1, \dots, s\}$.*

1.3 Gröbner basis

Let $S = K[x_1, \dots, x_n]$ and $\mathrm{Mon}(S)$ be the set of monomials of S. If $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$ then we set $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \dots x_n^{a_n}$. We recall that a *monomial order* on S is a total order $\leq$ on $\mathrm{Mon}(S)$ such that:

1. $1 \leq u$ for all $u \in \mathrm{Mon}(S)$;
2. if $u < v$ and $w \in \mathrm{Mon}(S)$, then $uw < vw$.

Example 1.3.1. Let $\mathbf{x}^{\mathbf{a}}, \mathbf{x}^{\mathbf{b}} \in \mathrm{Mon}(S)$. We provide the following well-known examples of monomial orders on S induced by $x_1 > \dots > x_n$.

1. *Pure lexicographic order* $<_{\mathrm{lex}}$: $\mathbf{x}^{\mathbf{a}} <_{\mathrm{lex}} \mathbf{x}^{\mathbf{b}}$, if the left-most non-zero component of $\mathbf{a} - \mathbf{b}$ is negative.
2. *(Graded) Lexicographic order* $<_{\mathrm{grlex}}$: $\mathbf{x}^{\mathbf{a}} <_{\mathrm{grlex}} \mathbf{x}^{\mathbf{b}}$, if either $\sum_{i=1}^n a_i < \sum_{i=1}^n b_i$ or if $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$ and the left-most non-zero component of $\mathbf{a} - \mathbf{b}$ is negative.
3. *(Graded) Reverse lexicographic order* $<_{\mathrm{grevlex}}$: $\mathbf{x}^{\mathbf{a}} <_{\mathrm{grevlex}} \mathbf{x}^{\mathbf{b}}$, if either $\sum_{i=1}^n a_i < \sum_{i=1}^n b_i$ or if $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$ and the right-most non-zero component of $\mathbf{a} - \mathbf{b}$ is positive.

Let $<$ be a monomial order on S and f be a non-null polynomial in S. Then f can be written as $f = \sum_{\mathbf{a} \in \mathbb{N}^n} \alpha_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}$, where only a finite number of $\alpha_{\mathbf{a}} \in K$ is non-null. We call *support* of f the finite set $\mathrm{supp}(f) = \{\mathbf{x}^{\mathbf{a}} : \alpha_{\mathbf{a}} \neq 0\}$; moreover, $\mathrm{supp}(f) = \emptyset$ if and only if $f = 0$. We denote by $\mathrm{in}_<(f)$ the largest monomial in $\mathrm{supp}(f)$ with respect to $<$, and call it the *initial monomial* of f. The coefficient β of $\mathrm{in}_<(f)$ in f is called the *leading coefficient* of f with respect to $<$, and $\beta\,\mathrm{in}_<(f)$ is called the *leading term* of f.
Let I be a non-zero ideal of S. The *initial ideal* of I is the monomial ideal $\mathrm{in}_<(I)$ generated by $\mathrm{in}_<(f)$ for all $f \in I$ with $f \neq 0$. In the case that $I = (0)$, we assume trivially $\mathrm{in}_<(I) = (0)$.

Remark 1.3.2. In general if $I = (g_1, \dots, g_m)$ then it is not true that $\mathrm{in}_<(I) = (\mathrm{in}_<(g_1), \dots, \mathrm{in}_<(g_m))$. In fact, consider the ideal I in $S[x_1, x_2, x_3]$ generated by $f = x_1x_3 - x_2^2$ and $g = x_1x_2x_3 - x_2x_3^2$. With respect to the reverse lexicographic order, denoted for simplicity by $<$, we have $\mathrm{in}_<(f) = x_2^2$ and $\mathrm{in}_<(g) = x_1x_2x_3$. Moreover observe that $h = (-x_1x_3)f + (-x_2)g = x_2^2x_3^2 - x_1^2x_3^2$, so $\mathrm{in}_<(h) = x_1^2x_3^2 \in \mathrm{in}_<(I)$ but $\mathrm{in}_<(h) \notin (x_2^2, x_1x_2x_3)$.
Anyway, $\mathrm{in}_<(I)$ is generated by a finite numbers of initial monomials, since it is a monomial ideal (see [16, Corollary 1.10])

Definition 1.3.3. Let I be an ideal in S and let $<$ be a monomial order on S. We say that elements $g_1, \dots, g_m \in I$ form a *Gröbner basis* of I with respect to the monomial order $<$ if $\mathrm{in}_<(I) = (\mathrm{in}_<(g_1), \dots, \mathrm{in}_<(g_m))$.

Note that a Gröbner basis of I always exists because $\mathrm{in}_<(I)$ a is finitely generated ideal. Moreover, it is not unique because if $\{g_1, \dots, g_m\}$ is a Gröbner basis of I with respect to $<$ then $\{g_1, \dots, g_m, g_1 + g_m\}$ is also.

Theorem 1.3.4. *Let I be an ideal in S and let $<$ be a monomial order on S. Suppose that $g_1, \dots, g_m \in I$ form a Gröbner basis of I with respect to a monomial order $<$. Then $g_1, \dots, g_m$ form a system of generators of I.*

Theorem 1.3.5. *Let $f, g_1, \dots, g_m \in S$ where g_i is a non-zero polynomial in S for all $i \in [m]$. Given a monomial order $<$, there exist $q_1, \dots, q_m, r \in S$ such that*

$$f = q_1 g_1 + \dots + f_m g_m + r$$

and the following conditions are satisfied:

1. *If $r \neq 0$ and if $u \in supp(r)$, then none of the initial monomials $in_<(g_1), \dots, in_<(g_m)$ divides u;*

2. $\mathrm{in}_<(q_i g_i) \leq \mathrm{in}_<(f)$ *for all $i \in [m]$.*

The expression $f = q_1 g_1 + \dots + q_m g_m + r$ satisfying the two conditions mentioned in Theorem 1.3.5 is called a *standard expression* of f, and r is called a *remainder* of f with respect to $g_1, \dots, g_m$. In such a case, we say that *f reduces to r with respect to $g_1, \dots, g_m$.*
Observe that f may have different standard expressions with respect to $g_1, \dots, g_m$. Consider $S = K[x_1, x_2, x_3]$, let $f = x_1^2 x_2 - x_3^3$, $g = x_1 - x_3$, $h = x_2$ and $<$ be the graded reverse-lexicographic order. Then $f = (x_1 x_2 + x_2 x_3) g + (x_3^2) h - x_3^3$ and $f = 0g + (x_1^2) h - x_3^3$ are two different standard expressions of f.

Proposition 1.3.6 (Buchberger's criterion)**.** *Let I be an ideal in S and let $<$ be a monomial order on S. Suppose that $g_1, \dots, g_m \in I$ form a Gröbner basis of I with respect to the monomial order $<$. Then each polynomial $f \in S$ has a unique remainder with respect to $g_1, \dots, g_m$. As a consequence, a polynomial f belongs to I if and only if f reduces to 0 with respect to $g_1, \dots, g_m$.*

Buchberger's criterion is one of the most important tool in Gröbner basis theory. It allows to check whether a generating set of an ideal is a Gröbner basis. To explain this criterion we need to introduce the so-called S-polynomials.
Let $<$ be a monomial order on S. Consider f and g two polynomials in S. The S-polynomial of f and g with respect to $<$ is defined by

$$S(f,g) = \frac{\mathrm{lcm}(\mathrm{in}_<(f), \mathrm{in}_<(g))}{\beta\, \mathrm{in}_<(f)} f - \frac{\mathrm{lcm}(\mathrm{in}_<(f), \mathrm{in}_<(g))}{\gamma\, \mathrm{in}_<(g)} g$$

where β and γ are respectively the leading coefficients of f and g.

Theorem 1.3.7. *Let $I = (g_1, \dots, g_m)$ be a non-null ideal in S and let $<$ be a monomial order on S. The following conditions are equivalent:*

1. *$g_1, \dots, g_m$ is a Gröbner basis of I with respect to $<$;*

2. *$S(g_i, g_j)$ reduces to 0 with respect to $g_1, \dots, g_m$, for all $i < j$.*

Proposition 1.3.8. *Let $<$ be a monomial order on S and let $f, g \in S$ such that* $\mathrm{in}_<(f)$ *and* $\mathrm{in}_<(g)$ *are relatively prime, that is* $\gcd(\mathrm{in}_<(f), \mathrm{in}_<(g)) = 1$. *Then $S(f,g)$ reduces to 0 with respect to f, g.*

The Buchberger's criterion allows to build a Gröbner basis of an ideal I starting from any system of generators. In fact, you have to follow the following steps of the so-called *Buchberger's algorithm*. Let I be an ideal of S and $\mathcal{G}$ be any system of generators of I.

1. *Step 1:* For each pair (g_i, g_j) of distinct generators in $\mathcal{G}$ we compute the S-polynomial and a remainder of it, called $r_{i,j}$.

2. *Step 2:* If all S-polynomials reduce to 0, then the algorithm ends and $\mathcal{G}$ is a Gröbner basis of I, otherwise we add one of the non-zero remainders r_{ij} to our system of generators, setting $\mathcal{G} = \mathcal{G} \cup \{r_{ij}\}$, and we go back to Step 1.

This algorithm ends obviously after a finite number of steps, since every strictly ascending sequence of ideals is finite. Indeed, when we add a non-zero remainder of an S-polynomial to $\mathcal{G}$, the monomial ideal $(\mathrm{in}_<(g) : g \in \mathcal{G})$ becomes strictly larger, but the ascending chain defined by these monomial ideals is finite.

Definition 1.3.9. Let I be an ideal in S and let $<$ be a monomial order on S. Suppose that $\mathcal{G} = \{g_1, \ldots, g_m\}$ is a Gröbner basis of I with respect to the monomial order $<$. We say that $\mathcal{G}$ is *reduced* if:

1. the leading coefficient of g_i is 1, for all $i \in [m]$;

2. no monomials in $\mathrm{supp}(g_j)$ is divisible by $\mathrm{in}_<(g_i)$, for all $i \neq j$.

Theorem 1.3.10. *Each ideal $I \subset S$ has a unique reduced Gröbner basis.*

Proposition 1.3.11. *Let I be a non-zero ideal of S and $<$ be a monomial order on S.*

1. *The set of monomials not in* $\mathrm{in}_<(I)$ *forms a K-basis of S/I.*

2. *If I is a graded ideal, then* $\dim_K I_j = \dim_K(\mathrm{in}_< I)_j$, *for all j. As a consequence, S/I and $S/\mathrm{in}_<(I)$ have the same Hilbert function and hence* $\dim S/I = \dim S/\mathrm{in}_<(I)$.

Example 1.3.12. In $S = K[x_1, x_2, x_3]$ consider the ideal I generated by $g_1 = x_1x_2 - x_3$ and $g_2 = x_1x_3 - x_2$. Let $<$ be the graded reverse lexicographic order on S. Then $\mathrm{in}_<(g_1) = x_1x_2$ and $\mathrm{in}_<(g_2) = x_1x_3$. We compute the reduce Gröbner basis of I with respect to $<$.

1. First of all, we compute the S-polynomial of g_1 and g_1:

$$S(g_1, g_2) = \frac{x_1x_2x_3}{x_1x_2}(x_1x_2 - x_3) - \frac{x_1x_2x_3}{x_1x_3}(x_1x_3 - x_2) = x_2^2 - x_3^2.$$

   Observe that $x_2^2 - x_3^2$ does not reduce to 0 modulo $\{g_1, g_2\}$, so we set $g_3 = x_2^2 - x_3^2$ and we add it to the set of generators of I.

2. We need now the S-polynomials of g_1, g_3 and g_2, g_3.

$$S(g_1, g_3) = -x_1x_3^2 + x_2x_3 = -x_3g_1$$

$$S(g_2, g_3) = -x_1x_3^3 + x_2^3 = -x_3^2g_2 - x_2g_3$$

Since $S(g_1, g_3)$ and $S(g_2, g_3)$ reduce to 0 with respect to $\{g_1, g_2, g_3\}$, then $\{g_1, g_2, g_3\}$ is a Gröbner basis of I with respect to $<$. Observe that it is also reduced.

1.4 Normal domains, semigroup rings and toric ideals

We start this section introducing the notion of integral closure of R and consequently the class of normal domains.
An R-algebra B is a ring B with a fixed ring homomorphism $\phi : R \longrightarrow B$. In fact, in B we can define an R-module structure with the operation defined by the multiplication of $\phi(a)$ and b in B, for all $a \in R$ and $b \in B$. A particular case is when $R \subseteq B$ or, in other words, when B is an *extension ring* of R.

Definition 1.4.1. Let B be an extension of R and let $y \in B$. We say that y is *integral over* R if there exists a monic polynomial with coefficients in R

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

such that $f(y) = 0$.

The set of elements of B that are integral over R is called the *integral closure of* R *in* B and it is denoted by $\overline{R}^B$.

Definition 1.4.2. We say that B is *integral over* R if $B = \overline{R}^B$.

Definition 1.4.3. We say that R is *integrally closed in* B if $R = \overline{R}^B$.

Definition 1.4.4. Let R be a domain and $Q(R)$ be its field of fractions. We say that R is a *normal domain* if R is integrally closed in $Q(R)$.

It can be proved that if R is a unique factorization domain (UFD) and $Q(R)$ is its field of fractions, then the integral elements of $Q(R)$ over R are precisely the elements of R, that is any UFD is a normal domain.

Example 1.4.5. $\mathbb{Z}$ and $\mathbb{Z}[X_1, \ldots, X_n]$ are UFDs, so they are examples of normal domains.

Following [3] we provide now the definition of an affine semigroup and of a toric ideal of a semigroup ring with some interesting well known results.
Let $\{\mathbf{h}_1, \ldots, \mathbf{h}_q\}$ be a subset of $\mathbb{Z}^n$ and $\mathcal{H}$ be the sub-monoid of the additive group $\mathbb{Z}^n$ generated by $\mathbf{h}_1, \ldots, \mathbf{h}_q$, so

$$\mathcal{H} = \{a_1\mathbf{h}_1 + \cdots + a_q\mathbf{h}_q : a_i \in \mathbb{N},\ \forall i \in [q]\}.$$

$\mathcal{H}$ is called the *affine semigroup* generated by $\{\mathbf{h}_1, \ldots, \mathbf{h}_q\}$.
Let $\mathcal{H} \subset \mathbb{Z}^n$ be an affine semigroup generated by $\{\mathbf{h}_1, \ldots, \mathbf{h}_q\}$ and $K[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$ be the *Laurent polynomial ring*. We denote by $K[\mathcal{H}]$ the subring of the Laurent polynomial ring generated over K by the monomials $\mathbf{x}^{\mathbf{h}_i}$, for $i \in [q]$. Observe that $\mathbf{x}^{\mathbf{a}} \in K[\mathcal{H}]$ if and only if $\mathbf{a} \in \mathcal{H}$. The K-algebra $K[\mathcal{H}]$ is called the *semigroup ring* of $\mathcal{H}$. For instance if $\mathcal{H} = \mathbb{N}(1,2) + \mathbb{N}(2,1)$, then $K[\mathcal{H}] = K[x_1x_2^2, x_1^2x_2]$.
Let $R = K[t_1, \ldots, t_q]$ be the polynomial ring in the indeterminates $t_1, \ldots, t_q$ and define the surjective K-algebra homomorphism $\phi : R \to K[\mathcal{H}]$ by $\phi(t_i) = \mathbf{x}^{\mathbf{h}_i}$, for $i \in [q]$. Its kernel $P_{\mathcal{H}}$, called the *toric ideal* of $\mathcal{H}$, is a graded ideal of R and it is also prime since $K[\mathcal{H}] \cong R/P_{\mathcal{H}}$ and $K[\mathcal{H}]$ is a domain.

Definition 1.4.6. We recall that a binomial f in S is given by the difference of two distinct monomials in S. An ideal $I \subset S$ is said to be a *binomial ideal* if it is generated by binomials.

Let $\pi : \mathbb{N}^q \to \mathcal{H}$ be the homomorphism of semigroups defined by $\pi(\mathbf{u}) = \sum_{i=1}^{q} u_i\mathbf{h}_i$ where $\mathbf{u} = (u_1, \ldots, u_q) \in \mathbb{N}^q$. Observe easily that $\phi(\mathbf{t}^{\mathbf{u}}) = \mathbf{x}^{\pi(\mathbf{u})}$ for $\mathbf{t}^{\mathbf{u}} \in R$.

Proposition 1.4.7. *Let $\mathcal{H}$ be an affine semigroup generated by $\mathbf{h}_1, \ldots, \mathbf{h}_p$ and $P_{\mathcal{H}}$ be its toric ideal. Then the set of binomials*

$$\{\mathbf{t}^{\mathbf{u}} - \mathbf{t}^{\mathbf{v}} : \mathbf{u}, \mathbf{v} \in \mathbb{N}^q, \pi(\mathbf{u}) = \pi(\mathbf{v})\}$$

generates $P_{\mathcal{H}}$ as a K-vector space. In particular, $P_{\mathcal{H}}$ is a binomial ideal.

We give some interesting results about the reduced Gröbner basis of binomial ideals.

Proposition 1.4.8. *Let I be a binomial ideal of S and $<$ be a monomial order on S. Then the reduced Gröbner basis of I with respect to $<$ consists of binomials.*

Theorem 1.4.9. *Let I be a binomial ideal of S. The following are equivalent:*

1. *I is a toric ideal of some semigroup ring;*

2. *I is a prime ideal.*

Definition 1.4.10. A binomial $\mathbf{t^u} - \mathbf{t^v} \in P_{\mathcal{H}}$ is called *primitive* if there is no other binomial $\mathbf{t^r} - \mathbf{t^s} \in P_{\mathcal{H}}$ such that $\mathbf{t^r}$ divides $\mathbf{t^u}$ and $\mathbf{t^s}$ divides $\mathbf{t^v}$.

Proposition 1.4.11. *Let I be a prime binomial ideal of S and $<$ be a monomial order on S. Then the reduced Gröbner basis of I with respect to $<$ consists of primitive binomials.*

Now, we define the normal affine semigroup and we state the nice relation with normal rings established by Hochster.

Definition 1.4.12. An affine semigroup $\mathcal{H} \subset \mathbb{Z}^q$ is said to be *normal* if $m\mathbf{g} \in \mathcal{H}$, for some $\mathbf{g} \in \mathbb{Z}^q$ and $m > 0$, then $\mathbf{g} \in \mathcal{H}$.

Theorem 1.4.13. *Let $\mathcal{H}$ be an affine semigroup and $K[\mathcal{H}]$ be its semigroup ring. The following are equivalent:*

1. *$\mathcal{H}$ is a normal semigroup;*

2. *$K[\mathcal{H}]$ is a normal ring.*

Theorem 1.4.14. *[3, Theorem 6.3.5] Let $\mathcal{H}$ be a normal semigroup. Then its semigroup ring $K[\mathcal{H}]$ is Cohen-Macaulay.*

Theorem 1.4.15. *[46] Let $\mathcal{H}$ be an affine semigroup and $<$ be a monomial order on R such that $\text{in}_<(P_{\mathcal{H}})$ is a squarefree monomial ideal. Then $K[\mathcal{H}]$ is a normal ring.*

It follows that if $\mathcal{H}$ is an affine semigroup and there exists a monomial order $<$ on R such that $\text{in}_<(P_{\mathcal{H}})$ is a squarefree monomial ideal, then $K[\mathcal{H}]$ is a normal Cohen-Macaulay ring.

We conclude this section recalling some basics on the toric ideals attached to edge rings. First of all we recall some useful notions of graph theory.
A *graph* G is an ordered pair of finite sets as (V, E) such that E is a subset of the set of unordered pairs of V. V is called the set of *vertices* of G and E is the set of *edges* of G. A graph not containing any *loop*, which is an unordered pair as $\{v, v\}$, is said to be *simple*. A *walk* of length t in G is a sequence of edges of G, as

$$\Gamma = \{\{v_{i_1}, v_{i_2}\}, \{v_{i_2}, v_{i_3}\}, \dots, \{v_{i_{t-2}}, v_{i_{t-1}}\}, \{v_{i_{t-1}}, v_{i_t}\}\}.$$

If $v_{i_1} = v_{i_t}$, then the walk Γ is called a *closed walk*. A *path* is a walk with all its vertices distinct. We say that G is *connected* if for every pair of vertices there exists a path connecting them.
Let G be a connected and simple finite graph with vertices $x_1, \dots, x_n$ and $E(G) = \{t_1, \dots, t_d\}$. Let $R = K[x_1, \dots, x_n]$ and $S = K[t_1, \dots, t_d]$ be the polynomial rings over a field K associated to $V(G)$ and $E(G)$. The *edge ring* $K[G]$ of G is the K-subalgebra

$K[\{x_i x_j : \{i,j\} \in E(G)\}]$ of S. Define the following surjective K-algebra homomorphism

$$\psi : S \longrightarrow K[G]$$
$$\psi(t_k) = x_i x_j \qquad \text{where } t_k = \{i,j\} \in E(G)$$

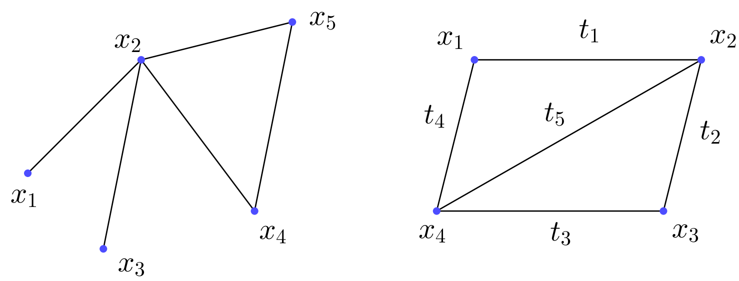
The kernel of ψ is the *toric ideal* of $K[G]$ and it is denoted by I_G.
Consider a closed walk as $\Gamma = \{t_{h_1}, t_{h_2}, \dots, t_{h_{2q}}\}$ where $t_{h_z} \in E(G)$ for all $z \in [2q]$. Hence we set

$$f_\Gamma = \prod_{k=1}^{q} t_{i_{2k-1}} - \prod_{k=1}^{q} t_{i_{2k}}$$

which belongs to I_G.

Theorem 1.4.16. *Let G be a graph. The toric ideal I_G is generated by f_Γ, where Γ is an even closed walk of G.*

Example 1.4.17. Consider the graph G_1 in Figure 1.1 (A). The edge ring of G_1 is $K[x_1x_2, x_2x_3, x_2x_4, x_2x_5, x_5x_4]$. The toric ideal of G_1 is the null ideal, since there does not exist any even closed walk in G_1.



(A) G_1 (B) G_2

FIGURE 1.1

On the other hand, consider now the graph G_2 displayed in Figure 1.1 (B). The edge ring of G_2 is $K[x_1x_2, x_2x_3, x_3x_4, x_4x_1, x_2x_4]$. The unique even closed walk is $\Gamma = (t_1, t_2, t_3, t_4)$, so the toric ideal of G_2 is the ideal in $K[t_1, t_2, t_3, t_4, t_5]$ generated by $t_1t_3 - t_2t_4$.

A *cycle* of length t is a closed walk of the form

$$C = \{\{v_{i_1}, v_{i_2}\}, \{v_{i_2}, v_{i_3}\}, \dots, \{v_{i_{t-2}}, v_{i_{t-1}}\}, \{v_{i_{t-1}}, v_{i_t}\}\},$$

with $v_{i_k} \neq v_{i_l}$ for all $1 \leq k < l \leq t-1$. A chord of a cycle C is an edge of the form $e = \{v_{i_k}, v_{i_l}\}$, where $1 \leq k < l \leq t-1$, with $e \notin E(C)$. A minimal cycle is a cycle with no chord. A graph is called *weakly chordal* if every cycle of length greater than 4 has a chord.
We conclude providing the following result which will be useful in Sub-section 3.2.

Theorem 1.4.18 ([34], [36]). *Let G be a bipartite graph. If G is weakly chordal then the associated toric ideal I_G is minimally generated by quadratic binomials attached to the cycles of G of length 4.*

Chapter 2

Binomial ideals of collections of cells and their primality

In [37] Ayesha Asloob Qureshi establishes a connection between collections of cells and Commutative Algebra, attaching to a collection of cells $\mathcal{P}$ the ideal generated by all inner 2-minors of $\mathcal{P}$. In this chapter we define the ideals generated by inner 2-minors associated to a collection of cells in $\mathbb{Z}^2$ and we study their primality.
Firstly, in the section 2.1 we introduce some basics on collections of cells and polyominoes and later we define the K-algebras associated to them following [37]. In section 2.2 we introduce a new kind of non-simple polyominoes, called *closed paths*, and we discuss their primality. More precisely, we characterize the primality of the polyomino ideal of a closed path by the non-existence of *zig-zag walks* of $\mathcal{P}$.

2.1 Collections of cells, polyominoes and inner minors

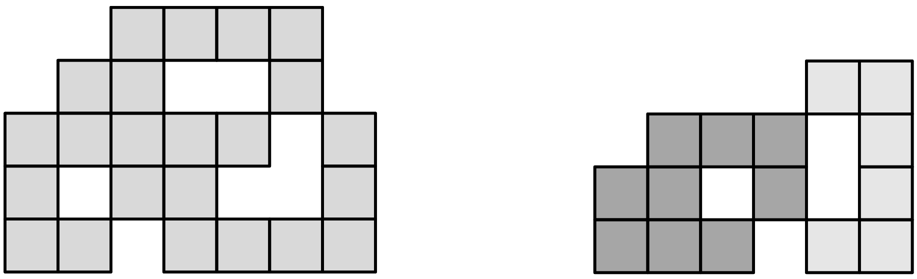
In this section we introduce collections of cells and polyominoes to which binomial ideals will be attached. For this purpose we need to introduce some concepts and notations.
Consider the natural partial order on $\mathbb{R}^2$: given $(i,j),(k,l) \in \mathbb{R}^2$, we say $(i,j) \leq (k,l)$ if $i \leq k$ and $j \leq l$. Let $a = (i,j), b = (k,l) \in \mathbb{Z}^2$. The set $[a,b] = \{(r,s) \in \mathbb{Z}^2 : i \leq r \leq k,\ j \leq s \leq l\}$ is called an *interval* of $\mathbb{Z}^2$. We define the *closure* of $[a,b]$ the set $\overline{[a,b]} = \{x \in \mathbb{R}^2 : a \leq x \leq b\}$. If $i < k$ and $j < l$, we say that $[a,b]$ is a *proper* interval. The elements a,b are called the *diagonal corners* and $c = (i,l)$, $d = (k,j)$ the *anti-diagonal corners* of $[a,b]$. If $j = l$ (or $i = k$) we say that a and b are in *horizontal* (or *vertical*) *position*. An elementary interval of the form $C = [a, a+(1,1)]$ is a *cell* with *lower left corner* a. The elements a, $a+(0,1)$, $a+(1,0)$ and $a+(1,1)$ are called the *vertices* or *corners* of C and the sets $\{a, a+(1,0)\}$, $\{a+(1,0), a+(1,1)\}$, $\{a+(0,1), a+(1,1)\}$ and $\{a, a+(0,1)\}$ are called the *edges* of C. We denote the set of the vertices and the edges of C respectively by $V(C)$ and $E(C)$.
Let C and D be two distinct cells of $\mathbb{Z}^2$. A *walk* from C to D is a sequence $\mathcal{C} : C = C_1, \dots, C_m = D$ of cells of $\mathbb{Z}^2$ such that $C_i \cap C_{i+1}$ is an edge of C_i and C_{i+1} for $i = 1, \dots, m-1$. If in addition $C_i \neq C_j$ for all $i \neq j$, then $\mathcal{C}$ is called a *path* from C to D. If $\mathcal{C}_1 : A_1, \dots, A_m$ and $\mathcal{C}_2 : B_1, \dots, B_n$ are two walks such that $A_m = B_1$, then the union of $\mathcal{C}_1$ and $\mathcal{C}_2$ is defined by the walk $A_1, \dots, A_{m-1}, A_m, B_2, \dots, B_n$ and it is denoted by $\mathcal{C}_1 \cup \mathcal{C}_2$.

Remark 2.1.1. In general, a walk $\mathcal{C} : C = C_1, \dots, C_m = D$ from C to D contains a path between C and D, that is there exist a path $\mathcal{F}$ from C to D such that every cell of $\mathcal{F}$ is a cell of $\mathcal{C}$. It can be proved by induction on the number m of cells of $\mathcal{C}$. If $m = 2$, then $\mathcal{C} : C = C_1, C_2 = D$ is obviously a path. Let $m > 2$ and suppose that any walk from C to D consisting of k cells, with $k < m$, contains a path between these cells. Suppose that not all the cells of $\mathcal{C}$ are distinct, so there exist $i, j \in \{1, \dots, m\}$ such that $C_i = C_j$, with $j > i$. Consider the sequence $\mathcal{C}' : C = C_1, \dots, C_{i-1}, C_j, \dots, C_m = D$, consisting of all the cells in $\mathcal{C}$ except $C_i, C_{i+1}, \dots, C_{j-1}$. $\mathcal{C}'$ is a walk from C to D having less than m cells, so applying the inductive hypothesis on $\mathcal{C}'$ we have a path from C to D contained in $\mathcal{C}'$, that is contained also in $\mathcal{C}$.

Let $\mathcal{P}$ be a non-empty collection of cells in $\mathbb{Z}^2$. We denote the set of the vertices of $\mathcal{P}$ by $V(\mathcal{P}) = \bigcup_{C \in \mathcal{P}} V(C)$ and the set of the edges of $\mathcal{P}$ by $E(\mathcal{P}) = \bigcup_{C \in \mathcal{P}} E(C)$. Let C and D be two cells of $\mathcal{P}$. We say that C and D are *connected* if there exists a walk $\mathcal{C} : C = C_1, \dots, C_m = D$ such that $C_i \in \mathcal{P}$ for all $i = 1, \dots, m$. We denote by (a_i, b_i) the lower left corner of C_i for all $i = 1, \dots, m$ and we observe that a walk can change direction in one of the following ways:

1. North, if $(a_{i+1} - a_i, b_{i+1} - b_i) = (0, 1)$ for some $i = 1, \dots, m-1$;
2. South, if $(a_{i+1} - a_i, b_{i+1} - b_i) = (0, -1)$ for some $i = 1, \dots, m-1$;
3. East, if $(a_{i+1} - a_i, b_{i+1} - b_i) = (1, 0)$ for some $i = 1, \dots, m-1$;
4. West, if $(a_{i+1} - a_i, b_{i+1} - b_i) = (-1, 0)$ for some $i = 1, \dots, m-1$.

Let $\mathcal{P}$ be a non-empty, finite collection of cells in $\mathbb{Z}^2$. We say that $\mathcal{P}$ is a *polyomino* if any two cells of $\mathcal{P}$ are connected. For instance, see Figure 2.1 (A). We say that $\mathcal{P}$ is *weakly connected* if for any two cells C and D in $\mathcal{P}$ there exists a sequence of cells $\mathcal{C} : C = C_1, \dots, C_m = D$ of $\mathcal{P}$ such that $V(C_i) \cap V(C_{i+1}) \neq \emptyset$ for all $i = 1, \dots, m-1$. Let $\mathcal{P}'$ be a subset of cells of $\mathcal{P}$. $\mathcal{P}'$ is called a *connected component* of $\mathcal{P}$ if $\mathcal{P}'$ is a polyomino and it is maximal with respect to the set inclusion, that is if $A \in \mathcal{P} \setminus \mathcal{P}'$ then $\mathcal{P}' \cup \{A\}$ is not a polyomino. For instance, see Figure 2.1 (B).



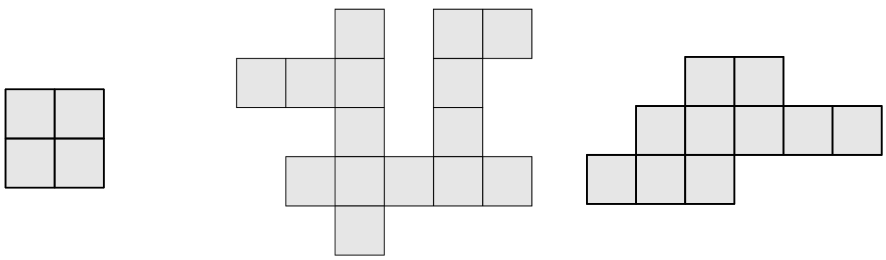
(A) (B)

FIGURE 2.1: A polyomino and a weakly connected collection of cells.

We say that $\mathcal{P}$ is *simple* if for any two cells C and D of $\mathbb{Z}^2$, not in $\mathcal{P}$, there exists a path $\mathcal{C} : C = C_1, \dots, C_m = D$ such that $C_i \notin \mathcal{P}$ for all $i = 1, \dots, m$. For example, the polyomino and the weakly connected collection of cells in Figure 2.1 are not simple. A finite collection of cells $\mathcal{H}$ not in $\mathcal{P}$ is a *hole* of $\mathcal{P}$ if any two cells F and G of $\mathcal{H}$ are connected by a path $\mathcal{F} : F = F_1, \dots, F_t = G$ such that $F_j \in \mathcal{H}$ for all $j = 1, \dots, t$ and $\mathcal{H}$ is maximal with respect to set inclusion. Observe that each hole of a collection $\mathcal{P}$ of cells is a simple polyomino and $\mathcal{P}$ is simple if and only if it has not any hole. Moreover, it is easy to see that a non-simple polyomino has a finite number of holes. We say that a cell E of $\mathbb{Z}^2$ is *external* to $\mathcal{P}$ if it satisfies one of the two following conditions: $E \notin \mathcal{P} \cup \mathcal{H}_1 \cup \dots \cup \mathcal{H}_n$ if $\mathcal{P}$ is a non-simple polyomino and $\mathcal{H}_1, \dots, \mathcal{H}_n$ are the holes of $\mathcal{P}$, or $E \notin \mathcal{P}$ if $\mathcal{P}$ is a simple polyomino. The set of the cells of $\mathbb{Z}^2$ external to $\mathcal{P}$ is called the *exterior* of $\mathcal{P}$. If $\mathcal{U}$ is the exterior of $\mathcal{P}$, then we observe that any two cells of $\mathcal{U}$ are connected in $\mathcal{U}$. We say that an edge of a cell of $\mathcal{P}$ is a *border edge* if it is not an edge of any other cell of $\mathcal{P}$. A *horizontal border edge of* $\mathcal{P}$ is defined to be a horizontal edge interval of $\mathcal{P}$ consisting of border edges of cells of $\mathcal{P}$. Similarly we define the *vertical border edge of* $\mathcal{P}$. The union of the closures of the border edges of $\mathcal{P}$ is called *perimeter* of $\mathcal{P}$.

Let A and B be two cells of $\mathbb{Z}^2$ and let $a = (i, j)$ and $b = (k, l)$ be the lower left corners of A and B, with $a \leq b$. The *cell interval*, denoted by $[A, B]$, is the set of the cells of $\mathbb{Z}^2$ with lower left corner (r, s) such that $i \leqslant r \leqslant k$ and $j \leqslant s \leqslant l$. If (i, j) and (k, l) are in horizontal position, we say that the cells A and B are in horizontal position. Similarly, we define two cells in vertical position. Let A and B be two cells of $\mathcal{P}$ in vertical or horizontal position. The cell interval $[A, B]$ is called a *block of* $\mathcal{P}$ *of length* n if any cell C of $[A, B]$ belongs to $\mathcal{P}$ and $|[A, B]| = n$. The cells A and B are called *extremal* cells of $[A, B]$. The block $[A, B]$ is *maximal* if there does not exist any block $[A', B']$ of $\mathcal{P}$ such that $[A, B] \subset [A', B']$. Moreover if A and B are in vertical (resp. horizontal) position, then $[A, B]$ is also called a *maximal vertical (resp. horizontal) block* of $\mathcal{P}$. An interval $[a, b]$ with $a = (i, j)$, $b = (k, j)$ and $i < k$ is called a *horizontal edge interval* of $\mathcal{P}$ if the sets $\{(\ell, j), (\ell + 1, j)\}$ are edges of cells of $\mathcal{P}$ for all $\ell = i, \dots, k - 1$. In addition, if $\{(i - 1, j), (i, j)\}$ and $\{(k, j), (k + 1, j)\}$ do not belong to $E(\mathcal{P})$, then $[a, b]$ is called a *maximal* horizontal edge interval of $\mathcal{P}$. We define similarly a *vertical edge interval* and a *maximal* vertical edge interval. Observe that a lattice interval of $\mathbb{Z}^2$ identifies a cell interval of $\mathbb{Z}^2$ and vice versa, so if I is an interval of $\mathbb{Z}^2$ we denote by $\mathcal{P}(I)$ the cell interval associated to I. A proper interval $[a, b]$ is called an *inner interval* of $\mathcal{P}$ if all cells of $[a, b]$ belong to $\mathcal{P}$.

A polyomino $\mathcal{P}$ is *thin* if it does not contain the square tetromino in Figure 2.2 (A). For instance in Figure 2.2 (B), you can see a thin polyomino. A polyomino $\mathcal{P}$ is called *row* (resp. *column*) *convex* if for any two cells A and B of $\mathcal{P}$ in horizontal position (resp. vertical position) then cell interval $[A, B]$ is contained in $\mathcal{P}$. If $\mathcal{P}$ is row and column convex, then $\mathcal{P}$ is called a *convex polyomino* (see Figure 2.2 (C)). A convex polyomino $\mathcal{P}$ is said to be *L-convex* if any two cells of $\mathcal{P}$ can be connected by a path having at most a change of direction.



(A) (B) (C)

FIGURE 2.2

Let $\mathcal{P}$ be a non-empty finite collection of cells in $\mathbb{Z}^2$. We always assume that the smallest interval of $\mathbb{Z}^2$ containing $V(\mathcal{P})$ is $[(1, 1), (m, n)]$. Let K be a field and $S_{\mathcal{P}} = K[x_v \mid v \in V(\mathcal{P})]$. Consider a proper interval $[a, b]$ of $\mathbb{Z}^2$, with a,b diagonal corners and c,d anti-diagonal ones. We attach the binomial $x_a x_b - x_c x_d$ to $[a, b]$ and if $[a, b]$ is an inner interval then the binomial $x_a x_b - x_c x_d$ is called an *inner 2-minor* of $\mathcal{P}$. We denote by $I_{\mathcal{P}} \subset S_{\mathcal{P}}$ the ideal in $S_{\mathcal{P}}$ generated by all the inner 2-minors of $\mathcal{P}$. We set also $K[\mathcal{P}] = S_{\mathcal{P}}/I_{\mathcal{P}}$, that is the coordinate ring of $\mathcal{P}$. If $\mathcal{P}$ is a polyomino, the ideal $I_{\mathcal{P}}$ is called the *polyomino ideal of* $\mathcal{P}$.

Example 2.1.2. Consider the polyomino $\mathcal{P}$ in Figure 2.3.

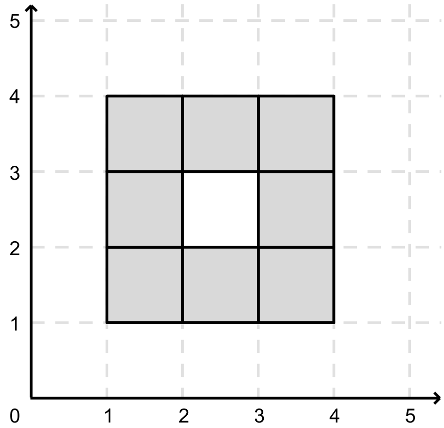


FIGURE 2.3: A polyomino $\mathcal{P}$.

Let K be a field. The polyomino ring $S_{\mathcal{P}}$ attached to $\mathcal{P}$ is given by

$$K[x_{11}, x_{21}, x_{31}, x_{41}, x_{12}, x_{22}, x_{32}, x_{42}, x_{13}, x_{23}, x_{33}, x_{43}, x_{14}, x_{24}, x_{34}, x_{44}]$$

and the polyomino ideal $I_{\mathcal{P}}$ of $\mathcal{P}$ is generated by the following twenty binomials:

$$\begin{gathered}
x_{34}x_{13} - x_{33}x_{14},\ x_{32}x_{21} - x_{31}x_{22},\ x_{42}x_{11} - x_{41}x_{12},\ x_{24}x_{12} - x_{22}x_{14},\\
x_{44}x_{32} - x_{42}x_{34}, x_{24}x_{13} - x_{23}x_{14},\ x_{44}x_{33} - x_{43}x_{34},\ x_{32}x_{11} - x_{31}x_{12},\\
x_{44}x_{23} - x_{43}x_{24},\ x_{22}x_{11} - x_{21}x_{12},\ x_{42}x_{31} - x_{41}x_{32},\ x_{23}x_{11} - x_{21}x_{13},\\
x_{43}x_{31} - x_{41}x_{33},\ x_{44}x_{13} - x_{43}x_{14},\ x_{34}x_{23} - x_{33}x_{24},\ x_{42}x_{21} - x_{41}x_{22},\\
x_{23}x_{12} - x_{22}x_{13},\ x_{24}x_{11} - x_{21}x_{14},\ x_{44}x_{31} - x_{41}x_{34},\ x_{43}x_{32} - x_{42}x_{33}.
\end{gathered}$$

Just as an example, note that the binomial $x_{11}x_{33} - x_{13}x_{31}$ is not a generator of $I_{\mathcal{P}}$ since $[(1,1),(3,3)]$ is not an inner interval of $\mathcal{P}$.

2.2 On the primality of closed path polyominoes

One of the most exciting challenges is to characterize the primality of $I_{\mathcal{P}}$ depending upon the shape of $\mathcal{P}$. Although the question seems very simple, it is very difficult to give a complete characterization of prime polyomino ideals. The primality of simple polyominoes is proved in [24], showing that the class of simple polyominoes coincides with that one of balanced polyominoes and using the primality of the polyomino ideal of a balanced one (see [25]). Independently of this, the same result is shown in [39], by identifying the coordinate ring of a simple polyomino with the edge ring of a bipartite and weakly chordal graph. Nowadays, the study of the primality is applied to multiply connected polyominoes, which are polyominoes with one or more holes. In [28] and [41], the authors discuss a family of prime polyominoes, obtained by removing a convex polyomino from a rectangle, which generalizes the class of "rectangle minus rectangle", introduced in [42]. In [32] the primality of polyomino ideals is studied using the Gröbner basis and the lattice ideals. In [31] the authors study the primality of *grid* polyominoes and, in particular, they introduce a particular sequence of inner intervals of $\mathcal{P}$, called a zig-zag walk, and they prove that $\mathcal{P}$ does not contain zig-zag walks if $I_{\mathcal{P}}$ is prime. It seems that the non-existence of zig-zag walks in a polyomino could characterize its primality [Conjecture 4.6, [31]].

Definition 2.2.1. A *zig-zag walk* of $\mathcal{P}$ is a sequence $\mathcal{W} : I_1, \dots, I_\ell$ of distinct inner intervals of $\mathcal{P}$ where, for all $i = 1, \dots, \ell$, the interval I_i has either diagonal corners v_i, z_i and anti-diagonal corners u_i, v_{i+1} or anti-diagonal corners v_i, z_i and diagonal corners u_i, v_{i+1}, such that:

1. $I_1 \cap I_\ell = \{v_1 = v_{\ell+1}\}$ and $I_i \cap I_{i+1} = \{v_{i+1}\}$, for all $i = 1, \dots, \ell - 1$;
2. v_i and v_{i+1} are on the same edge interval of $\mathcal{P}$, for all $i = 1, \dots, \ell$;
3. for all $i, j \in \{1, \dots, \ell\}$ with $i \neq j$, there exists no inner interval J of $\mathcal{P}$ such that z_i, z_j belong to J.

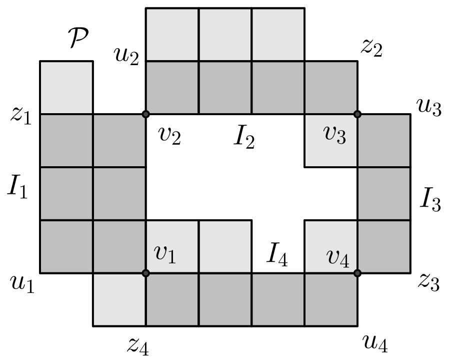


FIGURE 2.4: An example of a zig-zag walk of $\mathcal{P}$.

Inspired by the conjecture that characterizes the primality of a polyomino ideal by non-existence of zig-zag walks, we introduce a new class of non-simple polyominoes, called *closed paths*, and we characterize the primality of their associated ideal by zig-zag walks in [4].

2.2.1 Closed path polyominoes and zig-zag walks

Let us start with the definition of closed path polyomino and with some important geometric results about these kind of non-simple polyominoes.

Definition 2.2.2. A polyomino $\mathcal{P}$ is called a *closed path* if it is a sequence of cells $A_1, \dots, A_n, A_{n+1}$, $n > 5$, such that:

1. $A_1 = A_{n+1}$;
2. $A_i \cap A_{i+1}$ is a common edge, for all $i = 1, \dots, n$;
3. $A_i \neq A_j$, for all $i \neq j$ and $i, j \in \{1, \dots, n\}$;
4. For all $i \in \{1, \dots, n\}$ and for all $j \notin \{i-2, i-1, i, i+1, i+2\}$ then $A_i \cap A_j = \emptyset$, where $A_{-1} = A_{n-1}$, $A_0 = A_n$, $A_{n+1} = A_1$ and $A_{n+2} = A_2$.

Intuitively, a closed path is a path in which the two ends meet and the cells have a common edge only with the previous and next ones. Roughly speaking, it is similar to a pearl necklace on a table. The assumption $n > 5$ is not restrictive, in fact it is known that all polyominoes with less than 6 cells are simple polyominoes (see for instance [18]), so they are well known for what concerns the primality of $I_{\mathcal{P}}$ and other properties of such an ideal.

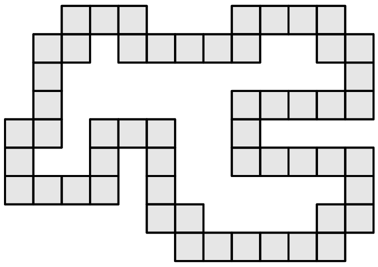


FIGURE 2.5: A closed path.

Remark 2.2.3. Let $\mathcal{P}$ be a closed path and $A_1, A_2, \dots, A_n, A_{n+1} = A_1$ the sequence of cells of $\mathcal{P}$ having the properties in Definition 2.2.2. Let $i \in \{1, \dots, n\}$ and consider the cells A_{i-1}, A_i, A_{i+1}. Up to reflections or rotations we have only one of the two arrangements described in Figure 2.6 (A) and (B).

(A) (B)

FIGURE 2.6

1. Referring to Figure 2.6(A), without loss of generality, we can suppose that $i = 1$, so $i + 1 = 2$ and $i - 1 = n$, otherwise it suffices to rename the indices. We prove that C, D, E, F do not belong to $\mathcal{P}$. Suppose that E belongs to $\mathcal{P}$. Since $E \cap A_1 \neq \emptyset$, from condition (4) of Definition 2.2.2 we have that $E = A_3$ or $E = A_{n-1}$, which contradicts (2) of Definition 2.2.2, because $E \cap A_2$ and $E \cap A_n$ are not edges. So E does not belong to $\mathcal{P}$. The same holds for the cells C and F by similar arguments. Moreover, from condition (4) of Definition 2.2.2 it follows that D is not in $\mathcal{P}$. By similar arguments it is possible to show that the cells C and D as in Figure 2.6(B) do not belong to $\mathcal{P}$.

2. Observe that for every cell H not belonging to $\mathcal{P}$ and for every cell A belonging to $\mathcal{P}$ and not placed as the cell A_i in Figure 2.6 (A) there exists a path of cells $H = H_1, \dots, H_m$ not belonging to $\mathcal{P}$ such that $H_m \cap A$ is an edge of H_m and A. In fact it is possible to consider a walk $\mathcal{C}_1 : H = F_1, \dots, F_r = G$ of cells not in $\mathcal{P}$ linking H to a cell G not in $\mathcal{P}$ and having an edge in common with a cell A_k of $\mathcal{P}$. We may assume that $k = 1$, otherwise it suffices to rename the indices. If $A = A_1$, then $\mathcal{C}_1$ is a walk of cells not in $\mathcal{P}$ such that $F_r \cap A$ is an edge of F_r and A, and by Remark 2.1.1 we obtain a desired path. If $A \neq A_1$, then we can consider another walk $\mathcal{C}_2 : G = G_1, \dots, G_t$ such that $G_j \notin \mathcal{P}$ for all $j = 1, \dots, t$ and $G_t \cap A$ is an edge of G_t and A, obtained travelling along the perimeter of $\mathcal{P}$ with the condition (2) of the Definition 2.2.2 and using the point (1) of this Remark. Considering the walk $\mathcal{C}_1 \cup \mathcal{C}_2$, we have a desired path by Remark 2.1.1.

Lemma 2.2.4. *Let $\mathcal{P}$ be a closed path. Then $\mathcal{P}$ contains a block of length at least 3.*

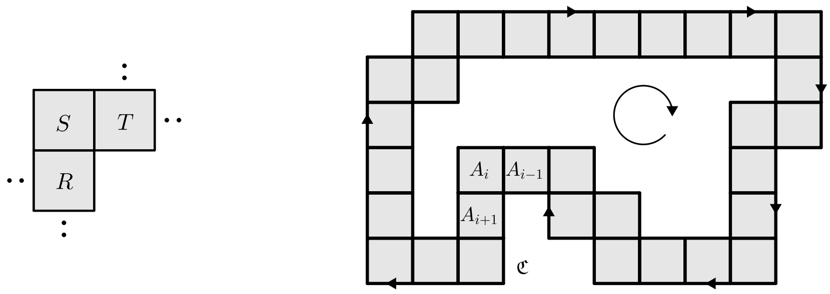
Proof. We suppose that $\mathcal{P}$ does not contain any block of length $n \geqslant 3$. We fix a cell A of $\mathcal{P}$ with lower left corner a. After a shift of coordinates, we may assume that $a = (1,1)$. Since $\mathcal{P}$ is a closed path, there exists a cell A_2, which has an edge in common with A. We may assume that the lower left corner of A_2 is $a_2 = (2,1)$. $\mathcal{P}$ is a closed path, then there exists a cell B_2, different from A, such that $A_2 \cap B_2$ is an edge of A_2 and B_2. If the lower left corner of B_2 is $(3,1)$, then $\{A, A_2, B_2\}$ is a block of length three, a contradiction. We may assume that the lower left corner of B_2 is $b_2 = (2,2)$. Continuing these arguments, we find a sequence of cells of $\mathcal{P}$, namely $A, A_2, B_2, \dots, A_m, B_m, \dots$, where the lower left corners of A_m and B_m are respectively $a_m = (m, m-1)$ and $b_m = (m,m)$, for all $m \geqslant 2$. Since $\mathcal{P}$ is a closed path, there exists $\overline{m} \in \mathbb{N} \setminus \{0,1\}$ such that $A_{\overline{m}} = A$ or $B_{\overline{m}} = A$, that is $a_{\overline{m}} = (1,1)$ or $b_{\overline{m}} = (1,1)$. It is a contradiction because $a_m > (1,1)$ and $b_m > (1,1)$, for all $m \geqslant 2$. $\square$

According to [24], we recall that a *rectilinear polygon* is a polygon whose edges meet orthogonally and it is called *simple* if there does not exist any self-intersection. In particular if $\mathfrak{C}$ is a rectilinear polygon, then the area bounded by $\mathfrak{C}$ is called the *interior* of $\mathfrak{C}$.

Proposition 2.2.5. *Let $\mathcal{P} : A_1, \dots, A_n, A_{n+1}$ be a closed path. Then the following hold:*

1. *$\mathcal{P}$ is a non-simple polyomino.*
2. *$\mathcal{P}$ has a unique hole.*
3. *Let $\mathcal{P}'$ be the polyomino consisting of all the cells of $\mathcal{P}$ except $A_i, A_{i+1}, \dots, A_{i+r}$ for some $i \in \{1, \dots, n\}$ and $1 \leq r < n-1$, where all indices are reduced modulo n. Then $\mathcal{P}'$ is a simple polyomino.*

Proof. 1) Firstly we show that there exist two cells not belonging to $\mathcal{P}$ and a simple rectilinear polygon $\mathfrak{C}$, consisting of the union of the closures of certain border edges of $\mathcal{P}$, such that the two cells are both neither in the interior of $\mathfrak{C}$ nor in the exterior of $\mathfrak{C}$. Consider a change of direction of $\mathcal{P}$ consisting of the cells R, S and T and we do opportune rotations of $\mathcal{P}$ in order to have $\{R, S, T\}$ as in Figure 2.7 (A). We set $S = A_1$ and, walking clockwise along the path, we label the cells of $\mathcal{P}$ increasingly from A_1 to A_n. It is not restrictive to assume $R = A_n$ and $T = A_2$. Observe that a such labelling induces a natural orientation along the perimeter of $\mathcal{P}$. Let $\mathfrak{C}$ be the union of the closures of the border edges of $\mathcal{P}$ having the following property: if r is a border edge of a cell A_i then $\overline{r} \in \mathfrak{C}$ if it has the cell A_i on its right with respect to the fixed orientation on the perimeter of $\mathcal{P}$. We prove that $\mathfrak{C}$ is a simple



(A)

(B) The arrows indicate the clockwise orientation of $\mathfrak{C}$.

FIGURE 2.7

rectilinear polygon. Observe that $\mathfrak{C}$ is the union of orthogonal line segments by construction, so if $\mathfrak{C}$ is a polygon then it is also rectilinear. We show firstly that $\mathfrak{C}$ is a polygon. We denote by r_1 the border edge of A_1 having a vertex in common with A_2. Let $\overline{r}_1, \overline{r}_2, \dots$ be the sequence of the closures of the border edges belonging to $\mathfrak{C}$, obtained following the clockwise orientation of the perimeter of $\mathcal{P}$ starting from $\overline{r}_1$. For all $i \in \{2, \dots, n-1\}$ considering three consecutive cells A_{i-1}, A_i and A_{i+1} of $\mathcal{P}$, the possible arrangements of $\overline{r}_j$ and $\overline{r}_{j+1}$ are displayed in Figure 2.8, up to just rotations. Then it is easy to see that $\overline{r}_j \cap \overline{r}_{j+1}$ is exactly a common endpoint of the two segments $\overline{r}_j$ and $\overline{r}_{j+1}$ for all j. Moreover, since A_n and A_1 have an edge in common, there exists $m \in \mathbb{N}$ such that r_m is the border edge of A_1 where $r_m \cap r_1$ is the upper left corner of A_1, so $\overline{r}_m \cap \overline{r}_1$ is a common endpoint of $\overline{r}_m$ and $\overline{r}_1$. Therefore,

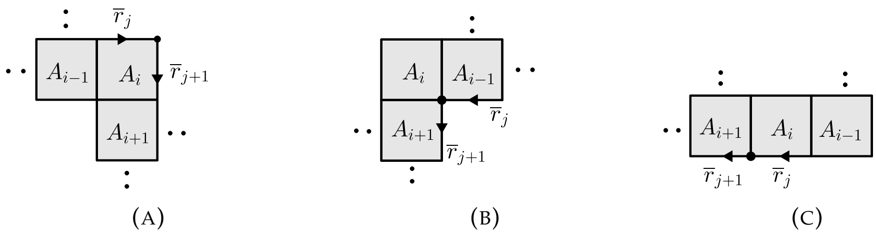


FIGURE 2.8

$\mathfrak{C}$ is a rectilinear polygon. We prove that $\mathfrak{C}$ is simple. First of all, we recall that the clockwise orientation along $\mathcal{P}$ induces an analogous orientation along the polygon $\mathfrak{C}$. By contradiction we suppose that $\mathfrak{C}$ is not simple, so there exists a self-intersection. Considering the orientation of $\mathfrak{C}$, we can distinguish exactly three cases up to just rotations, described in Figure 2.9, where the lines a, b and c belong to $\mathfrak{C}$. In the first case in Figure 2.9 (A) we obtain that the common edge of A_i and A_{i+1} belongs to $\mathfrak{C}$, but this is a contradiction since $\mathfrak{C}$ contains only border edges. The same contradiction rises in the second and third case, considering respectively the common edge of A_{i+1} and A_j as in Figure 2.9 (B), and the common edge of A_i and A_j as in Figure 2.9 (C). Therefore $\mathfrak{C}$ is a simple rectilinear polygon; for instance, see Figure 2.7 (B). In

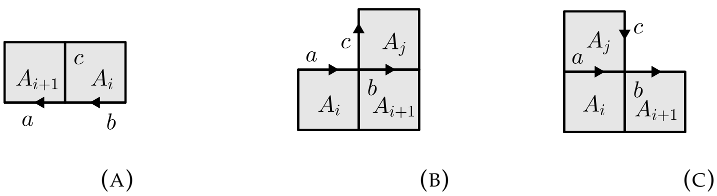


FIGURE 2.9

general it is easy to see geometrically that, walking clockwise along the perimeter of a rectilinear simple polygon, the interior of the polygon is on the right of the perimeter. Hence the cells of $\mathcal{P}$ are all situated in the interior of $\mathfrak{C}$. By Lemma 2.2.4 we can consider a part of $\mathcal{P}$ arranged as in Figure 2.10 (A), up to just rotations. By Remark 2.2.3 (1) we have that C and D do not belong to $\mathcal{P}$. We prove that C and D are neither both internal or both external to the polygon bounded by $\mathfrak{C}$. We denote by r_C and r_D the edges respectively of C and D that are border edges of A_i. We observe that either $\overline{r}_C \in \mathfrak{C}$ or $\overline{r}_D \in \mathfrak{C}$. We may assume that $\overline{r}_C \in \mathfrak{C}$, so $\overline{r}_C$ belongs to an edge of $\mathfrak{C}$, whose orientation goes from South to North, with reference to Figure 2.10 (B). In such a case C is external to the polygon bounded by $\mathfrak{C}$. We prove that

D is in the interior of $\mathfrak{C}$. Suppose by contradiction that D is external to the polygon bounded by $\mathfrak{C}$, so D is on the left of $\mathfrak{C}$ with respect to its orientation. In such a case, the only possibility is that A_i is on the right of $\mathfrak{C}$ with respect the orientation of $\mathfrak{C}$ and $\overline{r}_D \in \mathfrak{C}$. Therefore $\overline{r}_D$ belongs to another edge of $\mathfrak{C}$, whose orientation goes from North to South. Let A be the cell at North of A_i. The situation described above is summarized in Figure 2.10 (B). Walking along the edge of $\mathfrak{C}$ containing $\overline{r}_C$ we have

(A) (B)

FIGURE 2.10

that $A = A_{i+1}$. Walking along the edge of $\mathfrak{C}$ containing $\overline{r}_D$ we have that $A = A_{i-1}$. Then we have $A_{i+1} = A_{i-1}$, that is a contradiction with (3) of Definition 2.2.2. By similar arguments, if we assume that $\overline{r}_D \in \mathfrak{C}$ then C and D are respectively internal and external to $\mathfrak{C}$. We assume without loss of generality that C is internal to $\mathfrak{C}$ and D is external to $\mathfrak{C}$.

Suppose that $\mathcal{P}$ is a simple polyomino. Then there exists a path $\mathcal{F} : F_1, F_2, \dots, F_t$, which connects C and D, and F_k does not belong to $\mathcal{P}$ for all $k \in \{1, \dots, t\}$. Since C is internal to $\mathfrak{C}$ and D is external to $\mathfrak{C}$, there exist $k \in \{1, \dots, t-1\}$ and a border edge r of a cell F of $\mathcal{P}$ such that $E(F_k) \cap E(F_{k+1}) = \{r\}$. We observe that F, F_k and F_{k+1} are three cells of $\mathbb{Z}^2$ such that they have the edge r in common and $F_k \neq F_{k+1}$ because F_k,F_{k+1} belong to $\mathcal{F}$. Then either $F = F_k$ or $F = F_{k+1}$. But it is a contradiction because $F \in \mathcal{P}$ and $F \notin \mathcal{P}$ at the same time. Therefore $\mathcal{P}$ is a non-simple polyomino.

2) Suppose that $\mathcal{P}$ has more than one hole. In particular we can assume that $\mathcal{P}$ has two holes $\mathcal{H}_1$ and $\mathcal{H}_2$. Then there exist three cells B_1, C_1, D_1 of $\mathbb{Z}^2$ such that $B_1 \in \mathcal{H}_1$, $C_1 \in \mathcal{H}_2$ and D_1 is in the exterior of $\mathcal{P}$. In particular there does not exist any path of cells not belonging to $\mathcal{P}$ and linking B_1 to C_1, B_1 to D_1 and C_1 to D_1. By Lemma 2.2.4 we can consider a part of $\mathcal{P}$ arranged as in Figure 2.6(B). Considering the cells B_1 and A_i, we have by Remark 2.2.3 (2) that there exists a path $\mathcal{C}_1 : B_1, \dots, B_m$ of cells not in $\mathcal{P}$ such that $B_m \cap A_i$ is an edge of B_m and A_i. The same holds for C_1, A_i and D_1, A_i, hence there exist two paths $\mathcal{C}_2 : C_1, \dots, C_n$ and $\mathcal{C}_3 : D_1, \dots, D_r$ of cells not in $\mathcal{P}$ such that $C_n \cap A_i$ is an edge of C_n and A_i and $D_r \cap A_i$ is an edge of D_r and A_i. For the shape of this configuration then, among those paths, there are two, for instance $\mathcal{C}_1$ and $\mathcal{C}_2$, having C or D as the last cells. Let $\mathcal{C}_2^{rev}$ be the path obtained by $\mathcal{C}_2$ inverting the order of the cells, that is $\mathcal{C}_2^{rev} : C_1', \dots, C_n'$ where $C_i' = C_{n-i+1}$ for all $i = 1, \dots, n$. So, by Remark 2.1.1 we have that $\mathcal{C}_1 \cup \mathcal{C}_2^{rev}$ contains a path of cells not belonging to $\mathcal{P}$ linking B_1 to C_1, that is a contradiction.

3) Assume that $r = 1$ and suppose that A_i, A_{i+1} are arranged as in Figure 2.6(A). We can suppose that E belongs to the hole of $\mathcal{P}$ and D is in the exterior of $\mathcal{P}$. Let H_1, H_2 be two cells not belonging to $\mathcal{P}'$. Suppose that H_1 belongs to the hole of $\mathcal{P}$ and H_2 is exterior to $\mathcal{P}$, then there exist two paths $\mathcal{C}_1, \mathcal{C}_2$ of cells not belonging to $\mathcal{P}$ (so, not belonging to $\mathcal{P}'$) linking H_1 to E and D to H_2 respectively. We set $\mathcal{C}' : E, A_i, A_{i+1}, D$. Therefore, by Remark 2.1.1 we have that $\mathcal{C}_1 \cup \mathcal{C}' \cup \mathcal{C}_2$ contains a path of cells not belonging to $\mathcal{P}'$ linking H_1 to H_2. We obtain easily the same conclusion if both H_1, H_2 belong to the hole, or both H_1, H_2 are in the exterior of $\mathcal{P}$ and if one between H_1 or

H_2 is the cell A_i or A_{i+1}. By similar arguments we obtain the same conclusion in case A_i, A_{i+1} are arranged as in Figure 2.6(B). So if $r = 1$ then $\mathcal{P}'$ is a simple polyomino. The case $r > 1$ can be proved by similar arguments. □

Definition 2.2.6. Let $\mathcal{P}$ be a polyomino. A path of five cells A_1, A_2, A_3, A_4, A_5 of $\mathcal{P}$ is called an *L-configuration* if the two sequences A_1, A_2, A_3 and A_3, A_4, A_5 go in two orthogonal directions.

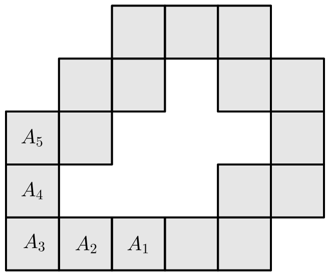


FIGURE 2.11: A closed path with an L-configuration.

Proposition 2.2.7. *Let $\mathcal{P}$ be a closed path. If $\mathcal{P}$ has at least an L-configuration, then $\mathcal{P}$ contains no zig-zag walks.*

Proof. We suppose that $\mathcal{P}$ contains a zig-zag walk $\mathcal{W} : I_1, \dots, I_\ell$. Let A_1, A_2, A_3, A_4, A_5 be an L-configuration. We denote by a, b the diagonal corners of A_3 and by c, d the anti-diagonal ones. We may suppose that $A_2 \cap A_3 = \{a, d\}$ and $A_3 \cap A_4 = \{d, b\}$, since similar arguments can be used in the other cases. Since $I_i \cap I_{i+1} \neq \emptyset$ for all $i \in \{1, \dots, \ell - 1\}$, there exists $r \in \{1, \dots, \ell\}$ such that $A_1, A_2 \in \mathcal{P}(I_r)$ and $A_4, A_5 \in \mathcal{P}(I_s)$, where $s = r + 1$ or $s = r - 1$, with $I_0 = I_\ell$ and $I_{\ell+1} = I_1$. We may suppose that $s = r + 1$ (see Figure 2.12). We prove that $v_{r+1} = d$.

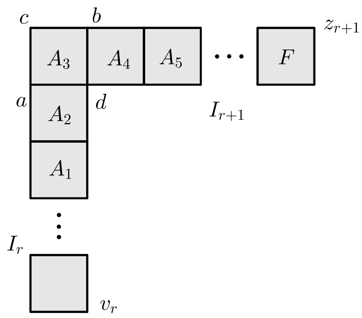


FIGURE 2.12

If $v_{r+1} \neq d$, then $\{v_{r+1}, d\} \subseteq I_r \cap I_{r+1}$, that is a contradiction. Since $v_{r+1} = d$ and $\mathcal{P}(I_r) \supseteq \{A_2\}$, the anti-diagonal corner z_r of I_r is equal to the vertex a of A_3. Let F be the cell of $\mathcal{P}$ such that $\mathcal{P}(I_{r+1}) = [A_4, F]$. Then $[z_r, z_{r+1}] = V([A_3, F])$. $V([A_3, F])$ is an inner interval of $\mathcal{P}$ such that z_r, z_{r+1} belong to it. This is a contradiction. □

Remark 2.2.8. Notice that it is possible to build closed paths, which contain no L-configurations and no zig-zag walks; see Figure 2.13.

Remark 2.2.9. If $\mathcal{P}$ is a closed path and $\mathcal{B}_1, \mathcal{B}_2$ are two maximal horizontal (or vertical) blocks of $\mathcal{P}$, then $|V(\mathcal{B}_1) \cap V(\mathcal{B}_2)| = 2$ or $V(\mathcal{B}_1) \cap V(\mathcal{B}_2) = \emptyset$. If

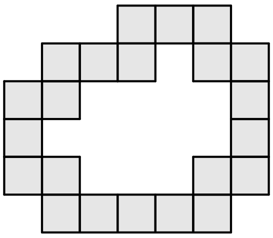


FIGURE 2.13: A closed path without any L-configuration.

$V(\mathcal{B}_1) \cap V(\mathcal{B}_2) = \{a, b\}$ then it is an edge belonging to $E(\mathcal{B}_1) \cap E(\mathcal{B}_2)$. Observe also that $\mathcal{P}$ is a union of blocks, not necessarily maximal, with the properties described above.

Definition 2.2.10. Let $\mathcal{P}$ be a polyomino. Let $\mathcal{B} = \{\mathcal{B}_i\}_{i=1,\dots,n}$ be a set of maximal horizontal (or vertical) blocks with length at least two, with $V(\mathcal{B}_i) \cap V(\mathcal{B}_{i+1}) = \{a_i, b_i\}$, $a_i \neq b_i$ for all $i = 1, \dots, n-1$. We say that $\mathcal{B}$ is a *ladder of n steps* if $[a_i, b_i]$ is not on the same edge interval of $[a_{i+1}, b_{i+1}]$ for all $i = 1, \dots, n-2$.

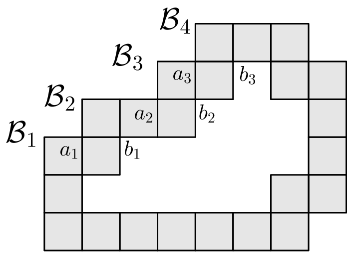


FIGURE 2.14: A closed path with a ladder of 4 steps.

Proposition 2.2.11. *Let $\mathcal{P}$ be a closed path. If $\mathcal{P}$ has a ladder of at least three steps, then $\mathcal{P}$ contains no zig-zag walks.*

Proof. Let $\mathcal{B} = \{\mathcal{B}_i\}_{i=1,\dots,n}$ be a ladder of n steps. We may assume that $n = 3$; for $n > 3$ the arguments are similar. We can suppose $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ are in horizontal position and the ladder is going up, otherwise we can reduce to this case by reflections or rotations (see Figure 2.15). Let a, b, c, d be the vertices of $\mathcal{P}$ such that

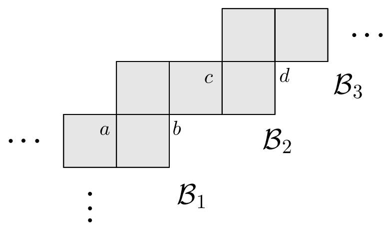


FIGURE 2.15

$V(\mathcal{B}_1) \cap V(\mathcal{B}_2) = \{a, b\}$ and $V(\mathcal{B}_2) \cap V(\mathcal{B}_3) = \{c, d\}$. We assume that $\mathcal{P}$ contains a zig-zag walk $\mathcal{W} : I_1, \dots, I_\ell$. We suppose that there exists $i \in \{1, \dots, \ell\}$ such that $\mathcal{P}(I_i) \subseteq \mathcal{B}_1$, $\mathcal{P}(I_{i+1}) \subseteq \mathcal{B}_2$ and $\mathcal{P}(I_{i+2}) \subseteq \mathcal{B}_3$. One of the following cases can occur:

1. $I_i \cap I_{i+1} = \{a\}$ and $I_{i+1} \cap I_{i+2} = \{c\}$;
2. $I_i \cap I_{i+1} = \{a\}$ and $I_{i+1} \cap I_{i+2} = \{d\}$;
3. $I_i \cap I_{i+1} = \{b\}$ and $I_{i+1} \cap I_{i+2} = \{c\}$;
4. $I_i \cap I_{i+1} = \{b\}$ and $I_{i+1} \cap I_{i+2} = \{d\}$.

If the first one occurs, then a, c should be on the same edge interval, a contradiction. The arguments are similar in the other cases.
Let A_1 and A_2 be the cells, belonging respectively to $\mathcal{B}_1$ and $\mathcal{B}_2$, which have the edge $\{a, b\}$ in common. Now we suppose that there exists $j \in \{1, \dots, \ell\}$ such that $\mathcal{P}(I_j)$ contains A_1 and A_2, that is $I_j = V([A_1, A_2])$. Then there exists $r \in \{1, \dots, \ell\}$ such that $\mathcal{P}(I_r)$ contains at least a cell in $\mathcal{B}_2 \cup \mathcal{B}_3$, where $r = j - 1$ or $r = j + 1$, with $I_0 = I_\ell$ and $I_{\ell+1} = I_1$. We may suppose that $r = j + 1$. If $\mathcal{B}_2$ contains at least three cells then there does not exist any interval $I \subseteq V(\mathcal{B}_2) \cup V(\mathcal{B}_3)$ such that $I_j \cap I$ is a vertex. In particular $|I_j \cap I_{j+1}| \neq 1$, that is a contradiction. If $\mathcal{B}_2$ contains two cells then the only possibility to have $|I_j \cap I_{j+1}| = 1$ is $v_{j+1} = c$. Moreover, in such a case, v_j is the lower left corner of A_1; so v_j and v_{j+1} do not belong to the same edge interval, that is a contradiction to the definition of a zig-zag walk. If there exists $j \in \{1, \dots, \ell\}$ such that $\mathcal{P}(I_j)$ contains the cells B_2 of $\mathcal{B}_2$ and B_3 of $\mathcal{B}_3$, that have in common the edge $\{c, d\}$, similar arguments lead to a contradiction. □

Remark 2.2.12. We note it is possible to build closed paths, which contain no ladders of $n \geq 2$ steps and no zig-zag walks; see Figure 2.16.

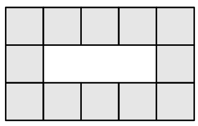


FIGURE 2.16: A closed path without any ladder.

2.2.2 Toric representations of the polyomino ideals of closed paths having an L-configuration or a ladder of at least three steps

Before giving the opportune toric representations of closed paths with an L-configuration or a ladder of at least three steps, we recall some notations and definitions contained in [41]. Moreover, we provide a more general version of [41, Lemma 2.2], which is very useful for our purpose. A binomial $f = f^+ - f^-$ in a binomial ideal $J \subset S_\mathcal{P}$ is called *redundant* if it can be expressed as a linear combination of binomials in J of lower degree. A binomial is called *irredundant* if it is not redundant. Moreover, we denote by V_f^+ the set of the vertices v, such that x_v appears in f^+, and by V_f^- the set of the vertices v, such that x_v appears in f^-.

Lemma 2.2.13. *Let $\mathcal{P}$ be a polyomino and $\phi : S_\mathcal{P} \to T$ a ring homomorphism with T an integral domain. Let $J = \ker \phi$ and $f = f^+ - f^-$ be a binomial in J with $\deg f \geq 3$. Suppose that:*

- *$I_\mathcal{P} \subseteq J$;*
- *$\phi(x_r) \neq 0$ for all $r \in V(\mathcal{P})$.*

If there exist three vertices $p, q \in V_f^+$ and $r \in V_f^-$ such that p, q are diagonal (respectively anti-diagonal) corners of an inner interval and r is one of the anti-diagonal (respectively diagonal) corners of the inner interval, then f is redundant in J.

Proof. Let I be the inner interval of $\mathcal{P}$, such that p,q are the diagonal vertices and r is an anti-diagonal one. We denote by s the other corner of I. We set $f_I = x_p x_q - x_r x_s$ and $f_J = x_s \frac{f^+}{x_p x_q} - \frac{f^-}{x_r}$. We have:

$$f = f^+ - f^- = \left(x_p x_q - x_r x_s\right) \frac{f^+}{x_p x_q} + x_r \left(x_s \frac{f^+}{x_p x_q} - \frac{f^-}{x_r} \right) = f_I \frac{f^+}{x_p x_q} + x_r f_J.$$

Since $I_{\mathcal{P}} \subseteq J$, it follows that $f_I \in J$. Since $f, f_I \in J$, we have $x_r f_J \in J$. Moreover $\phi(x_r) \neq 0$ and T is a domain, so $f_J \in J$. We observe that $\deg f_I$ and $\deg f_J$ are strictly less than $\deg f$, so we have the desired conclusion. □

Observe that the same claim of the previous result holds also if $p, q \in V_f^-$ and $r \in V_f^+$, by the same argument.

Let $\mathcal{P}$ be a closed path with an L-configuration, consisting of the sequence of cells A_1, A_2, A_3, A_4, A_5. We denote by a, b the diagonal corners of A_3 and by c, d the anti-diagonal ones. We may suppose that $A_2 \cap A_3 = \{b, d\}$ and $A_3 \cap A_4 = \{c, b\}$, otherwise we can consider opportune reflections or rotations in order to have such an L-configuration. We also set $A_3 = A$ (see Figure 2.17).

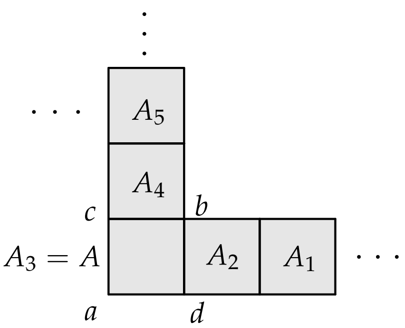


FIGURE 2.17

Let $\{V_i\}_{i \in I}$ be the sets of the maximal vertical edge intervals of $\mathcal{P}$ and $\{H_j\}_{j \in J}$ be the set of the maximal horizontal edge intervals of $\mathcal{P}$. Let $\{v_i\}_{i \in I}$ and $\{h_j\}_{j \in J}$ be the set of the variables associated respectively to $\{V_i\}_{i \in I}$ and $\{H_j\}_{j \in J}$. Let w be another variable different from v_i and h_j, $i \in I$ and $j \in J$. We define the following map:

$$\begin{aligned} \alpha : V(\mathcal{P}) &\longrightarrow K[\{v_i, h_j, w\} : i \in I, j \in J] \\ r &\longmapsto v_i h_j w^k \end{aligned}$$

with $r \in V_i \cap H_j$, $k = 0$ if $r \notin V(A)$, and $k = 1$, if $r \in V(A)$.
The toric ring, denoted by $T_{\mathcal{P}}$, is $K[\alpha(v) : v \in V(\mathcal{P})]$. We denote by $S_{\mathcal{P}}$ the polynomial ring $K[x_r : r \in V(\mathcal{P})]$ and we consider the following surjective ring homomorphism

$$\begin{aligned} \phi : S_{\mathcal{P}} &\longrightarrow T_{\mathcal{P}} \\ \phi(x_r) &= \alpha(r) \end{aligned}$$

The toric ideal $J_{\mathcal{P}}$ is the kernel of ϕ.

Proposition 2.2.14. *Let $\mathcal{P}$ be a closed path with an L-configuration. Then $I_{\mathcal{P}} \subseteq J_{\mathcal{P}}$.*

Proof. Let f be a binomial that is a generator of $I_{\mathcal{P}}$. Then there exists an inner interval $[p,q]$ of $\mathcal{P}$, such that $f = x_p x_q - x_r x_s$, where r,s are the anti-diagonal corners of $[p,q]$. We prove that $f \in J_{\mathcal{P}}$. Since $[p,q]$ is an inner interval, the vertices p, r and q, s are respectively on the same maximal vertical edge intervals and, similarly, the vertices p, s and q, r are respectively on the same maximal horizontal edge intervals. If $[p,q] \cap A = \emptyset$, then it is clear that $f \in J_{\mathcal{P}}$. If $[p,q] = A$, then p,q are the diagonal corners of A and r,s are the anti-diagonal ones, so $f \in J_{\mathcal{P}}$. If $[p,q] \cap A \neq \emptyset$ and $[p,q] \neq A$, then a corner of $[p,q]$ belongs to A and another one is not in A. We may assume that $p \in A$, in particular that $p = a$. Then $q \notin V(A)$, otherwise $[p,q] = A$. Since r and s are the anti-diagonal corners of $[p,q]$, then $r = c$ and $s \notin A$. It follows that $f \in J_{\mathcal{P}}$. Similar arguments hold in the other cases. $\square$

By Proposition 2.2.14 and the definition of $\phi : S_{\mathcal{P}} \to T_{\mathcal{P}}$, we can use Lemma 2.2.13 in the next Theorem, considering $J = J_{\mathcal{P}}$.

Theorem 2.2.15. *Let $\mathcal{P}$ be a closed path with an L-configuration. Then $I_{\mathcal{P}} = J_{\mathcal{P}}$.*

Proof. By Proposition 2.2.14 we have $I_{\mathcal{P}} \subseteq J_{\mathcal{P}}$. We prove that $J_{\mathcal{P}} \subseteq I_{\mathcal{P}}$, showing the following two facts:

1. every binomial of degree two in $J_{\mathcal{P}}$ belongs to $I_{\mathcal{P}}$;
2. every irredundant binomial in $J_{\mathcal{P}}$ is of degree two.

We prove (1). Let $f = x_p x_q - x_r x_s$ be a binomial in $J_{\mathcal{P}}$. Without loss of generality we can assume that p,q are the diagonal corners of the interval $[p,q]$. We denote by v_p, h_p and v_q, h_q the variables associated to the maximal horizontal and vertical edge intervals, which contain respectively p and q. Consider that $\phi(x_p x_q) = w^k v_p h_p v_q h_q = \phi(x_r x_s)$ with $k \in \{0,1,2\}$. The only possibility is that r,s are the anti-diagonal corners of $[p,q]$ and that $[p,r]$, $[p,s]$, $[s,q]$ and $[r,q]$ are edge intervals of $\mathcal{P}$. By contradiction, we assume that $[p,q]$ is not an inner interval of $\mathcal{P}$, in particular there exists a set of cells of $[p,q]$ that do not belong to $\mathcal{P}$. Since $[p,r]$, $[p,s]$, $[s,q]$ and $[r,q]$ are edge intervals in $\mathcal{P}$, then $[p,q]$ contains the hole $\mathcal{H}$ of $\mathcal{P}$. In this case, the only possible arrangement of the cells of $\mathcal{P}$ consists in having at least one of the corners p,q,r and s in A. We may assume that $p \in A$. Then w divides $\phi(x_p)\phi(x_q)$ and so w divides $\phi(x_r)$ or $\phi(x_s)$. From $w|\phi(x_r)$ (resp. $w|\phi(x_s)$) it follows that $r \in A$ (resp. $s \in A$). Since $\mathcal{H} \subseteq [p,q]$, we have either r or s does not belong to A, so it is a contradiction. Hence $[p,q]$ is an inner interval of $\mathcal{P}$.
We prove (2). We suppose that there exists a binomial f in $J_{\mathcal{P}}$ with $\deg f \geq 3$, such that f is irredundant. We suppose that every variable of f is in $\{x_a : a \in V(\mathcal{P}) \backslash V(A)\}$. We denote by $\mathcal{P}'$ the simple polyomino obtained by removing the cells having vertices in common with A. We define the map ϕ' as the restriction of ϕ on $K[x_a : a \in V(\mathcal{P}) \backslash V(A)]$ and we denote by $J_{\mathcal{P}'}$ the kernel of ϕ'. By Theorem 2.2 in [39], we have that $J_{\mathcal{P}'} = I_{\mathcal{P}'}$, where $I_{\mathcal{P}'}$ is the polyomino ideal associated to $\mathcal{P}'$. We observe that f is a binomial in $J_{\mathcal{P}'}$. Since $J_{\mathcal{P}'} \subset J_{\mathcal{P}}$ and f is irredundant in $J_{\mathcal{P}}$, then f is irredundant in $J_{\mathcal{P}'}$. Then f is irredundant in $I_{\mathcal{P}'}$, that is a contradiction. It follows that there exists at least one variable in f, that corresponds to a vertex of A. We recall that $f = f^+ - f^- \in J_{\mathcal{P}}$, so $\phi(f^+) = \phi(f^-)$. We may suppose that there exists $v_1 \in A$, such that x_{v_1} divides f^+, that is $v_1 \in V_f^+$. Then w divides $\phi(f^+) = \phi(f^-)$, so there exists $v_1' \in A$, such that $x_{v_1'}$ divides f^-, that is $v_1' \in V_f^-$. If $v_1 = v_1'$, then

$f = x_{v_1}(\tilde{f}^+ - \tilde{f}^-)$, where $\tilde{f}^+ - \tilde{f}^- \in J_\mathcal{P}$ and $\deg(\tilde{f}^+ - \tilde{f}^-) < \deg f$, a contradiction. Then $v_1 \neq v_1'$. Let V_{v_1} and H_{v_1} be the maximal vertical and horizontal edge intervals of $\mathcal{P}$, which contain v_1. Then v_{v_1} divides $\phi(f^+) = \phi(f^-)$, so there exists $v_2' \in V_{v_1}$ such that $x_{v_2'}$ divides f^-. Moreover h_{v_1} divides $\phi(f^+) = \phi(f^-)$, so there exists $v_3' \in H_{v_1}$ such that $x_{v_3'}$ divides f^-. Let $V_{v_1'}$ and $H_{v_1'}$ be the maximal vertical and horizontal edge intervals of $\mathcal{P}$, which contain v_1'. Then $v_{v_1'}$ divides $\phi(f^-) = \phi(f^+)$, so there exists $v_2 \in V_{v_1'}$ such that x_{v_2} divides f^+. Moreover $h_{v_1'}$ divides $\phi(f^-) = \phi(f^+)$, so there exists $v_3 \in H_{v_1'}$ such that x_{v_3} divides f^+. The following cases could occur:

(I) v_1 and v_1' are on the same vertical edge interval of $\mathcal{P}$. For the structure of $\mathcal{P}$,

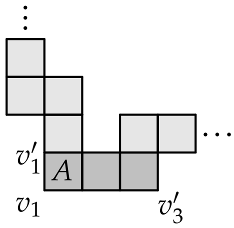


FIGURE 2.18

either v_3 or v_3' is a vertex which identifies an inner interval of $\mathcal{P}$ along with v_1 and v_1' (see Figure 2.18). From Lemma 2.2.13 a contradiction follows.

(II) v_1 and v_1' are on the same horizontal interval of $\mathcal{P}$. For the structure of $\mathcal{P}$,

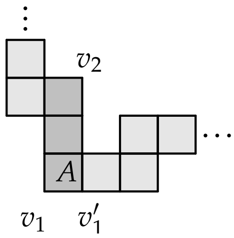


FIGURE 2.19

either v_2 or v_2' is a vertex which identifies an inner interval of $\mathcal{P}$ along with v_1 and v_1' (see Figure 2.19). As before, by Lemma 2.2.13, we have a contradiction.

(III) v_1 and v_1' are the diagonal corners of A. We may suppose that $v_1 = a$ and $v_1' = b$. We prove that v_3' cannot be an anti-diagonal corner of A. If v_3' is an anti-diagonal corner of A, then $v_3' = d$. For the structure of $\mathcal{P}$, either v_2 or v_2' is a vertex which identifies an inner interval of $\mathcal{P}$ respectively with v_1 or v_3'. If $[v_1, v_2]$ is an inner interval, then we have a contradiction, applying Lemma 2.2.13 to v_1, v_3', v_2. If the interval with anti-diagonal corners v_2', v_1' is an inner interval, then we have a contradiction, by Lemma 2.2.13 applied to v_2', v_3', v_1. By similar arguments, v_3, v_2 and v_2' cannot be anti-diagonal vertices of A.
For the structure of $\mathcal{P}$, either v_3 or v_3' is a vertex which identifies an inner interval of $\mathcal{P}$ respectively with v_1 or v_1'. We assume that $[v_1, v_3]$ is an inner interval of $\mathcal{P}$. We denote by g, h the anti-diagonal corners of $[v_1, v_3]$. For the structure of $\mathcal{P}$, either v_2 or v_2' is such that the interval identified by g, v_2 or v_1', v_2' is inner to $\mathcal{P}$. We assume that $[g, v_2]$ is an inner interval of $\mathcal{P}$ (see Figure 2.20).

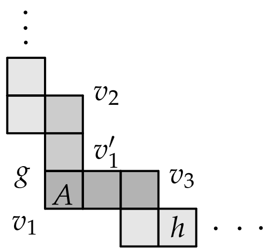


FIGURE 2.20

Then:

$$f = f^+ - f^- = \frac{f^+}{x_{v_1}x_{v_3}}(x_{v_1}x_{v_3} - x_g x_h) + \frac{f^+}{x_{v_1}x_{v_3}}x_g x_h - f^-.$$

Since $[v_1, v_3]$ is an inner interval of $\mathcal{P}$, then $x_{v_1}x_{v_3} - x_g x_h \in I_{\mathcal{P}} \subseteq J_{\mathcal{P}}$. We set $\tilde{f} = \frac{f^+}{x_{v_1}x_{v_3}}x_g x_h - f^-$, $f_1 = \frac{f^+}{x_{v_1}x_{v_3}}x_g x_h$ and $f_2 = f^-$, so $\tilde{f} = f_1 - f_2$. We observe that $\tilde{f} \in J_{\mathcal{P}}$, $x_{v_2}x_g$ divides f_1 and $x_{v_1'}$ divides f_2. Since $v_2, g \in V_{\tilde{f}}^+$ and $v_1' \in V_{\tilde{f}}^-$, from Lemma 2.2.13 it follows that $\tilde{f}$ is redundant in $J_{\mathcal{P}}$. Then f in redundant in $J_{\mathcal{P}}$, that is a contradiction. By similar arguments we can have the same conclusion in the other cases.

(IV) v_1 and v_1' are anti-diagonal corners of A. By arguments as in the previous case, we deduce that this one is not possible.

Then f is a redundant binomial in $J_{\mathcal{P}}$. In conclusion we have $J_{\mathcal{P}} \subseteq I_{\mathcal{P}}$, hence $J_{\mathcal{P}} = I_{\mathcal{P}}$. □

Corollary 2.2.16. *Let $\mathcal{P}$ be a closed path with an L-configuration. Then $I_{\mathcal{P}}$ is prime.*

Let $\mathcal{B} = \{\mathcal{B}_i\}_{i=1,\dots,m}$ be a maximal ladder of m steps, $m > 2$. After some convenient reflections or rotations of $\mathcal{P}$, we can suppose that $\mathcal{B}_1, \dots, \mathcal{B}_m$ are in horizontal position and the ladder is going down. We suppose that the block $\mathcal{B}_{m-1}$ is made up of n cells, which we denote $A_1, \dots, A_n$ from left to right. We also denote by a_i the lower left corner of A_i, for all $i = 1, \dots, n$. Let A be the cell of $\mathcal{B}_m$, having an edge in common with A_n. We denote by a, b the diagonal corners of A and by d the other anti-diagonal corner (see Figure 2.21).

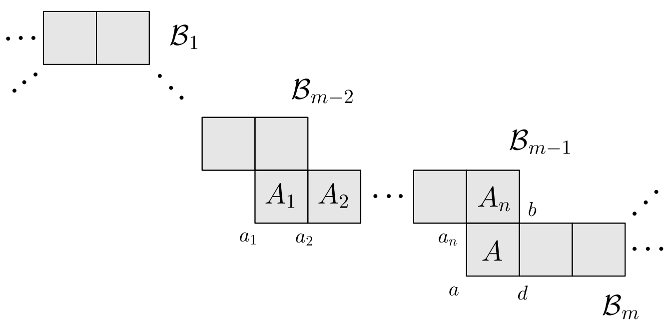


FIGURE 2.21

We also set $L_{\mathcal{B}} = \{a_1, \dots, a_n, d, a, b\}$. As before, we denote by $\{V_i\}_{i \in I}$ the set of the maximal edge intervals of $\mathcal{P}$ and by $\{H_j\}_{j \in J}$ the set of the maximal horizontal edge intervals of $\mathcal{P}$. Let $\{v_i\}_{i\in I}$ and $\{h_j\}_{j\in J}$ be the sets of the variables associated respectively to $\{V_i\}_{i\in I}$ and $\{H_j\}_{j\in J}$. Let $\mathcal{H}$ be the hole of $\mathcal{P}$ and w be another variable. We define the following map:

$$\begin{aligned}\alpha : V(\mathcal{P}) &\longrightarrow K[\{v_i, h_j, w\} : i \in I, j \in J]\\ r &\longmapsto v_i h_j w^k\end{aligned}$$

with $V_i \cap H_j = \{r\}$ and where $k = 0$, if $r \notin L_{\mathcal{B}}$, and $k = 1$, if $r \in L_{\mathcal{B}}$.
We denote by $T_{\mathcal{P}}$ the toric ring $K[\alpha(v) : v \in V(\mathcal{P})]$ and by $J_{\mathcal{P}}$ the kernel of the following surjective ring homomorphism:

$$\begin{aligned}\phi : S_{\mathcal{P}} &\longrightarrow T_{\mathcal{P}}\\ \phi(x_r) &= \alpha(r)\end{aligned}$$

Proposition 2.2.17. *Let $\mathcal{P}$ be a closed path with a ladder of m steps ($m > 2$). Then $I_{\mathcal{P}} \subseteq J_{\mathcal{P}}$.*

Proof. Let f be a binomial that is a generator of $I_{\mathcal{P}}$. Then there exists an inner interval $[p, q]$ of $\mathcal{P}$, such that $f = x_p x_q - x_r x_s$, where r, s are the anti-diagonal corners of $[p, q]$. If $[p, q] \cap L_{\mathcal{B}} = \emptyset$, then $f \in J_{\mathcal{P}}$. We suppose that $[p, q] \cap L_{\mathcal{B}} \neq \emptyset$. If $p, q \in L_{\mathcal{B}}$, then $[p, q] = A$, so $f \in J_{\mathcal{P}}$. If $p \in L_{\mathcal{B}}$ and $q \notin L_{\mathcal{B}}$, we have that either r or s belongs to $L_{\mathcal{B}}$ for the structure of $\mathcal{P}$, so $f \in J_{\mathcal{P}}$. The case $p \notin L_{\mathcal{B}}$ and $q \in L_{\mathcal{B}}$ is not possible by construction. Then the desired conclusion follows. □

By Proposition 2.2.17 and the definition of $\phi : S_{\mathcal{P}} \to T_{\mathcal{P}}$, we can use Lemma 2.2.13 in the next theorem, considering $J = J_{\mathcal{P}}$.

Theorem 2.2.18. *Let $\mathcal{P}$ be a closed path with a ladder of m steps ($m > 2$). Then $I_{\mathcal{P}} = J_{\mathcal{P}}$.*

Proof. Let $\mathcal{B} = \{\mathcal{B}_i\}_{i=1,\dots,m}$ be a maximal ladder of m steps, $m > 2$, where $\mathcal{B}_1, \dots, \mathcal{B}_m$ are in horizontal position and the ladder is going down. By Proposition 2.2.17, we have $I_{\mathcal{P}} \subseteq J_{\mathcal{P}}$. Similar arguments as in (1) of Theorem 2.2.15 allow us to prove that every binomial of degree two in $J_{\mathcal{P}}$ belongs to $I_{\mathcal{P}}$. We prove that every irredundant binomial in $J_{\mathcal{P}}$ is of degree two. We suppose that there exists a binomial f in $J_{\mathcal{P}}$ with $\deg f \geq 3$, such that f is irredundant. We prove that in f there are not any variables associated to the vertices of $L_{\mathcal{B}}$. We suppose that there exists $v_1 \in L_{\mathcal{B}}$, such that x_{v_1} divides f^+, that is $v_1 \in V_f^+$. As in the proof of Theorem 2.2.15, we can find a vertex $v_1' \in L_{\mathcal{B}} \cap V_f^-$, two vertices $v_2', v_3' \in V_f^-$ which are respectively on the same maximal vertical and horizontal edge intervals of $\mathcal{P}$ containing v_1, and two vertices $v_2, v_3 \in V_f^+$ which are respectively on the same vertical and horizontal edge intervals of $\mathcal{P}$ containing v_1'. The following cases could occur:

(I) v_1 and v_1' are on the same vertical edge interval of $\mathcal{P}$. For the structure of $\mathcal{P}$ either v_3 or v_3' is a vertex which identifies an inner interval of $\mathcal{P}$ along with v_1 and v_1' (see Figure 2.22). Lemma 2.2.13 leads to a contradiction.

(II) v_1 and v_1' are on the same horizontal edge interval of $\mathcal{P}$. If $\{v_1, v_1'\} = \{a, d\}$ or $\{v_1, v_1'\} = \{a_n, b\}$ or $\{v_1, v_1'\} \subseteq \{a_1, \dots, a_{n-1}\}$ with $n > 2$, then either v_2 or v_2' is a vertex which identifies an inner interval along with v_1 and v_1'. By using Lemma 2.2.13, we have a contradiction. We suppose that $v_1 \in \{a_1, \dots, a_{n-1}\}$ and $v_1' \in \{a_n, b\}$ or vice versa. We may assume that $v_1' = b$, because similar arguments hold when $v_1' = a_n$. If $v_2 \notin L_{\mathcal{B}}$, then we have a contradiction, using Lemma 2.2.13 to the vertices v_1, v_1' and v_2. Let v_2 be in $L_{\mathcal{B}}$; in particular the

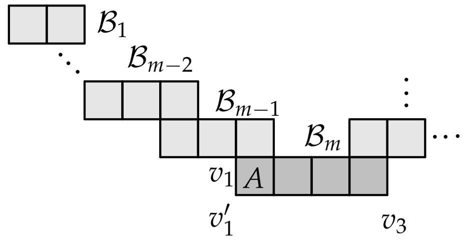


FIGURE 2.22

only possibility is $v_2 = d$. Let h_{v_2} be the variable associated with the horizontal interval of v_2. Then h_{v_2} divides $\phi(f^+) = \phi(f^-)$, so we have two possibilities. The first one is $v_2 \in V_f^-$, so $f = x_{v_2}(\tilde{f}^+ - \tilde{f}^-)$, that is f is not irredundant. Alternatively, there exists $\tilde{v} \in V_f^-$ such that $\tilde{v}$ is in the same horizontal edge interval of v_2; in particular f is not irredundant by Lemma 2.2.13 applied to the vertices $v_1', v_2, \tilde{v}$. In both cases we have a contradiction.

(III) v_1 and v_1' are not on the same horizontal or vertical edge intervals of $\mathcal{P}$. If they are diagonal or anti-diagonal vertices of A, then we have a contradiction, by similar arguments as in the last case (III) of Theorem 2.2.15. We suppose that $v_1 \in \{a_1, \dots, a_{n-1}\}$ and $v_1' \in \{a, d\}$ (or vice versa). We may assume that $v_1' = d$, because similar arguments holds when $v_1' = a$. The vertex v_2 does not belong to $L_{\mathcal{B}}$, otherwise we have a contradiction as in the previous case, so $[v_1, v_2]$ is an inner interval of $\mathcal{P}$. We denote by g, h the anti-diagonal vertices of $[v_1, v_2]$. We observe that $v_3 \notin L_{\mathcal{B}}$, otherwise we have a contradiction using the usual considerations to vertices v_2, v_3, v_1'. Then h, v_3 identify an inner interval of $\mathcal{P}$, with v_1' as diagonal corner (see Figure 2.23).

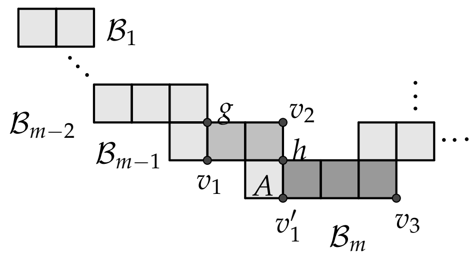


FIGURE 2.23

Then:

$$f = f^+ - f^- = \frac{f^+}{x_{v_1}x_{v_2}}(x_{v_1}x_{v_2} - x_g x_h) + \frac{f^+}{x_{v_1}x_{v_2}} x_g x_h - f^-.$$

Since $[v_1, v_2]$ is an inner interval of $\mathcal{P}$, then $x_{v_1}x_{v_2} - x_g x_h \in I_{\mathcal{P}} \subseteq J_{\mathcal{P}}$. We set $\tilde{f} = \frac{f^+}{x_{v_1}x_{v_2}} x_g x_h - f^-$, $f_1 = \frac{f^+}{x_{v_1}x_{v_2}} x_g x_h$ and $f_2 = f^-$, so $\tilde{f} = f_1 - f_2$. We observe that $\tilde{f} \in J_{\mathcal{P}}$, $x_{v_3}x_h$ divides f_1 and $x_{v_1'}$ divides f_2. Since $v_3, h \in V_{\tilde{f}}^+$ and $v_1' \in V_{\tilde{f}}^-$, by Lemma 2.2.13, we have that $\tilde{f}$ is redundant in $J_{\mathcal{P}}$. Then f is redundant in $J_{\mathcal{P}}$, that is a contradiction.

Summarizing, in f there are not any variables associated to any vertices of $L_{\mathcal{B}}$. We denote by b_i the upper right corner of the cell A_i, for all $i = 1, \dots, n$. We prove that in f there is no variable associated to a vertex in $\{b_2, \dots, b_n\}$. We suppose that there exists $i \in \{2, \dots, n\}$ such that x_{b_i} divides f^+. Let V_{b_i} be the maximal vertical edge interval of $\mathcal{P}$ such that $b_i \in V_{b_i}$. Since $b_i \in V_f^+$, there exists a vertex $v \in V_{b_i} \setminus \{b_i\}$, such that x_v divides f^-. For the structure of $\mathcal{P}$, the vertex v belongs to $L_{\mathcal{B}}$ and $v \in V_f^-$, that is a contradiction. Now we set $\mathcal{F} = L_{\mathcal{B}} \cup \{b_2, \dots, b_n\}$. In conclusion, in f there are only variables x_v, such that $v \in V(\mathcal{P}) \setminus \mathcal{F}$. We denote by $\mathcal{P}'$ the simple polyomino consisting of the cells of $\mathcal{P}$ that do not have a vertex in $L_{\mathcal{B}}$, and by $I_{\mathcal{P}'}$ the polyomino ideal associated to $\mathcal{P}'$. By Theorem 2.2 in [39], we have that $J_{\mathcal{P}'} = I_{\mathcal{P}'}$. Moreover f is a binomial in $J_{\mathcal{P}'}$. Since $J_{\mathcal{P}'} \subset J_{\mathcal{P}}$ and f is irredundant in $J_{\mathcal{P}}$, then f is irredundant in $J_{\mathcal{P}'}$. It follows that f is irredundant in $I_{\mathcal{P}'}$, that is a contradiction. In conclusion we have $J_{\mathcal{P}} \subseteq I_{\mathcal{P}}$. □

Corollary 2.2.19. *Let $\mathcal{P}$ be a closed path with a ladder of m steps ($m > 2$). Then $I_{\mathcal{P}}$ is prime.*

2.2.3 Characterization of prime closed paths by zig-zag walks

Let $\mathcal{P}$ be a polyomino. In [31] the authors have shown that if $I_{\mathcal{P}}$ is prime then $\mathcal{P}$ contains no zig-zag walks and they have conjectured that it is a sufficient condition for the primality of $I_{\mathcal{P}}$. We recall that the rank of $\mathcal{P}$, denoted by $\mathrm{rank}(\mathcal{P})$, is the number of the cells of $\mathcal{P}$. Using a computational method, they have shown that the conjecture is verified for $\mathrm{rank}(\mathcal{P}) \leq 14$. Here we prove that the conjecture is true for the class of closed paths.

Proposition 2.2.20. *Let $\mathcal{P}$ be a closed path and suppose that $\mathcal{P}$ has no zig-zag walks. Then $\mathcal{P}$ has an L-configuration or a ladder of at least three steps.*

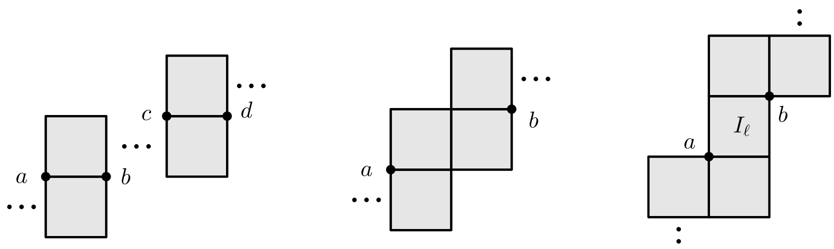
Proof. The structure of $\mathcal{P}$ assures that there exists at least a sequence of distinct inner intervals $I_1, \dots, I_\ell$ such that $|I_i \cap I_{i+1}| = 1$ for all $i = 1, \dots, \ell - 1$ and $|I_\ell \cap I_1| = 1$. Let $I_1 \cap I_\ell = \{v_1 = v_{\ell+1}\}$ and $I_i \cap I_{i+1} = \{v_{i+1}\}$ with $i \in \{1, \dots, \ell - 1\}$. Suppose that v_ℓ and v_1 are not in the same edge interval. After appropriate reflections or rotations, we can suppose that I_ℓ is a horizontal interval having v_ℓ and v_1 as diagonal corners. Let $\mathcal{B}$ be the maximal horizontal block of $\mathcal{P}$ containing $\mathcal{P}(I_\ell)$. We examine all possible different cases.

- $\mathcal{B}$ contains at least three cells. We can suppose that $\mathcal{P}$ has no L-configurations, otherwise we have finished. Then a part of the polyomino has the shape of Figure 2.24(A), where $v_\ell \in \{a, b\}$ and $v_1 \in \{c, d\}$. So we have a ladder of at least three steps.

- $\mathcal{B} = \mathcal{P}(I_\ell)$ and it contains exactly two cells. Then we are in the case of Figure 2.24(B), where $v_\ell = a$ and $v_1 = b$. We have again a ladder of at least three steps.

- $\mathcal{B} = \mathcal{P}(I_\ell)$ is a cell. Under the assumption that $\mathcal{P}$ has no L-configurations, we are in the case of Figure 2.24(C). In particular $\mathcal{P}$ has a ladder of at least three steps.

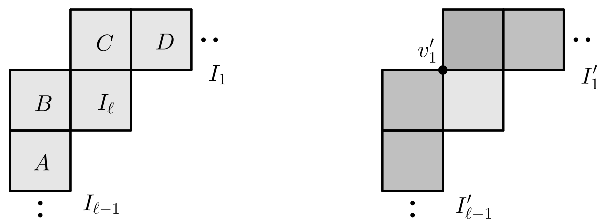
It remains to consider the case in which I_ℓ is a cell and $\mathcal{B}$ contains two cells. We prove that also in this case we obtain that $\mathcal{P}$ contains an L-configuration or a ladder of at least three steps. After an appropriate reflection, we can reduce to the case in



(A) (B) (C)

FIGURE 2.24

Figure 2.25(A). Observe that if there is a cell in the direction West with respect to the cell A (that is the first cell of $I_{\ell-1}$), or in the direction North with respect to the cell D (that is the first cell of I_1), then $\mathcal{P}$ has a ladder of at least three steps. So we can suppose that $\mathcal{P}$ has an adjacent cell to A in direction South and an adjacent cell to D in direction East. In such a case we can define another sequence of intervals $I_1',\dots,I_{\ell-1}'$, with $I_{\ell-1}' = I_{\ell-1}\cup B$, $I_1' = I_1 \cup C$ and $I_i' = I_i$ for $i\in\{2,\dots,\ell-2\}$; in particular we denote $I_1'\cap I_{\ell-1}' = \{v_1' = v_\ell'\}$ and $I_i'\cap I_{i+1}' = \{v_{i+1}'\}$ for $i\in\{1,\dots,\ell-2\}$. So, we are in the situation of Figure 2.25(B). Now suppose that v_1' and v_2' are not in the same edge interval. It is not difficult to see that in this case we are again in the situation of Figure 2.24(A). So we can assume that v_1' and v_2' are in the same edge interval. The same conclusion can be obtained for the vertices v_2' and v_3' and



(A) (B)

FIGURE 2.25

so on. Therefore we can reduce the proof to the case that in the initial sequence of intervals $I_1,\dots,I_\ell$ the vertices v_i and v_{i+1} belong to the same edge interval for every $i\in\{1,\dots,\ell\}$. Since $\mathcal{P}$ has no zig-zag walks, there exist z_i and z_j vertices of an inner interval J of $\mathcal{P}$, such that v_i and z_i are the diagonal or anti-diagonal corners of I_i, and v_j and z_j are the diagonal or anti-diagonal corners of I_j. Because of the structure of $\mathcal{P}$ the only possibilities are $j=i+1$ or $j=i-1$. We can assume that $j=i+1$ and v_{i+i} is a diagonal corner of I_{i+1}, so v_i is an anti-diagonal corner of I_i. Let $\mathcal{B}_i,\mathcal{B}_{i+1}$ be the maximal blocks of $\mathcal{P}$ containing $\mathcal{P}(I_i)$ and $\mathcal{P}(I_{i+1})$ respectively. Observe that $\mathcal{B}_i$ and $\mathcal{B}_{i+1}$ are not both in horizontal or vertical position, since J is an interval of $\mathcal{P}$, that is a closed path. So we can assume that $\mathcal{B}_i$ is in vertical position and $\mathcal{B}_{i+1}$ is in horizontal position. Observe that each block has at least three cells; in particular we refer to Figure 2.26 for the arrangement of this situation, observing that some appropriate cells with dashed lines must belong to the polyomino. In particular $\mathcal{B}_i\cup\mathcal{B}_{i+1}$ contains an L-configuration.

$\square$

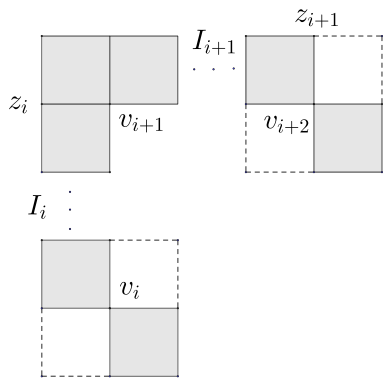


FIGURE 2.26

By Proposition 2.2.7, Proposition 2.2.11 and Proposition 2.2.20 we deduce that having an L-configuration or a ladder of at least three steps is a necessary and sufficient condition in order to have no zig-zag walks for a closed path. Now we are ready to state and to prove the following result of this work.

Theorem 2.2.21. *Let $\mathcal{P}$ be a closed path. $I_{\mathcal{P}}$ is prime if and only if $\mathcal{P}$ contains no zig-zag walks.*

Proof. The necessary condition is shown in [31, Corollary 3.6]. The sufficient one follows from the Proposition 2.2.20, Corollary 2.2.16 and Corollary 2.2.19. □

Chapter 3

Generalizations of closed paths, simple and weakly connected collections of cells and their primality

Actually the arguments in the proofs of the results explained so far can provide also the primality for a larger class of polyominoes. In this chapter we introduce three new classes of non-simple polyominoes, which can be viewed as generalizations of closed paths and we study their primality. Moreover, we discuss also the primality of simple and weakly collections of cells and, as an application, we characterize all prime weakly-closed paths, a new kind of non-simple polyominoes. Closed paths and weakly ones are the only known polyominoes so far for which [31, Conjecture 4.6] is proved. The results contained in this chapter are included in the papers [4] and [8].

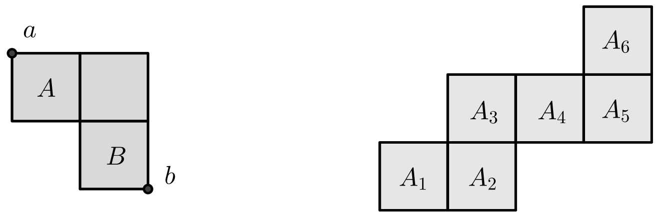
3.1 Primality of some classes of polyominoes joined by paths

In this section we define the $\mathcal{P}(\mathcal{S},\mathcal{C})$-polyominoes and the L-rectangles or the ladder-rectangles linked to a simple polyomino by two paths. Let us start introducing some useful definitions and notions.
We call a *L-triomino* any polyomino consisting of three cells not aligned; for instance see Figure 3.1 (A). Referring to the figure, we call *hooking vertices* the vertices a and b. Moreover we call *hooking edges* with respect to a (resp. b) the couple of edges of A (resp. B) that intersect at a (resp. at b).
A polyomino $\mathcal{C}$ is called an *open path* if it is a sequence of two or more cells $A_1,\dots,A_n$ such that:

1. $A_i \cap A_{i+1}$ is a common edge, for all $i = 1,\dots,n-1$;
2. $A_i \neq A_j$, for all $i \neq j$ and $i,j \in \{1,\dots,n\}$;
3. If $n > 2$, then $V(A_i) \cap V(A_j) = \emptyset$ for all $i \in \{1,\dots,n-2\}$ and for all $j \notin \{i,i+1,i+2\}$.

For instance see Figure 3.1. The edges of A_1 (resp. A_n), which do not belong to $E(A_2)$ (resp. $E(A_{n-1})$), are called *free edges*.

Definition 3.1.1. Let $\mathcal{S}$ be a simple polyomino, $\mathcal{C} : A_1,\dots,A_n$ be an open path and $\mathcal{T}_1$ and $\mathcal{T}_2$ be two L-triominoes. Moreover we denote by a_1, b_1 the hooking vertices of $\mathcal{T}_1$ and by a_2, b_2 the hooking vertices of $\mathcal{T}_2$. We denote by $\mathcal{P}(\mathcal{S},\mathcal{C})$ a polyomino satisfying the following conditions:



(A) (B)

FIGURE 3.1

1. $\mathcal{P}(\mathcal{S},\mathcal{C}) = \mathcal{S} \cup \mathcal{C} \cup \mathcal{T}_1 \cup \mathcal{T}_2$.
2. $V(\mathcal{S}) \cap V(\mathcal{C}) = \emptyset$ and $V(\mathcal{T}_1) \cap V(\mathcal{T}_2) = \emptyset$.
3. $E(\mathcal{S}) \cap E(\mathcal{T}_1) = \{V_1\}$ where V_1 is a hooking edge with respect to a_1 and $E(\mathcal{S}) \cap E(\mathcal{T}_2) = \{V_2\}$ where V_2 is a hooking edge with respect to a_2.
4. $E(\mathcal{C}) \cap E(\mathcal{T}_1) = \{W_1\}$ where $W_1 \in E(A_1)$ and it is is a hooking edge with respect to b_1, and $E(\mathcal{C}) \cap E(\mathcal{T}_2) = \{W_2\}$ where $W_2 \in E(A_n)$ and it is a hooking edge with respect to b_2.
5. $|V(\mathcal{C}) \cap V(\mathcal{T}_1)| = |V(\mathcal{S}) \cap V(\mathcal{T}_1)| = |V(\mathcal{S}) \cap V(\mathcal{T}_2)| = |V(\mathcal{C}) \cap V(\mathcal{T}_2)| = 2$.

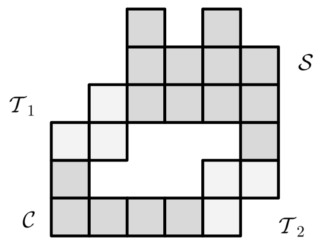


FIGURE 3.2: An example of $\mathcal{P}(\mathcal{S},\mathcal{C})$.

Remark 3.1.2. According to Proposition 2.2.5, it is easy to prove that a polyomino $\mathcal{P}(\mathcal{S},\mathcal{C})$, where $\mathcal{S}$ is a simple polyomino and $\mathcal{C}$ is an open path, is a non-simple polyomino and has only one hole. Moreover the polyomino consisting of all the cells of $\mathcal{P}(\mathcal{S},\mathcal{C})$, except two or more adjacent cells of $\mathcal{C}$, is a simple polyomino.

Theorem 3.1.3. *Let $\mathcal{P} = \mathcal{P}(\mathcal{S},\mathcal{C})$ be a polyomino with $\mathcal{S}$ a simple polyomino and $\mathcal{C}$ an open path. Suppose that $\mathcal{C}$ contains an L-configuration or a ladder of at least three steps. Then $I_{\mathcal{P}}$ is a prime ideal.*

Proof. If $\mathcal{C}$ contains an L-configuration then by defining the toric ideal as in subsection 2.2.2 we obtain the claim following the same steps as in Proposition 2.2.14 and Theorem 2.2.15, since the structure of $\mathcal{P}$ allows it. If $\mathcal{C}$ contains a ladder of at least three steps the proof is similar, considering the toric ideal in Proposition 2.2.17 and Theorem 2.2.18. ☐

Remark 3.1.4. Observe that if $\mathcal{C}$ contains an L-configuration or a ladder of at least three steps then $\mathcal{P}(\mathcal{S},\mathcal{C})$ has no zig-zag walks. The converse is not true (see Figure 3.3), so it is an open question to ask what are the conditions allowing $\mathcal{P}(\mathcal{S},\mathcal{C})$ to have no zig-zag walks. In particular, we ask if the conjecture in [31] is true also for polyominoes like $\mathcal{P}(\mathcal{S},\mathcal{C})$.

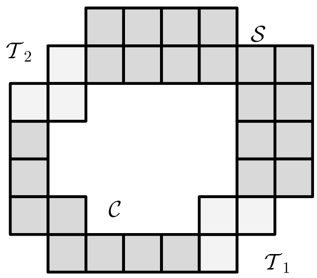


FIGURE 3.3

In [41] the author studied the polyomino ideal attached to a polyomino obtained by removing a convex polyomino from its ambient rectangle $\mathcal{R}$. Our idea is to build a non-simple polyomino adding two open paths and a simple polyomino to a rectangle $\mathcal{R}$.

Definition 3.1.5. Let $\mathcal{R}$ be a rectangle polyomino, associated to the interval $[(1,1),(m,n)]$, where $m \geq 4$ and $n \geq 2$. Let $\mathcal{S}$ be a simple polyomino, $\mathcal{P}_1 : C_1, \dots, C_t$ and $\mathcal{P}_2 : F_1, \dots, F_p$ be two open paths. A *rectangle linked to a simple polyomino by two paths*, denoted by $\mathcal{P}(\mathcal{R}, \mathcal{P}_1, S, \mathcal{P}_2)$, is a polyomino satisfying the following conditions, after opportune reflections or rotations:

1. $\mathcal{P}(\mathcal{R}, \mathcal{P}_1, S, \mathcal{P}_2) = \mathcal{R} \cup \mathcal{P}_1 \cup S \cup \mathcal{P}_2$.
2. $V(\mathcal{S}) \cap V(\mathcal{R}) = \emptyset$ and $V(\mathcal{P}_1) \cap V(\mathcal{P}_2) = \emptyset$.
3. The lower left corner of C_1 is $(1,n)$ and $V(\mathcal{P}_1) \cap V(\mathcal{R}) = \{(1,n),(2,n)\}$.
4. $E(C_t) \cap E(\mathcal{S}) = \{W\}$, where W is a free edge of C_t, and $|V(\mathcal{P}_1) \cap V(\mathcal{S})| = 2$.
5. $E(F_1) \cap E(\mathcal{S}) = \{Z\}$, where Z is a free edge of F_1, and $|V(\mathcal{P}_2) \cap V(\mathcal{S})| = 2$.
6. $E(F_p) \cap E(\mathcal{R}) = \{V\}$, where V is a free edge of F_p, and $|V(\mathcal{P}_2) \cap V(\mathcal{R})| = 2$.

Remark 3.1.6. On account of Proposition 2.2.5, a rectangle linked to a simple polyomino by two paths is not a simple polyomino and it has a unique hole. Let us denote by $\mathcal{P}_{\mathcal{R}}$ the collection of cells of $\mathcal{R}$, whose lower left corners are $(1,k)$ or $(2,k)$ for all $k = 1, \dots, n-1$, and by $\mathcal{P}_1^w$ the sequence of the first w cells of $\mathcal{P}_1$ for some $w \in \{1, \dots, t\}$. Then the polyomino consisting of all the cells of $\mathcal{P}$ except the cells of $\mathcal{P}_{\mathcal{R}} \cup \mathcal{P}_1^w$ is a simple polyomino.

Definition 3.1.7. Let $\mathcal{R}$, $\mathcal{S}$, $\mathcal{P}_1$ and $\mathcal{P}_2$ be as in the previous definition. A polyomino $\mathcal{P} = \mathcal{P}(\mathcal{R}, \mathcal{P}_1, S, \mathcal{P}_2)$ is called an *L-rectangle linked to a simple polyomino by two paths*, if

1. it satisfies all conditions in Definition 3.1.5;
2. the lower left corner of C_2 is $(1, n+1)$;
3. let V be the free edge of F_p such that $E(\mathcal{R}) \cap E(\mathcal{P}_2) = \{V\}$. Then $V \in \{\{(k,n),(k+1,n)\} : k = 3 \dots, m-1\} \cup \{\{(m,l),(m,l+1)\} : l = 1 \dots, n-1\} \cup \{\{(h,1),(h+1,1)\} : h = 3 \dots, m-1\}$.

Let V_1 and V_2 be the maximal vertical edge intervals of $\mathcal{P}$, which contain respectively the vertices $(1,n)$ and $(2,n)$. Denote by E_{V_1,V_2} the shortest maximal vertical edge interval between V_1 and V_2. Moreover, for all $k \in \{1,\dots,n\}$ let H_k be the maximal horizontal edge interval containing $(1,k)$, $\mathcal{F}_k$ the shortest one between H_k and H_{k+1} for each $k \in \{1,\dots,n-1\}$. We call $\mathcal{P}$ *good* if the following cells belong to $\mathcal{P}$:

- all cells having an edge in E_{V_1,V_2} and lying between V_1 and V_2;
- all cells having an edge in $\mathcal{F}_k$ and lying between H_k and H_{k+1}, for all $k \in \{1,\dots,n-1\}$.

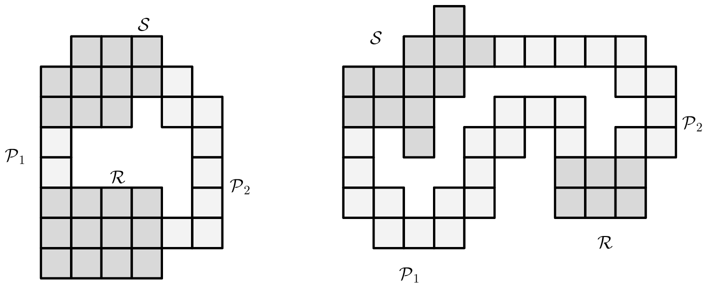


FIGURE 3.4: L-rectangles linked to a simple polyomino by two paths.

Referring to Figure 3.5, (A) and (B) are L-rectangles linked to a simple polyomino by two paths but not good, (C) is a good one, just as polyominoes in Figure 3.4 are.

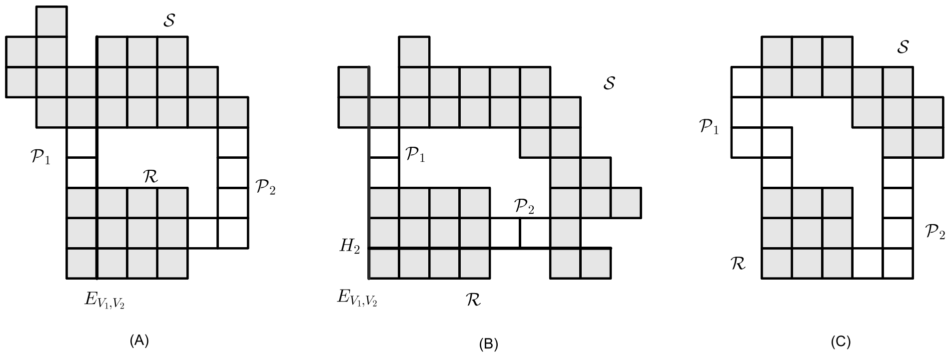


FIGURE 3.5

Proposition 3.1.8. *Let $\mathcal{P} = \mathcal{P}(\mathcal{R},\mathcal{P}_1,\mathcal{S},\mathcal{P}_2)$ be a good L-rectangle linked to a simple polyomino by two paths. Then $I_\mathcal{P}$ is prime.*

Proof. We denote by e the vertex $(2,n)$. We define the toric ideal $J_\mathcal{P}$ as done for closed paths with an L-configuration, where $V(A)$ is replaced by $A_e = \{v \in V(\mathcal{R}) : v \leq e\}$. By similar arguments as in Proposition 2.2.14 and Theorem 2.2.15, we have $I_\mathcal{P} = J_\mathcal{P}$, because of the good structure of $\mathcal{P}$. ☐

Definition 3.1.9. A polyomino $\mathcal{P}(\mathcal{R},\mathcal{P}_1,\mathcal{S},\mathcal{P}_2)$ is called a *ladder-rectangle linked to a simple polyomino by two paths*, if

1. it satisfies all conditions in Definition 3.1.5;
2. $\mathcal{P}_1$ contains two maximal horizontal blocks $[C_1, C_s]$ and $[C_{s+1}, C_q]$, where $2 \leq s < s+1 < q \leq t$, and the lower left corner of C_{s+1} is the upper left one of C_s;

3. the free edge V of F_p such that $E(\mathcal{R}) \cap E(\mathcal{P}_2) = \{V\}$ satisfies $V \in \{\{(k,n),(k+1,n)\} : k = 3,\dots,m-1\}$.

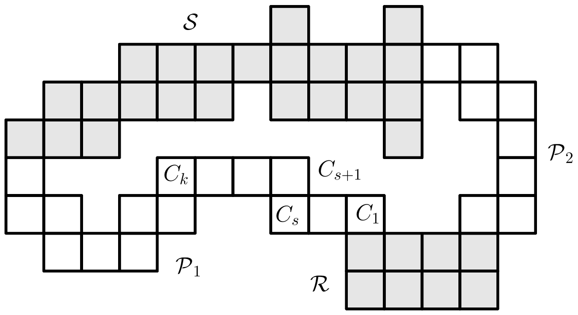


FIGURE 3.6: A ladder-rectangle linked to a simple polyomino by two paths.

Proposition 3.1.10. *Let $\mathcal{P} = \mathcal{P}(\mathcal{R},\mathcal{P}_1,\mathcal{S},\mathcal{P}_2)$ be a ladder-rectangle linked to a simple polyomino by two paths. Then $I_\mathcal{P}$ is prime.*

Proof. We denote by e the vertex $(2,n)$ and by a_i the lower left corner of the cell C_i of $\mathcal{P}_1$, for all $i \in \{1,\dots,s\}$. We define the toric ideal $J_\mathcal{P}$ as done for closed paths with a ladder having at least three steps, where $L_e = \{v \in V(\mathcal{R}) : v \leq e\} \cup \{a_2,\dots,a_s\}$. By similar arguments as in Proposition 2.2.17 and Theorem 2.2.18 we deduce that $I_\mathcal{P} = J_\mathcal{P}$. □

Remark 3.1.11. We observe that for the class of polyominoes $\mathcal{P}(\mathcal{R},\mathcal{P}_1,\mathcal{S},\mathcal{P}_2)$ the following:

1. $\mathcal{P}(\mathcal{R},\mathcal{P}_1,\mathcal{S},\mathcal{P}_2)$ is a good L-rectangle linked to a simple polyomino by two paths,

2. $\mathcal{P}(\mathcal{R},\mathcal{P}_1,\mathcal{S},\mathcal{P}_2)$ is a ladder-rectangle linked to a simple polyomino by two paths,

are sufficient conditions in order that it does not contain zig-zag walks. Necessary conditions to have no zig-zag walks and a positive answer to the conjecture in [31] for polyominoes like $\mathcal{P}(\mathcal{R},\mathcal{P}_1,\mathcal{S},\mathcal{P}_2)$ are open questions.

3.2 Primality of simple and weakly connected collections of cells

In this section we study the primality of weakly connected collections of cells and of a new class of non-simple polyominoes, called *weakly closed paths* ([8]). Firstly we define a bipartite graph $G(\mathcal{P})$ attached to a weakly connected and simple collection $\mathcal{P}$ of cells of $\mathbb{Z}^2$ and we show that the ideal of the inner 2-minors of $\mathcal{P}$ coincides with the toric ideal attached to the edge ring of $G(\mathcal{P})$. This result generalizes Theorem 3.10 of [39]. Finally we conjecture that absence of zig-zag walks in a weakly connected collection of cells characterizes its primality and, as an application, we characterize the primality of weakly closed paths.

Let $\mathcal{P}$ be a weakly connected collection of cells of $\mathbb{Z}^2$. Let $\{V_i\}_{i\in I}$ be the sets of the maximal vertical edge intervals of $\mathcal{P}$ and $\{H_j\}_{j\in J}$ be the set of the maximal horizontal edge intervals of $\mathcal{P}$. Let $\{v_i\}_{i\in I}$ and $\{h_j\}_{j\in J}$ be two sets of variables associated respectively to $\{V_i\}_{i\in I}$ and $\{H_j\}_{j\in J}$. We associate to $\mathcal{P}$ a bipartite graph $G(\mathcal{P})$, whose vertex set is $V(G(\mathcal{P})) = \{v_i\}_{i\in I} \sqcup \{h_j\}_{j\in J}$ and edge set is $E(G(\mathcal{P})) = \big\{\{v_i, h_j\} | V_i \cap H_j \in V(\mathcal{P})\big\}$. For instance, Figure 3.7 illustrates a collection of cells $\mathcal{P}$ on the left and its associated bipartite graph $G(\mathcal{P})$ on the right.

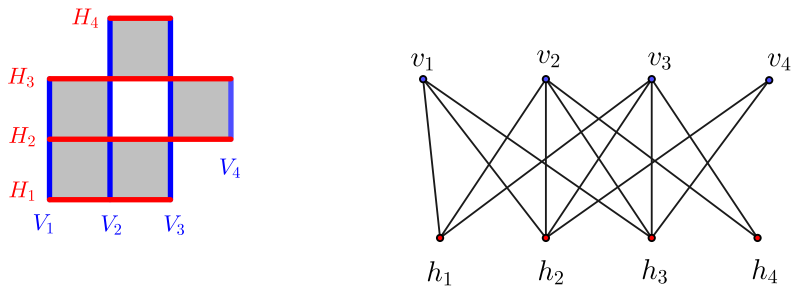


FIGURE 3.7

In the bipartite graph $G(\mathcal{P})$ a cycle $\mathcal{C}_{G(\mathcal{P})}$ of length $2r$ is a subset $\{v_{i_1}, h_{j_1}, \dots, v_{i_{r-1}}, h_{j_{r-1}}, v_{i_r}, h_{j_r}\}$ of distinct vertices of $V(G(\mathcal{P}))$ such that $\{v_{i_k}, h_{j_k}\}$ and $\{h_{j_k}, v_{i_{k+1}}\}$ belong to $E(G(\mathcal{P}))$ for all $k = 1, \dots, r$, where $i_{r+1} = i_1$. Since $\{v_{i_k}, h_{j_k}\} \in E(G(\mathcal{P}))$, $V_{i_k} \cap H_{j_k}$ is a vertex of $\mathcal{P}$ for all $k = 1, \dots, r$; similarly, since $\{h_{j_k}, v_{i_{k+1}}\} \in E(G(\mathcal{P}))$, $V_{i_{k+1}} \cap H_{j_k}$ is a vertex of $\mathcal{P}$ for all $k = 1, \dots, r$, where $i_{r+1} = i_1$. We can associate to each cycle $\mathcal{C}_{G(\mathcal{P})}$ in $G(\mathcal{P})$ the following binomial:

$$f_{\mathcal{C}_{G(\mathcal{P})}} = x_{V_{i_1} \cap H_{j_1}} \dots x_{V_{i_r} \cap H_{j_r}} - x_{V_{i_2} \cap H_{j_1}} \dots x_{V_{i_1} \cap H_{j_r}}$$

Following [25], we recall the definition of a cycle in $\mathcal{P}$. A cycle $\mathcal{C}_{\mathcal{P}}$ in $\mathcal{P}$ is a sequence $a_1, \dots, a_m$ of vertices of $\mathcal{P}$ such that:

1. $a_1 = a_m$;
2. $a_i \neq a_j$ for all $i \neq j$ with $i, j \in \{1, \dots, m-1\}$;
3. $[a_i, a_{i+1}]$ is a horizontal or vertical edge interval of $\mathcal{P}$ for all $i = 1, \dots, m-1$;
4. for all $i = 1, \dots, m$, if $[a_i, a_{i+1}]$ is a horizontal edge interval of $\mathcal{P}$, then $[a_{i+1}, a_{i+2}]$ is a vertical edge interval of $\mathcal{P}$ and vice versa, with $a_{m+1} = a_2$.

The vertices $a_1 \dots, a_{m-1}$ of $\mathcal{P}$ are called *vertices of* $\mathcal{C}_{\mathcal{P}}$ and we set $V(\mathcal{C}_{\mathcal{P}}) = \{a_1, \dots, a_{m-1}\}$. It follows from the definition of a cycle that m is odd, so we can consider the following binomial

$$f_{\mathcal{C}_{\mathcal{P}}} = \prod_{k=1}^{(m-1)/2} x_{a_{2k-1}} - \prod_{k=1}^{(m-1)/2} x_{a_{2k}}$$

and we can attach to each cycle $\mathcal{C}_{\mathcal{P}}$ in $\mathcal{P}$ the binomial $f_{\mathcal{C}_{\mathcal{P}}}$. Moreover, a cycle in $\mathcal{P}$ is called *primitive* if each maximal edge interval of $\mathcal{P}$ contains at most two vertices of $\mathcal{C}_{\mathcal{P}}$.

Remark 3.2.1. Arguing as in Section 1 of [39], a cycle $\mathcal{C}_{G(\mathcal{P})} = \{v_{i_1}, h_{j_1}, v_{i_2}, h_{j_2} \dots, v_{i_{r-1}}, h_{j_{r-1}}, v_{i_r}, h_{j_r}\}$ of the bipartite graph $G(\mathcal{P})$ associated to $\mathcal{P}$ defines a primitive cycle $\mathcal{C}_{\mathcal{P}} : V_{i_1} \cap H_{j_1}, V_{i_2} \cap H_{j_1}, V_{i_2} \cap H_{j_2}, \dots, V_{i_r} \cap H_{j_r}, V_{i_1} \cap H_{j_r}, V_{i_1} \cap H_{j_1}$ in $\mathcal{P}$ and vice versa. Moreover, we have also $f_{\mathcal{C}_{G(\mathcal{P})}} = f_{\mathcal{C}_{\mathcal{P}}}$.

Recall that a graph is called *weakly chordal* if every cycle of length greater than 4 has a chord. According to [39], if $\mathcal{C}_{\mathcal{P}} : a_1, \dots, a_m$ is a cycle in $\mathcal{P}$ then $\mathcal{C}_{\mathcal{P}}$ has a *self-crossing* if there exist two indices $i, j \in \{1, \dots, m-1\}$ such that:

1. $a_i, a_{i+1} \in V_k$ and $a_j, a_{j+1} \in H_l$ for some $k \in I$ and $l \in J$;
2. $a_i, a_{i+1}, a_j, a_{j+1}$ are all distinct;
3. $V_k \cap H_l \neq \emptyset$.

In such a case, as in Section 2 of [39], if $\mathcal{C}_{\mathcal{P}}$ is a primitive cycle in $\mathcal{P}$ having a self-crossing, then $\mathcal{C}_{G(\mathcal{P})}$ has a chord.

Moreover in [39] the authors show that the polyomino ideal attached to a simple polyomino is the toric ideal of the edge ring of the weakly chordal graph $G(\mathcal{P})$. Now, we give a generalization of these results, which will be useful and crucial later.

Proposition 3.2.2. *Let $\mathcal{P}$ be a weakly connected and simple collection of cells. Then $G(\mathcal{P})$ is weakly chordal.*

Proof. We may assume that $\mathcal{P}$ has two connected components, denoted by $\mathcal{P}_1$ and $\mathcal{P}_2$. The arguments are similar if $\mathcal{P}$ has more than two connected components. Let $V(\mathcal{P}_1) \cap V(\mathcal{P}_2) = \{\tilde{v}\}$. Let $\mathcal{C}_{G(\mathcal{P})} = \{v_{i_1}, h_{j_1}, \dots, v_{i_r}, h_{j_r}\}$ be a cycle of $G(\mathcal{P})$ of length $2r$ with $r \geq 3$. By Remark 3.2.1 we obtain that $\mathcal{C}_{G(\mathcal{P})}$ defines a primitive cycle in $\mathcal{P}$

$$\mathcal{C}_{\mathcal{P}} : V_{i_1} \cap H_{j_1}, V_{i_2} \cap H_{j_1}, V_{i_2} \cap H_{j_2}, \dots, V_{i_r} \cap H_{j_r}, V_{i_1} \cap H_{j_r}, V_{i_1} \cap H_{j_1}$$

We set $a_1 = V_{i_1} \cap H_{j_1}, a_2 = V_{i_2} \cap H_{j_1}, \dots, a_{2r-1} = V_{i_r} \cap H_{j_r}, a_{2r} = V_{i_1} \cap H_{j_r}, a_{2r+1} = V_{i_1} \cap H_{j_1}$. We distinguish two different cases. Firstly, we suppose that all vertices of $\mathcal{C}_{\mathcal{P}}$ are either in $V(\mathcal{P}_1)$ or $V(\mathcal{P}_2)$. We may assume that $a_k \in V(\mathcal{P}_1)$ for all $k = 1, \dots, 2r$. We prove that $\mathcal{C}_{G(\mathcal{P})}$ has a chord. Observe that $\mathcal{P}_1$ is a simple polyomino, otherwise $\mathcal{P}$ is not a simple collection of cells. Consider the bipartite graph $G(\mathcal{P}_1)$ attached to $\mathcal{P}_1$. By Lemma 2.1 in [39] it follows that $G(\mathcal{P}_1)$ is weakly chordal, hence the cycle $\mathcal{C}_{G(\mathcal{P})}$ has a chord.
In the second case, we suppose that there exist two distinct vertices different from $\tilde{v}$, one belonging to $V(\mathcal{P}_1)$ and the other to $V(\mathcal{P}_2)$. We prove that $\mathcal{C}_{\mathcal{P}}$ has a self-crossing. We denote by $V_{\tilde{v}}$ and $H_{\tilde{v}}$ respectively the vertical and horizontal maximal edge intervals of $\mathcal{P}$, such that $V_{\tilde{v}} \cap H_{\tilde{v}} = \{\tilde{v}\}$. It is not restrictive to assume that $a_1 \in V(\mathcal{P}_1) \backslash \{\tilde{v}\}$. Let i be the smallest integer such that $a_i \in V(\mathcal{P}_1)$ and $a_{i+1} \in V(\mathcal{P}_2)$. We can assume that $[a_i, a_{i+1}]$ is a horizontal interval of $\mathcal{P}$, so $[a_i, a_{i+1}]$ is contained in $H_{\tilde{v}}$ and it is obvious that $\tilde{v} \in [a_i, a_{i+1}]$. We note that $a_{2r+1} = a_1 \in V(\mathcal{P}_1) \backslash \{\tilde{v}\}$. Then there exists $p \in \{i+2, \dots, 2r\}$ such that $\tilde{v} \in [a_p, a_{p+1}]$, with $a_p, a_{p+1} \notin \{\tilde{v}\}$. Moreover, we note that from the primitivity of $\mathcal{C}_{\mathcal{P}}$ it follows immediately that $[a_p, a_{p+1}] \subseteq V_{\tilde{v}}$. Hence we obtain that there exist two distinct indices $i, p \in \{1, \dots, 2r\}$ such that:

1. $a_i, a_{i+1} \in H_{\tilde{v}}$ and $a_p, a_{p+1} \in V_{\tilde{v}}$;
2. $a_i, a_{i+1}, a_p, a_{p+1}$ are all distinct because they are the vertices of a primitive cycle in $\mathcal{P}$;
3. $V_{\tilde{v}} \cap H_{\tilde{v}} \neq \emptyset$ because obviously $V_{\tilde{v}} \cap H_{\tilde{v}} = \{\tilde{v}\}$.

In conclusion, $\mathcal{C}_{\mathcal{P}}$ has a self-crossing and as a consequence $\mathcal{C}_{G(\mathcal{P})}$ has a chord. $\square$

We define the following map:

$$\begin{aligned}\alpha : V(\mathcal{P}) &\longrightarrow K[\{v_i, h_j\} : i \in I, j \in J]\\ r &\longmapsto v_i h_j\end{aligned}$$

with $r \in V_i \cap H_j$. The toric ring $K[\alpha(v) : v \in V(\mathcal{P})]$ can be viewed as the edge ring of $G(\mathcal{P})$ and it is denoted by $K[G(\mathcal{P})]$. Let S be the polynomial ring $K[x_r : r \in V(\mathcal{P})]$ and let us consider the following surjective ring homomorphism:

$$\begin{aligned}\phi : S &\longrightarrow K[G(\mathcal{P})]\\ \phi(x_r) &= \alpha(r)\end{aligned}$$

The toric ideal $J_{\mathcal{P}}$ is the kernel of ϕ. It is known from Theorem 1.4.18 that if the bipartite graph $G(\mathcal{P})$ is weakly chordal then the associated toric ideal $J_{\mathcal{P}}$ is minimally generated by quadratic binomials attached to the cycles of $G(\mathcal{P})$ of length 4.

Theorem 3.2.3. *Let $\mathcal{P}$ be a weakly connected and simple collection of cells. Then $I_{\mathcal{P}} = J_{\mathcal{P}}$.*

Proof. Assume that $\mathcal{P}$ consists of the connected components $\mathcal{P}_1, \dots, \mathcal{P}_m$, with $m \geq 1$. We prove firstly that $I_{\mathcal{P}} \subseteq J_{\mathcal{P}}$. Let f be a generator of $I_{\mathcal{P}}$, so there exists an inner interval $[a, b]$ of $\mathcal{P}$, such that $f = x_a x_b - x_c x_d$, where c, d are the anti-diagonal corners of $[a, b]$. It is clear that $\phi(x_a x_b) = \alpha(a)\alpha(b) = \alpha(c)\alpha(d) = \phi(x_c x_d)$, so $f \in J_{\mathcal{P}}$. Therefore $I_{\mathcal{P}} \subseteq J_{\mathcal{P}}$. We prove that $J_{\mathcal{P}} \subseteq I_{\mathcal{P}}$. By Proposition 3.2.2 the bipartite graph $G(\mathcal{P})$ attached to $\mathcal{P}$ is weakly chordal, so $J_{\mathcal{P}}$ is generated minimally by quadratic binomials attached to cycles of $G(\mathcal{P})$ of length 4. Let f be a generator of $J_{\mathcal{P}}$. Then there exists a cycle of $G(\mathcal{P})$ of length 4, $\mathcal{C}_{G(\mathcal{P})} : v_{i_1}, h_{j_1}, v_{i_2}, h_{j_2}$, such that $f = f_{\mathcal{C}_{G(\mathcal{P})}}$. By Remark 3.2.1 $\mathcal{C}_{G(\mathcal{P})}$ defines the following primitive cycle in $\mathcal{P}$:

$$\mathcal{C}_{\mathcal{P}} : V_{i_1} \cap H_{j_1}, V_{i_2} \cap H_{j_1}, V_{i_2} \cap H_{j_2}, V_{i_1} \cap H_{j_2}, V_{i_1} \cap H_{j_1}.$$

We set $a_1 = V_{i_1} \cap H_{j_1}, a_2 = V_{i_2} \cap H_{j_1}, a_3 = V_{i_2} \cap H_{j_2}, a_4 = V_{i_1} \cap H_{j_2}$ and we have $f = f_{\mathcal{C}_{G(\mathcal{P})}} = f_{\mathcal{C}_{\mathcal{P}}}$. Since $\mathcal{P}$ is a simple collection of cells and $f = f_{\mathcal{C}_{\mathcal{P}}}$, there exists $j \in \{1, \dots, m\}$ such that $a_i \in V(\mathcal{P}_j)$ for all $i = 1, 2, 3, 4$. Consider the map ϕ' as the restriction of ϕ to $K[x_a : a \in V(\mathcal{P}_j)]$ and we denote by $J_{\mathcal{P}_j}$ the kernel of ϕ'. By Theorem 2.2 in [39] it follows that $J_{\mathcal{P}_j} = I_{\mathcal{P}_j}$, where $I_{\mathcal{P}_j}$ is the polyomino ideal associated to $\mathcal{P}_j$. Hence we have $f \in J_{\mathcal{P}_j} = I_{\mathcal{P}_j} \subseteq I_{\mathcal{P}}$. Therefore $J_{\mathcal{P}} \subseteq I_{\mathcal{P}}$. $\square$

Remark 3.2.4. We observe that there exist weakly connected and non-simple collections of cells that are not prime. The collection of cells in Figure 3.8 (A) is non-simple and weakly connected with four connected components but it is not prime. Its non-primality follows by [37, case (2) of Theorem 3.2, Corollary 3.6]. Conversely, in Figure 3.8 (B) there is a weakly connected and non-simple collection of cells which is prime. For the proof of its primality we refer to Remark 3.3.6.

In according to previous arguments it is natural to generalize the conjecture given in [31] for weakly connected collections of cells.

Conjecture 3.2.5. *Let $\mathcal{P}$ be a weakly connected collection of cells. The following are equivalent:*

1. *$I_{\mathcal{P}}$ is prime;*
2. *$\mathcal{P}$ has no zig-zag walks.*

(A) (B)

FIGURE 3.8

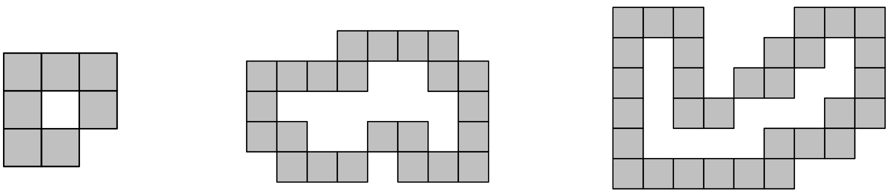
3.3 An application on the characterization of prime weakly closed paths

Here, we introduce a new class of polyominoes, which we call weakly closed path polyominoes. As an application of Theorem 3.2.3 and by using similar techniques of Section 2.2, we characterize all weakly closed paths having no zig-zag walks and their primality.

Definition 3.3.1. A finite non-empty collection of cells $\mathcal{P}$ is called a *weakly closed path* if it is a path of n cells $A_1, \dots, A_{n-1}, A_n = A_0$ with $n > 6$ such that:

1. $|V(A_0) \cap V(A_1)| = 1$;
2. $V(A_2) \cap V(A_0) = V(A_{n-1}) \cap V(A_1) = \emptyset$;
3. $V(A_i) \cap V(A_j) = \emptyset$ for all $i \in \{1, \dots, n\}$ and for all $j \notin \{i-2, i-1, i, i+1, i+2\}$, where the indices are reduced modulo n.

We call the unique vertex v_H in $V(A_0) \cap V(A_1)$ a *hooking corner*. Note that a weakly closed path is a non-simple polyomino having a unique hole. In Figure 3.9 there are some examples of weakly closed paths.
The difference between a closed path and a weakly closed path is subtle but quite deep. In fact in a closed path it is possible to order the cells in such a way that every cell has an edge in common with its consecutive cell. In a weakly closed path the same holds, with the exception of exactly two consecutive cells that have just a vertex in common. These polyominoes, as well as the closed paths, are particular thin polyominoes, which are polyominoes not containing the square tetromino. Moreover, observe that not all weakly closed paths can be obtained by removing a suitable cell from a closed path polyomino. In fact, let $\mathcal{P}$ be the weakly closed path in Figure 3.9 (C) and A and B be respectively the cells not in $\mathcal{P}$ having the hooking vertex respectively as lower right corner and upper left one. Then neither $\mathcal{P} \cup \{A\}$ nor $\mathcal{P} \cup \{B\}$ is a closed path polyomino.



(A) (B) (C)

FIGURE 3.9: Examples of weakly closed path polyominoes.

Let $\mathcal{P}$ be a polyomino. A *weak L-configuration* is a finite collection of cells of $\mathcal{P}$ such that:

1. it consists of a maximal horizontal (resp. vertical) block $[A, B]$ of length two, a vertical (resp. horizontal) block $[D, F]$ of length at least two and a cell C of $\mathcal{P}$, not belonging to $[A, B] \sqcup [D, F]$;
2. $V(C) \cap V([A, B]) = \{a_1\}$ and $V([D, F]) \cap V([A, B]) = \{a_2, b_2\}$, where $a_2 \neq b_2$;
3. $[a_2, b_2]$ is on the same maximal horizontal (resp. vertical) edge interval of $\mathcal{P}$ containing a_1 (see Figure 3.10).

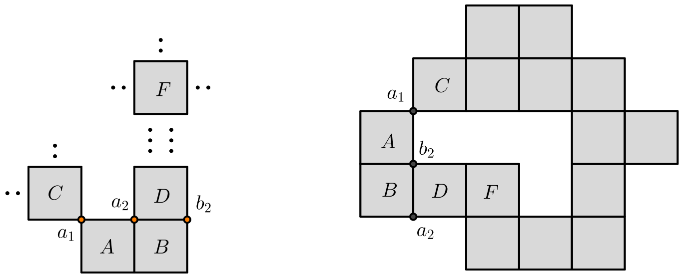


FIGURE 3.10: A Weak L-configuration and a polyomino containing a weak L-configuration

A finite collection of cells of $\mathcal{P}$, made up of a maximal horizontal (resp. vertical) block $[A, B]$ of $\mathcal{P}$ of length at least two and two distinct cells C and D of $\mathcal{P}$, not belonging to $[A, B]$, with $V(C) \cap V([A, B]) = \{a_1\}$ and $V(D) \cap V([A, B]) = \{a_2, b_2\}$ where $a_2 \neq b_2$, is called a *weak ladder* if $[a_2, b_2]$ is not on the same maximal horizontal (resp. vertical) edge interval of $\mathcal{P}$ containing a_1 (see Figure 3.11).

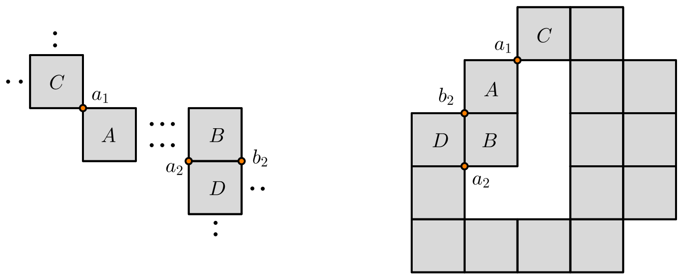


FIGURE 3.11: A Weak ladder and a polyomino containing a weak ladder

As introduced in Section 2.2, we say that a path of five cells C_1, C_2, C_3, C_4, C_5 of $\mathcal{P}$ is an *L-configuration* if the two sequences C_1, C_2, C_3 and C_3, C_4, C_5 go in two orthogonal directions. A set $\mathcal{B} = \{\mathcal{B}_i\}_{i=1,\dots,n}$ of maximal horizontal (or vertical) blocks of length at least two, with $V(\mathcal{B}_i) \cap V(\mathcal{B}_{i+1}) = \{a_i, b_i\}$ and $a_i \neq b_i$ for all $i = 1, \dots, n-1$, is called a *ladder of n steps* if $[a_i, b_i]$ is not on the same edge interval of $[a_{i+1}, b_{i+1}]$ for all $i = 1, \dots, n-2$. For instance, in Figure 3.12 we represent a polyomino with an L-configuration on the left and a polyomino having a ladder of three steps on the right.

Proposition 3.3.2. *Let $\mathcal{P}$ be a weakly closed path. If one of the following conditions holds:*

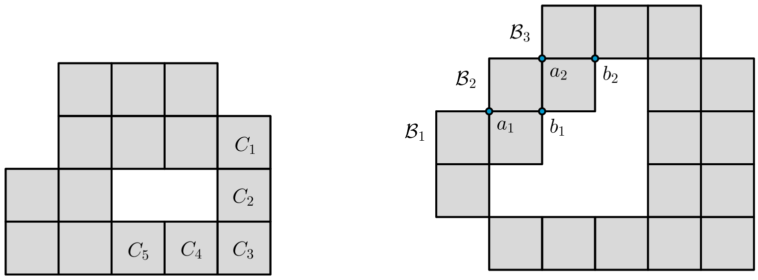


FIGURE 3.12: An example of L-configuration and horizontal ladder of three steps.

1. *$\mathcal{P}$ has a weak L-configuration,*
2. *$\mathcal{P}$ has a weak ladder,*
3. *$\mathcal{P}$ has an L-configuration,*
4. *$\mathcal{P}$ has a ladder of at least three steps,*

then $\mathcal{P}$ does not contain zig-zag walks.

Proof. (1) Suppose that $\mathcal{P}$ has a weak L-configuration. Assume that the weak L-configuration is as in the picture on the left in Figure 3.10, otherwise we apply suitable reflections or rotations in order to have it in such a position. Suppose that there exists a sequence $\mathcal{W} : I_1, \dots, I_\ell$ of distinct inner intervals of $\mathcal{P}$, where for all $i = 1, \dots, \ell$ the interval I_i has diagonal corners v_i, z_i and anti-diagonal corners u_i, v_{i+1}, such that $I_1 \cap I_\ell = \{v_1 = v_{\ell+1}\}$ and $I_i \cap I_{i+1} = \{v_{i+1}\}$, for all $i = 1, \dots, \ell - 1$. We may assume that v_i and v_{i+1} are on the same edge interval of $\mathcal{P}$ for all $i = 1, \dots, \ell$, otherwise we have finished. Observe that there exists $i \in \{1, \dots, \ell\}$ such that $C \in I_i$ and $I_i \cap I_r = \{a_1\}$ where $r = i - 1$ or $r = i + 1$. In such a case it is not restrictive to assume $1 < i < \ell - 1$ and $r = i + 1$. It follows from the shape of the weak L-configuration that $I_{i+1} \cap I_{i+2} = \{a_2\}$ and z_{i+1} is the lower right corner of A. The corner z_{i+2} of I_{i+2} belongs to $V([D, F])$ and is on the vertical edge interval of $\mathcal{P}$ containing b_2, so $[B, F]$ is an inner interval such that $z_{i+1}, z_{i+2} \in V([B, F])$. Therefore $\mathcal{P}$ cannot contain zig-zag walks.
(2) Suppose that $\mathcal{P}$ has a weak ladder. Assume that there exists a sequence $\mathcal{W} : I_1, \dots, I_\ell$ of distinct inner intervals of $\mathcal{P}$ such that $I_i \cap I_{i+1} = \{v_{i+1}\}$ for all $i = 1, \dots, \ell - 1$ and $I_1 \cap I_\ell = \{v_1 = v_{\ell+1}\}$. Then there exists $i \in \{1, \dots, n\}$ such that $C \in I_i$ and $I_i \cap I_r = \{a_1\}$ where $r = i - 1$ or $r = i + 1$. It is not restrictive to assume $1 < i < \ell - 1$ and $r = i + 1$. For the shape of the weak ladder we have either $I_{i+1} \cap I_{i+2} = \{a_2\}$ or $I_{i+1} \cap I_{i+2} = \{b_2\}$. Then v_{i+1} and v_{i+2} are not on the same edge interval of $\mathcal{P}$, so $\mathcal{P}$ cannot contain zig-zag walks.
(3) If $\mathcal{P}$ has an L-configuration, then we have the desired conclusion by arguing as done in (1).
(4) If $\mathcal{P}$ has a ladder of at least three steps, then the claim follows similarly as done in (2). □

Theorem 3.3.3. *Let $\mathcal{P}$ be a weakly closed path. The following conditions are equivalent:*

1. *$\mathcal{P}$ has an L-configuration or a ladder of at least three steps or a weak L-configuration or a weak ladder;*
2. *$\mathcal{P}$ does not contain zig-zag walks.*

Proof. The sufficient condition follows immediately from Proposition 3.3.2. We prove the necessary one arguing by contradiction. Suppose that $\mathcal{P}$ has no L-configuration, no ladder of at least three steps, no weak L-configuration and no weak ladder and we show how it is possible to find a zig-zag walk in $\mathcal{P}$. We may assume that v_H is respectively the lower right corner of A_0 and the upper left corner of A_1. Let $\mathcal{B}_1$ be the maximal horizontal or vertical block of $\mathcal{P}$ containing A_1. We may assume that $\mathcal{B}_1$ is in horizontal position, because similar arguments hold in the other case. We set $I_1 = V(\mathcal{B}_1)$ and $v_1 = v_H$ and z_1 as anti-diagonal corners. Let A_m be the cell of $\mathcal{P}$ such that $[A_1, A_m] = \mathcal{B}_1$ for some $m \in \{2, \ldots, n\}$. For the cell A_{m+1} the following cases are possible:

1. A_{m+1} is at East of A_m. It is a contradiction to the maximality of $\mathcal{B}_1$;
2. A_{m+1} is at South of A_m. Then $\{A_0\} \cup \mathcal{B}_1 \cup \{A_{m+1}\}$ is a weak ladder, so it is a contradiction;
3. A_{m+1} is at West of A_m. Then $A_{m+1} = A_{m-1}$, so it is a contradiction to Definition 3.3.1.

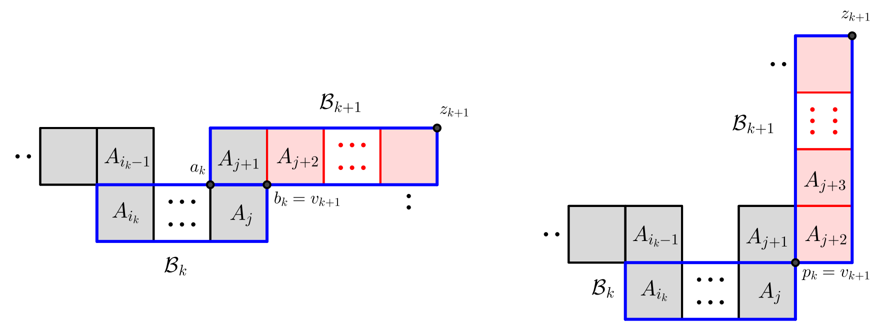
Necessarily A_{m+1} is at North of A_m. Now we consider the cell A_{m+2}, and we examine its positions with respect to A_{m+1}:

1. A_{m+2} is at West of A_{m+1}. It is a contradiction to (3) of Definition 3.3.1;
2. A_{m+2} is at North of A_{m+1}. Then $\{A_0\} \cup \mathcal{B}_1 \cup [A_{m+1}, A_{m+2}]$ is a weak L-configuration if $|\mathcal{B}_1| = 2$ or $\mathcal{B}_1 \cup [A_{m+1}, A_{m+2}]$ contains an L-configuration if $|\mathcal{B}_1| > 2$, so we have a contradiction in both cases;
3. A_{m+2} is at South of A_{m+1}. Then $A_{m+2} = A_m$, so it is a contradiction.

Necessarily A_{m+2} is at East of A_{m+1}. We observe that the cell A_{m+3} can be at North or at East of A_{m+2}. If A_{m+3} is at North of A_{m+2}, then by previous arguments A_{m+4} is also at North of A_{m+3}, so we denote by $\mathcal{B}_2$ the maximal vertical block containing $\{A_{m+2}, A_{m+3}, A_{m+4}\}$ and $V(\mathcal{B}_1) \cap V(\mathcal{B}_2) = \{p_1\}$; in such a case we set $I_2 = V(\mathcal{B}_2)$ having $v_2 = p_1$ and z_2 as diagonal corners. If A_{m+3} is at East of A_{m+2}, then A_{m+4} is also at East of A_{m+3}, so we denote by $\mathcal{B}_2$ the maximal horizontal block containing $\{A_{m+2}, A_{m+3}, A_{m+4}\}$ and $V(\mathcal{B}_1) \cap V(\mathcal{B}_2) = \{a_1, b_1\}$, with $a_1 < b_1$; in such a case we set $I_2 = V(\mathcal{B}_2 \setminus \{A_{m+1}\})$ having $v_2 = b_1$ and z_2 as diagonal corners. In both cases $|\mathcal{B}_2| \geq 3$. Let $\mathcal{B}$ be the maximal block containing A_n and let A_p be the other extremal cell of $\mathcal{B}$ for some $p \leq n-1$. If $\mathcal{B}$ is in vertical position, then A_{p-1} is at East of A_p and A_{p-2} is at North of A_{p-1} by similar arguments. Similarly, if $\mathcal{B}$ is in horizontal position, then A_{p-1} is at South of A_p and A_{p-2} is at West of A_{p-1}. Moreover it is easy to see that A_{p-2} is a cell of a maximal block of $\mathcal{P}$ of length at least three, denoted by $\mathcal{B}_f$.

Now, starting from $\mathcal{B}_2$, we define inductively a sequence of maximal blocks of $\mathcal{P}$ and, as a consequence, a sequence of inner intervals of $\mathcal{P}$. Let $\mathcal{B}_k$ be a maximal block of $\mathcal{P}$ of length at least three. We may assume that $\mathcal{B}_k$ is in horizontal position and that there exist A_{i_k} and $A_{i_{k+1}}$ with $i_k < i_{k+1}$ such that $\mathcal{B}_k = [A_{i_k}, A_{i_k+1}]$, otherwise we can apply appropriate reflections or rotations. For convenience we set $j = i_k + 1$. In order to define $\mathcal{B}_{i+1}$, we distinguish two cases, which depend on the position of A_{i_k-1} with respect to A_{i_k}. Assume that A_{i_k-1} is at North of A_{i_k} and observe that A_{i_k-2} is necessarily at West of A_{i_k-1}, otherwise $\{A_{i_k-2}, A_{i_k-1}\} \cup \mathcal{B}_k$ contains an L-configuration or Definition 3.3.1 is contradicted. Consider the cell A_{j+1}, so A_{j+1} is at North of A_j. In fact, if A_{j+1} is at South of A_j, then either $\{A_{i_k-2}, A_{i_k-1}\} \cup \mathcal{B}_k \cup \{A_{j+1}, A_{j+2}\}$ is a ladder of three steps or $\mathcal{B}_k \cup \{A_{j+1}, A_{j+2}\}$ contains an L-configuration or Definition 3.3.1 is contradicted. By similar arguments we deduce that A_{j+2} is at East of A_{j+1}. Now we can define the maximal block $\mathcal{B}_{k+1}$, depending on the position of A_{j+3}.

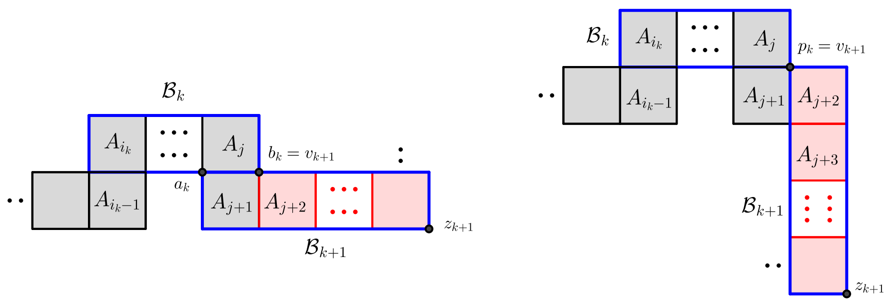
- If A_{j+3} is at East of A_{j+2}, then we denote by $\mathcal{B}_{k+1}$ the maximal horizontal block of $\mathcal{P}$ containing $\{A_{j+1}, A_{j+2}, A_{j+3}\}$. In such a case $|V(\mathcal{B}_k) \cap V(\mathcal{B}_{k+1})| = 2$ and we set $V(\mathcal{B}_k) \cap V(\mathcal{B}_{k+1}) = \{a_k, b_k\}$ with $a_k < b_k$. Hence we put $I_{k+1} = V(\mathcal{B}_{k+1} \setminus \{A_{j+1}\})$ and $v_{k+1} = b_k$, z_{k+1} as diagonal corners (see Figure 3.13 (A)).
- If A_{j+3} is at North of A_{j+2}, then A_{j+4} is at North of A_{j+3}, otherwise we have a contradiction, since $B_k \cup \{A_{j+1}, A_{j+2}\} \cup \{A_{j+3}, A_{j+4}\}$ would be a ladder of three steps. So we denote by $\mathcal{B}_{k+1}$ the maximal vertical block of $\mathcal{P}$ containing $\{A_{j+2}, A_{j+3}, A_{j+4}\}$. In such a case $|V(\mathcal{B}_k) \cap V(\mathcal{B}_{k+1})| = 1$ and we set $V(\mathcal{B}_k) \cap V(\mathcal{B}_{k+1}) = \{p_k\}$. Hence we put $I_{k+1} = V(\mathcal{B}_{k+1})$ having $v_{k+1} = p_k$, z_{k+1} as diagonal corners (see Figure 3.13 (B)).



(A) (B)

FIGURE 3.13

Assume that A_{i_k-1} is at South of A_{i_k}. By similar arguments, we can define the maximal block $\mathcal{B}_{k+1}$ and the inner interval I_{k+1}, which has v_{k+1} and z_{k+1} as anti-diagonal corners, as done in the previous case (see Figure 3.14). Observe that there exists a



(A) (B)

FIGURE 3.14

configuration in which $\mathcal{B}_k$ is in horizontal position and $\mathcal{B}_{k+1}$ is in vertical position, otherwise we have a contradiction to (1) of Definition 3.3.1. Starting from $k = 2$ and using the procedure described before, we define the sequence of maximal block $\mathcal{B}_2, \mathcal{B}_3, \dots$ and, since $\mathcal{P}$ is a weakly closed path, in particular a path from A_1 to A_{p-2}, then there exists $s \in \mathbb{N}$ such that $\mathcal{B}_s = \mathcal{B}_f$. We set $\mathcal{B}_{s+1} = \mathcal{B}$ and we observe that the only possible arrangements of the blocks $\mathcal{B}_s, \mathcal{B}_{s+1}$ and $\mathcal{B}_1$ are displayed in Figure 3.15

and 3.16. In particular, in the configurations of Figure 3.15 we put $I_{s+1} = V(\mathcal{B}_{s+1})$ having $v_{s+1} = p_s$, z_{s+1} as diagonal corners, and in the configurations of Figure 3.16 we put $I_{s+1} = V(\mathcal{B}_{s+1} \setminus \{A_p\})$ having $v_{s+1} = b_s$, z_{s+1} as diagonal corners.

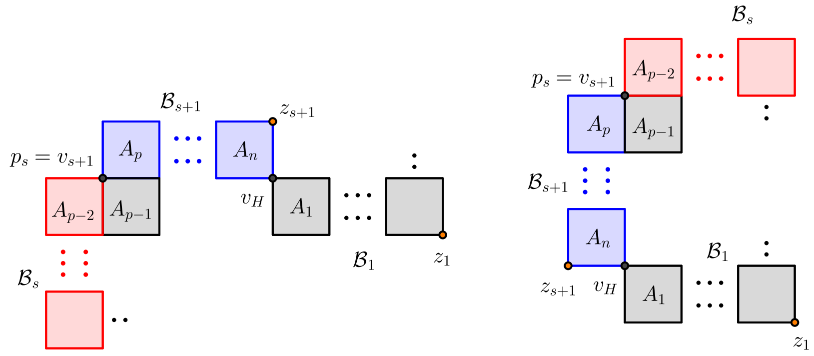


FIGURE 3.15

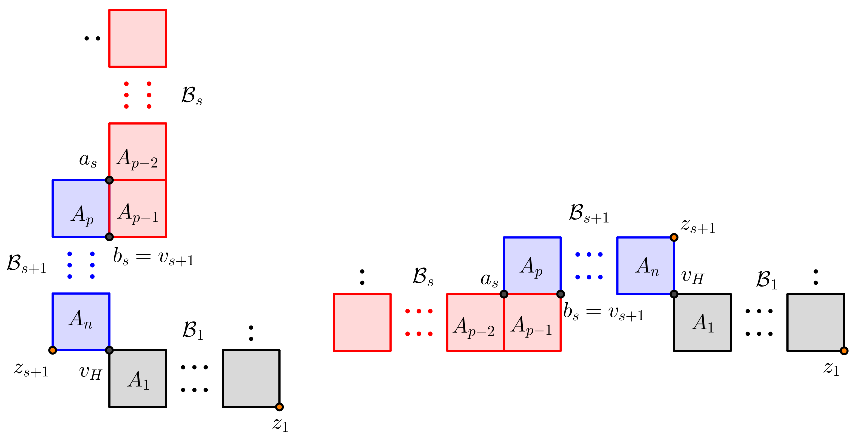
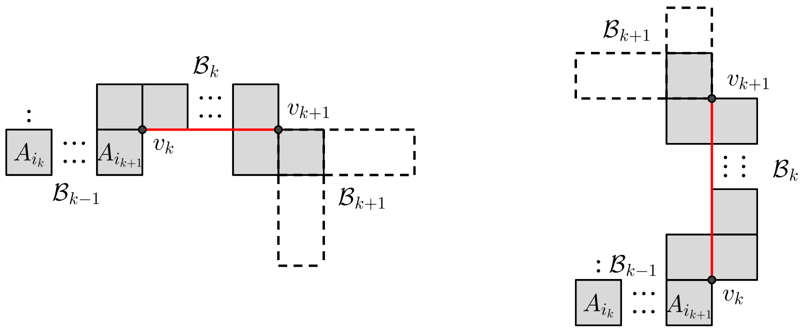


FIGURE 3.16

Hence there exists a sequence of maximal blocks $\mathcal{B}_1, \ldots, \mathcal{B}_s, \mathcal{B}_{s+1}$ of $\mathcal{P}$ with $V(\mathcal{B}_1) \cap V(\mathcal{B}_{s+1}) = \{v_H\}$ and a sequence $I_1, I_2, \ldots, I_s, I_{s+1}$ of inner intervals of $\mathcal{P}$ with $I_k \subseteq V(\mathcal{B}_k)$ for all $k = 1, \ldots, s+1$, having the properties described before. We prove that $\mathcal{W} : I_1, \ldots, I_s, I_{s+1}$ is a zig-zag walk of $\mathcal{P}$.

1. It is clear by the previous construction that $I_{s+1} \cap I_1 = \{v_H\}$ and $I_k \cap I_{k+1} = \{v_{k+1}\}$ for all $k = 1, \ldots, s$.

2. Let $k \in \{1, \ldots, s\}$. Firstly suppose that $k \in \{2, 3, \ldots, s-1\}$. Consider the blocks $\mathcal{B}_{k-1}$, $\mathcal{B}_k$ and $\mathcal{B}_{k+1}$. We may assume that $\mathcal{B}_{k-1}$ is in horizontal position and that there exist A_{i_k} and $A_{i_{k+1}}$ with $i_k < i_{k+1}$ such that $\mathcal{B}_{k-1} = [A_{i_k}, A_{i_{k+1}}]$ and $A_{i_k - 1}$ is at North of A_{i_k}, otherwise we can do opportune reflections or rotations. Assume that $\mathcal{B}_k$ is in horizontal position. By the construction of $\mathcal{B}_k$ and $\mathcal{B}_{k+1}$, we have the situation described in Figure 3.17 (A), where the dashed lines indicate the block $\mathcal{B}_{k+1}$ depending on its position. Therefore it follows that v_k and v_{k+1} are on the same edge interval of $\mathcal{P}$. The same holds if $\mathcal{B}_k$ is in vertical position, in particular see Figure 3.17 (B). If $k = 1$ or $k = s$, then we can consider the blocks $\mathcal{B}_s$, $\mathcal{B}_{s+1}$ and $\mathcal{B}_1$, so with reference to the Figures 3.15 and 3.16 the desired claim follows.



(A) (B)

FIGURE 3.17

3. Let $k, j \in \{1, \dots, s+1\}$ with $k < j$. If $k = 1$ and $j = s+1$ we have the situation in Figure 3.15 or 3.16, so the interval having z_1 and z_{s+1} as corners is not an inner interval of $\mathcal{P}$. If $j = k+1$, then we consider the blocks $\mathcal{B}_k$ and $\mathcal{B}_{k+1}$. We may assume that $\mathcal{B}_k$ is in horizontal position and that there exist A_{i_k} and $A_{i_{k+1}}$ with $i_k < i_{k+1}$ such that $\mathcal{B}_k = [A_{i_k}, A_{i_k+1}]$ and A_{i_k-1} is at South of A_{i_k}, otherwise we can do appropriate reflections or rotations. $\mathcal{B}_{k+1}$ is either in horizontal or vertical position. With reference to Figure 3.18, in both cases the interval having z_k and z_{k+1} as corners is not an inner interval of $\mathcal{P}$. If $j \neq k+1, k-1$, then the desired conclusion follows.

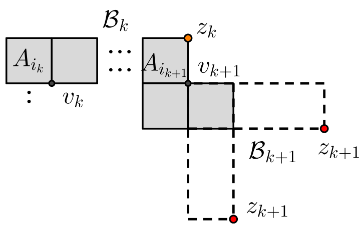


FIGURE 3.18

Therefore $\mathcal{W} : I_1, \dots, I_s$ is a zig-zag walk in $\mathcal{P}$. □

We introduce the following notation, which will be useful to prove more easily the next Proposition.

Definition 3.3.4. Let $\mathcal{P}$ be a non-simple polyomino with a unique hole $\mathcal{H}$. Let $\{V_i\}_{i\in I}$ be the set of the maximal edge intervals of $\mathcal{P}$ and $\{H_j\}_{j\in J}$ be the set of the maximal horizontal edge intervals of $\mathcal{P}$. Let $\{v_i\}_{i\in I}$ and $\{h_j\}_{j\in J}$ be the set of the variables associated respectively to $\{V_i\}_{i\in I}$ and $\{H_j\}_{j\in J}$. Let $\mathcal{H}$ be the hole of $\mathcal{P}$ and w be another variable. Let $\mathcal{I}$ be a subset of $V(\mathcal{P})$ and we define the following map:

$$\begin{aligned} \alpha : V(\mathcal{P}) &\longrightarrow K[\{v_i, h_j, w\} : i \in I, j \in J] \\ a &\longmapsto v_i h_j w^k \end{aligned}$$

where $a \in V_i \cap H_j$, $k = 0$ if $a \notin \mathcal{I}$, and $k = 1$ if $a \in \mathcal{I}$. The toric ring, denoted by $T_{\mathcal{P}}$, is $K[\alpha(v) : v \in V(\mathcal{P})]$. We consider the following surjective ring homomorphism $\phi : S \longrightarrow T_{\mathcal{P}}$ defined by $\phi(x_a) = \alpha(a)$ and the kernel of ϕ is the toric ideal denoted by $J_{\mathcal{P}}$.

Proposition 3.3.5. *Let $\mathcal{P}$ be a weakly closed path. If one of the following conditions holds:*

1. *$\mathcal{P}$ has an L-configuration,*
2. *$\mathcal{P}$ has a weak L-configuration,*
3. *$\mathcal{P}$ has a ladder of at least three steps,*
4. *$\mathcal{P}$ has a weak ladder,*

then $I_{\mathcal{P}}$ is prime.

Proof. (1) Suppose that $\mathcal{P}$ has an L-configuration $\{C_1, C_2, C_3, C_4, C_5\}$ and we can consider suitable reflections or rotations of $\mathcal{P}$ in order to have the L-configuration in the position of Figure 3.19. For convenience set $C_3 = A$. In order to have the desired

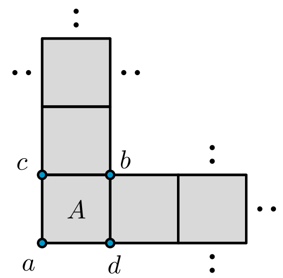


FIGURE 3.19

claim, it is sufficient to prove that $I_{\mathcal{P}} = J_{\mathcal{P}}$, where $J_{\mathcal{P}}$ is the toric ideal defined in Definition 3.3.4 with $\mathcal{I} = V(A)$. Observe that it follows easily by arguing as done in Theorem 2.2.15 and by applying Theorem 3.2.3.
(2) Let $\mathcal{L} = \{C\} \cup [A, B] \cup [D, F]$ be a weak L-configuration of $\mathcal{P}$. We consider opportune reflections or rotations of $\mathcal{P}$ in order to have $\mathcal{L}$ as in the picture on the left in Figure 3.10. By similar arguments as in case (1), we conclude that $I_{\mathcal{P}} = J_{\mathcal{P}}$, where $J_{\mathcal{P}}$ is the toric ideal defined in Definition 3.3.4 with $\mathcal{I} = V(B)$.
(3) Suppose that $\mathcal{P}$ has a ladder of at least three steps and let $\mathcal{B} = \{\mathcal{B}_i\}_{i=1,\dots,m}$ be a maximal ladder of m steps, with $m > 2$. We can consider suitable reflections or rotations of $\mathcal{P}$ in order to have the ladder as in Figure 3.20. We may assume that the block $\mathcal{B}_{m-1}$ consists of n cells which we denote by $B_1, \dots, B_n$ from left to right. Let b_i be the lower left corner of B_i for all $i = 1, \dots, n$ and let B be the cell of $\mathcal{B}_m$, having an edge in common with B_n. We denote by a, b the diagonal corners of B and by d the other anti-diagonal corner. We want to show that $I_{\mathcal{P}} = J_{\mathcal{P}}$, where $J_{\mathcal{P}}$ is the toric ideal defined in 3.3.4 with $\mathcal{I} = \{b_1, \dots, b_n, b, a, d\}$. We observe that it follows by arguing as done in Theorem 2.2.18 and by using Theorem 3.2.3.
(4) Suppose that $\mathcal{P}$ has a weak ladder. Let $\mathcal{L} = \{C, D\} \cup [A, B]$ be the weak ladder of $\mathcal{P}$, that we can assume being as in Figure 3.11. We may assume that the block $[A, B]$ is made up of n cells, with $n \geq 2$, which we denote by $C_1, \dots, C_n$ from left to right. Firstly, we assume that the block containing C is in vertical position. We denote by l_C and l_{C_1} respectively the lower left corner of C and C_1 (see Figure 3.21 (A)). By similar arguments as in case (1), we conclude that $I_{\mathcal{P}} = J_{\mathcal{P}}$, where $J_{\mathcal{P}}$ is the toric ideal defined at the beginning of this section with $\mathcal{I} = \{v_H, l_C, l_{C_1}\}$. Now, we assume that the block containing C is in horizontal position. We denote by r_C the upper right corner of C and by r_{C_i} the upper right corner of C_i for all $i = 1, \dots, n$. The conclusion $I_{\mathcal{P}} = J_{\mathcal{P}}$, where $J_{\mathcal{P}}$ is the toric ideal defined in Definition 3.3.4 with $\mathcal{I} = \{v_H, r_C, r_{C_1}, \dots, r_{C_n}\}$, follows as in the proof of case (3) (see Figure 3.21 (B)). $\square$

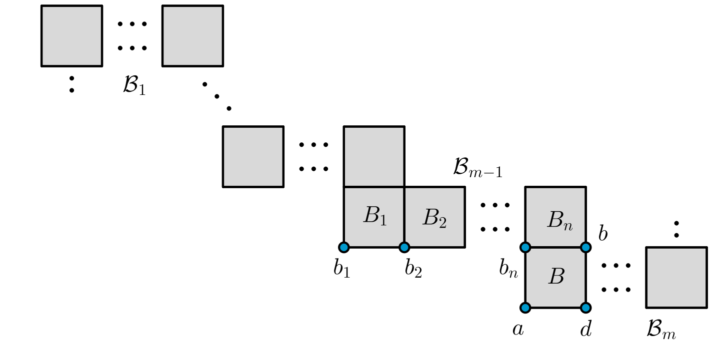


FIGURE 3.20

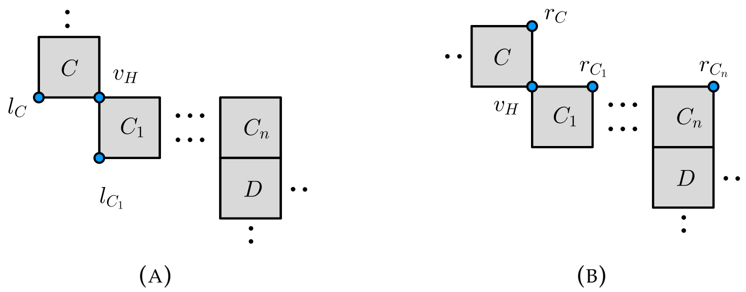


FIGURE 3.21

Remark 3.3.6. We observe that the weakly connected and non-simple collection of cells $\mathcal{P}$ in Figure 3.8 (B) is prime in fact by similar arguments as in case (1) of Proposition 3.3.5 it follows that $I_{\mathcal{P}} = J_{\mathcal{P}}$ where $J_{\mathcal{P}}$ is the toric ideal defined in Definition 3.3.4 with $\mathcal{I} = V(A)$.

Theorem 3.3.7. *Let $\mathcal{P}$ be a weakly closed path. $I_{\mathcal{P}}$ is prime if and only if $\mathcal{P}$ does not contain zig-zag walks.*

Proof. The necessary condition is proved in [31, Corollary 3.6]. The sufficient one follows from Theorem 3.3.3 and from Proposition 3.3.5. □

Chapter 4

Gröbner bases and Cohen-Macaulay property of polyomino ideals

A classical result in Commutative Algebra states that if I is a graded ideal of $K[x_1,\dots,x_n]$ such that $\mathrm{in}_<(I)$ is squarefree for some monomial order $<$, then I is a radical ideal. The radicality for polyomino ideals is still open because it seems very difficult to study the Gröbner basis for this kind of binomial ideals. More in general, an unresolved and very interesting question is if there always exists a monomial order $<$ for which the set of generators of the polyomino ideal forms the reduced Gröbner basis with respect to $<$.
In this chapter we study the Gröbner bases of polyomino ideals and we give a positive answer to the previous question for the class of closed path polyominoes. As a consequence we get that the coordinate ring of a closed path without zig-zag walks is a normal Cohen-Macaulay domain. Moreover, the closed paths having zig-zag walks provide an interesting class of non-prime but radical polyominoes.
All the results in this chapter are contained in [7].

4.1 Some conditions for the reduced Gröbner basis of polyomino ideal

In this section we provide some conditions on opportune monomial orders in order to the set of the inner 2-minors of a polyomino ideal forms the reduced Gröbner basis. First of all, we recall some known results which are useful along the section.

In [37] Qureshi defines two monomial orders and she gives a necessary and sufficient condition in order to reach the previous desired aim. Define a total order on the variables attached to $V(\mathcal{P})$ as follows: $x_a > x_b$ with $a=(i,j)$ and $b=(k,l)$, if $i>k$, or $i=k$ and $j>l$. Let $<^1_{\mathrm{lex}}$ be the lexicographical order induced by this order of the variables. Similarly, we denote by $<^2_{\mathrm{lex}}$ the lexicographical order induced by the total order of the variables defined as follows: $x_a > x_b$ with $a=(i,j)$ and $b=(k,l)$, if $i<k$, or $i=k$ and $j>l$.

Theorem 4.1.1. *[37, Theorem 4.1] Let $\mathcal{P}$ be a collection of cells. Then the set of inner 2-minors of $\mathcal{P}$ forms the reduced Gröbner basis with respect to $<^1_{\mathrm{lex}}$ if and only if for any two inner intervals $[a,b]$ and $[b,c]$ of $\mathcal{P}$, either $[e,c]$ or $[d,c]$ is an inner interval of $\mathcal{P}$, where d and e are the anti-diagonal corners of $[a,b]$ (see Figure 4.1).*
Moreover, the set of inner 2-minors of P forms the reduced Gröbner basis with respect to $<^2_{\mathrm{lex}}$ if and only if for any two inner intervals $[b,a]$ and $[d,c]$ of $\mathcal{P}$ with anti-diagonal corners e,f

and f, g as shown in Figure 4.1, either b, g or e, c are anti-diagonal corners of an inner interval of $\mathcal{P}$.

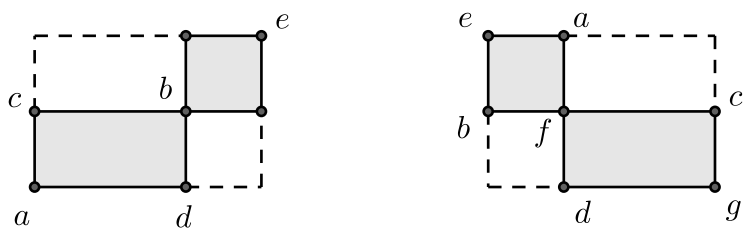


FIGURE 4.1

Moreover, if $\mathcal{P}$ is a weakly connected and simple collection of cells, from Theorem 3.2.3, we know that $I_{\mathcal{P}}$ is the toric ideal of an edge ring attached to a bipartite and weakly chordal graph. In such a case, the set of generators of $I_{\mathcal{P}}$ forms the reduced Gröbner basis with respect to a suitable monomial order defined in [34]. For all *convex* polyominoes we provided an algorithm developed in Macaulay2 to define the associated polyomino ring with this specific monomial order.

Now, we can state the conditions given in [7]. Let $\mathcal{P}$ be a non-empty collection of cells with $V(\mathcal{P}) = \{a_1, \dots, a_n\}$. We define a P-*order* to be a total order on the set $V(\mathcal{P})$. Observe that the monomial orderings defined in [32] and [37] are induced by specific P-orders.
If $<^{\mathsf{P}}$ is a P-order, we denote by $<^{\mathsf{P}}_{\mathrm{lex}}$ the lexicographic order induced by $<^{\mathsf{P}}$ on $S = K[x_v \mid v \in V(\mathcal{P})]$, that is the lexicographic order induced by the total order on the variables defined in the following way: $x_{a_i} <^{\mathsf{P}}_{\mathrm{lex}} x_{a_j}$ if and only if $a_i <^{\mathsf{P}} a_j$ for $i, j \in \{1, \dots, n\}$. If $f \in S$, we denote by $\mathrm{in}(f)$ the leading term of f with respect to $<^{\mathsf{P}}_{\mathrm{lex}}$.
Let $f, g \in I_{\mathcal{P}}$, we denote by $S(f,g)$ the S-polynomial of f, g with respect to $<^{\mathsf{P}}_{\mathrm{lex}}$. Let $\mathcal{G}$ be the set of all inner 2-minors of $\mathcal{P}$ (that is the set of generators of $I_{\mathcal{P}}$). We want to study some conditions on $<^{\mathsf{P}}$ in order to $S(f,g)$ reduces to 0 modulo $\mathcal{G}$.
First of all observe that if $[a,b]$ and $[\alpha,\beta]$ are two inner intervals and $[a,b] \cap [\alpha,\beta]$ does not contain any corner of $[a,b]$ and $[\alpha,\beta]$, then $\gcd(\mathrm{in}(f_{a,b}), \mathrm{in}(f_{\alpha,\beta})) = 1$ so $S(f_{a,b}, f_{\alpha,\beta})$ reduces to 0. So it suffices to study the remaining cases.
In the remainder of this section the inner intervals $[a,b]$ and $[\alpha,\beta]$ have respectively c, d and γ, δ as anti-diagonal corners, as in figure 4.2.

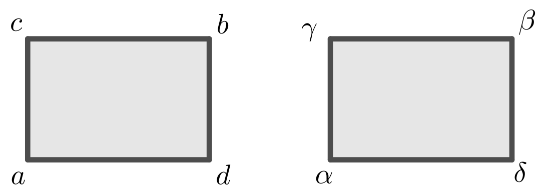


FIGURE 4.2

In the following lemmas we examine all possible cases in which $|\{a,b,c,d\} \cap \{\alpha,\beta,\gamma,\delta\}|$ is equal to 1 or 2.

Lemma 4.1.2. *Let $\mathcal{P}$ be a collection of cells and $[a,b]$ and $[\alpha,\beta]$ be two inner intervals such that $|\{a,b,c,d\}\cap\{\alpha,\beta,\gamma,\delta\}|=2$. Then $S(f_{a,b},f_{\alpha,\beta})$ reduces to 0 modulo $\mathcal{G}$ with respect to $<^{\mathsf{P}}_{\mathrm{lex}}$ for any* P*-order.*

Proof. We may assume that $\alpha=d$ and $\gamma=b$, because the other cases can be discussed similarly. If $\gcd(\mathrm{in}(f_{a,b}),\mathrm{in}(f_{\alpha,\beta}))=1$, then there is nothing to prove, so we have to distinguish the following cases.
First case: $\mathrm{in}(f_{a,b}) = x_a x_b$ and $\mathrm{in}(f_{\alpha,\beta}) = -x_b x_\delta$. In such a case $S(f_{a,b},f_{\alpha,\beta}) = -x_\delta x_d x_c + x_a x_\beta x_d = x_d(x_a x_\beta - x_c x_\delta)$. Observe that $f_{a,\beta}\in I_{\mathcal{P}}$. If $\mathrm{in}(S(f_{a,b},f_{\alpha,\beta})) = x_a x_\beta x_d$ then $\mathrm{in}(f_{a,\beta}) = x_a x_\beta$, so $S(f_{a,b},f_{\alpha,\beta})$ reduces to 0. If $\mathrm{in}(S(f_{a,b},f_{\alpha,\beta})) = -x_\delta x_d x_c$ then $\mathrm{in}(f_{a,\beta}) = -x_c x_\delta$, so $S(f_{a,b},f_{\alpha,\beta})$ reduces to 0 also in this case.
Second case: $\mathrm{in}(f_{a,b}) = -x_d x_c$ and $\mathrm{in}(f_{\alpha,\beta}) = x_\beta x_d$. In such a case $S(f_{a,b},f_{\alpha,\beta}) = -x_\beta x_a x_b + x_c x_b x_\delta = -x_b(x_a x_\beta - x_c x_\delta)$. As in the first case, if $\mathrm{in}(S(f_{a,b},f_{\alpha,\beta})) = -x_\beta x_a x_b$ then $\mathrm{in}(f_{a,\beta}) = x_a x_\beta$, so $S(f_{a,b},f_{\alpha,\beta})$ reduces to 0. If $\mathrm{in}(S(f_{a,b},f_{\alpha,\beta})) = +x_c x_b x_\delta$ then $\mathrm{in}(f_{a,\beta}) = -x_c x_\delta$, so $S(f_{a,b},f_{\alpha,\beta})$ reduces to 0 also in this case. □

Lemma 4.1.3. *Let $\mathcal{P}$ be a collection of cells and $[a,b]$ and $[\alpha,\beta]$ be two inner intervals with $\beta=b$ and $\gamma\in]c,b[$ (see Figure 4.3(A)). Let h be the vertex such that $[h,b]$ is the inner interval having d,γ as anti-diagonal corner and r be the vertex such that $[r,h]$ is the interval having a,α as anti-diagonal corner. Let $<^{\mathsf{P}}$ be a* P*-order on $V(\mathcal{P})$ and suppose that $\gcd(\mathrm{in}(f_{a,b}),\mathrm{in}(f_{\alpha,\beta}))\neq 1$. Then $S(f_{a,b},f_{\alpha,\beta})$ reduces to 0 modulo $\mathcal{G}$ with respect to $<^{\mathsf{P}}_{\mathrm{lex}}$ if and only if one of the following conditions occurs:*

1. *$x_a x_\gamma x_\delta <^{\mathsf{P}}_{\mathrm{lex}} x_\alpha x_c x_d$ and in addiction $h,\delta <^{\mathsf{P}} \alpha$ or $h,\delta <^{\mathsf{P}} d$;*
2. *$x_a x_\gamma x_\delta <^{\mathsf{P}}_{\mathrm{lex}} x_\alpha x_c x_d$, $\{r,h,a,\alpha\}$ is the set of vertices of an inner interval of $\mathcal{P}$ and in addiction $r,\gamma <^{\mathsf{P}} \alpha$ or $r,\gamma <^{\mathsf{P}} c$;*
3. *$x_\alpha x_c x_d <^{\mathsf{P}}_{\mathrm{lex}} x_a x_\gamma x_\delta$ and in addiction $h,c <^{\mathsf{P}} a$ or $h,c <^{\mathsf{P}} \gamma$;*
4. *$x_\alpha x_c x_d <^{\mathsf{P}}_{\mathrm{lex}} x_a x_\gamma x_\delta$, $\{r,h,a,\alpha\}$ is the set of vertices of an inner interval of $\mathcal{P}$ and in addiction $r,d <^{\mathsf{P}} a$ or $r,d <^{\mathsf{P}} \delta$.*

The same characterization holds for $S(f_{c,d},f_{\gamma,\delta})$, $S(f_{b,a},f_{b,\alpha})$ and $S(f_{d,c},f_{\delta,\gamma})$ considering all the rotations of the described configuration (see respectively Figure 4.3(B), Figure 4.3(C) and Figure 4.3(D)).

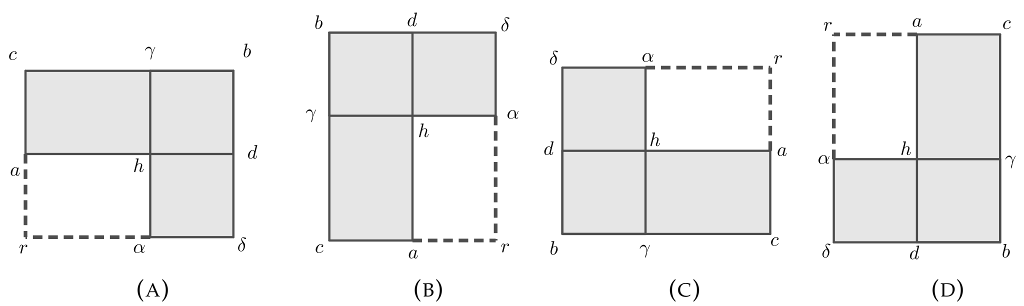


FIGURE 4.3

Proof. Observe that $\gcd(\mathrm{in}(f_{a,b}),\mathrm{in}(f_{\alpha,\beta}))\neq 1$ if and only if $\mathrm{in}(f_{a,b}) = x_a x_b$ and $\mathrm{in}(f_{\alpha,\beta}) = x_\alpha x_b$. Since $S(f_{a,b},f_{\alpha,\beta}) = -x_\alpha x_c x_d + x_a x_\gamma x_\delta$, we have two possibilities:
1) $\mathrm{in}(S(f_{a,b},f_{\alpha,\beta})) = -x_\alpha x_c x_d$, in particular $x_a x_\gamma x_\delta <^{\mathsf{P}}_{\mathrm{lex}} x_\alpha x_c x_d$. Observe that, since

$x_c x_d$ is not the leading term of $f_{a,b}$, in such a case the only possibilities for the reduction of $S(f_{a,b}, f_{\alpha,\beta})$ is through a first division by $f_{\alpha,d}$ if $\mathrm{in}(f_{\alpha,d}) = x_\alpha x_d$ or by $f_{r\gamma}$ if $\mathrm{in}(f_{r,\gamma}) = -x_\alpha x_c$. The first case is possible if and only if $(h, \delta <^{\mathsf{P}} \alpha) \vee (h, \delta <^{\mathsf{P}} d)$, and in such case indeed, after a little computation, $S(f_{a,b}, f_{\alpha,\beta})$ reduces by $f_{\alpha,d}$ to $x_\delta(x_a x_\gamma - x_c x_h) = x_\delta f_{a\gamma}$ and this one reduces to 0. For this case we obtain the condition (1) of this lemma. The second case is possible if and only if the condition (2) is satisfied, that is if $[r, h]$ is an inner interval of $\mathcal{P}$ and $(r, \gamma <^{\mathsf{P}} \alpha) \vee (r, \gamma <^{\mathsf{P}} c)$. In such a case in fact $S(f_{a,b}, f_{\alpha,\beta})$ reduces through $f_{r,\gamma}$ to $x_\gamma(x_a x_\delta - x_r x_d) = x_\gamma f_{a\delta}$ and this one reduces to 0.
2) $\mathrm{in}(S(f_{a,b}, f_{\alpha,\beta})) = x_a x_\gamma x_\delta$, in particular $x_\alpha x_c x_d <^{\mathsf{P}}_{\mathrm{lex}} x_a x_\gamma x_\delta$. We can argue as in the first part of this proof observing that, since $x_\gamma x_\delta$ is not the leading term of $f_{\alpha,\beta}$, in such a case the only possibilities for the reduction of $S(f_{a,b}, f_{\alpha,\beta})$ is through $f_{a,\gamma}$ if $\mathrm{in}(f_{a,\gamma}) = x_a x_\gamma$ or by $f_{r,d}$ if $\mathrm{in}(f_{r,d}) = -x_a x_\delta$. The first case is possible if and only if $(h, c <^{\mathsf{P}} a) \vee (h, c <^{\mathsf{P}} \gamma)$, that is the condition (3) holds, while the second is possible if and only if $[r, h]$ is an inner interval of $\mathcal{P}$ and $(r, d <^{\mathsf{P}} a) \vee (r, d <^{\mathsf{P}} \delta)$, that is the condition (4) is satisfied. In both cases $S(f_{a,b}, f_{\alpha,\beta})$ reduces to 0.
The last statement of this lemma is verified since the only effects of the rotation of a configuration are the different notations for the same intervals (for instance $[a, b]$ becomes $[c, d]$, $[b, a]$ or $[d, c]$) or the change of the sign of the binomials in the generators of $I_{\mathcal{P}}$. $\square$

The following four lemmas can be proved by the same arguments of Lemma 4.1.3, so we omit their proofs.

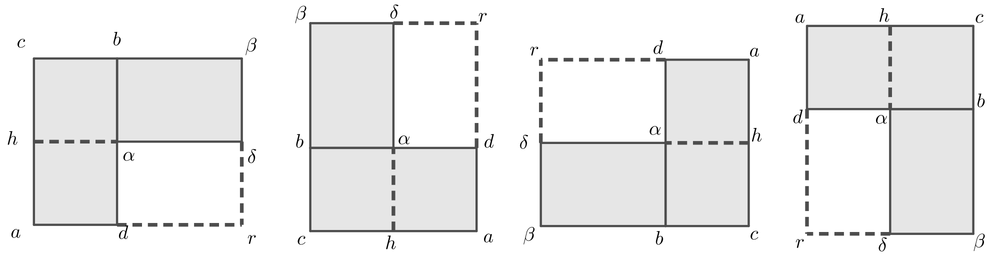
Lemma 4.1.4. *Let $\mathcal{P}$ be a collection of cells and $[a, b]$ and $[\alpha, \beta]$ be two inner intervals with $\gamma = b$ and $\alpha \in]d, b[$ (see Figure 4.4(A)). Let h be the vertex such that $[h, b]$ is the inner interval having c, α as anti-diagonal corner and r be the vertex such that r, α are the anti-diagonal corners of the interval $[d, \delta]$. Let $<^{\mathsf{P}}$ be a P-order on $V(\mathcal{P})$ and suppose that $\gcd(\mathrm{in}(f_{a,b}), \mathrm{in}(f_{\alpha,\beta})) \neq 1$. Then $S(f_{a,b}, f_{\alpha,\beta})$ reduces to 0 modulo $\mathcal{G}$ with respect to $<^{\mathsf{P}}_{\mathrm{lex}}$ if and only if one of the following conditions occurs:*

1. *$x_a x_\alpha x_\beta <^{\mathsf{P}}_{\mathrm{lex}} x_\delta x_c x_d$ and in addiction $h, \beta <^{\mathsf{P}} c$ or $h, \beta <^{\mathsf{P}} \delta$;*
2. *$x_a x_\alpha x_\beta <^{\mathsf{P}}_{\mathrm{lex}} x_\delta x_c x_d$, $\{d, \delta, \alpha, r\}$ is the set of vertices of an inner interval of $\mathcal{P}$ and in addiction $r, \alpha <^{\mathsf{P}} \delta$ or $r, \alpha <^{\mathsf{P}} d$;*
3. *$x_\delta x_c x_d <^{\mathsf{P}}_{\mathrm{lex}} x_a x_\alpha x_\beta$ and in addiction $h, d <^{\mathsf{P}} a$ or $h, d <^{\mathsf{P}} \alpha$;*
4. *$x_\delta x_c x_d <^{\mathsf{P}}_{\mathrm{lex}} x_a x_\alpha x_\beta$, $\{d, \delta, \alpha, r\}$ is the set of vertices of an inner interval of $\mathcal{P}$ and in addiction $r, c <^{\mathsf{P}} a$ or $r, c <^{\mathsf{P}} \beta$.*

The same characterization holds for $S(f_{c,d}, f_{\gamma,\delta})$, $S(f_{b,a}, f_{\beta,\alpha})$ and $S(f_{d,c}, f_{\delta,\gamma})$ considering all the rotations of the described configuration (see respectively Figure 4.4(B), Figure 4.4(C) and Figure 4.4(D)).

Lemma 4.1.5. *Let $\mathcal{P}$ be a collection of cells and $[a, b]$ and $[\alpha, \beta]$ be two inner intervals with $\alpha = c$ and $b \in]\alpha, \delta[$ (see Figure 4.5(A)). Let h be the vertex such that h, δ are the diagonal corners of the inner interval $[b, \beta]$ and r be the vertex such that r, b are the anti-diagonal corners of the interval $[d, \delta]$. Let $<^{\mathsf{P}}$ be a P-order on $V(\mathcal{P})$ and suppose that $\gcd(\mathrm{in}(f_{a,b}), \mathrm{in}(f_{\alpha,\beta})) \neq 1$. Then $S(f_{a,b}, f_{\alpha,\beta})$ reduces to 0 modulo $\mathcal{G}$ with respect to $<^{\mathsf{P}}_{\mathrm{lex}}$ if and only if one of the following conditions occurs:*

1. *$x_d x_\delta x_\gamma <^{\mathsf{P}}_{\mathrm{lex}} x_\beta x_a x_b$ and in addiction $h, \delta <^{\mathsf{P}} b$ or $h, \delta <^{\mathsf{P}} \beta$;*



(A) (B) (C) (D)

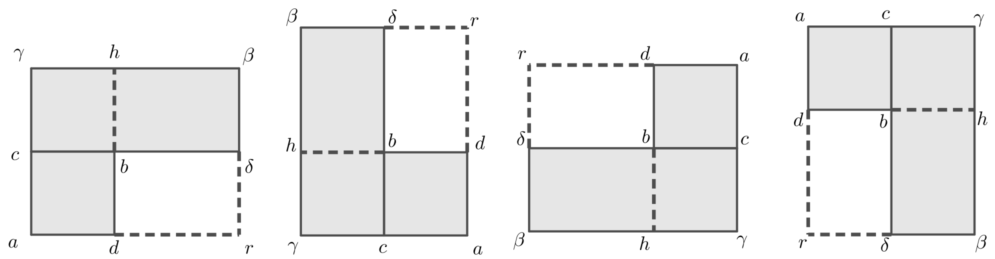
FIGURE 4.4

2. $x_d x_\delta x_\gamma <^{\mathsf{P}}_{\text{lex}} x_\beta x_a x_b$, $\{d,\delta,b,r\}$ *is the set of vertices of an inner interval of* $\mathcal{P}$ *and in addiction* $r,\gamma <^{\mathsf{P}} a$ *or* $r,\gamma <^{\mathsf{P}} \beta$;

3. $x_\beta x_a x_b <^{\mathsf{P}}_{\text{lex}} x_d x_\delta x_\gamma$ *and in addiction* $h,a <^{\mathsf{P}} d$ *or* $h,a <^{\mathsf{P}} \gamma$;

4. $x_\beta x_a x_b <^{\mathsf{P}}_{\text{lex}} x_d x_\delta x_\gamma$, $\{d,\delta,b,r\}$ *is the set of vertices of an inner interval of* $\mathcal{P}$ *and in addiction* $r,b <^{\mathsf{P}} d$ *or* $r,b <^{\mathsf{P}} \delta$.

The same characterization holds for $S(f_{c,d}, f_{\gamma,\delta})$, $S(f_{b,a}, f_{\beta,\alpha})$ *and* $S(f_{d,c}, f_{\delta,\gamma})$ *considering all the rotations of the described configuration (see respectively Figure 4.5(B), Figure 4.5(C) and Figure 4.5(D)).*



(A) (B) (C) (D)

FIGURE 4.5

Lemma 4.1.6. *Let* $\mathcal{P}$ *be a collection of cells and* $[a,b]$ *and* $[\alpha,\beta]$ *be two inner intervals with* $\gamma = c$ *and* $\delta \in]a,b[$ *(see Figure 4.6(A)). Let* $[h,r]$ *be the inner interval having* d,δ *as anti-diagonal corners. Let* $<^{\mathsf{P}}$ *be a* P*-order on* $V(\mathcal{P})$ *and suppose that* $\gcd(\text{in}(f_{a,b}), \text{in}(f_{\alpha,\beta})) \neq 1$. *Then* $S(f_{a,b}, f_{\alpha,\beta})$ *reduces to 0 modulo* $\mathcal{G}$ *with respect to* $<^{\mathsf{P}}_{\text{lex}}$ *if and only if one of the following conditions occurs:*

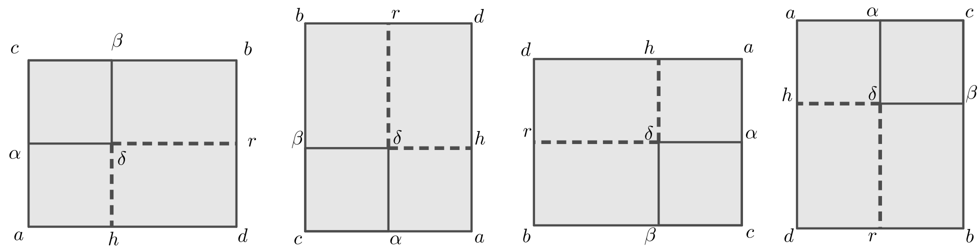
1. $x_d x_\alpha x_\beta <^{\mathsf{P}}_{\text{lex}} x_\delta x_a x_b$ *and in addiction* $h,\alpha <^{\mathsf{P}} a$ *or* $h,\alpha <^{\mathsf{P}} \delta$;

2. $x_d x_\alpha x_\beta <^{\mathsf{P}}_{\text{lex}} x_\delta x_a x_b$ *and in addiction* $r,\beta <^{\mathsf{P}} \delta$ *or* $r,\beta <^{\mathsf{P}} b$;

3. $x_\delta x_a x_b <^{\mathsf{P}}_{\text{lex}} x_d x_\alpha x_\beta$ *and in addiction* $r,a <^{\mathsf{P}} \alpha$ *or* $r,a <^{\mathsf{P}} d$;

4. $x_\delta x_a x_b <^{\mathsf{P}}_{\text{lex}} x_d x_\alpha x_\beta$ *and in addiction* $h,b <^{\mathsf{P}} d$ *or* $h,b <^{\mathsf{P}} \beta$.

The same characterization holds for $S(f_{c,d}, f_{\gamma,\delta})$, $S(f_{b,a}, f_{\beta,\alpha})$ *and* $S(f_{d,c}, f_{\delta,\gamma})$ *considering all the rotations of the described configuration (see respectively Figure 4.6(B), Figure 4.6(C) and Figure 4.6(D)).*



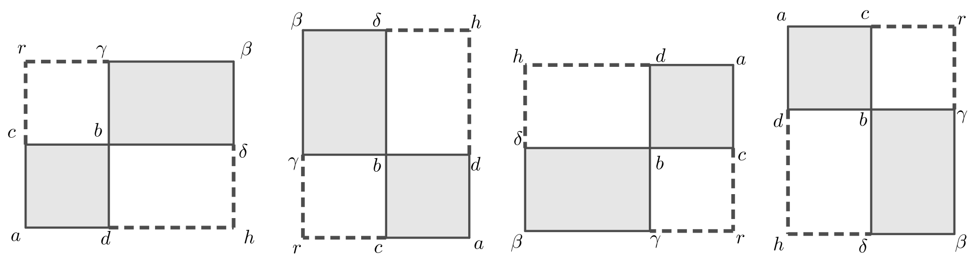
(A) (B) (C) (D)

FIGURE 4.6

Lemma 4.1.7. *Let $\mathcal{P}$ be a collection of cells and $[a,b]$ and $[\alpha,\beta]$ be two inner intervals with $\alpha = b$ and $\beta \notin [a,b]$ (see Figure 4.7(A)). Let h, r be the anti-diagonal corners, different to b, respectively of the intervals $[d,\delta]$ and $[c,\gamma]$. Let $<^{\mathrm{P}}$ be a* P*-order on $V(\mathcal{P})$ and suppose that $\gcd(\mathrm{in}(f_{a,b}),\mathrm{in}(f_{\alpha,\beta})) \neq 1$. Then $S(f_{a,b}, f_{\alpha,\beta})$ reduces to 0 modulo $\mathcal{G}$ with respect to $<^{\mathrm{P}}_{\mathrm{lex}}$ if and only if one of the following conditions occurs:*

1. *$x_a x_\gamma x_\delta <^{\mathrm{P}}_{\mathrm{lex}} x_\beta x_d x_c$, $\{d,\delta,b,h\}$ is the set of vertices of an inner interval of $\mathcal{P}$ and in addiction $h,\gamma <^{\mathrm{P}} \beta$ or $h,\gamma <^{\mathrm{P}} d$;*
2. *$x_a x_\gamma x_\delta <^{\mathrm{P}}_{\mathrm{lex}} x_\beta x_d x_c$, $\{c,\gamma,b,r\}$ is the set of vertices of an inner interval of $\mathcal{P}$ and in addiction $r,\delta <^{\mathrm{P}} \beta$ or $r,\delta <^{\mathrm{P}} c$;*
3. *$x_\beta x_d x_c <^{\mathrm{P}}_{\mathrm{lex}} x_a x_\gamma x_\delta$, $\{c,\gamma,b,r\}$ is the set of vertices of an inner interval of $\mathcal{P}$ and in addiction $r,d <^{\mathrm{P}} a$) or $r,d <^{\mathrm{P}} \gamma$;*
4. *$x_\beta x_d x_c <^{\mathrm{P}}_{\mathrm{lex}} x_a x_\gamma x_\delta$, $\{d,\delta,b,h\}$ is the set of vertices of an inner interval of $\mathcal{P}$ and in addiction $c,h <^{\mathrm{P}} a$ or $c,h <^{\mathrm{P}} \delta$.*

The same characterization holds for $S(f_{c,d}, f_{\gamma,\delta})$, $S(f_{b,a}, f_{\beta,\alpha})$ and $S(f_{d,c}, f_{\delta,\gamma})$ considering all the rotations of the described configuration (see respectively Figure 4.7(B), Figure 4.7(C) and Figure 4.7(D)).



(A) (B) (C) (D)

FIGURE 4.7

4.2 Gröbner basis and Cohen-Macaulay property of closed path polyominoes

In this section we introduce some monomial orders for the class of closed path polyominoes and we prove that the set of the generators of the polyomino ideal attached to a closed path forms the reduced Gröbner basis with respect to these monomial orders. Hence $I_{\mathcal{P}}$ is radical, for all closed path $\mathcal{P}$ (for a reference see [26, Corollary 2.2]). Moreover, it is known that the polyomino ideal attached to a closed path containing an L-configuration or a ladder of at least three steps, equivalently having no zig-zag walks, is prime. As a consequence, we obtain that the coordinate ring of a closed path having no zig-zag walks is a normal Cohen-Macaulay domain.

Let $\mathcal{P}$ be a polyomino. We examine four special configurations of cells of a polyomino, that permit us when $\mathcal{P}$ is a closed path to provide some particular subsets $Y \subset V(\mathcal{P})$ for which we can define the following P-order. In according to Qureshi (see [37]) we recall that $a <^1 b$ if and only if, for $a = (i,j)$ and $b = (k,l)$, $i < k$, or $i = k$ and $j < l$.

Definition 4.2.1. Let $Y \subset V(\mathcal{P})$. We define the P-order $<^Y$ in the following way:

$$a <^Y b \Leftrightarrow \begin{cases} a \notin Y \text{ and } b \in Y \\ a, b \notin Y \text{ and } a <^1 b \\ a, b \in Y \text{ and } a <^1 b \end{cases}$$

for $a, b \in V(\mathcal{P})$.

We call an *W-pentomino with middle cell A* a collection of cells of $\mathcal{P}$ consisting of an horizontal block $\mathcal{B}_1 = [A_1, B_1]$ of rank two, a vertical block $\mathcal{B}_2 = [A_2, B_2]$ of rank two and a cell A not belonging to $\mathcal{B}_1 \cup \mathcal{B}_2$, such that $V(\mathcal{B}_1) \cap V(\mathcal{B}_2) = \{w\}$ and where w is the lower right corner of A. Moreover, if $\mathcal{W}$ is a W-pentomino with middle cell A, we denote with x_W the left upper corner of A, with y_W the lower right corner of B_1 and with z_W the lower right corner of A_2. See Figure 5.3.

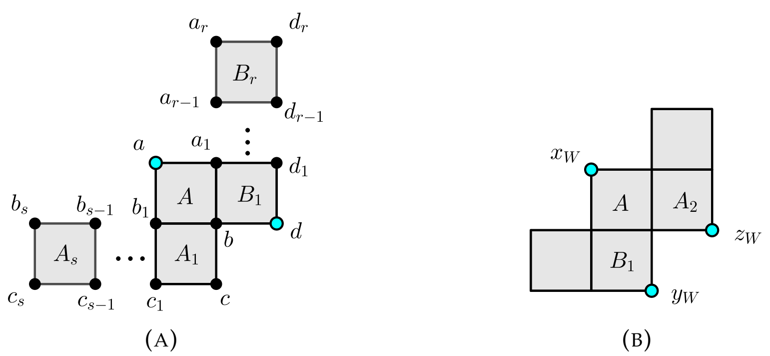
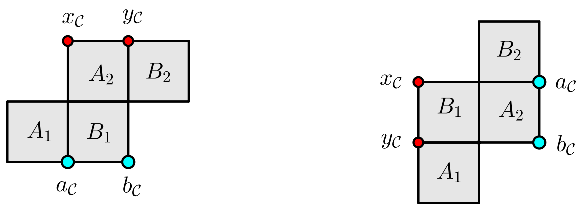


FIGURE 4.8: W-pentomino

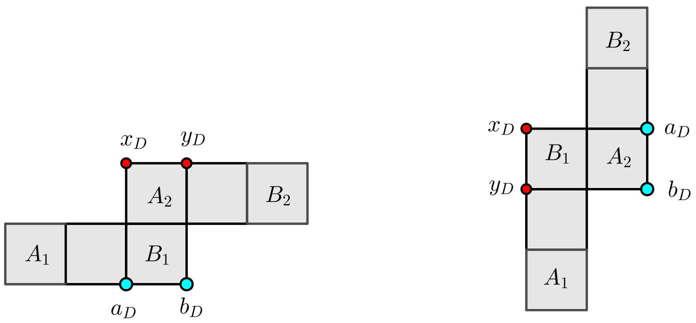
We call an *LD-horizontal (vertical) skew tetromino* a collection of cells of $\mathcal{P}$ consisting of two horizontal (vertical) blocks of rank two $\mathcal{B}_1 = [A_1, B_1]$ and $\mathcal{B}_2 = [A_2, B_2]$ such that $V(B_1) \cap V(A_2) = \{w_1, w_2\}$ and w_1, w_2 are right and left upper (lower and upper right) corners of B_1. Moreover, if $\mathcal{C}$ is an LD-horizontal (vertical) skew tetromino, we denote with $x_{\mathcal{C}}, y_{\mathcal{C}}$ the left and right upper corners of A_2 (the upper and lower left corners of B_1), and with $a_{\mathcal{C}}, b_{\mathcal{C}}$ the left and right lower corners of B_1 (the upper and lower right corners of A_2). See Figure 4.9.
We call an *LD-horizontal (vertical) skew hexomino* a collection of cells of $\mathcal{P}$ consisting of two horizontal (vertical) blocks of rank three $\mathcal{B}_1 = [A_1, B_1]$ and $\mathcal{B}_2 = [A_2, B_2]$ such that $V(B_1) \cap V(A_2) = \{w_1, w_2\}$ and w_1, w_2 are respectively the right and left upper (lower and upper right) corners of B_1. Moreover, if $\mathcal{D}$ is an LD-horizontal (vertical) skew tetromino, we denote by x_D, y_D the left and right upper corners of A_2



(A) (B)

FIGURE 4.9: LD-horizontal skew tetromino (A) and LD-vertical skew tetromino (B)

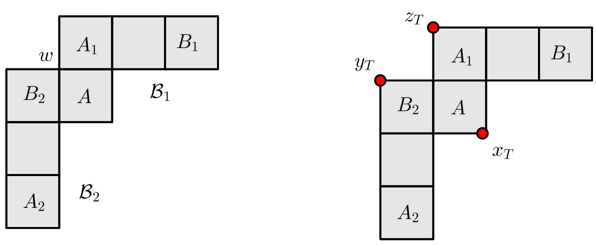
(the upper and lower left corners of B_1), and by a_D, b_D the the left and right upper corners of B_1 (the upper and lower right corners of A_2). See Figure 4.10.



(A) (B)

FIGURE 4.10: LD-horizontal skew hexomino (A) and LD-vertical skew hexomino (B)

We call an *RW-heptomino with middle cell* A a collection of cells of $\mathcal{P}$ consisting of an horizontal block $\mathcal{B}_1 = [A_1, B_1]$ of rank three, a vertical block $\mathcal{B}_2 = [A_2, B_2]$ of rank three and a cell A not belonging to $\mathcal{B}_1 \cup \mathcal{B}_2$, such that $V(\mathcal{B}_1) \cap V(\mathcal{B}_2) = \{w\}$ and where w is the upper left corner of A. Moreover, if $\mathcal{T}$ is an RW-pentomino with middle cell A, we denote by x_T the right lower corner of A, with y_T the left upper corner of B_2 and by z_T the left upper corner of A_1. See Figure 4.11.



(A) (B)

FIGURE 4.11: RW-heptomino

Theorem 4.2.2. *Let $\mathcal{P}$ be a closed path polyomino not containing any W-pentomino. Let $\mathcal{R}$ be the set of all LD-horizontal and vertical skew tetrominoes contained in $\mathcal{P}$ and let $Y =$*

$\bigcup_{\mathcal{C}\in\mathcal{R}}\{x_\mathcal{C}, y_\mathcal{C}\}$. *Then $\mathcal{G}$ is the reduced Gröbner basis of $I_\mathcal{P}$ with respect to the monomial order* $<^Y_{\text{lex}}$.

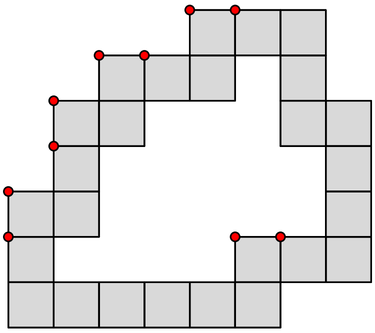
Proof. Let $f = x_p x_q - x_r x_s$ and $g = x_u x_v - x_w x_z$ be the two binomials attached respectively to the inner intervals $[p,q]$ and $[u,v]$ of $\mathcal{P}$. We prove that $S(f,g)$ reduces to 0 modulo $\mathcal{G}$ with respect to $<^Y_{\text{lex}}$, examining all possible cases on $\{p,q,r,s\} \cap \{u,v,w,z\}$.
The case $\{p,q,r,s\} \cap \{u,v,w,z\} = \emptyset$ is trivial. If $|\{p,q,r,s\} \cap \{u,v,w,z\}| = 2$, then the claim follows from Lemma 4.1.2. Assume that $|\{p,q,r,s\} \cap \{u,v,w,z\}| = 1$ and that $[p,q]$ is not contained in $[u,v]$ or vice versa. Suppose that $q = v$. For the structure of $\mathcal{P}$ we may assume that $s \in]z,v[$ and $w \in]r,q[$, so there exists $k \in \{1,\dots,n\}$ such that $A_k = [p,q] \cap [u,v]$. Let A_{k-1} be the cell of $\mathcal{P}_{[p,q]}$ adjacent to A_k. If A_{k-2} is at North of A_{k-1} then we have the conclusion from (1) of Lemma 4.1.3. If A_{k-2} is at West of A_{k-1} then the claim follows either by being $\gcd(\text{in}(f),\text{in}(g)) = 1$ or by applying (1) of Lemma 4.1.3 if $\gcd(\text{in}(f),\text{in}(g)) \neq 1$. The cases $r = w$, $p = u$ and $s = z$ can be proved similarly to the previous ones. Suppose that $q = w$. We may assume that $u \in]s,q[$, because the arguments are similar when $s \in]u,w[$. Let A_k be the cell of $\mathcal{P}$ having r, u as anti-diagonal corners and we denote by A_{k-1} and A_{k+1} respectively the cells of $\mathcal{P}_{[p,q]}$ and $\mathcal{P}_{[u,v]}$ adjacent to A_k. If $\{A_{k-2}, A_{k-1}, A_k, A_{k+1}\}$ is an LD-vertical skew tetromino or $\{A_{k-2}, A_{k-1}, A_k, A_{k+1}, A_{k+2}\}$ is an L-configuration then $\gcd(\text{in}(f),\text{in}(g)) = 1$. If $\{A_{k-1}, A_k, A_{k+1}, A_{k+2}\}$ is an LD-horizontal skew tetromino, then $\gcd(\text{in}(f),\text{in}(g)) = x_w$ and applying (1) of Lemma 4.1.4 we have the desired conclusion. Similar arguments hold in the cases $s = u$, $v = r$ and $z = p$. Suppose $q = u$ and let A_k and A_{k+2} be the cells of $\mathcal{P}$ having respectively q as upper right and lower left corner. If $\{A_{k-1}, A_k, A_{k+1}, A_{k+2}\}$ or $\{A_k, A_{k+1}, A_{k+2}, A_{k+3}\}$ is an LD-vertical skew tetromino, then $\gcd(\text{in}(f),\text{in}(g)) = 1$. If $\{A_{k-1}, A_k, A_{k+1}, A_{k+2}, A_{k+3}\}$ is an L-configuration, the claim follows either by $\gcd(\text{in}(f),\text{in}(g)) = 1$ or by applying Lemma 4.1.7 if $\gcd(\text{in}(f),\text{in}(g)) \neq 1$. If $\{A_{k-1}, A_k, A_{k+1}, A_{k+2}, A_{k+3}\}$ is not an L-configuration and does not contain an LD-vertical skew tetromino, then the only two possibilities are that either $\{A_{k-1}, A_k, A_{k+1}, A_{k+2}\}$ or $\{A_k, A_{k+1}, A_{k+2}, A_{k+3}\}$ is an LD-horizontal skew tetromino. In both cases $\gcd(\text{in}(f),\text{in}(g)) = 1$, in particular in the first case the claim follows since $\mathcal{P}$ has not any W-pentomino, so A_{k+3} is at East of A_{k+2}. The other cases $s = w$, $z = r$ or $v = p$ can be proved by similar arguments. Finally, it is easy to see that in such cases $\mathcal{G}$ is also the reduced Gröbner basis of $I_\mathcal{P}$. □

In Figure 4.12 (A) is shown an example of polyomino satisfying Theorem 4.2.2.

Remark 4.2.3. In [32] the authors introduced the class of *thin* polyominoes, that consists of all polyominoes not containing the configuration whose shape is a square made up of four cells. Such a class can be viewed as a generalization of closed paths. We observe that the conclusion of the previous theorem does not hold in general for thin polyominoes, using the same monomial order. In fact, we can consider the thin polyomino in Figure 4.12(B) and it is not difficult to show that the S-polynomial associated to the marked intervals does not reduce to 0.

Remark 4.2.4. By the same arguments, the statement of Theorem 4.2.2 holds also for $Y = \bigcup_{\mathcal{C}\in\mathcal{R}}\{a_\mathcal{C}, b_\mathcal{C}\}$.

Theorem 4.2.5. *Let $\mathcal{P}$ be a closed path polyomino not containing any RW-heptomino. Let $\mathcal{R}_1$ be the set of all LD-horizontal and vertical skew hexominoes contained in $\mathcal{P}$ and let $\mathcal{R}_2$ be the set of all W-pentominoes contained in $\mathcal{P}$. Let* $Y = (\bigcup_{\mathcal{D}\in\mathcal{R}_1}\{a_\mathcal{D}, b_\mathcal{D}\}) \cup$


(A)

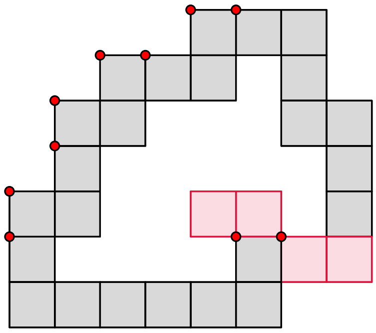

(B)

FIGURE 4.12: The highlighted points belong to Y

$(\bigcup_{\mathcal{W}\in\mathcal{R}_2}\{x_W, y_W\})$. *Then $\mathcal{G}$ is the reduced Gröbner basis of $I_\mathcal{P}$ with respect to the monomial order $<^Y_{\text{lex}}$.*

Proof. Let $f = x_p x_q - x_r x_s$ and $g = x_u x_v - x_w x_z$ be the two binomials attached respectively to the inner intervals $[p,q]$ and $[u,v]$ of $\mathcal{P}$. We discuss the case $|\{p,q,r,s\}\cap\{u,v,w,z\}| = 1$, where $[p,q]$ is not contained in $[u,v]$ or vice versa. The cases $q = v$, $r = w$, $p = u$ and $s = z$, as well as $q = w$, $s = u$, $v = r$ and $z = p$, can be proved as in Theorem 4.2.2. Suppose $q = u$ and let A_k and A_{k+2} be the cells of $\mathcal{P}$ having respectively q as upper right and lower left corner. If $\{A_{k-1}, A_k, A_{k+1}, A_{k+2}, A_{k+3}\}$ is an L-configuration the claim follows either if $\gcd(\text{in}(f),\text{in}(g)) = 1$ or by applying Lemma 4.1.7 if $\gcd(\text{in}(f),\text{in}(g)) \neq 1$. If $\{A_{k-2}, A_{k-1}, A_k, A_{k+1}, A_{k+2}, A_{k+3}\}$ or $\{A_{k-1}, A_k, A_{k+1}, A_{k+2}, A_{k+3}, A_{k+4}\}$ is an LD-horizontal or vertical skew hexomino, then $\gcd(\text{in}(f),\text{in}(g)) = 1$. Since there does not exist any RW-heptomino, the last possibilities consist in being A_{k-1}, A_k, A_{k+1} or A_{k+2} the middle cell of a W-pentomino. In all these cases we have the desired conclusion either if $\gcd(\text{in}(f),\text{in}(g)) = 1$ or by applying Lemma 4.1.7 if $\gcd(\text{in}(f),\text{in}(g)) \neq 1$. The cases $s = w$, $z = r$ or $v = p$ can be proved by similar arguments. □

Remark 4.2.6. With the same arguments, the statement of Theorem 4.2.5 holds also considering $Y = (\bigcup_{\mathcal{D}\in\mathcal{R}_1}\{a_D, b_D\}) \cup (\bigcup_{\mathcal{W}\in\mathcal{R}_2}\{x_W, z_W\})$.

Given a closed path polyomino $\mathcal{P}$ containing both W-pentominoes and RW-heptominoes, our aim is to find a P-order $<^Y$, for a suitable set $Y \subset V(\mathcal{P})$, such that $\mathcal{G}$ is the Gröbner basis of $I_\mathcal{P}$ with respect to the monomial order $<^Y_{\text{lex}}$. We are going to define the set Y by combining the previous construction and the highlighted points in Figures 5.3, 4.10, 4.11, and proceeding iteratively from the structure of the polyomino and the arrangement of the cells. In order to simplify notations and writings, we summarize in the table in Figure 4.13 the arrangements with highlighted points already introduced in the previous definitions that are useful to define the new set Y. We build up the set Y using the algorithm explained below, for which it is also important to consider the configurations described in Figure 4.14 and Figure 4.15.

Algorithm 4.2.7. Let $\mathcal{P}$ be a closed path polyomino, whose sequence of cells is $A_1, A_2, \dots, A_n, A_{n+1}$ (with $A_1 = A_{n+1}$) and containing both W-pentominoes and RW-heptominoes. Let $i, j \in \{1, 2, \dots, n, n+1\}$ with $i < j$. We define $Y_{i,j} \subset V(\mathcal{P})$ be the set provided by the algorithmic scheme described below:

1. Start with $Y_{i,j} = \emptyset$.

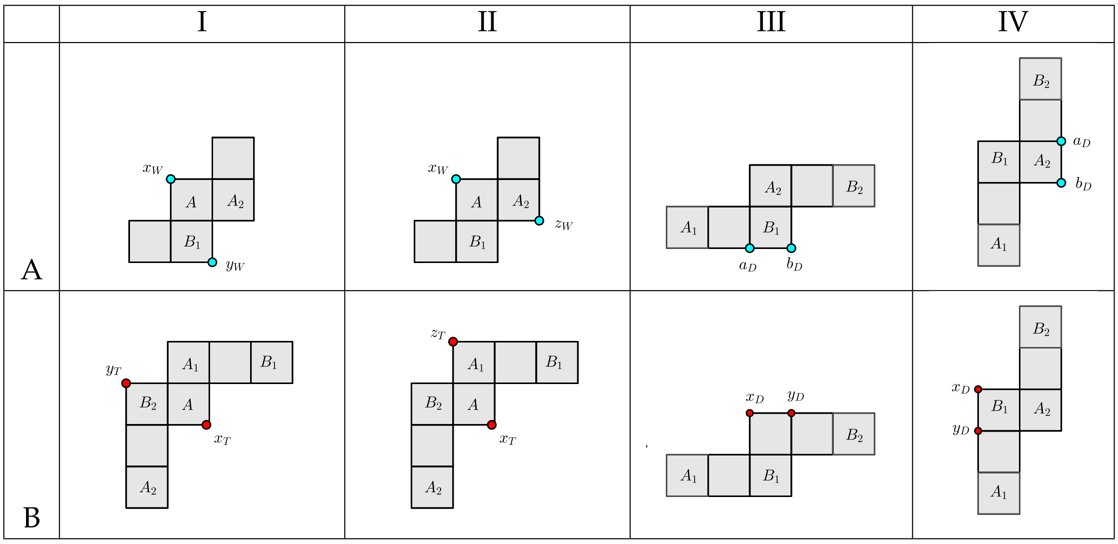


FIGURE 4.13

2. Define $\mathcal{Q} = \{q \in \{i, \dots, j\} \mid A_q$ is the middle cell of a RW-heptomino$\}$.
3. If $\mathcal{Q} \neq \emptyset$ define $q_1 = \min \mathcal{Q}$, otherwise define $q_1 = j$.
4. FOR $k \in \{i, \dots, q_1\}$ DO:
   (a) IF A_k is the middle cell of a W-pentomino THEN $Y_{i,j} = Y_{i,j} \cup \{x_W, z_W\}$ with reference to II-A of Figure 4.13.
   (b) IF $A_k, A_{k+1}, \dots, A_{k+6}$ is a sequence of cells of an LD-horizontal skew hexomino THEN $Y_{i,j} = Y_{i,j} \cup \{a_D, b_D\}$ with reference to III-A of Figure 4.13.
   (c) IF $A_k, A_{k+1}, \dots, A_{k+6}$ is a sequence of cells of an LD-vertical skew hexomino THEN $Y_{i,j} = Y_{i,j} \cup \{a_D, b_D\}$ with reference to IV-A of Figure 4.13.
5. Define $\mathcal{R} = \{r \in \{q_1 + 1, \dots, j\} \mid A_r$ is the middle cell of a W-pentomino$\}$.
6. If $\mathcal{R} \neq \emptyset$ define $r_1 = \min \mathcal{R}$, otherwise define $r_1 = j$.
7. Define $Q = q_1$ and $R = r_1$.
8. Consider the RW-heptomino with middle cell A_Q and let $M = \max\{m \in \{i, \dots, Q\} \mid A_m \cap Y_{i,j} \neq \emptyset\}$.
9. FOR $k \in \{Q, \dots, R\}$ DO:
   (a) IF A_k is the middle cell of an RW-heptomino THEN
   IF A_M and A_Q do not occur as in the configurations of Figure 4.14
   THEN $Y_{i,j} = Y_{i,j} \cup \{x_T, y_T\}$ with reference to I-B of Figure 4.13
   ELSE $Y_{i,j} = Y_{i,j} \cup \{x_T, z_T\}$ with reference to II-B of Figure 4.13.
   (b) IF $A_k, A_{k+1}, \dots, A_{k+6}$ is a sequence of cells of an LD-horizontal skew hexomino THEN $Y_{i,j} = Y_{i,j} \cup \{x_D, y_D\}$ with reference to III-B of Figure 4.13.
   (c) IF $A_k, A_{k+1}, \dots, A_{k+6}$ is a sequence of cells of an LD-vertical skew hexomino THEN $Y_{i,j} = Y_{i,j} \cup \{x_D, y_D\}$ with reference to IV-B of Figure 4.13.
   (d) IF $R = j$ THEN RETURN $Y_{i,j}$.
10. Define $\mathcal{Q} = \{q \in \{r_1 + 1, \dots, j\} \mid A_q$ is the middle cell of a RW-heptomino$\}$.

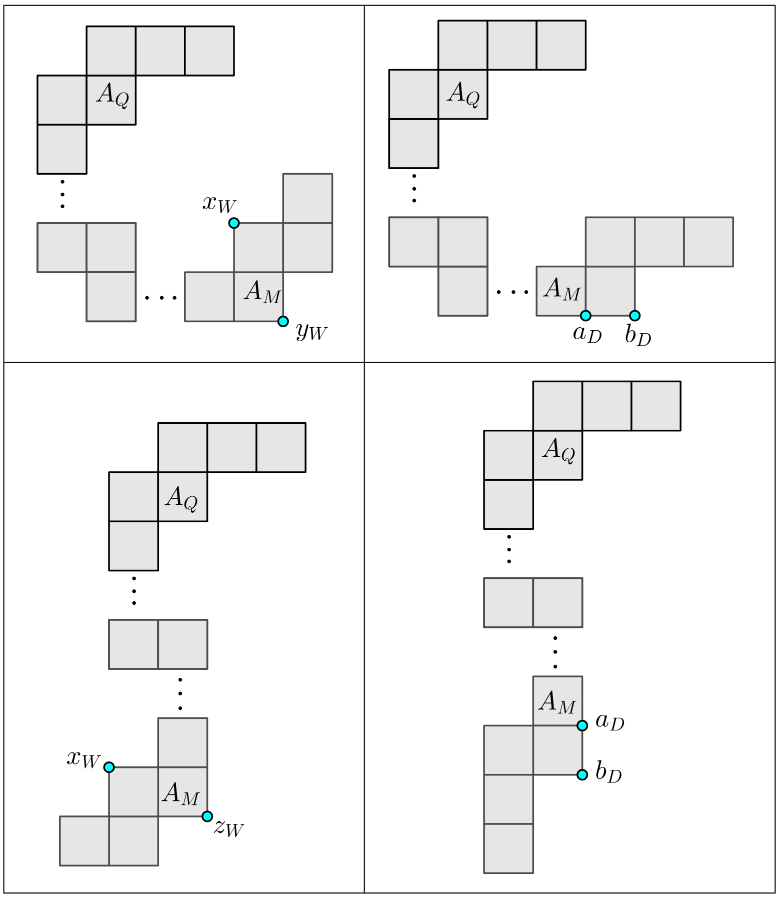


FIGURE 4.14: Conflicting configurations with I-B

11. If $\mathcal{Q} \neq \emptyset$ define $q_2 = \min \mathcal{Q}$, otherwise define $q_2 = j$.

12. Define $R = r_1$ and $Q = q_2$.

13. Consider the W-pentomino with middle cell A_R and let $M = \max\{m \in \{i, \dots, R\} \mid A_m \cap Y_{i,j} \neq \emptyset\}$.

14. FOR $k \in \{R, \dots, Q\}$ DO:

    (a) IF A_k is the middle cell of a W-pentomino THEN

    IF A_M and A_Q do not occur as in the configurations of Figure 4.15
    THEN $Y_{i,j} = Y_{i,j} \cup \{x_W, y_W\}$ with reference to I-A of Figure 4.13
    ELSE $Y_{i,j} = Y_{i,j} \cup \{x_W, z_W\}$ with reference to II-A of Figure 4.13.

    (b) IF $A_k, A_{k+1}, \dots, A_{k+6}$ is a sequence of cells of an LD-horizontal skew hexomino THEN $Y_{i,j} = Y_{i,j} \cup \{a_D, b_D\}$ with reference to III-A of Figure 4.13.

    (c) IF $A_k, A_{k+1}, \dots, A_{k+6}$ is a sequence of cells of an LD-vertical skew hexomino THEN $Y_{i,j} = Y_{i,j} \cup \{a_D, b_D\}$ with reference to IV-A of Figure 4.13.

    (d) IF $Q = j$ THEN RETURN $Y_{i,j}$.

15. $\ell = 2$.

16. WHILE $\ell > 1$ DO

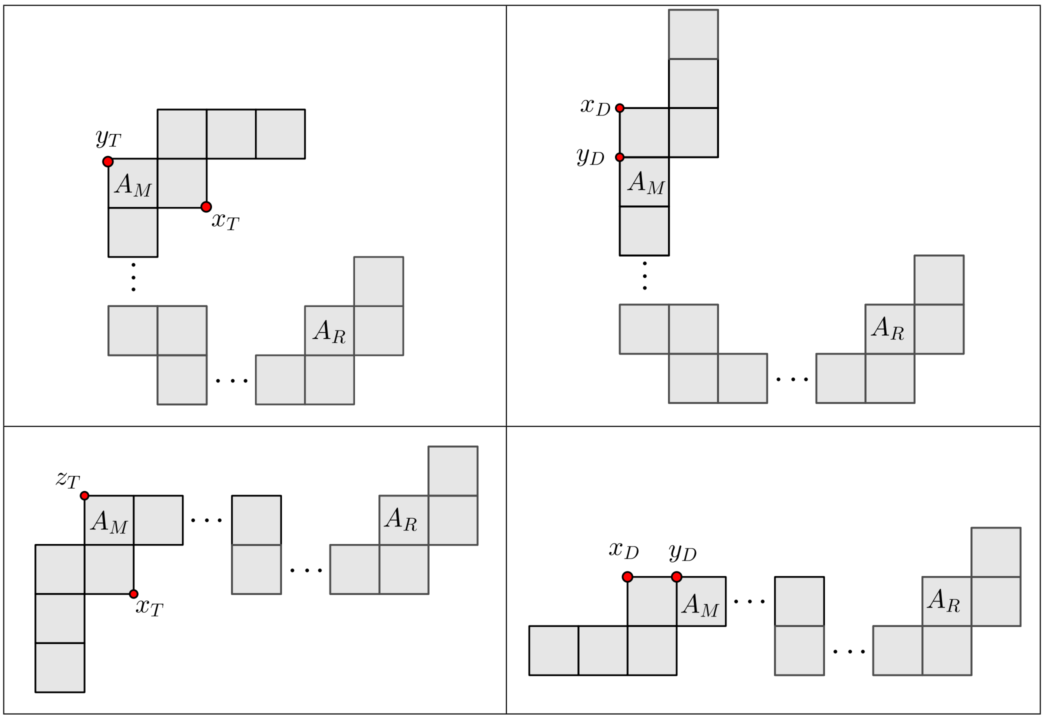


FIGURE 4.15: Conflicting configurations with I-A

(a) Define

$$\mathcal{R} = \{r \in \{q_\ell + 1, \dots, j\} \mid A_r \text{ is the middle cell of a W-pentomino}\}$$

(b) If $\mathcal{R} \neq \emptyset$ define $r_\ell = \min \mathcal{R}$, otherwise define $r_\ell = j$.
(c) Define $Q = q_\ell$ and $R = r_\ell$.
(d) $M = \max\{m \in \{i, \dots, Q\} \mid A_m \cap Y_{i,j} \neq \emptyset\}$.
(e) Execute the instructions in (9).
(f) Define

$$\mathcal{Q} = \{q \in \{r_\ell + 1, \dots, j\} \mid A_q \text{ is the middle cell of a RW-heptomino}\}$$

(g) If $\mathcal{Q} \neq \emptyset$ define $q_{\ell+1} = \min \mathcal{Q}$, otherwise define $q_{\ell+1} = j$.
(h) Define $R = r_\ell$ and $Q = q_{\ell+1}$.
(i) $M = \max\{m \in \{i, \dots, R\} \mid A_m \cap Y_{i,j} \neq \emptyset\}$.
(j) Execute the instructions in in (14).
(k) $\ell = \ell + 1$.

17. END

Observe that, since $r_\ell < r_{\ell+1}$ and $q_\ell < q_{\ell+1}$ for all $\ell \in \mathbb{N}$ then there exists $\overline{\ell}$ such that $r_{\overline{\ell}} = j$ or $q_{\overline{\ell}} = j$, so the procedure stops and the set $Y_{i,j}$ is returned.

Definition 4.2.8. Let $\mathcal{P}$ be a closed path polyomino containing both W-pentominoes and RW-heptominoes. Consider a W-pentomino $\mathcal{W}$ of $\mathcal{P}$ and suppose that $\mathcal{W}$ contains the cells A_1, A_2, A_3, A_4 and A_5, labelled bottom up as in Figure 4.16. We put $L = Y_{2,n+1}$. In Figure 4.17 we make in evidence, for instance, the points belonging to L.

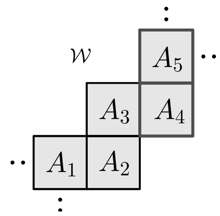


FIGURE 4.16

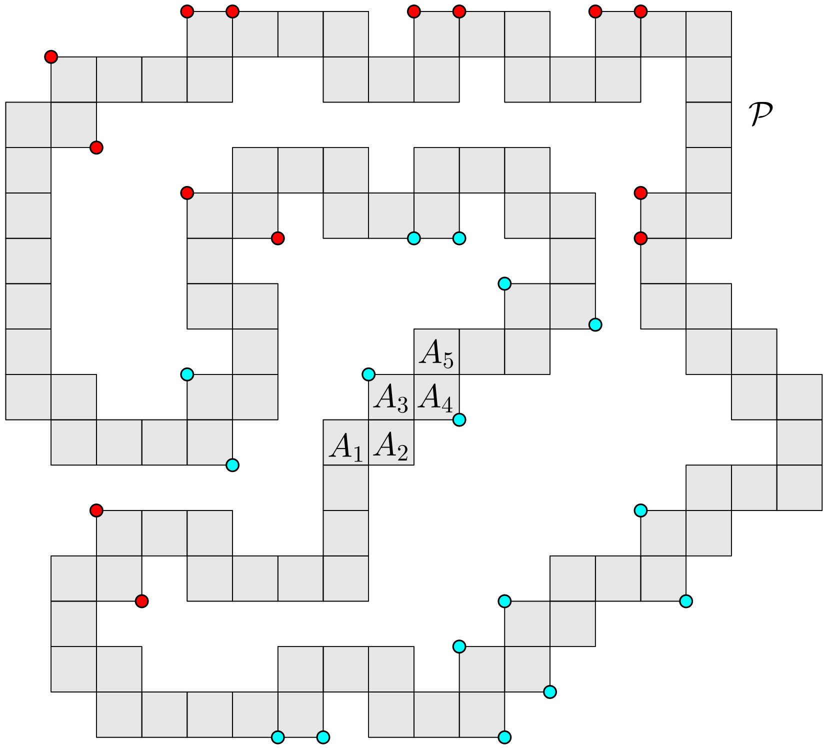


FIGURE 4.17: The set $Y_{2,n+1}$ consists of the highlighted points

Theorem 4.2.9. *Let $\mathcal{P}$ be a closed path. Suppose that $\mathcal{P}$ contains a W-pentomino and an RW-heptomino and let L be the set given in Definition 4.2.8. Then $\mathcal{G}$ is the reduced Gröbner basis of $I_\mathcal{P}$ with respect to $<_{\text{lex}}^L$.*

Proof. Let f and g be the two binomials attached respectively to the inner intervals $[p,q]$ and $[u,v]$ of $\mathcal{P}$. It suffices to show that $S(f,g)$ reduces to 0 modulo $\mathcal{G}$ in every case. Observe that the desired claim follows from Definitions 4.2.7 and 4.2.8, arguing as in Theorem 4.2.5. In fact, we always have that either $\gcd(\text{in}(f),\text{in}(g)) = 1$ or, if $\gcd(\text{in}(f),\text{in}(g)) \neq 1$, it is sufficient to apply the previous lemmas. $\square$

Theorem 4.2.10. *Let $\mathcal{P}$ be a closed path polyomino having an L-configuration or a ladder of at least three steps, or equivalently having no zig-zag walks. Then $K[\mathcal{P}]$ is a normal Cohen-Macaulay domain.*

Proof. From Theorem 4.2.9 we obtain that there exists a monomial order $\prec$ such that $\mathcal{G}$ is the Gröbner basis of $I_\mathcal{P}$ with respect to $\prec$, in particular $I_\mathcal{P}$ admits a squarefree initial ideal with respect to some monomial order. Since $\mathcal{P}$ has an L-configuration or a ladder of three steps, from Subsection 2.2.3 we have that $I_\mathcal{P}$ is a toric ideal. By Theorem 1.4.15 we obtain that $K[\mathcal{P}]$ is normal and by Theorem 1.4.14 we obtain that $K[\mathcal{P}]$ is Cohen-Macaulay. $\square$

Theorem 4.2.11. *Let $\mathcal{P}$ be a closed path polyomino. Then $I_{\mathcal{P}}$ is radical.*

Proof. It follows from the fact that a graded ideal I is radical if $\mathrm{in}_<(I)$ is squarefree for some monomial order $<$. □

Remark 4.2.12. In [25] the authors proved that if $\mathcal{P}$ is a balanced polyomino, equivalently $\mathcal{P}$ is simple, then the universal Gröbner basis is squarefree. In general this fact does not hold for a non-simple polyomino. Consider the closed path $\mathcal{P}$ in Figure 4.18. Let $\{V_1, V_2, V_3, V_4\}$ and $\{H_1, H_2, H_3, H_4\}$ be respectively the sets of the maximal

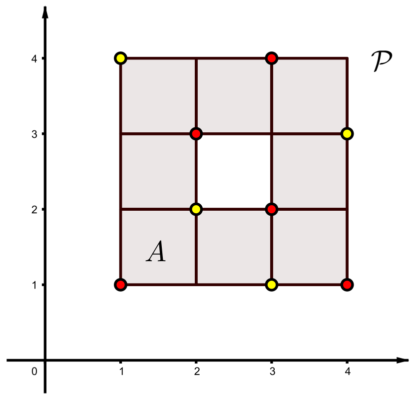


FIGURE 4.18

vertical and horizontal edge intervals of $\mathcal{P}$ such that $r = (i,j) \in V_i \cap H_j$, and let $\{v_1, v_2, v_3, v_4\}$ and $\{h_1, h_2, h_3, h_4\}$ be the associated sets of the variables. Let w be another variable different from v_i and h_j. We recall from Theorem 2.2.15 that $I_{\mathcal{P}} = J_{\mathcal{P}}$, where $J_{\mathcal{P}}$ is the kernel of ϕ, defined as

$$\phi : K[x_{ij} : (i,j) \in V(\mathcal{P})] \longrightarrow K[\{v_i, h_j, w\} : i,j \in \{1,2,3,4\}]$$
$$\phi(x_{ij}) = v_i h_j w^k$$

where $k = 0$ if $(i,j) \notin A$, and $k = 1$, if $(i,j) \in A$.
Consider the binomial $f = x_{11}x_{23}x_{32}x_{34}x_{41} - x_{14}x_{22}x_{31}^2x_{43}$ attached to the vertices in red and yellow. Observe that $f \in I_{\mathcal{P}}$ because $\phi(x_{11}x_{23}x_{32}x_{34}x_{41}) = \phi(x_{14}x_{22}x_{31}^2x_{43})$. We show that f is primitive, that is there does not exist any binomial $g = g^+ - g^-$ in $I_{\mathcal{P}}$ with $g \neq f$ such that $g^+ | x_{11}x_{23}x_{32}x_{34}x_{41}$ and $g^- | x_{14}x_{22}x_{31}^2x_{43}$. Suppose by contradiction that there exists such a binomial. Observe that $2 < \deg(g) < 5$, since $f \neq g$ and all binomials of degree two satisfying the primitive conditions are not inner 2-minors. It is sufficient to prove that x_{11} (resp. x_{22}) cannot divide g^+ (resp. g^-). If that happens, then w divides $\phi(g^+)$, which is equal to $\phi(g^-)$, so x_{22} divides g^-. Since $g \in I_{\mathcal{P}} = J_{\mathcal{P}}$, in particular $\phi(g^+) = \phi(g^-)$, we obtain that $g^+ = x_{11}x_{23}x_{32}x_{34}x_{41}$ and $g^- = x_{14}x_{22}x_{31}^2x_{43}$ from easy calculations. Hence $f = g$, a contradiction. In conclusion we have that f is a primitive binomial of $I_{\mathcal{P}}$. Since for a toric ideal the universal Gröbner basis coincides with the Graver basis (see [47]), the primitive binomials of $I_{\mathcal{P}}$ form the universal Gröbner basis $\mathcal{G}$ of $I_{\mathcal{P}}$. Since f is a primitive binomial of $I_{\mathcal{P}}$, it follows that $\mathcal{G}$ is not squarefree. Anyway $I_{\mathcal{P}}$ is a radical ideal which admits a squarefree initial ideal with a different monomial ordering, for instance with respect to $<^1_{\mathrm{lex}}$, since the set of generator of $I_{\mathcal{P}}$ is the reduced Gröbner basis by [37, Theorem 4.1].

Chapter 5

The Hilbert-Poincaré series and the rook polynomial of some non-simple polyominoes

In this chapter we study the Castelnuovo-Mumford regularity and, in particular, the Hilbert-Poincaré series of some classes of non-simple polyominoes relating them to a particular polynomial in $\mathbb{Z}[t]$, called the *rook polynomial*. As a consequence, a characterization of the Gorenstein property for closed paths without zig-zag walks is given. The regularity, the Hilbert-Poincaré series and the Gorenstein property are studied for several class of polyominoes. For references we recall in particular [1], [17], [37], [38] and [40]. In the next sections, we show in detail all results contained in [6].

5.1 Hilbert-Poincaré series of $(\mathcal{L}, \mathcal{C})$-polyominoes

In this section we define the class of $(\mathcal{L}, \mathcal{C})$-polyominoes and we provide an explicit formula for the Hilbert-Poincaré series of the related coordinate rings, depending on the Hilbert-Poincaré series of some polyominoes obtained eliminating specific cells. In such case we compute also the Krull dimension of the coordinate ring of a polyomino belonging to this class.

Definition 5.1.1. Let $\mathcal{L}$ be the union of the two cell intervals $[A, A_r]$, consisting of the cells $A, A_1, \dots, A_r$, and $[A, B_s]$, consisting of the cells $A, B_1, \dots, B_s$, where A, A_r and A, B_s are respectively in horizontal and vertical position with $r, s \geq 2$. We denote by a, b and c, d respectively the diagonal and anti-diagonal corners of A, by d_i and a_i respectively the upper left and upper right corners of B_i for $i \in [s]$ and by b_j and c_j respectively the upper and lower right corners of A_j for $j \in [r]$. Let $\mathcal{C}$ be a polyomino. We say that a polyomino $\mathcal{P}$ is an *$(\mathcal{L}, \mathcal{C})$-polyomino* if $\mathcal{P} = \mathcal{L} \sqcup \mathcal{C}$ and it satisfies one and only one of the following four conditions (see also Figure 5.1):

(1) $V(\mathcal{L}) \cap V(\mathcal{C}) = \{a_{s-1}, a_s, b_{r-1}, b_r\}$;

(2) $V(\mathcal{L}) \cap V(\mathcal{C}) = \{a_{s-1}, a_s, c_{r-1}, c_r\}$;

(3) $V(\mathcal{L}) \cap V(\mathcal{C}) = \{d_{s-1}, d_s, b_{r-1}, b_r\}$;

(4) $V(\mathcal{L}) \cap V(\mathcal{C}) = \{d_{s-1}, d_s, c_{r-1}, c_r\}$;

If $\mathcal{P}$ is an $(\mathcal{L}, \mathcal{C})$-polyomino, the following related polyominoes will be used along the paper:

- $\mathcal{P}_1 = \mathcal{P} \backslash [A, A_r]$;

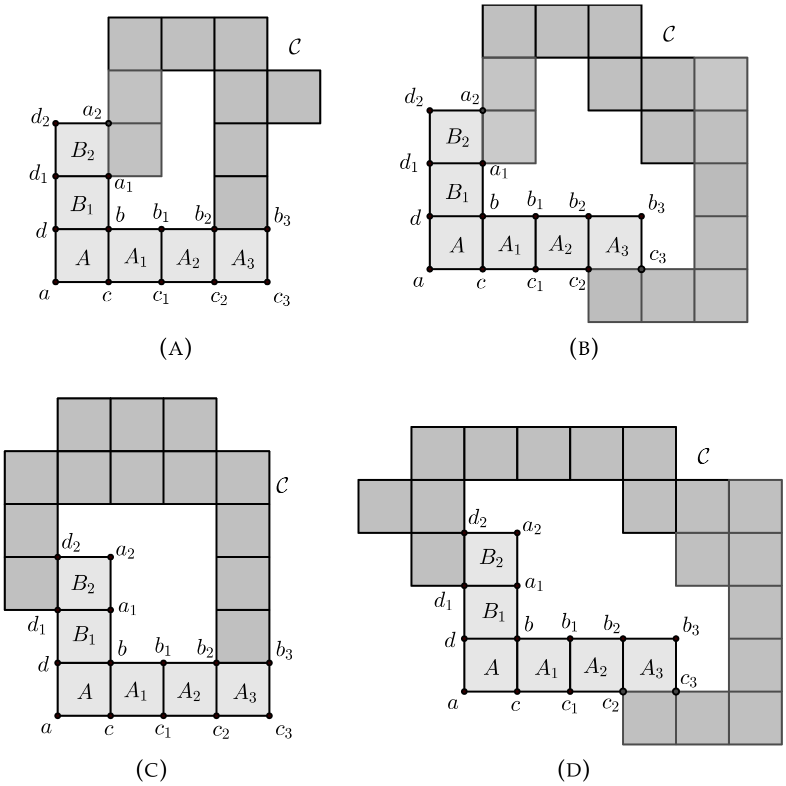


FIGURE 5.1: Examples of the different cases of $(\mathcal{L},\mathcal{C})$-polyominoes

- $\mathcal{P}_2 = \mathcal{P}\backslash[A, B_s]$;
- $\mathcal{P}_3 = \mathcal{P}\backslash([A, A_r] \cup [A, B_s]) = \mathcal{C}$;
- $\mathcal{P}_4 = \mathcal{P}\backslash\{A, A_1, B_1\}$;
- $\mathcal{P}_1' = \mathcal{P}\backslash[A_1, A_r]$;
- $\mathcal{P}_2' = \mathcal{P}\backslash[B_1, B_s]$.

Lemma 5.1.2. *Let $\mathcal{P}$ be an $(\mathcal{L},\mathcal{C})$-polyomino. Then $S_\mathcal{P}/(I_\mathcal{P}, x_a, x_d, x_c, x_b) \cong K[\mathcal{P}_4]$.*

Proof. Observe that $I_\mathcal{P}$ can be written in the following way

$$I_\mathcal{P} = I_{\mathcal{P}_4} + (x_a x_b - x_b x_c) + \sum_{i=1}^{r}(x_a x_{a_i} - x_c x_{d_i}) + \sum_{i=1}^{r}(x_d x_{a_i} - x_b x_{d_i}) +$$
$$\sum_{i=1}^{s}(x_a x_{b_i} - x_d x_{c_i}) + \sum_{i=1}^{r}(x_c x_{b_i} - x_b x_{c_i}).$$

It follows that $(I_\mathcal{P}, x_a, x_d, x_c, x_b) = (I_{\mathcal{P}_4}, x_a, x_d, x_c, x_b)$, in particular

$$S_\mathcal{P}/(I_\mathcal{P}, x_a, x_d, x_c, x_b) = S_\mathcal{P}/(I_{\mathcal{P}_4}, x_a, x_d, x_c, x_b) \cong S_{\mathcal{P}_4}/I_{\mathcal{P}_4} = K[\mathcal{P}_4]$$

. □

Proposition 5.1.3. *Let $\mathcal{P}$ be an $(\mathcal{L},\mathcal{C})$-polyomino. If $I_\mathcal{P}$ is prime, then $K[\mathcal{P}_i]$ and $K[\mathcal{P}_j']$ are domains for $i \in [4]$ and $j \in \{1,2\}$.*

Proof. We may assume that $\mathcal{P}$ is an $(\mathcal{L},\mathcal{C})$-polyomino such that $V(\mathcal{L})\cap V(\mathcal{C}) = \{a_{s-1},a_s,b_{r-1},b_r\}$, since similar arguments can be used in the other cases. We prove that $K[\mathcal{P}_1]$ is a domain. Observe that $I_\mathcal{P}$ is a toric ideal since $I_\mathcal{P}$ is a prime binomial ideal. Then there exists a map $\phi : S_\mathcal{P} \to K[t_1^{\pm 1},\dots,t_d^{\pm 1}]$ with $x_{ij} \mapsto \mathbf{t}^{\mathbf{a}_{ij}}$ for all $(i,j) \in V(\mathcal{P})$ such that $I_\mathcal{P} = \ker\phi$. Let $\mathcal{V} = \{a,c,c_1,\dots,c_r,b_1,\dots,b_{r-2}\}$, we define $\phi_\mathcal{V} : S_\mathcal{P} \to K[t_1^{\pm 1},\dots,t_d^{\pm 1}]$, by $\phi_\mathcal{V}(x_v) = 0$ if $v \in \mathcal{V}$ and $\phi_\mathcal{V}(x_v) = \phi(x_v)$ otherwise. Put $J_\mathcal{V} := (I_\mathcal{P},\{x_v \mid v \in \mathcal{V}\})$, we prove that $J_\mathcal{V} = \ker\phi_\mathcal{V}$. If $f \in \ker\phi_\mathcal{V}$, we can write $f = \tilde{f} + \beta g$ where $\beta \in S_\mathcal{P}$, $g \in (\{x_v \mid v \in \mathcal{V}\})$ and $\tilde{f}$ not containing variables in the set $\{x_v \mid v \in \mathcal{V}\}$. Since $\phi_\mathcal{V}(f) = 0$, we have $\phi(\tilde{f}) = 0$, so $\tilde{f} \in \ker\phi = I_\mathcal{P}$. For the other inclusion it suffices to prove that $I_\mathcal{P} \subseteq \ker\phi_\mathcal{V}$. In such a case observe that, for this configuration, if $f = x_{i_1}x_{i_2} - x_{j_1}x_{j_2}$ is a generator of $I_\mathcal{P}$ then $\{x_{i_1},x_{i_2}\}\cap\{x_v \mid v \in \mathcal{V}\} \neq \emptyset$ if and only if $\{x_{j_1},x_{j_2}\}\cap\{x_v \mid v \in \mathcal{V}\} \neq \emptyset$, so in all possible cases we have $\phi_\mathcal{V}(f) = 0$. Therefore $J_\mathcal{V} = \ker\phi_\mathcal{V}$ and $J_\mathcal{V}$ is a prime ideal. As in Lemma 5.1.2, we have also that $J_\mathcal{V} = (I_{\mathcal{P}_1}, x_a, x_c, x_{c_i}, x_{b_j} : i \in [r], j \in [r-2])$, and $K[\mathcal{P}_1] \cong S_\mathcal{P}/J$ is a domain. The proof for this case is done. For the other polyominoes the proof is analogue, considering:

- for $\mathcal{P}_2$ the set $\mathcal{V} = \{a,d,d_1,\dots,d_s,a_1,\dots,a_{s-2}\}$;
- for $\mathcal{P}_3$ the set $\mathcal{V} = \{a,b,c,d,c_1,\dots,c_r,d_1,\dots,d_s,a_1,\dots,a_{s-2},b_1,\dots,b_{r-2}\}$;
- for $\mathcal{P}_4$ the set $\mathcal{V} = \{a,b,c,d\}$;
- for $\mathcal{P}_1'$ the set $\mathcal{V} = \{b_1,\dots,b_{r-2},c_1,\dots,c_r\}$;
- for $\mathcal{P}_2'$ the set $\mathcal{V} = \{a_1,\dots,a_{s-2},d_1,\dots,d_s\}$.

□

If $\mathcal{P}$ is an $(\mathcal{L},\mathcal{C})$-polyomino, our aim is to provide a formula for the Hilbert-Poincaré series of $K[\mathcal{P}]$, involving the Hilbert-Poincaré series of $K[\mathcal{P}_1]$, $K[\mathcal{P}_2]$, $K[\mathcal{P}_3]$ and $K[\mathcal{P}_4]$ in the hypotheses that $K[\mathcal{P}]$ is an integral domain. In particular, if (i_1,i_2,i_3,i_4) is a permutation of the set $\{a,b,c,d\}$, our strategy consists in considering the following four short exact sequences:

$$0 \longrightarrow S_\mathcal{P}/(I_\mathcal{P}:x_{i_1}) \longrightarrow S_\mathcal{P}/I_\mathcal{P} \longrightarrow S_\mathcal{P}/(I_\mathcal{P},x_{i_1}) \longrightarrow 0$$
$$0 \longrightarrow S_\mathcal{P}/((I_\mathcal{P},x_{i_1}):x_{i_2}) \longrightarrow S_\mathcal{P}/(I_\mathcal{P},x_{i_1}) \longrightarrow S_\mathcal{P}/(I_\mathcal{P},x_{i_1},x_{i_2}) \longrightarrow 0$$
$$0 \longrightarrow S_\mathcal{P}/((I_\mathcal{P},x_{i_1},x_{i_2}):x_{i_3}) \longrightarrow S_\mathcal{P}/(I_\mathcal{P},x_{i_1},x_{i_2}) \longrightarrow S_\mathcal{P}/(I_\mathcal{P},x_{i_1},x_{i_2},x_{i_3}) \longrightarrow 0$$
$$0 \longrightarrow S_\mathcal{P}/((I_\mathcal{P},x_{i_1},x_{i_2},x_{i_3}):x_{i_4}) \longrightarrow S_\mathcal{P}/(I_\mathcal{P},x_{i_1},x_{i_2},x_{i_3}) \longrightarrow S_\mathcal{P}/(I_\mathcal{P},x_a,x_d,x_c,x_b) \longrightarrow 0$$

From the exact sequences above, we will obtain the Hilbert-Poincaré series of $S_\mathcal{P}/I_\mathcal{P}$ by a repeated application of Proposition 1.1.15 and considering in each case a suitable permutation (i_1,i_2,i_3,i_4) of the set $\{a,b,c,d\}$ in order to compute the Hilbert-Poincaré series of the rings in the intermediate steps. To reach our aim we provide several preliminary lemmas, distinguishing the different possibilities for the set $V(\mathcal{L})\cap V(\mathcal{C})$.

Lemma 5.1.4. *Let $\mathcal{P}$ be an $(\mathcal{L},\mathcal{C})$-polyomino such that $V(\mathcal{L})\cap V(\mathcal{C}) = \{a_{s-1},a_s,b_{r-1},b_r\}$. Suppose that $I_\mathcal{P}$ is prime. Then:*

(1) $S_\mathcal{P}/((I_\mathcal{P},x_a):x_d) \cong K[\mathcal{P}_1]\otimes_K K[x_{b_1},\dots,x_{b_{r-2}}]$;

(2) $S_\mathcal{P}/((I_\mathcal{P},x_a,x_d):x_c) \cong K[\mathcal{P}_2]\otimes_K K[x_{a_1},\dots,x_{a_{s-2}}]$;

(3) $S_\mathcal{P}/((I_\mathcal{P},x_a,x_d,x_c):x_b) \cong K[\mathcal{P}_3]\otimes_K K[x_b,x_{a_1},\dots,x_{a_{s-2}},x_{b_1},\dots,x_{b_{r-2}}]$.

Proof. (1) Firstly observe that $I_{\mathcal{P}}$ can be written in the following way:

$$I_{\mathcal{P}} = I_{\mathcal{P}_1} + (x_a x_b - x_c x_d) + \sum_{i=1}^{s}(x_a x_{a_i} - x_c x_{d_i}) + \sum_{j=1}^{r}(x_a x_{b_j} - x_d x_{c_j}) +$$
$$\sum_{j=1}^{r}(x_c x_{b_j} - x_b x_{c_j}) + \sum_{\substack{k,l\in[r]\\ k<l}}(x_{c_k} x_{b_l} - x_{c_l} x_{b_k}) +$$
$$(\{x_{c_{r-1}} x_v - x_{c_r} x_u \,|\, [c_{r-1}, v] \in \mathcal{I}(\mathcal{P}), u = v - (1,0)\}),$$

Now we describe the ideal $(I_{\mathcal{P}}, x_a)$ in $S_{\mathcal{P}}$:

$$(I_{\mathcal{P}}, x_a) = (I_{\mathcal{P}_1}, x_a) + (x_c x_d) + \sum_{i=1}^{s}(x_c x_{d_i}) + \sum_{j=1}^{r}(x_d x_{c_j}) +$$
$$\sum_{j=1}^{r}(x_c x_{b_j} - x_b x_{c_j}) + \sum_{\substack{k,l\in[r]\\ k<l}}(x_{c_k} x_{b_l} - x_{c_l} x_{b_k}) +$$
$$(\{x_{c_{r-1}} x_v - x_{c_r} x_u \,|\, [c_{r-1}, v] \in \mathcal{I}(\mathcal{P}), u = v - (1,0)\}).$$

We prove that $(I_{\mathcal{P}}, x_a) : x_d = (I_{\mathcal{P}_1}, x_a) + (x_c) + \sum_{i=1}^{r}(x_{c_i})$. It follows trivially from the previous equality that $(I_{\mathcal{P}}, x_a) : x_d \supseteq (I_{\mathcal{P}_1}, x_a) + (x_c) + \sum_{i=1}^{r}(x_{c_i})$. Let $f \in S_{\mathcal{P}}$ such that $x_d f \in (I_{\mathcal{P}}, x_a)$. Then

$$x_d f = g + \alpha x_a + \beta x_c x_d + \sum_{i=1}^{s}\gamma_i x_c x_{d_i} + \sum_{j=1}^{r}\delta_j x_d x_{c_j} + \sum_{i=j}^{r}\omega_j (x_c x_{b_j} - x_b x_{c_j}) +$$
$$+ \sum_{\substack{k,l\in[r]\\ k<l}} \nu_{kl}(x_{c_k} x_{b_l} - x_{c_l} x_{b_k}) + \sum_{\substack{[c_{r-1},v]\in\mathcal{I}(\mathcal{P})\\ u=v-(1,0)}} \lambda_v (x_{c_{r-1}} x_v - x_{c_r} x_u),$$

where $g \in \mathcal{I}_{\mathcal{P}_1}, \alpha, \beta, \gamma_i, \delta_j, \omega_j, \nu_{k,l}\lambda_v \in S_{\mathcal{P}}$ for all $i,k \in [s], j,l \in [r]$ and for all $v \in V(\mathcal{P})$ such that $[c_{r-1}, v] \in \mathcal{I}(\mathcal{P})$. As a consequence:

$$x_d\left(f - \beta x_c - \sum_{j=1}^{r}\delta_j x_{c_j}\right) = g + \alpha x_a + \sum_{i=1}^{s}(\gamma_i x_{d_i}) x_c + \sum_{i=j}^{r}(\omega_j x_{b_j}) x_c - \sum_{i=j}^{r}(\omega_j x_b) x_{c_j} +$$
$$+ \sum_{\substack{k,l\in[r]\\ k<l}} (\nu_{kl} x_{b_l}) x_{c_k} - \sum_{\substack{k,l\in[r]\\ k<l}} (\nu_{kl} x_{b_k}) x_{c_l} + \left(\sum_{\substack{[c_{r-1},v]\in\mathcal{I}(\mathcal{P})\\ u=v-(1,0)}} \lambda_v x_v\right) x_{c_{r-1}} +$$
$$- \left(\sum_{\substack{[c_{r-1},v]\in\mathcal{I}(\mathcal{P})\\ u=v-(1,0)}} \lambda_v x_u\right) x_{c_r}.$$

Hence we obtain that $x_d(f - \beta x_c - \sum_{j=1}^{r}\delta_j x_{c_j}) \in I_{\mathcal{P}_1} + (x_a) + (x_c) + \sum_{i=1}^{r}(x_{c_i})$. Since $K[\mathcal{P}_1]$ is a domain (Proposition 5.1.3) and $a, c, c_i \notin V(\mathcal{P}_1)$ for all $i \in [r]$ then $I_{\mathcal{P}_1} + (x_a) + (x_c) + \sum_{i=1}^{r}(x_{c_i})$ is a prime ideal in $S_{\mathcal{P}}$. Since $x_d \notin I_{\mathcal{P}_1}$, we have $f - \beta x_c - \sum_{j=1}^{r}\delta_j x_{c_j} \in I_{\mathcal{P}_1} + (x_a) + (x_c) + \sum_{i=1}^{r}(x_{c_i})$, so $f \in I_{\mathcal{P}_1} + (x_a) + (x_c) + \sum_{i=1}^{r}(x_{c_i})$, that is $(I_{\mathcal{P}}, x_a) : x_d \subseteq (I_{\mathcal{P}_1}, x_a) + (x_c) + \sum_{i=1}^{r}(x_{c_i})$. In conclusion we have $(I_{\mathcal{P}}, x_a) : x_d = (I_{\mathcal{P}_1}, x_a) + (x_c) + \sum_{i=1}^{r}(x_{c_i})$ and as a consequence $S_{\mathcal{P}}/((I_{\mathcal{P}}, x_a) : x_d) = S_{\mathcal{P}}/(I_{\mathcal{P}_1} + (x_a, x_c, x_{c_1}, \dots, x_{c_r})) \cong S_{\mathcal{P}_1}/I_{\mathcal{P}_1} \otimes_K K[x_v \mid v \in V(\mathcal{P} \setminus \mathcal{P}_1)]/(x_a, x_c, x_{c_1}, \dots, x_{c_r}) = K[\mathcal{P}_1] \otimes_K K[x_{b_1}, \dots, x_{b_{r-2}}]$. The claim (1) is proved.
(2) By similar computations as in the first part of (1) we can prove that $(I_{\mathcal{P}}, x_a, x_d) : x_c = (I_{\mathcal{P}_2}, x_a, x_d) + \sum_{i=1}^{s}(x_{d_i})$, so claim (2) follows by using similar arguments as in

the last part in (1).
(3) The argument is the same, considering that $(I_{\mathcal{P}}, x_a, x_d, x_c) : x_b = (I_{\mathcal{P}_3}, x_a, x_d, x_c) + \sum_{i=1}^{s}(x_{d_i}) + \sum_{j=1}^{r}(x_{c_i})$ can be proved using computations similar to the previous cases. □

In the previous result we examine, for a $(\mathcal{L},\mathcal{C})$-polyomino, the case $V(\mathcal{L}) \cap V(\mathcal{C}) = \{a_{s-1}, a_s, b_{r-1}, b_r\}$. In order to examine the other cases we need other preliminary results involving the polyominoes $\mathcal{P}'_1$ and $\mathcal{P}'_2$.

Lemma 5.1.5. *Let $\mathcal{P}$ be an $(\mathcal{L},\mathcal{C})$-polyomino. Then*

(1) $S_{\mathcal{P}'_2}/(I_{\mathcal{P}'_2}, x_b, x_c) \cong K[\mathcal{P}_2]$. Moreover, if $I_{\mathcal{P}_2}$ is a prime ideal then $(I_{\mathcal{P}'_2}, x_b, x_c)$ is a prime ideal of $S_{\mathcal{P}}$.

(2) $S_{\mathcal{P}'_1}/(I_{\mathcal{P}'_1}, x_b, x_d) \cong K[\mathcal{P}_1]$. Moreover, if $I_{\mathcal{P}_1}$ is a prime ideal then $(I_{\mathcal{P}'_1}, x_b, x_d)$ is a prime ideal of $S_{\mathcal{P}}$.

Proof. (1) Let $\mathcal{R}$ be the polyomino obtained from the cells of $\mathcal{P}_2$ and renaming the vertices b and c respectively by d and a, in particular $S_{\mathcal{R}} = K[x_v \mid v \in V(\mathcal{P}'_2) \setminus \{b, c\}]$. Observe that

$$I_{\mathcal{P}'_2} = I_{\mathcal{R}} + (x_a x_b - x_c x_d) + \sum_{i=1}^{r}(x_c x_{b_i} - x_b x_{c_i})$$

So $(I_{\mathcal{P}'_2}, x_b, x_c) = (I_{\mathcal{R}}, x_b, x_c)$ and in particular $S_{\mathcal{P}'_2}/(I_{\mathcal{P}'_2}, x_b, x_c) = S_{\mathcal{P}'_2}/(I_{\mathcal{R}}, x_b, x_c) \cong S_{\mathcal{R}}/I_{\mathcal{R}} = K[\mathcal{R}] \cong K[\mathcal{P}_2]$, since x_b, x_c do not belong to the support of any element of $I_{\mathcal{R}}$ and observing that, apart from the name of the vertices involved, $\mathcal{R} = \mathcal{P}_2$. Furthermore $S_{\mathcal{P}}/(I_{\mathcal{P}'_2}, x_b, x_c) \cong S_{\mathcal{P}'_2}/(I_{\mathcal{P}'_2}, x_b, x_c) \otimes_K K[x_v \mid v \in V(\mathcal{P} \setminus \mathcal{P}'_2)] \cong K[\mathcal{P}_2] \otimes_K K[x_v \mid v \in V(\mathcal{P} \setminus \mathcal{P}'_2)]$, so also the last claim follows.
(2) The result can be obtained arguing as in the proof of (1). Indeed the arrangements involved in these situations can be considered the same up to one reflection and one rotation. □

Lemma 5.1.6. *Let $\mathcal{P}$ be an $(\mathcal{L},\mathcal{C})$-polyomino such that $V(\mathcal{L}) \cap V(\mathcal{C}) = \{d_{s-1}, d_s, b_{r-1}, b_r\}$. Suppose that $I_{\mathcal{P}}$ is prime. Then:*

(1) $S_{\mathcal{P}}/((I_{\mathcal{P}}, x_c) : x_b) \cong K[\mathcal{P}_1] \otimes_K K[x_{b_1}, \dots, x_{b_{r-2}}]$;

(2) $S_{\mathcal{P}}/((I_{\mathcal{P}}, x_b, x_c) : x_a) \cong K[\mathcal{P}_2] \otimes_K K[x_{d_1}, \dots, x_{d_{s-2}}]$;

(3) $S_{\mathcal{P}}/((I_{\mathcal{P}}, x_a, x_b, x_c) : x_d) \cong K[\mathcal{P}_3] \otimes_K K[x_d, x_{b_1}, \dots, x_{b_{r-2}}, x_{d_1}, \dots, x_{d_{s-2}}]$.

Proof. Arguing as in Lemma 5.1.4, we obtain the equalities of the following ideals:

(1) $(I_{\mathcal{P}}, x_c) : x_b = (I_{\mathcal{P}_1}, x_c) + (x_a) + \sum_{i=1}^{r}(x_{c_i})$

(2) $(I_{\mathcal{P}}, x_b, x_c) : x_a = (I_{\mathcal{P}'_2}, x_b, x_c) + \sum_{i=1}^{s}(x_{a_i})$.

(3) $(I_{\mathcal{P}}, x_a, x_b, x_c) : x_d = (I_{\mathcal{P}_3}, x_a, x_b, x_c) + \sum_{i=1}^{s}(x_{a_i}) + \sum_{i=1}^{r}(x_{c_i})$.

In particular, the second equality above holds since $(I_{\mathcal{P}'_2}, x_b, x_c)$ is a prime ideal by Lemma 5.1.5. By the same lemma we have also $S_{\mathcal{P}'_2}/(I_{\mathcal{P}'_2}, x_b, x_c) \cong K[\mathcal{P}_2]$, from which claim (2) derives. For the sake of completeness we provide its proof.
Observe that $I_{\mathcal{P}}$ can be written in the following way:

$$I_{\mathcal{P}} = I_{\mathcal{P}'_2} + \sum_{i=1}^{s}(x_a x_{a_i} - x_c x_{d_i}) + \sum_{i=1}^{s}(x_d x_{a_i} - x_b x_{d_i}) + \sum_{\substack{k,l \in [s] \\ k<l}}(x_{d_k} x_{a_l} - x_{d_l} x_{a_k}) +$$
$$+ (\{x_{a_s} x_v - x_{a_{s-1}} x_u | [v, a_s] \in \mathcal{I}(\mathcal{P}), u = v + (0,1)\}),$$

It follows:

$$(I_{\mathcal{P}}, x_b, x_c) = (I_{\mathcal{P}_2'}, x_b, x_c) + \sum_{i=1}^{s}(x_a x_{a_i}) + \sum_{i=1}^{s}(x_d x_{a_i}) + \sum_{\substack{k,l\in[s]\\k<l}} (x_{d_k}x_{a_l} - x_{d_l}x_{a_k})$$
$$+ (\{x_{a_s}x_v - x_{a_{s-1}}x_u | [v, a_s] \in \mathcal{I}(\mathcal{P}), u = v + (0,1)\}),$$

We prove that $(I_{\mathcal{P}}, x_b, x_c) : x_a = (I_{\mathcal{P}_2'}, x_b, x_c) + \sum_{i=1}^{s}(x_{a_i})$. From the previous equality it follows that $(I_{\mathcal{P}}, x_b, x_c) : x_a \supseteq (I_{\mathcal{P}_2'}, x_b, x_c) + \sum_{i=1}^{s}(x_{a_i})$. Let $f \in S$ such that $x_a f \in (I_{\mathcal{P}}, x_b, x_c)$. Then

$$x_a f = g + \sum_{i=1}^{s} \gamma_i x_a x_{a_i} + \sum_{j=1}^{s} \delta_j x_d x_{a_j} + \sum_{\substack{k,l\in[s]\\k<l}} \nu_{kl}(x_{d_k}x_{a_l} - x_{d_l}x_{a_k}) +$$
$$+ \sum_{\substack{[v,a_s]\in\mathcal{I}(\mathcal{P})\\u=v+(0,1)}} \lambda_v (x_{a_s}x_v - x_{a_{s-1}}x_u),$$

where $g \in (\mathcal{I}_{\mathcal{P}_2'}, x_b, x_c)$, $\gamma_i, \delta_j, \nu_{k,l}\lambda_v \in S_{\mathcal{P}}$ for all $i,k,j,l \in [s]$ and for all $v \in V(\mathcal{P})$ such that $[v, a_s] \in \mathcal{I}(\mathcal{P})$. As a consequence:

$$x_a\left(f - \sum_{i=1}^{s}\gamma_i x_{a_i}\right) = g + \sum_{j=1}^{s}(\delta_j x_d)x_{a_j} + \sum_{\substack{k,l\in[s]\\k<l}} (\nu_{kl}x_{d_k})x_{a_l} - \sum_{\substack{k,l\in[s]\\k<l}} (\nu_{kl}x_{d_l})x_{a_k} +$$
$$+ \left(\sum_{\substack{[v,a_s]\in\mathcal{I}(\mathcal{P})\\u=v+(0,1)}} \lambda_v x_v\right)x_{a_s} - \left(\sum_{\substack{[v,a_s]\in\mathcal{I}(\mathcal{P})\\u=v+(0,1)}} \lambda_v x_u\right)x_{a_{s-1}}.$$

Hence we obtain that $x_a(f - \sum_{i=1}^{s}\gamma_i x_{a_i}) \in (I_{\mathcal{P}_2'}, x_b, x_c) + \sum_{i=1}^{s}(x_{a_i})$. Since $(I_{\mathcal{P}_2'}, x_b, x_c)$ is prime and $a_i \notin V(\mathcal{P}_2')$ for all $i \in [s]$ then $(I_{\mathcal{P}_2'}, x_b, x_c) + \sum_{i=1}^{s}(x_{a_i})$ is a prime ideal in $S_{\mathcal{P}}$. By being $x_a \notin I_{\mathcal{P}_2'}$, we have $f - \sum_{i=1}^{s}\gamma_i x_{a_i} \in (I_{\mathcal{P}_2'}, x_b, x_c) + \sum_{i=1}^{s}(x_{a_i})$, so $f \in (I_{\mathcal{P}_2'}, x_b, x_c) + \sum_{i=1}^{s}(x_{a_i})$, that is $(I_{\mathcal{P}}, x_b, x_c) : x_a \subseteq (I_{\mathcal{P}_2'}, x_b, x_c) + \sum_{i=1}^{s}(x_{a_i})$. In conclusion we have $(I_{\mathcal{P}}, x_b, x_c) : x_a = (I_{\mathcal{P}_2'}, x_b, x_c) + \sum_{i=1}^{s}(x_{a_i})$ and as a consequence $S_{\mathcal{P}}/((I_{\mathcal{P}}, x_b, x_c) : x_a) = S_{\mathcal{P}}/(I_{\mathcal{P}_2'}, x_b, x_c) + (x_{a_1}, \dots, x_{a_s})) \cong S_{\mathcal{P}_2}/(I_{\mathcal{P}_2'}, x_b, x_c) \otimes_K K[x_v \mid v \in V(\mathcal{P} \setminus \mathcal{P}_1)]/(x_{a_1}, \dots, x_{a_s}) \cong K[\mathcal{P}_2] \otimes_K K[x_{d_1}, \dots, x_{d_{s-2}}]$. □

We omit to provide the analogous result for the case $V(\mathcal{L}) \cap V(\mathcal{C}) = \{a_{s-1}, a_s, c_{r-1}, c_r\}$. In fact, we can reduce it to the case examined in the previous Lemma up to a rotation and a reflection.

Lemma 5.1.7. *Let $\mathcal{P}$ be an $(\mathcal{L},\mathcal{C})$-polyomino such that $V(\mathcal{L}) \cap V(\mathcal{C}) = \{d_{s-1}, d_s, c_{r-1}, c_r\}$. Suppose that $I_{\mathcal{P}}$ is prime. Then:*

(1) $S_{\mathcal{P}}/((I_{\mathcal{P}}, x_b) : x_c) \cong K[\mathcal{P}_1] \otimes_K K[x_{c_1}, \dots, x_{c_{r-2}}]$;

(2) $S_{\mathcal{P}}/((I_{\mathcal{P}}, x_b, x_c) : x_d) \cong K[\mathcal{P}_2] \otimes_K K[x_{d_1}, \dots, x_{d_{s-2}}]$;

(3) $S_{\mathcal{P}}/((I_{\mathcal{P}}, x_b, x_c, x_d) : x_a) \cong K[\mathcal{P}_3] \otimes_K K[x_d, x_{c_1}, \dots, x_{c_{r-2}}, x_{d_1}, \dots, x_{d_{s-2}}]$.

Proof. The claims follow reasoning as in Lemma 5.1.4, obtaining the equalities of the following ideals:

(1) $(I_{\mathcal{P}}, x_b) : x_c = (I_{\mathcal{P}_1'}, x_b, x_d) + \sum_{i=1}^{r}(x_{b_i})$

(2) $(I_{\mathcal{P}}, x_b, x_c) : x_d = (I_{\mathcal{P}_2'}, x_b, x_c) + \sum_{i=1}^{s}(x_{a_i})$.

(3) $(I_{\mathcal{P}}, x_b, x_c, x_d) : x_a = (I_{\mathcal{P}_3}, x_b, x_c, x_d) + \sum_{i=1}^{s}(x_{a_i}) + \sum_{i=1}^{r}(x_{b_i})$.

In particular, the first equality follows from the primality of $(I_{\mathcal{P}'_1}, x_b, x_d)$, the second equality follows from the primality of $(I_{\mathcal{P}'_2}, x_b, x_c)$, both by Lemma 5.1.5. □

Theorem 5.1.8. *Let $\mathcal{P}$ be an $(\mathcal{L},\mathcal{C})$-polyomino. Suppose that $I_{\mathcal{P}}$ is prime. Then:*

$$\mathrm{HP}_{K[\mathcal{P}]}(t) = \frac{1}{1-t}\mathrm{HP}_{K[\mathcal{P}_4]}(t) + \frac{t}{1-t}\left[\frac{\mathrm{HP}_{K[\mathcal{P}_1]}(t)}{(1-t)^{r-2}} + \frac{\mathrm{HP}_{K[\mathcal{P}_2]}(t)}{(1-t)^{s-2}} + \frac{\mathrm{HP}_{K[\mathcal{P}_3]}(t)}{(1-t)^{s+r-3}}\right]$$

Proof. Assume that $V(\mathcal{L}) \cap V(\mathcal{C}) = \{a_{s-1}, a_s, b_{r-1}, b_r\}$. Consider the following four short exact sequences:

$$0 \longrightarrow S_{\mathcal{P}}/(I_{\mathcal{P}} : x_a) \longrightarrow S_{\mathcal{P}}/I_{\mathcal{P}} \longrightarrow S_{\mathcal{P}}/(I_{\mathcal{P}}, x_a) \longrightarrow 0$$
$$0 \longrightarrow S_{\mathcal{P}}/((I_{\mathcal{P}}, x_a) : x_d) \longrightarrow S_{\mathcal{P}}/(I_{\mathcal{P}}, x_a) \longrightarrow S_{\mathcal{P}}/(I_{\mathcal{P}}, x_a, x_d) \longrightarrow 0$$
$$0 \longrightarrow S_{\mathcal{P}}/((I_{\mathcal{P}}, x_a, x_d) : x_c) \longrightarrow S_{\mathcal{P}}/(I_{\mathcal{P}}, x_a, x_d) \longrightarrow S_{\mathcal{P}}/(I_{\mathcal{P}}, x_a, x_d, x_c) \longrightarrow 0$$
$$0 \longrightarrow S_{\mathcal{P}}/((I_{\mathcal{P}}, x_a, x_d, x_c) : x_b) \longrightarrow S_{\mathcal{P}}/(I_{\mathcal{P}}, x_a, x_d, x_c) \longrightarrow S_{\mathcal{P}}/(I_{\mathcal{P}}, x_a, x_d, x_c, x_b) \longrightarrow 0$$

Since $I_{\mathcal{P}} : x_a = I_{\mathcal{P}}$, because $I_{\mathcal{P}}$ is prime, the claim easily follows by repeated applications of Proposition 1.1.15 and from Proposition 1.1.16, Lemma 5.1.2 and Lemma 5.1.4.
If $V(\mathcal{L}) \cap V(\mathcal{C}) = \{d_{s-1}, d_s, b_{r-1}, b_r\}$ the formula is obtained referring to Lemma 5.1.6 by a suitable permutation of the set $\{a, b, c, d\}$. For symmetry, we obtain the claim also for the case $V(\mathcal{L}) \cap V(\mathcal{C}) = \{a_{s-1}, a_s, c_{r-1}, c_r\}$.
Finally, if $V(\mathcal{L}) \cap V(\mathcal{C}) = \{d_{s-1}, d_s, c_{r-1}, c_r\}$ we use again the same argument together with Lemma 5.1.7. □

Corollary 5.1.9. *Let $\mathcal{P}$ be an $(\mathcal{L},\mathcal{C})$-polyomino, suppose that $I_{\mathcal{P}}$ is a prime ideal and $\mathcal{C}$ is a simple polyomino. Then:*

$$\mathrm{HP}_{K[\mathcal{P}]}(t) = \frac{h_{K[\mathcal{P}_4]}(t) + t\left[h_{K[\mathcal{P}_1]}(t) + h_{K[\mathcal{P}_2]}(t) + (1-t)h_{K[\mathcal{P}_3]}(t)\right]}{(1-t)^{|V(\mathcal{P})| - \operatorname{rank}\mathcal{P}}}$$

In particular $K[\mathcal{P}]$ has Krull dimension $|V(\mathcal{P})| - \operatorname{rank}\mathcal{P}$.

Proof. Since $\mathcal{C}$ is a simple polyomino, then $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ and $\mathcal{P}_4$ are simple polyominoes, so we have that $K[\mathcal{P}_j]$ is a normal Cohen-Macaulay domain of dimension $|V(\mathcal{P}_j)| - \operatorname{rank}\mathcal{P}_j$ for $j \in \{1,2,3,4\}$ from [25, Corollary 3.3] and [24, Theorem 2.1]. We put $|V(\mathcal{P})| = n$ and $\operatorname{rank}\mathcal{P} = p$. Observe that

- $|V(\mathcal{P}_1)| = n - 2r$ and $\operatorname{rank}\mathcal{P}_1 = p - r - 1$, so $|V(\mathcal{P}_1)| - \operatorname{rank}\mathcal{P}_1 = n - p - r + 1$;
- $|V(\mathcal{P}_2)| = n - 2s$ and $\operatorname{rank}\mathcal{P}_2 = p - s - 1$, so $|V(\mathcal{P}_2)| - \operatorname{rank}\mathcal{P}_2 = n - p - s + 1$;
- $|V(\mathcal{P}_3)| = n - 2s - 2r$ and $\operatorname{rank}\mathcal{P}_3 = p - r - s - 1$, so $|V(\mathcal{P}_3)| - \operatorname{rank}\mathcal{P}_3 = n - p - s - r + 1$;
- $|V(\mathcal{P}_4)| = n - 4$ and $\operatorname{rank}\mathcal{P}_4 = p - 3$, so $|V(\mathcal{P}_4)| - \operatorname{rank}\mathcal{P}_4 = n - p - 1$.

Then $n - p = |V(\mathcal{P}_4)| - \operatorname{rank}\mathcal{P}_4 + 1 = |V(\mathcal{P}_1)| - \operatorname{rank}\mathcal{P}_1 + (r-2) + 1 = |V(\mathcal{P}_2)| - \operatorname{rank}\mathcal{P}_2 + (s-2) + 1$ and $|V(\mathcal{P}_3)| - \operatorname{rank}\mathcal{P}_3 + (s+r-3) + 1 = n - p - 1$. Therefore the formula for $\mathrm{HP}_{K[\mathcal{P}]}(t)$ in the statement follows from Theorem 5.1.8 after an easy computation. Finally, let $h(t)$ be the polynomial in the numerator of the formula. By [3, Corollary 4.1.10], observe that $h(1) = h_{K[\mathcal{P}_4]}(1) + h_{K[\mathcal{P}_1]}(1) + h_{K[\mathcal{P}_2]}(1) > 0$, so $(1-t)$ does not divide $h(t)$, hence $K[\mathcal{P}]$ has Krull dimension $|V(\mathcal{P})| - \operatorname{rank} \mathcal{P}$. $\square$

5.2 Hilbert-Poincaré series of prime closed path polyominoes without L-configurations

In this section we suppose that $\mathcal{P}$ is a prime closed path polyomino having no L-configurations, so $\mathcal{P}$ contains a ladder of at least three steps ([4, Section 6]). Let $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ be three maximal horizontal blocks of a ladder of n steps in $\mathcal{P}$, $n \geq 3$. Without loss of generality, we can assume that there does not exist a maximal block $\mathcal{K} \neq \mathcal{B}_2, \mathcal{B}_3$ of $\mathcal{P}$ such that $\{\mathcal{K}, \mathcal{B}_1, \mathcal{B}_2\}$ is a ladder of three steps. Moreover, applying suitable reflections or rotations of $\mathcal{P}$, we can suppose that the orientation of the ladder is right/up, as in Figure 5.2.

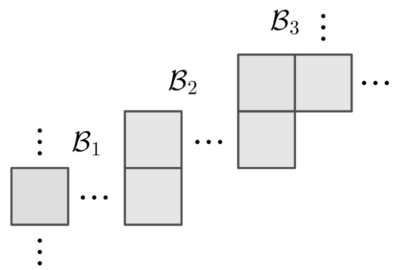


FIGURE 5.2

Our aim is to study the Hilbert-Poincaré series of the coordinate ring of $\mathcal{P}$. We split our arguments in two cases. In the first case we suppose that at least one block between $\mathcal{B}_1$ or $\mathcal{B}_2$ contains exactly two cells, in the second one assume that $\mathcal{B}_1$ and $\mathcal{B}_2$ contain at least three cells.

Assume that at least one block between $\mathcal{B}_1$ or $\mathcal{B}_2$ contains exactly two cells. We start with some preliminary definitions that we adopt throughout this subsection. Let $\mathcal{W}$ be a collection of cells consisting of an horizontal block $[A_s, A_1]$ of rank at least two, containing the cells $A_s, A_{s-1}, \dots, A_1$, a vertical block $[B_1, B_r]$ of rank at least two, containing the cells $B_1, B_2, \dots, B_r$, and a cell A not belonging to $[A_s, A_1] \cup [B_1, B_r]$, such that $V([A_s, A_1]) \cap V([B_1, B_r]) = \{b\}$, where b is the lower right corner of A. Moreover we denote the left upper corner of A by a, the lower right corner of B_1 by d, the lower right corner of A_1 by c. Moreover, let b_i and c_i be respectively the left upper and lower corners of A_i for $i \in [s]$, let a_j and d_j be respectively the left and the right upper corners of B_j for $j \in [r]$ (Figure 5.3).
Since $\mathcal{P}$ has no L-configurations, it is trivial to check that $\mathcal{P}$ contains a collection of cells $\mathcal{W}$ such that $[A_s, A_1]$ and $[B_1, B_r]$ are maximal blocks of $\mathcal{P}$. In particular, if $\mathcal{M}$ is the collection of cells such that $\mathcal{P} = \mathcal{W} \sqcup \mathcal{M}$, then we call $\mathcal{W}$:

- *1-Configuration*, if $V(\mathcal{W}) \cap V(\mathcal{M}) = \{c_{s-1}, c_s, d_{r-1}, d_r\}$;
- *2-Configuration*, if $V(\mathcal{W}) \cap V(\mathcal{M}) = \{b_{s-1}, b_s, d_{r-1}, d_r\}$.

Observe that just one of the following cases can occur:

(1) $|\mathcal{B}_1| = |\mathcal{B}_2| = 2$. In such a case $s = 2$ and $r = 2$, $\mathcal{B}_1 = [A_2, A_1]$ and $\mathcal{B}_2 = [A, B_1]$, so we obtain an 1-Configuration.

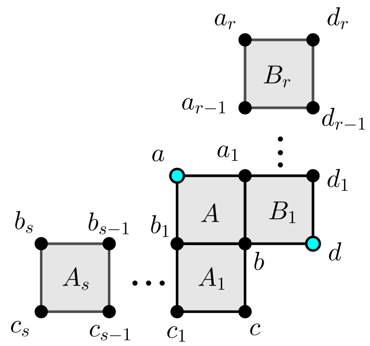


FIGURE 5.3: A collection of cells $\mathcal{W}$

(2) $|\mathcal{B}_1| > 2$ and $|\mathcal{B}_2| = 2$. In such a case $s > 2$ and $r = 2$, $\mathcal{B}_1 = [A_s, A_1]$ and $\mathcal{B}_2 = [A, B_1]$, so we have an 1-Configuration or a 2-Configuration depending on $\mathcal{M} \cap \{A_s\}$.

(3) $|\mathcal{B}_1| = 2$ and $|\mathcal{B}_2| > 2$. In such a case, after a suitable rotation and reflection, consider a new ladder where $\mathcal{B}_1 = [A_1, A]$ and $\mathcal{B}_2 = [B_1, B_r]$, $s \geq 2$ and $r > 2$. Let C be a cell of $\mathcal{P}$ such that $I := [C, A_1]$ is a maximal block of $\mathcal{P}$. Therefore we obtain an 1-Configuration or a 2-Configuration depending on the position of the cell of $\mathcal{P} \backslash I$ adjacent to C.

The following related polyominoes will be essential in this subsection:

- $\mathcal{Q} = \mathcal{P} \setminus \{A\}$;
- $\mathcal{Q}_1 = \mathcal{P} \setminus \{A, A_1, B_1\}$;
- $\mathcal{R}_1 = \mathcal{Q} \setminus \{B_1\}$;
- $\mathcal{R}_2 = \mathcal{Q} \setminus \{B_1, \dots, B_s\}$;
- $\mathcal{F}_1 = \mathcal{Q} \setminus \{A_1, \dots, A_s\}$;
- $\mathcal{F}_2 = \mathcal{Q} \setminus \{A_1, B_1, \dots, B_s\}$.

Let $<^1$ be the total order on $V(\mathcal{P})$ defined as $u <^1 v$ if and only if, for $u = (i, j)$ and $v = (k, l)$, $i < k$, or $i = k$ and $j < l$. Let $Y \subset V(\mathcal{P})$ and let $<^Y_{\text{lex}}$ be the lexicographic order in $S_{\mathcal{P}}$ induced by the following order on the variables of $S_{\mathcal{P}}$:

$$\text{for } u, v \in V(\mathcal{P}) \qquad x_u <^Y_{\text{lex}} x_v \Leftrightarrow \begin{cases} u \notin Y \text{ and } v \in Y \\ u, v \notin Y \text{ and } u <^1 v \\ u, v \in Y \text{ and } u <^1 v \end{cases}$$

Considering Figure 5.3, from Theorem 4.2.9 we know that there exists a set $L \subset V(\mathcal{P})$, with $a, d \in L$ and $b, c, a_1, b_1, c_1, d_1 \notin L$, such that the set of generators of $I_{\mathcal{P}}$ forms the reduced Gröbner basis of $I_{\mathcal{P}}$ with respect to $<^L_{\text{lex}}$. Furthermore, in the case of 1-Configuration also $d_2, \dots d_r \notin L$. For convenience we denote such a monomial order by $\prec_{\mathcal{P}}$. Moreover, let $\prec_{\mathcal{Q}}$, $\prec_{\mathcal{Q}_1}$, $\prec_{\mathcal{R}_1}$, $\prec_{\mathcal{R}_2}$ be the monomial orders induced from $\prec_{\mathcal{P}}$ respectively on the rings $S_{\mathcal{Q}}$, $S_{\mathcal{Q}_1}$, $S_{\mathcal{R}_1}$, $S_{\mathcal{R}_2}$. The following proposition will be useful.

Proposition 5.2.1. *Let $\mathcal{P}$ be a closed path polyomino containing a collection of cells of type $\mathcal{W}$. Then the set of the inner 2-minors of $\mathcal{Q}$ is the reduced Gröbner basis of $I_{\mathcal{Q}}$ with respect to the monomial order $\prec_{\mathcal{Q}}$. The same holds for the polyominoes $\mathcal{Q}_1$, $\mathcal{R}_1$ and $\mathcal{R}_2$ considering respectively the monomial orders $\prec_{\mathcal{Q}_1}$, $\prec_{\mathcal{R}_1}$ and $\prec_{\mathcal{R}_2}$.*

Proof. Let f, g be two generators of $I_{\mathcal{Q}} \subset I_{\mathcal{P}}$. Since every S-polynomial $S(f,g)$ reduces to zero in $I_{\mathcal{P}}$ then the conditions in lemmas in Section 4.1 are satisfied for the collection of cells $\mathcal{P}$. Apart from the occurrences $f = x_b x_{c_1} - x_c x_{b_1}$ and $g = x_b x_{d_1} - x_d x_{a_1}$, in which the leading terms of f and g have the greatest common divisor equal to 1, the other conditions of the mentioned lemmas do not involve the cell A. So the same conditions hold also for the collection of cells $\mathcal{Q}$, hence $S(f,g)$ reduces to zero also in $I_{\mathcal{Q}}$. By the same argument also the second claim in the statement holds. □

Remark 5.2.2. Observe that $\mathcal{Q}_1, \mathcal{R}_1, \mathcal{R}_2, \mathcal{F}_1$ and $\mathcal{F}_2$ are simple polyominoes, so their related coordinate rings are normal Cohen-Macaulay domains whose Krull dimension is given by the difference between the number of vertices and the number of cells of the fixed polyomino (see [25, Corollary 3.3] and [24, Theorem 2.1]). The polyomino $\mathcal{Q}$ is not simple but it is a weakly closed path and it is easy to see that $\mathcal{Q}$ has a weak ladder in the cases which we are studying. Therefore $I_{\mathcal{Q}}$ is a prime ideal (equivalently $K[\mathcal{Q}]$ is a domain) from Proposition 3.3.5. Moreover, from Proposition 5.2.1 and arguing as in the proof of Theorem 4.2.10 we also obtain that $K[\mathcal{Q}]$ is a normal Cohen-Macaulay domain.

We are going to use all these introductory facts in the proofs of the next results. With abuse of notation we refer to $\mathrm{in}(I_{\mathcal{P}})$, $\mathrm{in}(I_{\mathcal{Q}})$, $\mathrm{in}(I_{\mathcal{Q}_1})$, $\mathrm{in}(I_{\mathcal{R}_1})$, $\mathrm{in}(I_{\mathcal{R}_2})$ respectively for the initial ideals of $I_{\mathcal{P}}$ with respect to $\prec_{\mathcal{P}}$, of $I_{\mathcal{Q}}$ with respect to $\prec_{\mathcal{Q}}$, of $I_{\mathcal{Q}_1}$ with respect to $\prec_{\mathcal{Q}_1}$, of $I_{\mathcal{R}_1}$ with respect to $\prec_{\mathcal{R}_1}$ and of $I_{\mathcal{R}_2}$ with respect to $\prec_{\mathcal{R}_2}$.

Proposition 5.2.3. *Let $\mathcal{P}$ be a closed path polyomino containing a collection of cells of type $\mathcal{W}$. Then*

$$\mathrm{HP}_{K[\mathcal{P}]}(t) = \mathrm{HP}_{K[\mathcal{Q}]}(t) + \frac{t}{1-t}\mathrm{HP}_{K[\mathcal{Q}_1]}(t)$$

Proof. Observe that:

$$\begin{aligned}
I_{\mathcal{P}} &= I_{\mathcal{Q}} + (x_{b_1}x_{a_1} - x_a x_b) + (x_{b_1}x_{d_1} - x_a x_d) + (x_{c_1}x_{a_1} - x_a x_c),\\
I_{\mathcal{P}} &= I_{\mathcal{Q}_1} + (x_{b_1}x_{a_1} - x_a x_b) + (x_{b_1}x_{d_1} - x_a x_d) + (x_{c_1}x_{a_1} - x_a x_c) +\\
&+ \sum_{i=1}^{r}(x_b x_{d_i} - x_d x_{a_i}) + \sum_{i=1}^{s}(x_b x_{c_i} - x_c x_{b_i}).
\end{aligned}$$

From Proposition 5.2.1 we obtain:

$$\begin{aligned}
&\mathrm{in}(I_{\mathcal{P}}) = \mathrm{in}(I_{\mathcal{Q}}) + (x_a x_b) + (x_a x_d) + (x_a x_c),\\
&\mathrm{in}(I_{\mathcal{P}}) = \mathrm{in}(I_{\mathcal{Q}_1}) + (x_a x_b) + (x_a x_d) + (x_a x_c) + (\{\max_{\prec_{\mathcal{P}}}\{x_b x_{d_i}, x_d x_{a_i}\} : i \in [r]\}) +\\
&+ (\{\max_{\prec_{\mathcal{P}}}\{x_b x_{c_i}, x_c x_{b_i} : i \in [s]\}).
\end{aligned}$$

From the above equalities it is not difficult to see that:

- $(\mathrm{in}(I_{\mathcal{P}}), x_a) = (\mathrm{in}(I_{\mathcal{Q}}), x_a)$, in particular $S_{\mathcal{P}}/(\mathrm{in}(I_{\mathcal{P}}), x_a) = S_{\mathcal{P}}/(\mathrm{in}(I_{\mathcal{Q}}), x_a) \cong S_{\mathcal{Q}}/\mathrm{in}(I_{\mathcal{Q}})$.
- $\mathrm{in}(I_{\mathcal{P}}) : x_a = (\mathrm{in}(I_{\mathcal{Q}_1}), x_b, x_c, x_d)$ (see for instance [22, Proposition 1.2.2]), in particular $S_{\mathcal{P}}/(\mathrm{in}(I_{\mathcal{P}}) : x_a) = S_{\mathcal{P}}/(\mathrm{in}(I_{\mathcal{Q}_1}), x_b, x_c, x_d) \cong S_{\mathcal{Q}_1}/\mathrm{in}(I_{\mathcal{Q}_1}) \otimes_K K[x_a]$.

Consider the following exact sequence:

$$0 \longrightarrow S_{\mathcal{P}}/(\mathrm{in}(I_{\mathcal{P}}) : x_a) \longrightarrow S_{\mathcal{P}}/\mathrm{in}(I_{\mathcal{P}}) \longrightarrow S_{\mathcal{P}}/(\mathrm{in}(I_{\mathcal{P}}), x_a) \longrightarrow 0$$

Since for every graded ideal I of a standard graded K-algebra S and for every monomial order $<$ on S it is verified that S/I and $S/\mathrm{in}_<(I)$ have the same Hilbert function (see [22, Corollary 6.1.5]), then from the above computations and from Propositions 1.1.15 and 1.1.16 we obtain $\mathrm{HP}_{K[\mathcal{P}]}(t) = \mathrm{HP}_{K[\mathcal{Q}]}(t) + \frac{t}{1-t}\mathrm{HP}_{K[\mathcal{Q}_1]}(t)$. □

We observed that $\mathcal{Q}$ is not a simple polyomino. Our aim is to provide a formula for the Hilbert-Poincaré series of $K[\mathcal{P}]$ involving the Hilbert-Poincaré series related to the coordinate rings of simple polyominoes. By the previous result, since $\mathcal{Q}_1$ is a simple polyomino, we have to study the Hilbert-Poincaré series of $K[\mathcal{Q}]$. We examine 1-Configuration and 2-Configuration separately.

Theorem 5.2.4. *Let $\mathcal{P}$ be a closed path polyomino containing a collection of cells of type $\mathcal{W}$ with the occurrence of 1-Configuration. Then*

$$\mathrm{HP}_{K[\mathcal{P}]}(t) = \frac{h_{K[\mathcal{R}_1]}(t) + t\left[h_{K[\mathcal{R}_2]}(t) + h_{K[\mathcal{Q}_1]}(t)\right]}{(1-t)^{|V(\mathcal{P})|-\operatorname{rank}\mathcal{P}}}$$

In particular, the Krull dimension of $K[\mathcal{P}]$ is $|V(\mathcal{P})| - \operatorname{rank}\mathcal{P}$.

Proof. Observe that:

$$\begin{aligned}
I_{\mathcal{Q}} &= I_{\mathcal{R}_1} + \sum_{i=1}^{r}(x_b x_{d_i} - x_d x_{a_i}),\\
I_{\mathcal{Q}} &= I_{\mathcal{R}_2} + \sum_{i=1}^{r}(x_b x_{d_i} - x_d x_{a_i}) + \sum_{\substack{k,l\in[r]\\ k<l}}(x_{a_k}x_{d_l} - x_{a_l}x_{d_k}) +\\
&+ (\{x_{a_{r-1}}x_v - x_{a_r}x_u \mid [a_{r-1}, v] \in \mathcal{I}(\mathcal{Q}), u = v-(0,1)\}).
\end{aligned}$$

From Proposition 5.2.1 we obtain:

$$\begin{aligned}
\mathrm{in}(I_{\mathcal{Q}}) &= \mathrm{in}(I_{\mathcal{R}_1}) + \sum_{i=1}^{r}(x_d x_{a_i}),\\
\mathrm{in}(I_{\mathcal{Q}}) &= \mathrm{in}(I_{\mathcal{R}_2}) + \sum_{i=1}^{r}(x_d x_{a_i}) + (\{\max_{\prec_{\mathcal{Q}}}\{x_{a_k}x_{d_l}, x_{a_l}x_{d_k}\} \mid k,l \in [r], k<l\}) +\\
&+ (\{\max_{\prec_{\mathcal{Q}}}\{x_{a_{r-1}}x_v, x_{a_r}x_u\} \mid [a_{r-1}, v] \in \mathcal{I}(\mathcal{Q}), u = v-(0,1)\}).
\end{aligned}$$

From the above equalities is not difficult to see that:

- $(\mathrm{in}(I_{\mathcal{Q}}), x_d) = (\mathrm{in}(I_{\mathcal{R}_1}), x_d)$, in particular $S_{\mathcal{Q}}/(\mathrm{in}(I_{\mathcal{Q}}), x_d) = S_{\mathcal{Q}}/(\mathrm{in}(I_{\mathcal{R}_1}), x_d) \cong S_{\mathcal{R}_1}/\mathrm{in}(I_{\mathcal{R}_1})$.
- $\mathrm{in}(I_{\mathcal{Q}}) : x_d = \mathrm{in}(I_{\mathcal{R}_2}) + \sum_{i=1}^{r}(x_{a_i})$, in particular $S_{\mathcal{Q}}/(\mathrm{in}(I_{\mathcal{Q}}) : x_d) \cong S_{\mathcal{R}_2}/\mathrm{in}(I_{\mathcal{R}_2}) \otimes_K K[x_{d_1}, \dots, x_{d_{r-2}}]$.

So, arguing as in the proof of Proposition 5.2.3, we obtain $\mathrm{HP}_{K[\mathcal{Q}]}(t) = \mathrm{HP}_{K[\mathcal{R}_1]}(t) + t \cdot \frac{\mathrm{HP}_{K[\mathcal{R}_2]}(t)}{(1-t)^{r-2}}$. Combining such an equality with the claim of Proposition 5.2.3 we have:

$$\mathrm{HP}_{K[\mathcal{P}]}(t) = \mathrm{HP}_{K[\mathcal{R}_1]}(t) + t \cdot \left(\frac{\mathrm{HP}_{K[\mathcal{R}_2]}(t)}{(1-t)^{r-2}} + \frac{\mathrm{HP}_{K[\mathcal{Q}_1]}(t)}{1-t}\right)$$

Set $|V(\mathcal{P})| = n$ and $\operatorname{rank}\mathcal{P} = p$. Observe that

- $|V(\mathcal{R}_1)| = n-2$ and $\operatorname{rank}\mathcal{R}_1 = p-2$, so $|V(\mathcal{R}_1)| - \operatorname{rank}\mathcal{R}_1 = n-p$ and this is the Krull dimension of $K[\mathcal{R}_1]$ since $\mathcal{R}_1$ is simple;
- $|V(\mathcal{R}_2)| = n-2r+1$ and $\operatorname{rank}\mathcal{P}_2 = p-r-1$, so $|V(\mathcal{R}_2)| - \operatorname{rank}\mathcal{R}_2 = n-p-r+2$ and this is the Krull dimension of $K[\mathcal{R}_2]$;
- $|V(\mathcal{Q}_1)| = n-4$ and $\operatorname{rank}\mathcal{Q}_1 = p-3$, so $|V(\mathcal{Q}_1)| - \operatorname{rank}\mathcal{Q}_1 = n-p-1$ and this is the Krull dimension of $K[\mathcal{Q}_1]$.

Therefore, by easy computations, we obtain the formula for $\mathrm{HP}_{K[\mathcal{P}]}(t)$ in the statement. Finally, because of the Cohen-Macaulay property of $K[\mathcal{R}_1]$, $K[\mathcal{R}_2]$ and $K[\mathcal{Q}_1]$ and by [3, Corollary 4.1.10], we have that $h_{K[\mathcal{R}_1]}(1) + h_{K[\mathcal{R}_2]}(1) + h_{K[\mathcal{Q}_1]}(1) > 0$, so $\dim K[\mathcal{P}] = |V(\mathcal{P})| - \operatorname{rank}\mathcal{P}$. □

Now we want to study the 2-Configuration. In such a case we do not need to use the initial ideals.

Theorem 5.2.5. *Let $\mathcal{P}$ be a closed path polyomino containing a collection of cells of type $\mathcal{W}$ with the occurence of 2-Configuration. Then*

$$\mathrm{HP}_{K[\mathcal{P}]}(t) = \frac{(1+t)h_{K[\mathcal{Q}_1]}(t) + t\big[h_{K[\mathcal{F}_1]}(t) + h_{K[\mathcal{F}_2]}(t)\big]}{(1-t)^{|V(\mathcal{P})|-\operatorname{rank}\mathcal{P}}}$$

In particular, the Krull dimension of $K[\mathcal{P}]$ is $|V(\mathcal{P})| - \operatorname{rank}\mathcal{P}$.

Proof. Arguing as in Lemma 5.1.4 we obtain the following equalities:

(1) $I_{\mathcal{Q}} : x_c = I_{\mathcal{Q}}$;

(2) $(I_{\mathcal{Q}}, x_c) : x_b = I_{\mathcal{F}_1} + (x_c) + \sum_{i=1}^{s}(x_{c_i})$;

(3) $(I_{\mathcal{Q}}, x_b, x_c) : x_d = I_{\mathcal{F}_2} + (x_b, x_c) + \sum_{i=1}^{r}(x_{a_i})$.

(4) $(I_{\mathcal{Q}}, x_b, x_c, x_d) = (I_{\mathcal{Q}_1}, x_b, x_c, x_d)$

Again by the same arguments of Lemma 5.1.4 we obtain also the following:

(1) $S_{\mathcal{Q}}/(I_{\mathcal{Q}} : x_c) = K[\mathcal{Q}]$;

(2) $S_{\mathcal{Q}}/((I_{\mathcal{Q}}, x_c) : x_b) \cong K[\mathcal{F}_1] \otimes_K K[x_{b_1}, \dots, x_{b_{s-2}}]$;

(3) $S_{\mathcal{Q}}/((I_{\mathcal{Q}}, x_b, x_c) : x_d) \cong K[\mathcal{F}_2] \otimes_K K[x_d, x_{d_1}, \dots, x_{d_{r-2}}]$;

(4) $S_{\mathcal{Q}}/(I_{\mathcal{Q}}, x_b, x_c, x_d) \cong K[\mathcal{Q}_1]$

Now considering the suitable exact sequences and arguing as in Theorem 5.1.8, the following holds:

$$\mathrm{HP}_{K[\mathcal{Q}]}(t) = \frac{1}{1-t}\mathrm{HP}_{K[\mathcal{Q}_1]} + \frac{t}{1-t}\left[\frac{\mathrm{HP}_{K[\mathcal{F}_1]}(t)}{(1-t)^{s-2}} + \frac{\mathrm{HP}_{K[\mathcal{F}_2]}(t)}{(1-t)^{r-1}}\right]$$

So, from Theorem 5.2.3 we have:

$$\mathrm{HP}_{K[\mathcal{P}]}(t) = \frac{1+t}{1-t}\mathrm{HP}_{K[\mathcal{Q}_1]} + \frac{t}{1-t}\left[\frac{\mathrm{HP}_{K[\mathcal{F}_1]}(t)}{(1-t)^{s-2}} + \frac{\mathrm{HP}_{K[\mathcal{F}_2]}(t)}{(1-t)^{r-1}}\right]$$

Finally we obtain our claims arguing as in the last part of the previous result (or also, for instance, as in Corollary 5.1.9). □

Assume that $\mathcal{B}_1$ and $\mathcal{B}_2$ contain at least three cells. Suppose that $\mathcal{B}_1 = [B_1, B]$ consists of the cells $B_1, \dots, B_r, B$, $r \geq 2$, and $\mathcal{B}_2 = [A, A_r]$ of the cells $A, A_1, \dots, A_s$, $s \geq 2$. We denote the upper and lower left corners of A by a, c respectively, the upper and lower right corners of A by b, d respectively, the left and right lower corners of B by f, g respectively, the upper and lower right corners of A_i by a_i, b_i respectively for $i \in [s]$, the lower and upper left corners of B_i by c_i, d_i respectively for $i \in [r]$. Considering our assumption on the ladder at the beginning of Section 5.2 and the fact that $\mathcal{P}$ has not any L-configuration, we have that $c_1, c_2 \notin V(\mathcal{P}) \setminus V(\mathcal{B}_1)$. The described arrangement is summarized in Figure 5.4.
For our purpose we need to introduce the following related polyominoes:

- $\mathcal{K}_1 = \mathcal{P} \backslash [B_1, B]$;
- $\mathcal{K}_2 = \mathcal{P} \backslash ([A, A_s] \cup \{B, B_r\})$;
- $\mathcal{K}_3 = \mathcal{P} \backslash ([B_1, B] \cup \{A\})$;
- $\mathcal{K}_4 = \mathcal{P} \backslash \{A, B, A_1, B_r\}$.

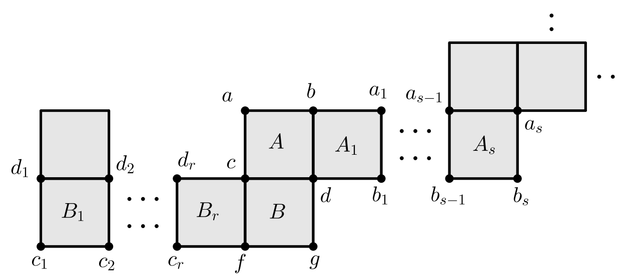


FIGURE 5.4

Lemma 5.2.6. *Let $\mathcal{P}$ be a closed path polyomino having a ladder of at least three steps satisfying the previous assumptions. Then the following hold:*

(1) $S_{\mathcal{P}}/(I_{\mathcal{P}} : x_g) \cong K[\mathcal{P}]$;

(2) $S_{\mathcal{P}}/((I_{\mathcal{P}}, x_g) : x_d) \cong K[\mathcal{K}_1] \otimes_K K[x_{d_3}, \dots, x_{d_r}]$;

(3) $S_{\mathcal{P}}/((I_{\mathcal{P}}, x_g, x_d) : x_b) \cong K[\mathcal{K}_2] \otimes_K K[x_a, x_b, x_{a_1}, \dots, x_{a_{s-2}}]$;

(4) $S_{\mathcal{P}}/((I_{\mathcal{P}}, x_g, x_d, x_b) : x_f) \cong S_{\mathcal{P}}/(I_{\mathcal{P}}, x_g, x_d, x_b)$;

(5) $S_{\mathcal{P}}/((I_{\mathcal{P}}, x_g, x_d, x_b, x_f) : x_c) \cong K[\mathcal{K}_3] \otimes_K K[x_{d_3}, \dots, x_{d_r}]$;

(6) $S_{\mathcal{P}}/((I_{\mathcal{P}}, x_g, x_d, x_b, x_f, x_c) : x_a) \cong K[\mathcal{K}_1] \otimes_K K[x_a, x_{a_1}, \dots, x_{a_{s-2}}]$;

(7) $S_{\mathcal{P}}/(I_{\mathcal{P}}, x_g, x_d, x_b, x_f, x_c, x_a) \cong K[\mathcal{K}_4]$.

Proof. To prove the isomorphisms in the statements $(1) - (7)$, it is enough to prove the following equalities:

(1) $I_{\mathcal{P}} : x_g = I_{\mathcal{P}}$;

(2) $(I_{\mathcal{P}}, x_g) : x_d = I_{\mathcal{K}_1} + (x_f, x_g) + \sum_{i=1}^{r}(x_{c_i})$;

(3) $(I_{\mathcal{P}}, x_g, x_d) : x_b = I_{\mathcal{K}_2} + (x_f, x_g, x_d, x_c) + \sum_{i=1}^{s}(x_{b_i})$;

(4) $(I_{\mathcal{P}}, x_g, x_d, x_b) : x_f = (I_{\mathcal{P}}, x_g, x_d, x_b)$;

(5) $(I_\mathcal{P}, x_g, x_d, x_b, x_f) : x_c = (I_{\mathcal{K}_1}, x_b, x_d) + (x_g, x_f) + \sum_{i=1}^{r}(x_{c_i})$;

(6) $(I_\mathcal{P}, x_g, x_d, x_b, x_f, x_c) : x_a = I_{\mathcal{K}_2} + (x_f, x_g, x_d, x_c, x_b) + \sum_{i=1}^{s}(x_{b_i})$;

(7) $(I_\mathcal{P}, x_g, x_d, x_b, x_f, x_c, x_a) = (I_{\mathcal{K}_4}, x_g, x_d, x_b, x_f, x_c, x_a)$.

In particular the equality (1) is trivial since $I_\mathcal{P}$ is prime, (2), (3) and (6), together with the related claims, can be proved as done in Lemma 5.1.4. The equality (5) and its related claim follow as in Lemma 5.1.6, considering also that $S_{\mathcal{K}_1}/(I_{\mathcal{K}_1}, x_b, x_d) \cong K[\mathcal{K}_3]$ and arguing as in Lemma 5.1.5. We obtain the equality (7) and its related claim as for Lemma 5.1.2.
Finally, in order to show 4), we prove that $(I_\mathcal{P}, x_g, x_d, x_b)$ is a prime ideal in $S_\mathcal{P}$. Let $\{V_i\}_{i\in I}$ be the set of the maximal edge intervals of $\mathcal{P}$ and $\{H_j\}_{j\in J}$ be the set of the maximal horizontal edge intervals of $\mathcal{P}$. Let $\{v_i\}_{i\in I}$ and $\{h_j\}_{j\in J}$ be the set of the variables associated respectively to $\{V_i\}_{i\in I}$ and $\{H_j\}_{j\in J}$. Let w be another variable and set $\mathcal{I} = \{f, c, d, g, b_1, \dots, b_s\} \subset V(\mathcal{P})$. We consider the following ring homomorphism

$$\phi : S_\mathcal{P} \longrightarrow K[\{v_i, h_j, w\} : i \in I, j \in J]$$

defined by $\phi(x_{ij}) = v_i h_j w^k$, where $(i,j) \in V_i \cap H_j$, $k = 0$ if $(i,j) \notin \mathcal{I}$, and $k = 1$ if $(i,j) \in \mathcal{I}$. From Theorem 2.2.18 we have $I_\mathcal{P} = \ker \phi$. Let $i' \in I$ such that $V_{i'}$ is the maximal edge interval of $\mathcal{P}$ containing b, d and g. We define $\psi : S_\mathcal{P} \to K[\{v_i, h_j, w\} : i \in I \backslash \{i'\}, j \in J]$ as $\psi(x_v) = \phi(x_v)$ if $v \in V(\mathcal{P}) \backslash \{b, d, g\}$, and $\psi(x_b) = \psi(x_d) = \psi(x_g) = 0$. It is not difficult to check that $(I_\mathcal{P}, x_d, x_b, x_g) \subseteq \ker \psi$. Let $f \in \ker \psi$. We can write $f = \tilde{f} + \beta x_b + \delta x_d + \gamma x_g$ where $\beta, \delta, \gamma \in S_\mathcal{P}$ and x_b, x_d, x_g are not variables of $\tilde{f}$. Since $\psi(f) = 0$, we have $\phi(\tilde{f}) = 0$, so $\tilde{f} \in \ker \phi = I_\mathcal{P}$. Hence $S_\mathcal{P}/(I_\mathcal{P}, x_b, x_d, x_g) \cong \mathrm{Im}(\psi)$, that is a domain since it is the subring of a domain. So $(I_\mathcal{P}, x_b, x_d, x_g)$ is prime in $S_\mathcal{P}$. □

Remark 5.2.7. If we suppose that $\mathcal{B}_2$ has just two cells (so $s = 1$), then $(I_\mathcal{P}, x_g, x_d, x_b)$ is not prime. In fact, set $b = a_0$, denote the cell adjacent to A_1 by C, and let b, p and q, a_1 be respectively the diagonal and anti-diagonal corners of C. Observe that in such a case $x_q x_{a_1} \in (I_\mathcal{P}, x_g, x_d, x_b)$ but $x_q, x_{a_1} \notin (I_\mathcal{P}, x_g, x_d, x_b)$.

Theorem 5.2.8. *Let $\mathcal{P}$ be a closed path polyomino having a ladder of at least three steps where $\mathcal{B}_1$ and $\mathcal{B}_2$ contain at least three cells. Then*

$$\mathrm{HP}_{K[\mathcal{P}]}(t) = \frac{h_{K[\mathcal{K}_4]}(t) + t\left[h_{K[\mathcal{K}_1]}(t) + 2 \cdot h_{K[\mathcal{K}_2]}(t) + h_{K[\mathcal{K}_3]}(t)\right]}{(1-t)^{|V(\mathcal{P})| - \operatorname{rank} \mathcal{P}}}$$

In particular $K[\mathcal{P}]$ has Krull dimension $|V(\mathcal{P})| - \operatorname{rank} \mathcal{P}$.

Proof. It follows from Lemma 5.2.6 considering the suitable exact sequences and arguing as done in Theorem 5.1.8 and Corollary 5.1.9. □

5.3 Rook polynomial and some consequences for the Gorensteiness

Let us start introducing a very important combinatorial tool, that is the *rook polynomial* of a polyomino. Let $\mathcal{P}$ be a polyomino. A *k-rook configuration* in $\mathcal{P}$ is a configuration of k rooks which are arranged in $\mathcal{P}$ in non-attacking positions.
The rook number $r(\mathcal{P})$ is the maximum number of rooks which can be placed in $\mathcal{P}$ in non-attacking positions. We denote by $\mathcal{R}(\mathcal{P}, k)$ the set of all k-rook configurations

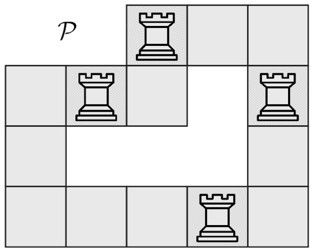


FIGURE 5.5: An example of a 4-rook configuration in $\mathcal{P}$.

in $\mathcal{P}$ and we set $r_k = |\mathcal{R}(\mathcal{P},k)|$ for all $k \in \{0,\dots,r(\mathcal{P})\}$, conventionally $r_0 = 1$. The *rook polynomial* of $\mathcal{P}$ is the polynomial $r_{\mathcal{P}}(t) = \sum_{k=0}^{r(\mathcal{P})} r_k t^k \in \mathbb{Z}[t]$. The latter seems to be related to the h-polynomial of the thin polyominoes and provides a very nice tool to study the Gorenstein property.

Nowadays a complete characterization of Gorensteiness is not still know for the coordinate rings of polyominoes and some partial results are given only for particular classes of polyominoes. In [1] and [37] the authors give a complete characterization respectively for convex polyominoes and for stack polyominoes. In [40] it is showed that if $\mathcal{P}$ is a simple thin polyomino then the h-polynomial $h(t)$ of $K[\mathcal{P}]$ is the rook polynomial and they characterize the Gorenstein simple thin polyominoes with the S-property.

Definition 5.3.1. Let $\mathcal{P}$ be a thin polyomino. A cell C is called *single* if there exists a unique maximal interval of $\mathcal{P}$ containing C. We say that $\mathcal{P}$ has the *S-property* if every maximal interval of $\mathcal{P}$ has only one single cell.

Finally, it is conjectured that a polyomino is thin if and only if $h(t) = r_{\mathcal{P}}(t)$. In this section we give also a partial support to this conjecture, since we provide an affirmative answer for closed paths. In this sense, in [30], it is also discussed this conjecture for a certain class of polyominoes, in particular it is proved that if $\mathcal{P}$ is a convex non-thin polyomino whose vertex set is a sublattice of $\mathbb{N}^2$ then $h(t) \neq r_{\mathcal{P}}(t)$. In [38] the authors introduce a particular equivalence relation on the rook complex of a simple polyomino $\mathcal{P}$ to define the *switching rook polynomial* and they conjecture that h-polynomial of $K[\mathcal{P}]$ is equal to the latter. Moreover they prove it for the class of parallelogram polyominoes (refer to [38, Section 2.3]) and implicitly for L-convex polyominoes, and by a computational method also for all simple polyominoes with rank at most eleven. Finally they characterize all Gorenstein parallelogram polyominoes.

The aim of this section is to give a complete characterization of the Gorenstein closed path polyominoes having no zig-zag walks. Firstly, we start showing how the rook polynomial is related to Hilbert-Poincaré series of the polyominoes considered in this chapter so far.

Proposition 5.3.2. *Let $\mathcal{P}$ be a $(\mathcal{L},\mathcal{C})$-polyomino. Let $\mathcal{P}_1,\mathcal{P}_2,\mathcal{P}_3,\mathcal{P}_4$ be the polyominoes in Section 3. Then:*

(1) $r(\mathcal{P}_1) = r(\mathcal{P}_2) = r(\mathcal{P}) - 1$;

(2) $r(\mathcal{P}_3) = r(\mathcal{P}) - 2$;

(3) $r(\mathcal{P}) - 2 \leq r(\mathcal{P}_4) \leq r(\mathcal{P})$.

Proof. (1) Let $\mathcal{P}_1 = \mathcal{P} \backslash [A, A_r]$. Once we fix a rook in a cell of $[A_1, \ldots, A_{r-1}]$, we cannot place another rook in $[A, A_r]$ in non-attacking position in $\mathcal{P}$, so $r(\mathcal{P}_1) = r(\mathcal{P}) - 1$. In a similar way it can be showed that $r(\mathcal{P}_2) = r(\mathcal{P}) - 1$.
(2) It follows by similar previous arguments on the intervals $[A, A_r]$ and $[A, B_s]$.
(3) Since $\mathcal{P} = \mathcal{P}_4 \cup [A, A_1] \cup [A, B_1]$, it is obvious that $r(\mathcal{P}_4) \leq r(\mathcal{P})$. Moreover, $\mathcal{P}_4 = \mathcal{P}_3 \cup [A_2, A_r] \cup [B_2, B_s]$, so $r(\mathcal{P}_3) \leq r(\mathcal{P}_4)$, that is $r(\mathcal{P}) - 2 \leq r(\mathcal{P}_4)$. In particular, observe that if $r, s > 3$ then $r(\mathcal{P}_4) = r(\mathcal{P})$, if either $r = 3$ or $s = 3$ then $r(\mathcal{P}_4) = r(\mathcal{P}) - 1$, and if $r, s = 3$ then $r(\mathcal{P}_4) = r(\mathcal{P}) - 2$. $\square$

Theorem 5.3.3. *Let $\mathcal{P}$ be a $(\mathcal{L}, \mathcal{C})$-polyomino. Suppose that $\mathcal{C}$ is a simple thin polyomino. Then $h_{K[\mathcal{P}]}(t)$ is the rook polynomial of $\mathcal{P}$. Moreover* $\mathrm{reg}(K[\mathcal{P}]) = r(\mathcal{P})$.

Proof. It is known that $h_{K[\mathcal{P}]}(t) = h_{K[\mathcal{P}_4]}(t) + t\big[h_{K[\mathcal{P}_1]}(t) + h_{K[\mathcal{P}_2]}(t) + (1-t)h_{K[\mathcal{P}_3]}(t)\big]$. We denote by $r_{\mathcal{P}_j}(t) = \sum_{k=0}^{r(\mathcal{P}_j)} r_k^{(j)} t^k$ the rook polynomial of $\mathcal{P}_j$. Since $\mathcal{C}$ is a simple thin polyomino, then $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ and $\mathcal{P}_4$ are simple thin polyominoes, so $h_{K[\mathcal{P}_j]}(t) = r_{\mathcal{P}_j}(t)$ for $j \in \{1, 2, 3, 4\}$. By Proposition 5.3.2 we have $\deg h_{K[\mathcal{P}]} = r(\mathcal{P})$. Then

$$h_{K[\mathcal{P}]}(t) = \sum_{k=0}^{r(\mathcal{P})} [r_k^{(4)} + r_{k-1}^{(1)} + r_{k-1}^{(2)} + r_{k-1}^{(3)} - r_{k-2}^{(3)}] t^k,$$

where we set $r_{-1}^{(j)}, r_{-2}^{(3)}, r_{r(\mathcal{P})-1}^{(3)}, r_k^{(4)}$ equal to 0, for all $j \in \{1, 2, 3\}$ and for $k \geq r(\mathcal{P}_4)$. We want to prove that $r_k^{(4)} + r_{k-1}^{(1)} + r_{k-1}^{(2)} + r_{k-1}^{(3)} - r_{k-2}^{(3)}$ is exactly the number of ways in which k rooks can be placed in $\mathcal{P}$ in non-attacking positions, for all $k \in \{0, \ldots, r(\mathcal{P})\}$. Fix $k \in \{0, \ldots, r(\mathcal{P})\}$. Observe that:

(1) $r_k^{(4)}$ can be viewed as the number of k-rook configurations in $\mathcal{P}$ such that no rook is placed on A, A_1 and B_1.

(2) Assume that a rook $\mathcal{T}$ is placed in A_1. Then we cannot place any rook on a cell of $[A, A_r]$, so $r_{k-1}^{(1)}$ is the number of all $(k-1)$-rook configurations in $\mathcal{P}_1$. Hence $r_{k-1}^{(1)}$ is the number of all k-rook configurations in $\mathcal{P}$ such that a rook is on A_1. Observe that there are some k-rook configurations in $\mathcal{P}$ in which a rook $\mathcal{T}' \neq \mathcal{T}$ is on B_1. Paraphrasing, note that $r_{k-1}^{(1)}$ is the number of all k-rook configurations in $\mathcal{P}$ such that $\mathcal{T}$ is on A_1 and $\mathcal{T}'$ is not on B_1 plus those ones where $\mathcal{T}$ is on A_1 and $\mathcal{T}'$ is on B_1.

(3) Assume that a rook $\mathcal{T}$ is placed in B_1. Arguing as before, $r_{k-1}^{(1)}$ is the number of all k-rook configurations in $\mathcal{P}$ such that $\mathcal{T}$ is on B_1 and $\mathcal{T}'$ is not on A_1 plus those ones where $\mathcal{T}$ is on B_1 and $\mathcal{T}'$ is on A_1.

(4) Assume that a rook is placed on A. Then we cannot place any rook on a cell of $[A, A_r] \cup [A, B_s]$, so $r_{k-1}^{(3)}$ is the number of all $(k-1)$-rook configurations in $\mathcal{P}_3$, that is the number of k-rook configurations in $\mathcal{P}$ such that a rook is placed on A.

(5) Fix a rook $\mathcal{T}$ in A_1 and another one $\mathcal{T}'$ in B_1. Then we cannot place any rook on a cell of $[A, A_r] \cup [A, B_s]$, so $r_{k-2}^{(3)}$ is the number of all $(k-2)$-rook configurations in $\mathcal{P}_3$. Hence $r_{k-2}^{(3)}$ is the number of all k-rook configurations in $\mathcal{P}$ such that a rook is on A_1 and another is on B_1.

From (1), (2), (3), (4) and (5) it follows that $r_k^{(4)} + r_{k-1}^{(1)} + r_{k-1}^{(2)} + r_{k-1}^{(3)} - r_{k-2}^{(3)}$ is the number of k-rook configurations in $\mathcal{P}$. $\square$

Proposition 5.3.4. *Let $\mathcal{P}$ be a closed path polyomino having a ladder of at least three steps where $\mathcal{B}_1$ and $\mathcal{B}_2$ contain at least three cells. Then $h_{K[\mathcal{P}]}(t)$ is the rook polynomial of $\mathcal{P}$ and* $\mathrm{reg}(K[\mathcal{P}]) = r(\mathcal{P})$.

Proof. It can be proved by similar arguments as in Proposition 5.3.2 that the rook numbers of $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$ and $\mathcal{K}_4$ satisfy the following:

(1) $r(\mathcal{K}_1) = r(\mathcal{P}) - 1$;

(2) $r(\mathcal{P}) - 2 \leq r(\mathcal{K}_2) \leq r(\mathcal{P}) - 1$;

(3) $r(\mathcal{K}_3) = r(\mathcal{P}) - 1$;

(4) $r(\mathcal{P}) - 2 \leq r(\mathcal{K}_4) \leq r(\mathcal{P})$.

We denote by $r_{\mathcal{K}_j}(t) = \sum_{k=0}^{r(\mathcal{K}_j)} r_k^{(j)} t^k$ the rook polynomial of $\mathcal{K}_j$, for $j = 1,2,3,4$. Observe that $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$ and $\mathcal{K}_4$ are simple thin polyominoes, so $h_{K[\mathcal{K}_j]}(t) = r_{\mathcal{K}_j}(t)$ for $j \in \{1,2,3,4\}$, and by the above formulas and Theorem 5.2.8 we have $\deg h_{K[\mathcal{P}]} = r(\mathcal{P})$. Moreover

$$h_{K[\mathcal{P}]}(t) = \sum_{k=0}^{r(\mathcal{P})} [r_k^{(4)} + r_{k-1}^{(1)} + 2r_{k-1}^{(2)} + r_{k-1}^{(3)}]t^k,$$

where we set $r_{-1}^{(j)}, r_k^{(4)}, r_l^{(2)}$ equal to 0, for all $j \in \{1,2,3\}$, for $k \geq r(\mathcal{K}_4)$ and $l \geq r(\mathcal{K}_2)$. Similarly as done in Theorem 5.3.3, we have that $r_k^{(4)} + r_{k-1}^{(1)} + 2r_{k-1}^{(2)} + r_{k-1}^{(3)}$ is the number of k-rook configurations in $\mathcal{P}$, for all $k \in \{0,\dots,r(\mathcal{P})\}$. In fact, let $k \in \{0,\dots,r(\mathcal{P})\}$. Observe that:

(1) $r_k^{(4)}$ is the number of k-rook configurations in $\mathcal{P}$ such that no rook is placed on A, A_1, B and B_r.

(2) Fix a rook $\mathcal{T}$ on B_r. Then $r_{k-1}^{(1)}$ is the number of all k-rook configurations in $\mathcal{P}$ such that $\mathcal{T}$ is on B_r. Observe that among these configurations, there are some k-rook configurations in which $\mathcal{T}' \neq \mathcal{T}$ is placed either in A or in A_1.

(3) Fix a rook $\mathcal{T}$ in B. Then $r_{k-1}^{(3)}$ is the number of all k-rook configurations in $\mathcal{P}$ such that $\mathcal{T}$ is on B. As before, among these configurations there are some k-rook configurations in which $\mathcal{T}' \neq \mathcal{T}$ is placed in A_1.

(4) Assume that a rook is placed in A (resp. A_1). Then $r_{k-1}^{(2)}$ is the number of all k-rook configurations in $\mathcal{P}$ such that $\mathcal{T}$ is on A (resp. A_1), and no rook is on a cell of $[A, A_s] \cup \{B, B_r\}$.

From (1), (2), (3) and (4) we have the desired conclusion. □

In order to complete the study of closed path polyominoes having no L-configuration, it remains to consider 1-Configuration and 2-Configuration introduced in the previous section. For such cases we mention only the analogous result, omitting the proof since the arguments are similar.

Proposition 5.3.5. *Let $\mathcal{P}$ be a closed path with a ladder of at least three steps where at least one block between $\mathcal{B}_1$ or $\mathcal{B}_2$ contains exactly two cells.Then $h_{K[\mathcal{P}]}(t)$ is the rook polynomial of $\mathcal{P}$ and* $\mathrm{reg}(K[\mathcal{P}]) = r(\mathcal{P})$.

Observing that a closed path having an L-configuration is an $(\mathcal{L}, \mathcal{C})$-polyomino with $\mathcal{C}$ a path of cells and gathering all the results above, we obtain the following general result.

Theorem 5.3.6. *Let $\mathcal{P}$ be a closed path having no zig-zag walks, equivalently having an L-configuration or a ladder of three steps. Then:*

(1) $K[\mathcal{P}]$ is a normal Cohen-Macaulay domain of Krull dimension $|V(\mathcal{P})| - \operatorname{rank} \mathcal{P}$;

(2) $h_{K[\mathcal{P}]}(t)$ is the rook polynomial of $\mathcal{P}$ and $\operatorname{reg}(K[\mathcal{P}]) = r(\mathcal{P})$.

At this point we are ready to provide the condition for the Gorenstein property of a closed path polyomino having no zig-zag walks.
Observe firstly that if $\mathcal{P}$ is a closed path polyomino, then $\mathcal{P}$ has the S-property if and only if every maximal block of $\mathcal{P}$ contains exactly three cells.

Theorem 5.3.7. *Let $\mathcal{P}$ be a closed path having no zig-zag walks. The following are equivalent:*

(1) $\mathcal{P}$ has the S-property;

(2) $K[\mathcal{P}]$ is Gorenstein.

Proof. If $\mathcal{P}$ has no zig-zag walks, then $K[\mathcal{P}]$ is a normal Cohen-Macaulay domain of Krull dimension $|V(\mathcal{P})| - \operatorname{rank} \mathcal{P}$ and $h_{K[\mathcal{P}]}(t) = r_{\mathcal{P}}(t) = \sum_{k=0}^{s} r_k t^k$, where $s = r(\mathcal{P})$. In such a case it is known from Theorem 1.2.14 that $K[\mathcal{P}]$ is Gorenstein if and only if $r_i = r_{s-i}$ for all $i = 0, \dots, s$.
(1) $\Rightarrow$ (2). Suppose that $\mathcal{P}$ has the S-property. Fix $i \in \{0, 1, \dots, r(\mathcal{P})\}$ and prove that $r_i = r_{s-i}$. Since $\mathcal{P}$ has the S-property, $\mathcal{P}$ consists of maximal cell intervals of rank three. If $i = 0$ then it is trivial that $r_0 = r_s = 1$. Assume $i \in [s-1]$. It is not restrictive to consider a part of $\mathcal{P}$ arranged as in Figure 5.6.

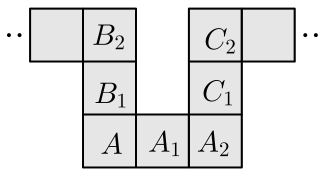


FIGURE 5.6

Define $\mathcal{P}_1 = \mathcal{P} \setminus \{A, A_1, A_2\}$, $\mathcal{P}_2 = \mathcal{P} \setminus \{A, A_1, A_2, C_1, C_2\}$ and $\mathcal{P}_3 = \mathcal{P} \setminus \{A, A_1, A_2, B_1, B_2\}$. We denote by $r_{\mathcal{P}_j}(t) = \sum_{k=0}^{r(\mathcal{P}_j)} r_k^{(j)} t^k$ the rook polynomial of $\mathcal{P}_j$. Observe that $r(\mathcal{P}_1) = r(\mathcal{P}) - 1 = s - 1$ and $r(\mathcal{P}_2) = r(\mathcal{P}_3) = r(\mathcal{P}) - 2 = s - 2$. By similar arguments as in Theorem 5.3.3, it is easy to prove that $r_k = r_k^{(1)} + r_{k-1}^{(1)} + r_{k-1}^{(2)} + r_{k-1}^{(3)}$ for all $k \in \{1, \dots, s\}$. Then

$$r_{s-i} = r_{s-i}^{(1)} + r_{s-i-1}^{(1)} + r_{s-i-1}^{(2)} + r_{s-i-1}^{(3)} = r_{(s-1)-(i-1)}^{(1)} + \\ + r_{(s-1)-i}^{(1)} + r_{(s-2)-(i-1)}^{(2)} + r_{(s-2)-(i-1)}^{(3)}.$$

Since $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ are simple thin polyominoes having the S-property, then by Theorem 4.2 of [40] we have: $r_{(s-1)-(i-1)}^{(1)} = r_{i-1}^{(1)}$, $r_{(s-1)-i}^{(1)} = r_i^{(1)}$, $r_{(s-1)-(i-2)}^{(2)} = r_{i-1}^{(2)}$ and $r_{(s-2)-(i-1)}^{(3)} = r_{i-1}^{(3)}$. Hence

$$r_{s-i} = r_{(s-1)-(i-1)}^{(1)} + r_{(s-1)-i}^{(1)} + r_{(s-2)-(i-1)}^{(2)} + r_{(s-2)-(i-1)}^{(3)} = r_{i-1}^{(1)} + \\ + r_i^{(1)} + r_{i-1}^{(2)} + r_{i-1}^{(3)} = r_i.$$

$(2) \Rightarrow (1)$. Assume that $K[\mathcal{P}]$ is Gorenstein, that is $r_i = r_{s-i}$ for all $i = 0, \dots, s$. We prove that $\mathcal{P}$ has the S-property. First of all, we observe that all the ranks of the maximal intervals of $\mathcal{P}$ cannot be greater than or equal to four. In fact, if there exists a maximal interval $I = [A, B]$ with $\operatorname{rank} I \geq 4$, then we can consider two distinct cells $C, D \in I \setminus \{A, B\}$. Hence we can obtain an s-rook configuration in $\mathcal{P}$ with a rook in C and another one with a rook in D, so $r_s \geq 2 > r_0 = 1$, that is a contradiction. In addition, in such a case, we can suppose that $\mathcal{P}$ has an L-configuration, otherwise it is not difficult to see that $\mathcal{P}$ has a subpolyomino as in Figure 5.3, and arguing as in the proof of the case $b) \Rightarrow c)$ (hypothesis (2)) of [40, Theorem 4.2], then $K[\mathcal{P}]$ is not Gorenstein. So, let $\{A, A_1, A_2, B_1, B_2\}$ be an L-configuration of $\mathcal{P}$, as in Figure 5.6. Consider $\mathcal{P}' = \mathcal{P} \setminus \{A, A_1, A_2\}$, which is a simple thin polyomino. Let $r_{\mathcal{P}'}(t) = \sum_{k=0}^{s'} r'_k t^k$ be the rook polynomial of $\mathcal{P}'$, where $s' = r(\mathcal{P}) - 1$. We prove that $\mathcal{P}'$ has the S-property. Suppose that $\mathcal{P}'$ has not the S-property so from the case $b) \Rightarrow c)$ of [40, Theorem 4.2] it follows that either $r'_{s'} > 1$ or $r'_{s'-1} > \operatorname{rank} \mathcal{P}'$. Both cases lead to a contradiction with $r_s = 1$ or $r_{s-1} = \operatorname{rank} \mathcal{P}$. By similar arguments we can prove that $\mathcal{P}'' = \mathcal{P} \setminus \{A, B_1, B_2\}$ is a simple thin polyomino having the S-property. Since $\mathcal{P}'$ and $\mathcal{P}''$ have the S-property, it follows trivially that also $\mathcal{P}$ has the S-property. $\square$

Chapter 6

A package for Macaulay2 to deal with the inner 2-minor ideals of collection of cells

In this chapter we explain the package `PolyominoIdeals` ([5]) for the computer algebra software `Macaulay2` ([19]). The purpose of this package is to define and manipulate the binomial ideals attached to collections of cells.

6.1 Functions and options

In this section we describe the functions provided in the package. First of all, consider that a collection of cells $\mathcal{P}$ is encoded, for the package, with a list of lists, where each list represents a cell of the collection and contains two lists representing the diagonal corners of the cell, the first for the lower left corner, the second for the upper right corner. For instance, the collection of cells in Figure 6.1 is encoded with the list `Q = {{{1, 1}, {2, 2}}, {{2, 1}, {3, 2}}, {{3, 1}, {4, 2}}, {{2, 2}, {3, 3}}, {{3, 2}, {4, 3}}, {{2, 3}, {3, 4}}}`.

6.1.1 polyoIdeal function

Let $\mathcal{P}$ be a polyomino and $I_{\mathcal{P}}$ be a polyomino ideal associated to $\mathcal{P}$. The `polyoIdeal` function available in the `PolyominoIdeals` package gives the generators of polyomino ideal $I_{\mathcal{P}}$ as output. The polynomial ring defined as $S_{\mathcal{P}} = K[x_v : v \in V(\mathcal{P})]$ is auto-declared in the `polyoIdeal` function and can be accessed with the command `ring(polyoIdeal(Q))` where Q is the input list which comprises of the diagonal corners of each cell in $\mathcal{P}$.

For example, consider the polyomino $\mathcal{P}$ in Figure 6.1 and, fixing the lower left corner A as $(1,1)$, we embed $\mathcal{P}$ with the list `Q = {{{1, 1}, {2, 2}}, {{2, 1}, {3, 2}}, {{3, 1}, {4, 2}}, {{2, 2}, {3, 3}}, {{3, 2}, {4, 3}}, {{2, 3}, {3, 4}}}`. Using the `polyoIdeal(Q)` function we obtain the binomials that generate the polyomino ideal.

```
Macaulay2, version 1.20
with packages: ConwayPolynomials, Elimination, IntegralClosure,
InverseSystems, Isomorphism, LLLBases, MinimalPrimes, OnlineLookup,
PrimaryDecomposition, ReesAlgebra, Saturation, TangentCone

i1 : loadPackage "PolyominoIdeals";
i2 : Q={{{1, 1}, {2, 2}}, {{2, 1}, {3, 2}}, {{3, 1}, {4, 2}}, {{2, 2},
       {3, 3}}, {{3, 2}, {4, 3}}, {{2, 3}, {3, 4}}};
i3 : I=polyoIdeal(Q);
```

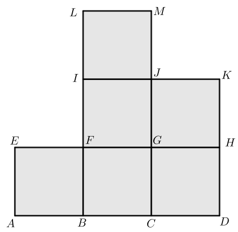


FIGURE 6.1: A polyomino.

```
i4 : g = gens I
o4 : | x_(4,3)x_(3,2)-x_(4,2)x_(3,3)  x_(2,2)x_(1,1)-x_(2,1)x_(1,2)
     ------------------------------------------------------------------
     x_(4,3)x_(2,1)-x_(4,1)x_(2,3)  x_(3,2)x_(2,1)-x_(3,1)x_(2,2)
     ------------------------------------------------------------------
     x_(4,3)x_(2,2)-x_(4,2)x_(2,3)  x_(3,3)x_(2,1)-x_(3,1)x_(2,3)
     ------------------------------------------------------------------
     x_(4,2)x_(1,1)-x_(4,1)x_(1,2)  x_(3,4)x_(2,1)-x_(3,1)x_(2,4)
     ------------------------------------------------------------------
     x_(3,3)x_(2,2)-x_(3,2)x_(2,3)  x_(4,2)x_(3,1)-x_(4,1)x_(3,2)
     ------------------------------------------------------------------
     x_(3,4)x_(2,2)-x_(3,2)x_(2,4)  x_(3,2)x_(1,1)-x_(3,1)x_(1,2)
     ------------------------------------------------------------------
     x_(4,3)x_(3,1)-x_(4,1)x_(3,3)  x_(3,4)x_(2,3)-x_(3,3)x_(2,4)
     ------------------------------------------------------------------
     x_(4,2)x_(2,1)-x_(4,1)x_(2,2) |
```

6.1.2 polyoMatrix function

Let $\mathcal{P}$ be a collection of cells and $[(p,q),(r,s)]$ be the smallest interval of $\mathbb{N}^2$ containing $\mathcal{P}$. The matrix $M(\mathcal{P})$ is a matrix having $s-q+1$ rows and $r-p+1$ columns with $M(\mathcal{P})_{i,j} = x_{(i,j)}$ if (i,j) is a vertex of $\mathcal{P}$, otherwise it is zero.
Consider the same polyomino given in Figure 6.1 encoded by `Q = {{{1, 1}, {2, 2}}, {{2, 1}, {3, 2}}, {{3, 1}, {4, 2}}, {{2, 2}, {3, 3}}, {{3, 2}, {4, 3}}, {{2, 3}, {3, 4}}}`.
The associated matrix is obtained using `polyoMatrix` function.

```
Macaulay2, version 1.20
with packages: ConwayPolynomials, Elimination, IntegralClosure,
InverseSystems, Isomorphism, LLLBases, MinimalPrimes, OnlineLookup,
PrimaryDecomposition, ReesAlgebra, Saturation, TangentCone

i1 : loadPackage "PolyominoIdeals";
i2 : Q={{{1, 1}, {2, 2}}, {{2, 1}, {3, 2}}, {{3, 1}, {4, 2}}, {{2, 2},
               {3, 3}}, {{3, 2}, {4, 3}}, {{2, 3}, {3, 4}}};
i3 : M=polyoMatrix(Q);
o3 : | 0       x_(2,4) x_(3,4) 0       |
     | 0       x_(2,3) x_(3,3) x_(4,3) |
     | x_(1,2) x_(2,2) x_(3,2) x_(4,2) |
     | x_(1,1) x_(2,1) x_(3,1) x_(4,1) |
```

The associated matrix for a collection of cells can help to order the variables to define a polynomial ring with another monomial order. In particular, this function is fundamental for coding the option when `RingChoice` has a different value by 1 (see Subsection 6.1.5).

6.1.3 polyoToric function

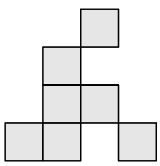
Let $\mathcal{P}$ be a weakly connected collection of cell. We introduce a suitable toric ideal attached to $\mathcal{P}$ based on that one given in [31] for polyominoes. Consider the following total order on $V(\mathcal{P})$: $a = (i,j) > b = (k,l)$, if $i > k$, or $i = k$ and $j > l$. If $\mathcal{H}$ is a hole of $\mathcal{P}$, then we call the lower left corner e of $\mathcal{H}$ the minimum, with respect to $<$, of the vertices of $\mathcal{H}$. Let $\mathcal{H}_1, \dots, \mathcal{H}_r$ be the holes of $\mathcal{P}$ and $e_k = (i_k, j_k)$ be the lower left corner of $\mathcal{H}_k$. For $k \in K = [r]$, we define the following subset $F_k = \{(i,j) \in V(\mathcal{P}) : i \leq i_k, j \leq j_k\}$. Denote by $\{V_i\}_{i \in I}$ the set of all the maximal vertical edge intervals of $\mathcal{P}$, and by $\{H_j\}_{j \in J}$ the set of all the maximal horizontal edge intervals of $\mathcal{P}$. Let $\{v_i\}_{i \in I}$, $\{h_j\}_{j \in J}$, and $\{w_k\}_{k \in K}$ be three sets of variables. We consider the map

$$\alpha : V(\mathcal{P}) \to K[h_i, v_j, w_k : i \in I, j \in J, k \in K]$$

$$a \to \prod_{a \in H_i \cap V_j} h_i v_j \prod_{a \in F_k} w_k$$

The toric ring $T_{\mathcal{P}}$ associated to $\mathcal{P}$ is defined as $T_{\mathcal{P}} = K[\alpha(a) : a \in V(\mathcal{P})]$. The homomorphism $\psi : S \to T_{\mathcal{P}}$ with $x_a \to \alpha(a)$ is surjective and the toric ideal $J_{\mathcal{P}}$ is the kernel of ψ. Observe that the latter generalizes in a natural way those ones given in [39] and in the previous sections on closed paths and weakly ones.
The function `PolyoToric(Q,H)` provides the toric ideal $J_{\mathcal{P}}$ defined before, where `Q` is the list encoding the collection of cells and `H` is the list of the lower left corners of the holes. It provides a nice tool to study the primality of the inner 2-minors ideal of weakly connected collections of cells. Here we illustrate some examples.

Example 6.1.1. Consider the simple and weakly-connected collection $\mathcal{P}$ of cells in Figure 6.2 (A), encoded by the list `Q={{{1, 1}, {2, 2}}, {{2, 2}, {3, 3}}, {{2, 1}, {3, 2}},{{3, 2}, {4, 3}},{{2, 3}, {3, 4}}, {{4, 1}, {5, 2}}, {{3, 4}, {4, 5}}}}`.
We can compute the ideal $I_{\mathcal{P}}$ using the function `polyoIdeal(Q)`, the toric ideal $J_{\mathcal{P}}$ with `polyoToric(Q,{})` and finally we make a comparison between the two ideals. We underline that to verify the equality, we need to bring the ideal `J=polyoToric(Q,{})` in the ring `R` of `polyoIdeal(Q)`, using the command `substitute(J,R)`. In according to the Theorem 3.2.3, we find that $I_{\mathcal{P}} = J_{\mathcal{P}}$.


(A)

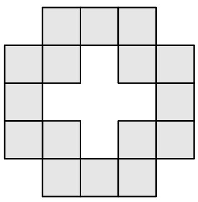

(B)

FIGURE 6.2

```
Macaulay2, version 1.20
with packages: ConwayPolynomials, Elimination, IntegralClosure,
InverseSystems, Isomorphism, LLLBases, MinimalPrimes, OnlineLookup,
PrimaryDecomposition, ReesAlgebra, Saturation, TangentCone

i1 : loadPackage "PolyominoIdeals";
i2 : Q={{{1, 1}, {2, 2}}, {{2, 2}, {3, 3}}, {{2, 1}, {3, 2}},{{3, 2},
```

```
          {4, 3}}, {{2, 3}, {3, 4}}, {{4, 1}, {5, 2}}, {{3, 4}, {4, 5}}};
i3 : I=polyoIdeal(Q);
i4 : J=polyoToric(Q,{});
i5 : R=ring I;
i6 : J=substitute(J,R);
o6 : Ideal of R
i7 : J==I
o7 = true
```

Consider the closed path polyomino $\mathcal{P}$ in Figure 6.2 (B). The polyomino ideal is not prime (see Subsection 2.2.3), so $I_{\mathcal{P}} \subset J_{\mathcal{P}}$ since $I_{\mathcal{P}} = (J_{\mathcal{P}})_2$ (Lemma 3.1, [31]). We can compute also the set of the binomials generating $J_{\mathcal{P}}$ but not $I_{\mathcal{P}}$.

```
Macaulay2, version 1.20
with packages: ConwayPolynomials, Elimination, IntegralClosure,
InverseSystems, Isomorphism, LLLBases, MinimalPrimes, OnlineLookup,
PrimaryDecomposition, ReesAlgebra, Saturation, TangentCone

i1 : loadPackage "PolyominoIdeals";
i2 : Q={{{2, 1}, {3, 2}}, {{2, 2}, {3, 3}}, {{1, 2}, {2, 3}}, {{1, 3},
        {2, 4}}, {{1, 4}, {2, 5}}, {{2, 4}, {3, 5}}, {{2, 5}, {3, 6}},
    {{3, 5}, {4, 6}}, {{4, 5}, {5, 6}}, {{4, 4}, {5, 5}}, {{5, 4},
    {6, 5}}, {{5, 3}, {6, 4}}, {{5, 2}, {6, 3}}, {{4, 2}, {5, 3}},
    {{4, 1}, {5, 2}}, {{3, 1}, {4, 2}}};
i3 : I=polyoIdeal(Q);
i4 : J=polyoToric(Q,{{2,3}});
i5 : R=ring I;
i6 : J=substitute(J,R);
i7 : J==I
o7 = false
i8 : select(first entries mingens J,f->first degree f>=3)
o8 = {x    x    x    x    - x    x    x    x    }
       6,5  5,1  2,6  1,2     6,2  5,6  2,1  1,5
```

6.1.4 The options `Field` and `TermOrder`

Let $\mathcal{P}$ be a collection of cells. The option `Field` for the function `polyIdeal` allows changing the base ring of the polynomial ring embedded in $I_{\mathcal{P}}$. One can choose every base ring that `Macaulay2` provides. The option `TermOrder` allows changing the monomial order of the ambient ring of $I_{\mathcal{P}}$ as given by the function `polyoIdeal`. In particular, by default, it provides the lexicographic order but one can replace it with other monomial orders defined in `Macaulay2`. See, for instance, the following example.

6.1.5 `RingChoice`: an option for the function `polyoIdeal`

Let $\mathcal{P}$ be a collection of cells. Recall that the definition of a ring in `Macaulay2` needs to provide, together with a base ring and a set of variables, also a monomial order. `RingChoice` is an option that allows choosing between two available rings that one can define into $I_{\mathcal{P}}$.
If `RingChoice` is equal to 1, or by default, the function `polyoIdeal` gives the ideal $I_{\mathcal{P}}$ in the polynomial ring $S_{\mathcal{P}} = K[x_a : a \in V(\mathcal{P})]$, where K is a field and the monomial order is defined by `Term order` induced by the following order of the variables: $x_a > x_b$ with $a = (i, j)$ and $b = (k, l)$, if $i > k$, or $i = k$ and $j > l$.
Now we describe what is the ambient ring in the case `RingChoice` has a value different from 1. Consider the edge ring $R = K[s_i t_j : (i, j) \in V(\mathcal{P})]$ associated to the

bipartite graph G with vertex set $\{s_1, \dots, s_m\} \cup \{t_1, \dots, t_n\}$ to $\mathcal{P}$ such that each vertex $(i, j) \in V(\mathcal{P})$ determines the edge $\{s_i, t_j\}$ in G. Let $S = K[x_{ij} : (i, j) \in V(\mathcal{P})$ and $\phi : S \to R$ be the K-algebra homomorphism defined by $\phi(x_{ij}) = s_i t_j$, for all $(i, j) \in V(\mathcal{P})$ and set $J_{\mathcal{P}} = \ker(\phi)$. From Theorem 2.1 of [37], we know that $I_{\mathcal{P}} = J_{\mathcal{P}}$, if $\mathcal{P}$ is a weakly connected and convex collection of cells. In such a case, from [34] we get that the generators of $I_{\mathcal{P}}$ form the reduced Gröbner basis with respect to a suitable order $<$, and in particular the initial ideal $\mathrm{in}_<(I_{\mathcal{P}})$ is squarefree and generated in degree two. Following the proof in [34], the implemented routine provides the polynomial ring $S_{\mathcal{P}}$ where the monomial order is $<$.

Example 6.1.2. The polyomino $\mathcal{P}$ in Figure 6.1 is convex. Using the options `RingChoice => 2` to define $I_{\mathcal{P}}$, the ambient ring of $I_{\mathcal{P}}$ is given by `PolyoRingConvex`. Hence the initial ideal is squarefree in degree two.

```
Macaulay2, version 1.20
with packages: ConwayPolynomials, Elimination, IntegralClosure,
InverseSystems, Isomorphism, LLLBases, MinimalPrimes, OnlineLookup,
PrimaryDecomposition, ReesAlgebra, Saturation, TangentCone

i1 : loadPackage "PolyominoIdeals";
i2 : Q={{{1, 1}, {2, 2}}, {{2, 1}, {3, 2}}, {{3, 1}, {4, 2}}, {{2, 2},
        {3, 3}}, {{3, 2}, {4, 3}}, {{2, 3}, {3, 4}}};
i3 : I=polyoIdeal(Q);
i4 : In= monomialIdeal(leadTerm(I))


o4 = monomialIdeal (x    x   ,x    x   ,x    x   ,x    x   ,x    x   ,x    x   ,
                     2,4  3,3  3,3  4,2  2,3  4,2  2,4  3,2  2,3  3,2  4,2  1,1
--------------------------------------------------------------------------------
x    x   , x    x   , x    x   , x    x   , x    x   , x    x   , x    x   ,
 3,2  1,1   2,2  1,1   3,3  4,1   2,3  4,1   3,2  4,1   2,2  4,1   2,4  3,1
--------------------------------------------------------------------------------
x    x   , x    x   )
 2,3  3,1   2,2  3,1

i5 : Q={{{1, 3}, {2, 4}}, {{2, 2}, {3, 3}}, {{2, 3}, {3, 4}}, {{2, 4},
    {3, 5}}, {{3, 4}, {4, 5}}, {{3, 3}, {4, 4}}, {{3, 2}, {4, 3}},
    {{3, 1}, {4, 2}}, {{3, 5}, {4, 6}}, {{4, 4}, {5, 5}}, {{4, 3},
    {5, 4}}, {{5, 4}, {6, 5}}};

i6 : I = polyoIdeal(Q,RingChoice=>2);

i7 : In= monomialIdeal(leadTerm(I))

o7 = monomialIdeal (x    x   , x    x   , x    x   , x    x   , x    x   , x    x
                    4,6  3,1  4,6  3,2  4,1  3,2  3,2  2,3  4,2  2,3  4,6  3,3
--------------------------------------------------------------------------------
x    x   , x    x   , x    x   , x    x   , x    x   , x    x   , x    x   ,
4,1  3,3  4,2  3,3  2,3  5,5  3,3  5,5  4,3  5,5  3,2  2,5  4,2  2,5
--------------------------------------------------------------------------------
x    x   , x    x   , x    x   , x    x   , x    x   , x    x   , x    x   ,
3,3  2,5  4,3  2,5  4,6  3,5  4,1  3,5  4,2  3,5  4,3  3,5  5,3  1,4
--------------------------------------------------------------------------------
x    x   , x    x   , x    x   , x    x   , x    x   , x    x   , x    x   ,
2,3  1,4  3,3  1,4  4,3  1,4  5,5  6,4  2,5  6,4  3,5  6,4  4,5  6,4
--------------------------------------------------------------------------------
x    x   , x    x   , x    x   , x    x   , x    x   , x    x   , x    x   ,
2,3  5,4  3,3  5,4  4,3  5,4  2,5  5,4  3,5  5,4  4,5  5,4  3,2  2,4
--------------------------------------------------------------------------------
x    x   , x    x   , x    x   , x    x   , x    x   , x    x   , x    x   ,
4,2  2,4  3,3  2,4  4,3  2,4  3,5  2,4  4,5  2,4  4,6  3,4  4,1  3,4
--------------------------------------------------------------------------------
```

```
x    x    , x    x    , x    x    )
4,2  3,4    4,3  3,4    4,5  3,4
```

6.2 Code of the package PolyominoIdeals.m2

We devote this section to show the code of the package.

```
newPackage(
"PolyominoIdeals",
Version => "1.0",
Date => "December 22, 2022",
Authors => {
        {
                Name => "Carmelo Cisto",
                Email => "ccisto@unime.it"
        },
        {
                Name => "Francesco Navarra",
                Email => "fnavarra@unime.it"
        },
        {
                Name => "Rizwan Jahangir",
                Email => "rizwan@sabanciuniv.edu",
                HomePage => "https://myweb.sabanciuniv.edu/rizwan"
        }
},
Headline => "binomial ideals of collections of cells",
Keywords => {"Binomial Ideals", "Inner 2-minor Ideals",
        "Polyomino Ideals"},
DebuggingMode => false,
Reload => true
)
export {
        "polyoIdeal",
        "polyoMatrix",
        "polyoToric",

        -- options
        "Field",
        "TermOrder",
        "RingChoice"
}
------------------------------------------------------------------------
-- Declaration of some variables
------------------------------------------------------------------------
x:=vars(23);
u:=vars(20);
v:=vars(21);
h:=vars(7);

------------------------------------------------------------------------
------------------------------ polyoVertices ---------------------------
------------------------------------------------------------------------
--
-- polyoVertices is a function which computes the set of
-- the vertices of the collection of cells.
--
-- A collection of cells is encoded here by a list Q,
```

```
-- whose elements are the lists of the diagonal corners
-- of the cells.
--
- For instance:
-
-                    __
-                 __|__|
-              __|__|
-             |__|__|        is encoded as Q={{{1,1},{2,2}},{{2,1},{3,2}},
-                                            {{2,2},{3,3}},{{3,3},{4,4}}}
-
------------------------------------------------------------------------

polyoVertices=(Q)->(
for i from 0 to #Q-1 do(
  if Q#i#1-Q#i#0 != {1,1} then
  error "The input list does not represent a collection of cells"
);
V:={};
for i from 0 to #Q-1 do(
  V=join(V,toList ({Q#i#0#0,Q#i#0#1}..{Q#i#1#0,Q#i#1#1}));
);
V=set V;
V=toList(V);

return V;
);


------------------------------------------------------------------------
--------------------------- polyoRingDefault ---------------------------
------------------------------------------------------------------------
--
-- The function polyoRing defines the ring attached to
-- a collection of cells, where the monomial order is
-- given by the order defined in the option Term order,
-- induced by the following order of the variables:
-- x_a > x_b with a=(i,j) and b=(k,l), if i > k, or
-- i = k and j > l.
--
------------------------------------------------------------------------

polyoRingDefault = method (Options=>{Field => QQ, TermOrder=>Lex})
polyoRingDefault List := opts -> Q -> (

V:=reverse(sort(polyoVertices(Q)));
Gen:={};
for i from 0 to #V-1 do(
  Gen=join(Gen,{x_(V#i#0,V#i#1)});
);
R:=(opts.Field)[Gen, MonomialOrder => opts.TermOrder];
return R;
);


------------------------------------------------------------------------
----------------------------- polyoMatrix ------------------------------
------------------------------------------------------------------------
--
-- The function polyoMatrix define the matrix attached to a
-- collection of cells P, where the smallest intervall
-- containing it is [(p,q),(r,s)]. The matrix has r-p
-- rows and s-q columns and the (i,j)-th entry is x_(i,j)
```

```
-- if (i,j) is a vertex of P, otherwise it is zero.
--
------------------------------------------------------------------------------

-- Define two functions to compute p,q,r and s from the list Q
-- enconding the collection of cells. The function Mv(Q)
-- computes the list {p,r}:

Mv=(Q)->(
V:={};
for i from 0 to #Q-1 do(
  V=join(V,toList{Q#i#1#1});
);
return toList{min(V)-1,max(V)};
);

-- The function Mh(Q) computes the list {q,s}:

Mh=(Q)->(
V:={};
for i from 0 to #Q-1 do(
  V=join(V,toList{Q#i#1#0});
);
return toList{min(V)-1,max(V)};
);

-- Define the function polyoMatrix, to compute the matrix.

polyoMatrix = method(TypicalValue=>Matrix)
polyoMatrix List := Q ->(
R:=polyoRing(Q);
V:=polyoVertices(Q);
Corners:={};
for i from 0 to #V-1 do(
  Corners=join(Corners,{(V#i#0,V#i#1)});
);
H:={};
Verti:=Mv(Q);
Orizon:=Mh(Q);
for j from Verti#0 to Verti#1 do(
  L:={};
  for i from Orizon#0 to Orizon#1 do(
    if member((i,j),Corners) then
    L=join(L,toList{x_(i,j)_R})
    else L=join(L,{0});
  );
  H=append(H,L);
);
H=reverse(H);
return matrix(H);
);


---------------------------------------------------------------------------------
------------------------------ polyoRingConvex ------------------------------
---------------------------------------------------------------------------------
--
-- The function polyoRingConvex returns the polynomial ring of a
-- collection of cells P with a new monomial order. In particular,
-- if P is a weakly connected and convex collections of cells then
-- polyoRingConvex defines a polynomial ring in which the monomial
-- order is defined as in the paper: H. Ohsugi and T. Hibi,
-- "Koszul bipartite graphs", Adv. Appl. Math. 22, 25-28, 1999.
```

```
-- We know that the generators of the binomial ideal associated
-- with a weakly connected and convex collections of cells forms
-- the reduced Groebner basis with respect to this order, and so
-- the initial ideal is squarefree and generated in degree two.
------------------------------------------------------------------------

-- vectorLessEqThan is a baby function which compares two vectors
-- in N^d, defining A < B if the rightmost nonzero component
-- of the vector A - B is negative.

vectorLessEqThan=(A,B)->(
Ar:=reverse(A);
Br:=reverse(B);
if Ar==Br then return true else
  for i from 0 to #A-1 do(
    if Ar#i<Br#i then break return true;
    if Ar#i>Br#i then break return false;
  );
);

-- Sub1 is a baby function which replaces 1 in the
-- non-null entries of a generic vector.

Sub1:=(M)->(
N:={};
for i from 0 to #M-1 do(
  if M#i!=0 then N=join(N,{1})
  else N=join(N,{0});
);
return N;
);

-- polyoMatrixReduced is a function which returns a new matrix from
-- polyoMatrix(Q) by switching rows or columns as done in the paper:
-- H. Ohsugi and T. Hibi, "Koszul bipartite graphs ", Adv. Appl.
-- Math. 22, 25-28, 1999.

polyoMatrixReduced=(Q)->(
PolyominoMat:=polyoMatrix(Q);
EntrateMat:=entries(PolyominoMat);
numberrow:=numgens(target(PolyominoMat));
MutMat:=mutableMatrix(PolyominoMat);
SubEntrateMat:={};
for k from 0 to numberrow-1 do(
  SubEntrateMat=join(SubEntrateMat,{Sub1(EntrateMat#k)});
);
for i from 0 to numberrow-1 do(
  for j from i to numberrow-1 do(
    if vectorLessEqThan(SubEntrateMat#i,SubEntrateMat#j)==false then(
    MutMat=rowSwap(MutMat,i,j);
    SubEntrateMat=switch(i,j,SubEntrateMat);
    );
  );
);
rowMutMat:=matrix(MutMat);
TMutMat:=transpose(rowMutMat);
SecondEntrateMat:=entries(TMutMat);
nc:=numgens(source(rowMutMat));
MutarowMutMat:=mutableMatrix(rowMutMat);
Sos:={};
for c from 0 to nc-1 do(
  Sos=join(Sos,{Sub1(SecondEntrateMat#c)});
);
```

```
for a from 0 to (nc-1) do(
  for b from a to (nc-1) do(
    if vectorLessEqThan(Sos#a,Sos#b)==false then(
    MutarowMutMat=columnSwap(MutarowMutMat,a,b);
    Sos=switch(a,b,Sos);
    );
  );
);
return matrix(MutarowMutMat);
);

-- polyoRingConvex defines a new polynomial ring.

polyoRingConvex = method(Options=>{Field => QQ})
polyoRingConvex List := opts -> Q -> (
PMR:=polyoMatrixReduced(Q);
EPMR:=entries(PMR);
numRow:=numgens(target(PMR));
numColumn:=numgens(source(PMR));
variables:={};
for i from 0 to numRow-1 do(
  for j from 0 to numColumn-1 do(
    if EPMR#i#j==0 then variables=join(variables,toList{})
    else variables=join(variables,toList{EPMR#i#j});
  );
);
Gens:=variables;
S:=(opts.Field)[Gens, MonomialOrder => RevLex, Global=> false];
return S;
);


--------------------------------------------------------------
------------------------- polyoRing --------------------------
--------------------------------------------------------------
-- The function polyoRing defines the ring for polyoIdeal.
-- Whether it is 1 or by default it returns the ideal computed
-- by polyoIdeal in the ambient ring given by polyoRingDefault.
-- With a value different by 1 it returns the ideal in the
-- ambient ring given by polyoRingConvex.
--------------------------------------------------------------

polyoRing = method (Options=>{Field => QQ, TermOrder=>Lex,
        RingChoice=>1})
polyoRing List := opts -> Q ->(
if opts.RingChoice==1 then
return polyoRingDefault(Q,Field=>opts.Field,
TermOrder=>opts.TermOrder)
else return polyoRingConvex(Q,Field=>opts.Field);
);

--------------------------------------------------------------
------------------------- polyoIdeal -------------------------
--------------------------------------------------------------
-- polyoIdeal is a function which returns the inner 2-minor ideal
-- attached to a collection of cells.
--
-- The option RingChoice with value 1 and by default returns the
-- ideal in the ambient ring given by polyoRingDefault. With a
-- value different by 1 it returns the ideal in the ambient ring
-- given by polyoRingConvex
--------------------------------------------------------------
```

```
--isInnerInterval is a function such that, if A and B are two
-- cells, it returns true if [A,B] is an inner interval of the
-- collection of cells, otherwise it returns false.

isInnerInterval=(A,B,Q)->(
C:=B-{1,1};
if C==A then return true;
if member({C,B},Q)==false then return false;
tag:=true;
for i from A#1+1 to B#1 do (
  for j from A#0 to B#0-1 do (
    if member({{j,i-1},{j+1,i}},Q)==false then return false;
  );
);
return tag;
);

polyoIdeal = method (Options=>{Field => QQ, TermOrder=>Lex,
        RingChoice=>1})
polyoIdeal List := opts -> Q ->(

R:=polyoRing(Q, Field=>opts.Field, TermOrder=>opts.TermOrder,
RingChoice=>opts.RingChoice);
InnerBinomials:={};
for i from 0 to #Q-1 do(
  lLowCorner := Q#i#0;
    for j from 0 to #Q-1 do(
    rUpCorner := Q#j#1;
    if lLowCorner#0<rUpCorner#0 and lLowCorner#1<rUpCorner#1 then (
      if isInnerInterval(lLowCorner,rUpCorner,Q) then (
      a:=lLowCorner#0;
      b:=lLowCorner#1;
      c:=rUpCorner#0;
      d:=rUpCorner#1;
      InnerBinomials=join(InnerBinomials,{x_(a,b)_R*x_(c,d)_R-x_(a,d)_R*x_(c,b)_R});
      );
    );
  );
);
InnerBinomials = set InnerBinomials;
InnerBinomials = toList InnerBinomials;
I:=ideal(InnerBinomials);
return I;
);


-------------------------------------------------------------------
-------------------------- polyoToric ----------------------------
-------------------------------------------------------------------
--Given a polyomino encoded by Q and the list H of the lower left
-- corners of each hole of the polyomino, the function polyoToric
-- returns the toric ideal as defined in the paper:
--Mascia, Rinaldo, Romeo, "Primality of multiply connected
-- polyominoes", Illinois J. Math. 64(3), 291-304, 2020.
-------------------------------------------------------------------

Leq2=(A,B)->(
if A#0<=B#0 and A#1<=B#1 then return true;
return false;
);

polyoToric = method(TypicalValue=>Ideal)
polyoToric(List, List) := (Q,H) -> (
```

```
V:=reverse(sort(polyoVertices(Q)));
Oriz:={};
Vert:={};
for i from 0 to #V-1 do(
  Oriz=join(Oriz,{V#i#0});
  Vert=join(Vert,{V#i#1});
);
Oriz=set Oriz;
Oriz=toList (Oriz);
Oriz=sort(Oriz);
Vert=set Vert;
Vert=toList (Vert);
Vert=sort(Vert);
VerInt:={};
for i from min(Oriz) to max(Oriz) do(
  j:=min(Vert);
  while j<max(Vert) do(
    L1:={};
    while member({{i,j},{i+1,j+1}},Q) or member({{i-1,j},{i,j+1}},Q)  do(
     L1=join(L1,{{i,j},{i,j+1}});
     j=j+1;
    );
  L1=set L1;
  L1=toList (L1);
  VerInt=join(VerInt,{L1});
  j=j+1;
  );
);
VerInt=delete({},VerInt);
OrInt:={};
for j from min(Vert) to max(Vert) do(
  i:=min(Oriz);
  while i<max(Oriz) do(
    L1:={};
     while member({{i,j},{i+1,j+1}},Q) or member({{i,j-1},{i+1,j}},Q)  do(
     L1=join(L1,{{i,j},{i+1,j}});
     i=i+1;
    );
  L1=set L1;
  L1=toList (L1);
  OrInt=join(OrInt,{L1});
  i=i+1;
  );
);
OrInt=delete({},OrInt);
Svar:={};
for i from 0 to #OrInt-1 do(
  Svar=join(Svar,{u_(i)});
);
for i from 0 to #VerInt-1 do(
  Svar=join(Svar,{v_(i)});
);
for i from 0 to #H-1 do(
  Svar=join(Svar,{h_(i)});
);
S:=QQ[Svar, MonomialOrder => Lex];
Im:={};
for i from 0 to #V-1 do(
  m:=1;
  for k from 0 to #OrInt-1 do(
    if  member(V#i,OrInt#k) then m=m*u_(k)_S;
  );
for k from 0 to #VerInt-1 do(
```

```
  if member(V#i,VerInt#k) then m=m*v_(k)_S;
);
for j from 0 to #H-1 do(
  if Leq2(V#i,H#j) then m=m*h_(j)_S;
);
Im=join(Im,{m});
);
T:=polyoRing(Q);
f:=map(S,T,Im);
J:= kernel f;
return J;
);
```

--- End of source code ---

Open questions and future works

We conclude this work showing some open questions and the future researches which can be dealt on this topic.
The radicality of the polyomino ideals gives an interesting problem. It seems that all polyomino ideals are radical, in fact if $\mathcal{P}$ is a polyomino and $<$ is the reverse lexicographical order on $S_{\mathcal{P}}$ induced by the ordering of the variables defined by $x_{ij} > x_{kl}$ if $j > l$, or, $j = l$ and $i > k$, then the Gröbner basis of $I_{\mathcal{P}}$ with respect to $<$ could be squarefree and consequently $I_{\mathcal{P}}$ radical, but a complete proof is not still given. For what concerning the study of the Gröbner basis, we wonder if it is possible to find a polyomino $\mathcal{P}$ for which the set of generators does not form the reduced Gröbner basis of $I_{\mathcal{P}}$ with respect to any monomial order. During "EMS Summer School of Combinatorial Commutative Algebra" held in Gebze (Turkey), that I attended, Prof. Takayuki Hibi remarked that this last question is very underrated although an answer seems very difficult to give.
As already said along the thesis, a complete characterization of the primality of $I_{\mathcal{P}}$ is not still known but the conjecture that states that the inner 2-minor ideal of a collection of cells is prime if and only the collection of cells does not contain any zig-zag walk is very promising. Try to give a complete proof of this conjecture is a very fascinating challenge as well as a complete characterization of the polyominoes whose coordinate ring is a normal Cohen-Macaulay domain or Gorenstein.
Another interesting problem is the study of the Hilbert-Poincaré series of $K[\mathcal{P}]$, in particular to provide a complete proof of the conjecture stating that if $\mathcal{P}$ is a simple polyomino then the h-polynomial of $K[\mathcal{P}]$ is equal to the switching rook polynomial of $\mathcal{P}$ (see [38]). Motived by these considerations, in [33] we investigate the conjecture also for non-simple polyominoes and we study the Hilbert-Poincaré series of frame polyominoes, which are polyominoes obtained removing a parallelogram polyomino from a rectangle polyomino. We establish a bijection between the set of non-attacking and non-switching rook arrangements in $\mathcal{P}$ and the facets of the shellable simplicial complex, having $\mathrm{in}_<(I_{\mathcal{P}})$ as Stanley-Reisner ideal. As a consequence we get that the h-polynomial of $K[\mathcal{P}]$ is the switching rook polynomial of $\mathcal{P}$ and the regularity of $K[\mathcal{P}]$ is the rook number of $\mathcal{P}$.
In [23] the authors extend the concept of König type to graded ideals in $K[x_1, \dots, x_n]$ and in [20] Herzog and Hibi prove that if $\mathcal{P}$ is a simple thin polyomino then $I_{\mathcal{P}}$ is of König type. In [14] we study the König type property for non-simple polyominoes and we prove that, for closed path polyominoes, the polyomino ideals are of König type. We remark also that not all the polyominoes have their ideals of König type. In particular, a class of polyominoes for which this property does not hold is given by the parallelogram polyominoes, combining the results in [20] and [38]. In general, it could be very interesting to give a complete classification of the polyominoes which have the König type property.
Another huge problem is the study of the primary decomposition of $I_{\mathcal{P}}$. To face this problem, in [9] we study the primary decomposition of a more general class of binomial ideals. In particular, we introduce the concept of *polyocollection*, a combinatorial object that generalizes the definitions of collection of cells and polyomino,

that can be used to compute a primary decomposition of non-prime polyomino ideals. Furthermore, we give a description of the minimal primary decomposition of non-prime closed path polyominoes. In particular, for such a class of polyominoes, we characterize the set of all zig-zag walks and show that the minimal prime ideals have a very nice combinatorial description. If we know that $I_{\mathcal{P}}$ is radical for a collection of cells $\mathcal{P}$ then [9, Theorem 3.12] will provide a primary decomposition of $I_{\mathcal{P}}$. Another property that could be investigated is the unmixedness of polyomino ideals, that we prove for closed paths in [9]. Moreover it could be interesting to see if the properties studied for the inner 2-minor ideals of a collection of cells can be extended for polyocollections. For what concerning the package `PolyominoIdeals` for `Macaulay2` some combinatorial functions could be implemented.

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