CHAPTER THREE

RESEARCH METHODOLOGY

Introduction

The previous two chapters provided a review of literature that began, in a broad sense, within the context of the van Hiele Theory and Class Inclusion. This review involved a detailed discussion concerning research specifically directed at geometrical class inclusion and an appropriate framework to assist in the exploration of this concept. In the light of the research themes that emerged in Chapter 2, a study was designed which investigates students' understandings of geometrical class inclusion concepts within the specific content areas of triangles and quadrilaterals.

This chapter outlines the methodologies devised and undertaken within this investigation. To describe the methodological considerations and provide a detailed description of the design, this chapter is divided into six sections. The first section provides a description of the context of the study, while the second section outlines the design inclusive of pilot study, participants, and development of the interview structure. The following three sections considers the methodological issues, data analysis plan and evaluation of the design. The final section is titled Conclusion.

CONTEXT OF THE STUDY

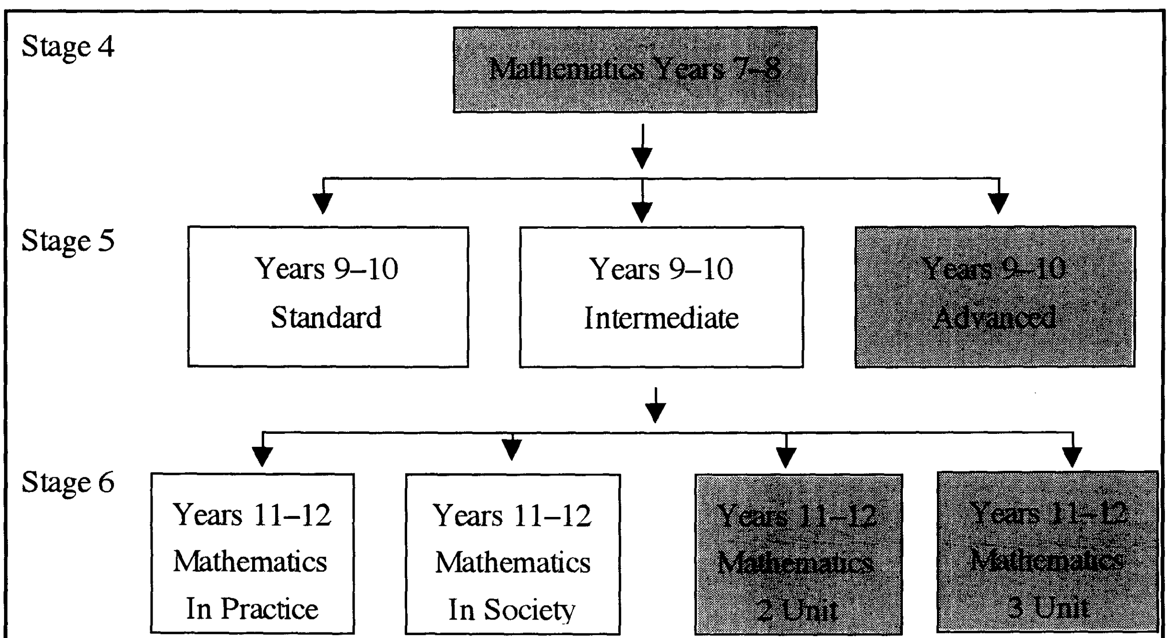
This section provides a detailed description of the contextual issues related to the current study. These contextual issues include the geographical setting in which the study is located, and mathematical background of the chosen sample.

Geographical Setting

Each of the students chosen to participate in this study attended secondary schools in the inland city, Armidale. While the population, approximately 22 000, places Armidale in the relatively small rural city category, the Armidale community remains unique in nature due to its strong educational focus. Historically, Armidale has become known as a 'university town,' as schooling and education are considered the main industry. As a result of this, Armidale has a relatively high proportion of professional residents with a changing and diverse population, and has gained the reputation of being a regional cultural centre. Educational facilities in Armidale include, the University of New England, TAFE, six secondary schools comprising of (two public, one systemic Catholic, and three private), and eight primary schools. The sample chosen for this study included secondary school students ranging from Years 8 to 12 (ages 12–18 years).

Secondary Mathematics Courses

The NSW secondary curriculum at the time of data collection comprised eight key learning areas, Mathematics being one of these. The NSW Mathematics Syllabus 7–12 is divided into 3 stages, as illustrated in Figure 3.1. While each course is organised into different content areas, the content areas are integrated throughout each year of the schooling. Thus Geometry, the focus content area of this study, is revisited each year by students through topics designed at the school level. The three stages in the secondary component of the Mathematics syllabus are Stage 4: Mathematics Years 7–8 (approximately age 12–14 years); Stage 5: Years 9–10 (approximately age 14–16 years); and Stage 6: Years 11–12 (approximately age 16–18 years). Below the figure is a general description of the secondary modules pertinent to this study and the Geometry content specified for each module.



Key: Shading denotes coursed from which the sample of students were chosen.

Figure 3. 1 NSW Secondary Mathematics Courses (at time of data collection)

Years 7–8 Mathematics (Stage 4)

The Years 7–8 Syllabus is organised as a single basic course divided into six strands, namely, Problem Solving, Geometry, Measurement, Statistics, Number, and Algebra. Implementation of this syllabus requires the teacher to interweave the six strands in order to provide a balanced program to meet the needs, interests, and abilities of their students. While the course content areas remain the same for all students, the depth of treatment to meet the appropriate level for various groups of students is determined by the school. "Geometry in Years 7–8 is to be treated informally through practical pupil activities" (Board of Secondary Education, New South Wales, 1988, p. 9). This strand focuses upon:

- the development of spatial visualisation skills
- the development of appropriate language and notational skills
- the aesthetic and functional uses of shape in the environment
- the accurate and effective use of geometrical instruments
- the investigation and discovery of geometric properties through handling and manipulating shapes, paper folding, cutting out, tracing, sketching, measuring and constructing.

(Board of Secondary Education, New South Wales, 1988, p. 9)

While the majority of objectives are described as informal, there is scope within the syllabus for more able students to progress to simple deductive reasoning, such as the application of congruent triangles to deduce properties. The properties of common plane shapes are treated informally through student investigation. Such investigations concern side lengths, angles, symmetry, diagonals, parallelism, and angle sums using concrete materials. The properties explored are also consolidated through measurement and construction activities. The classification of plane figures is divided into two areas, triangles (G3.6) and quadrilaterals (G3.7).

The content and skills concerning the classification of triangles include the ability to:

- classify triangles as
  - scalene
  - isosceles
  - equilateral
  - right angled
  - acute angled
  - obtuse angled
- demonstrate that a given triangle may belong to more than one class
- recognise types of triangles in composite figures
- describe triangles in terms of properties.

(Board of Secondary Education, New South Wales, 1988, p. 35)

The content and skills concerning the classification of quadrilaterals include the ability to:

- classify quadrilaterals as:
  - trapeziums
  - kites
  - parallelograms
  - rectangles
  - squares
  - rhombuses
- recognise quadrilaterals in composite figures
- describe quadrilaterals in terms of the properties of sides, angles, parallel lines, line and/or rotational symmetry, diagonals, intersection of diagonals.

(Board of Secondary Education, New South Wales, 1988, p. 37)

At this stage, definitions requiring minimal properties are not required; however, formal terms are introduced where appropriate.

Year 9–10 Mathematics

Stage 5 of the NSW Mathematics Syllabus (Years 9–10) acknowledges the various stages of development reached by students who have completed Stage 4 Mathematics. Due to differences in achievement and conceptual understanding of students, the syllabus is divided into three courses, namely, Advanced, Intermediate, and Standard. Each course has variations in "mathematical abstraction, depth of treatment and practicality" (Board of Studies, New South Wales, 1996a, p. 5). The Standard course encourages the development of basic mathematical skills and combines a thematic and topical approach. The Advanced course caters for those students who have a sound grasp of all content of the Years 7–8 syllabus and require the development of a more abstract approach to mathematical thinking. The Intermediate course, which in degree of complexity is between the other two courses, has elements of both.

Of particular interest to this study is the Geometry content contained in the Advanced and Intermediate courses. The Intermediate course reviews and develops many of the geometrical facts and relationships of the Year 7–8 Geometry strand. Through the use of various tools, such as pen and paper, construction equipment, graphics calculators, and computer software, students are placed in a situation where they consider real geometrical situations. An investigative approach is recommended to establish conditions for congruency, however, it is not intended that the students prove relationships and theorems formally involving deductive Geometry. The Advanced course is designed on the assumption that the students are competent with all Year 7–8 Geometry content, and aims to foster "an understanding of some principles of deductive reasoning and the use of these principles in the context of formal Geometry" (Board of Studies, New South Wales, 1996a, p. 45).

Pertinent to this study is the following geometrical content of the Advanced course (Board of Studies, New South Wales, 1996a). In relation to triangles and quadrilaterals, learning experiences should provide students with the opportunity to:

- prove and apply properties of triangles and quadrilaterals
- recognise and apply definitions, e.g. an isosceles triangle is a triangle in which there are two sides equal in length
- establish and apply theorems about triangles and quadrilaterals
- apply theorems and properties of triangles and quadrilaterals to solve problems, justifying the solutions using appropriate language

- use geometrical relationships and properties of triangles, quadrilaterals, and polygons to prove statements about geometrical figures.

The Intermediate course (Board of Studies, New South Wales, 1996b), in relation to triangles and quadrilaterals, provides students with the opportunity to:

- establish and apply properties of triangles and quadrilaterals
- describe triangles and quadrilaterals in terms of sides, angles, symmetry, parallelism, and diagonals
- apply properties of triangles and quadrilaterals to solve numerical problems, giving a reason for their solutions using appropriate language.

Year 11–12 Mathematics

The Year 11–12 Mathematics Syllabus (Stage 6) is divided into four courses, namely, 3 Unit/4 Unit, 2 Unit, Mathematics in Society (2 Unit General) and Mathematics in Practice. Of particular interest to this study are the 3 Unit/4 Unit and 2 Unit courses. These courses are calculus based and are undertaken by high achieving Mathematics students. The 2 Unit course is intended to promote an "understanding of and competence in some further aspects of Mathematics which are applicable to the real world" (Board of Senior School Studies, New South Wales, 1982, p. 1). "The 3 Unit course is intended to give these students a thorough understanding of, and competence in, aspects of mathematics including many which are applicable to the real world" (p. 1). Hence, the 2 Unit course is recommended for those students requiring Mathematics as a minor discipline at a tertiary level, while 3 Unit is considered a minimum basis for those students requiring Mathematics as a major discipline.

All the Geometry content of the 2 Unit course is included in the 3 Unit course, with extension items for the 3 Unit students and a greater depth of treatment expected. The 2 Unit material of interest to the relationships among figures and their properties includes:

- properties of special triangles and quadrilaterals
- definitions of triangles and quadrilaterals
- tests for special quadrilaterals
- application of above properties to simple theoretical problems requiring one or more steps of reasoning.

The syllabus notes refer specifically to definitions requiring an understanding of notions of class inclusion. For example, "A parallelogram should be defined as a quadrilateral with both pairs of opposite sides parallel and a rectangle as a parallelogram with one angle a right angle. A rhombus is a parallelogram with a pair of adjacent sides equal and a square is a rectangle with a pair of adjacent sides equal" (Board of Senior School Studies, 1982, p. 15).

In summary, the Secondary Mathematics Courses, namely, Years 7–8, Years 9–10 and Years 11–12, all include Geometry content as a significant component. While the material is at differing degrees of abstraction depending upon the Year level and course chosen, each Stage (4–6) has provision for students to explore the relationships between properties and their figures. This exploration progresses from an emphasis on informal investigations of figures and their properties, to formal definitions requiring the application of class inclusion concepts. Due to the integrated and spiralling nature of the secondary syllabus, notions of class inclusion are explored at varying degrees in each of the courses described above.

DESIGN OF THE STUDY

This section describes the methodology used to explore the global research themes for this study. An overview of the design structure devised to meet the research themes and a description of the sample chosen for the study is followed by a description of the pilot study including implications for the main study. Following these considerations, details of the development of the interview sequence are presented.

Overview of the Design

The research design developed for this study involved three main sections, referred to as Study 1, Study 2, and Study 3. Each of these studies involved material gathered from interviews involving secondary school students. While each of the studies investigates the research themes outlined in Chapter 2, Studies 1, 2, and 3 have individual research focuses that complement each other in addressing the global research themes of the study.

Study 1 was the first part of the main study and involved students selected from Years 8–11 who were currently in high achieving Mathematics classes. The purpose of this study was to explore students' understandings of the relationships among triangles and the relationships among triangle properties. The two main aims were: firstly, to provide a general framework based on initial identified characteristics of students' responses concerning an understanding of class inclusion notions in a geometrical sense; secondly, to consider students' responses in the light of the SOLO model with the aim to provide a deeper interpretation of the general framework and to assist in the explication of developmental pathways. As a result of these aims, a subsequent purpose of the study involved the determination of the SOLO model as an appropriate tool for providing insights into the van Hiele Theory. The results generated from this study provided baseline data for Study 2 and Study 3 in relation to how students' understanding of class inclusion concepts evolve.

Study 2 formed the second part of the main study and involved the same sample of students selected for Study 1. The purpose of this study was to extend the findings of Study 1, through the exploration of students' understandings of the relationships among quadrilaterals and among quadrilateral properties. This was achieved by placing the students in a more complex setting, the aim being to explicate a broader generalisation concerning developmental pathways leading to an understanding of geometrical class inclusion concepts. These results were also used to provide baseline data for Study 3.

Study 3 formed the final section of the main study and had two parts. The third study involved the initial sample, and a second intervention of fifty-percent of the initial sample of secondary students involved in Studies 1 and 2. The latter group of students were interviewed two years later (Intervention 2) utilising the same format as in Intervention 1 (initial interview). The purpose was of the longitudinal perspective to gather additional data to allow a quantitative comparative analysis which synthesised and compared responses to relationships among figures and relationships among properties in the light of two contexts, being triangles and quadrilaterals. In addition, the longitudinal effect on the development of students' understandings of the relationships among figures and their properties were addressed. Study 3 provided the opportunity to further validate the results of the previous studies, and to clarify students' responses.

Pilot Study

Before the study could begin, there was a need to explore through a pilot study the diversity of the range of responses offered by students when engaged in an interview involving tasks related to the relationships among figures and their properties. The pilot was also necessary to ascertain the appropriateness of particular student tasks for promoting student discussion concerning their identified relationships. This pilot was an essential prerequisite to the main studies (1, 2, and 3) as it provided information concerning the suitability of the tasks, question types, and timing of the interview process. The framework chosen for the pilot was guided by the research themes of the previous chapter.

Sample

The Pilot Study, which comprised an interview format divided into two sections, was implemented with four students (two females, two males) on an individual basis, one from each of Years 8, 9, 10, and 11. The students were chosen from the upper ability students in each class, from two schools. The upper ability students were chosen as it was necessary to explore students' responses who were demonstrating Level 2 and Level 3 (van Hiele) understanding of geometrical concepts.

Research issues

In particular, the following issues were the focus of the pilot investigation:

1. To identify diversity in the range of responses students offer when asked to:
   - classify triangles and quadrilaterals
   - provide minimum property descriptions of triangles and quadrilaterals
   - discuss notions of class inclusion.
2. To develop a simple structured interview sequence consisting of open-item questions with the flexibility to move into a clinical questioning sequence involving prompts and probes.
3. To investigate the use of student tasks incorporating diagrams and property cards as a catalyst for generating discussion concerning relationships among figures and properties.
4. To practise questioning techniques involving prompting and probing.

Pilot design

The pilot investigation had two main sections. Section 1 involved the relationships among figures, and Section 2 focused upon the relationships among properties. Section 1 began by asking the students to group and justify their groupings among seven different triangles, namely, equilateral, acute isosceles, obtuse isosceles, right isosceles, acute scalene, obtuse scalene, and right scalene. The same task was then repeated with six quadrilaterals, namely, square, parallelogram, rectangle, rhombus, kite, and trapezium. The students were provided with a diagram of the figures, each on a separate card, and were asked to sort the figures into smaller groups giving reasons for the groupings chosen. The students were not provided with the names of figures, and the diagrams included no markings or labelling to indicate equality of sides or angle size. Section 2 required the students to list the known properties of a named triangle, and a named quadrilateral, and then provide combinations of the minimum properties that would signify that shape. The proforma and diagrams utilised in the pilot are contained in Appendix A.

It became evident from Section 1 of the pilot study that the student tasks provided an effective catalyst for discussion. The interviewer was able to probe for further information concerning the justification of the groupings. Thus, the interview design involved a semi-structured interview sequence, which assisted in keeping the students 'on-task'. This structure resulted in some diverse responses, which were drawn from common questions across the sample, while the flexibility to move into clinical questioning provided further depth at an individual level.

Implications for the main Studies

Although the provision of individual figure cards allowed each student to work with an identical set of figures, it became evident that some students were working with some unfamiliar figures and orientations of the figures. This was evident in student discussion concerning shapes that they could not identify. Hence, while the figures chosen for the pilot were shown to be appropriate, in order to enable the students to work with familiar recalled information and to have a means for expressing this information, one change to the pilot involved the use of students' own diagrams. As a result of this change, the benefits of altering the task to include a tree diagram, rather than just groupings, was evident, thus providing a situation where all minor and major links could be expressed, and all class relationships could be illustrated pictorially and be used as a focal point for discussion. In addition, the pilot study was altered to allow the opportunity for the students to return to the same question on several occasions through opportunities to supplement figures and devise additional tree diagrams.

Section 2 of the pilot study illustrated the effectiveness of initiating discussion of the relationships among properties through the devising and justifying of minimum property descriptions/definitions. However it became evident that the students were providing limited responses which often involved spontaneous recall, rather than the utilisation of all known properties of the figure as a means for recording combinations already used. To address this issue, a recommended change to the main study included the student being given a set of property cards for triangles, and a set for quadrilaterals. From this set, the properties of a figure were chosen and then used for minimum property descriptions within the context of leaving clues for a friend to work out a particular shape. Thus the question remained open to include both minimum descriptions and definitions.

Participants for the Main Studies

The research sample for the main study initially comprised twenty-four students, six from each of Year 8 to Year 11 (ages 13 to 17), who were selected from two secondary schools in Armidale. The students were of above average ability and there were equal numbers of males and females. Purposeful sampling techniques were applied and the Head Mathematics Teacher at each of the schools selected the students. Factors taken into consideration included each student's mathematical ability and availability to participate in the study. The longitudinal phase of the main study, Study 3, involved twelve students from the original pool, initially from Years 8 and 9, being interviewed on the same task two years later when in Years 10 and 11. A summary of the sample is provided in Table 3.1 which outlines the numbers, and school year groupings of the students chosen.

Table 3.1 Overview of research sample for Studies 1, 2, and 3

PARTICIPANTS

Year Gender	8 M F	9 M F	10 M F	11 M F	Total
Studies 1,2 (Qual)	3* 3*	3* 3*	3 3	3 3	24
Studies 3A (Quant)	3* 3*	3* 3*	3 3 3* 3*	3 3 3* 3*	36
3B (Long)			3* 3*	3* 3*	12

Key: * indicates students that participated in both interventions of Study 3

Interview Structure

In light of the insights gained from the pilot study in regard to the appropriateness of the interview as the research instrument for this study, an interview sequence including student tasks was designed for this investigation. The student tasks and interview questions for the pilot were developed after initial exploratory tasks and questions relating to geometrical relationships were administered on a one-to-one basis to students ranging from Year 3 to Year 11. While this occurred on an informal basis, the responses to questions in the content areas of triangles and quadrilaterals concerning the relationships among figures and among their properties determined the appropriateness of the items, and the most suitable Year groupings to include in the study. As a result, a pilot study was implemented, as discussed above, and modifications made to the original interview proforma.

The interview proforma designed for Study 1 and 2 (Appendix B) involved two aspects to each study, these being: Relationships Among Figures (Triangles in the case of Study 1 and Quadrilaterals in the case of Study 2), and Relationships Among Properties.

In summary, Study 1 and Study 2 involved student responses gained from the following parts:

Study 1: Relationships Among Triangles
Relationships Among Triangle Properties

Study 2: Relationships Among Quadrilaterals
Relationships Among Quadrilateral Properties

Study 1 and Study 2 involved an identical sample of students and the data were collected for both studies during the same student interview session. Since the studies involved students from secondary schools, aged 13–17 years, it was necessary to gain consent from the Principal, Head Mathematics Teacher, and relevant Mathematics teachers from each of the schools. Once consent was obtained from the staff involved, a consent letter was sent to the student's home to gain parental/guardian and participant consent. The voluntary nature of involvement was also explained to the students at the commencement of each interview. Each interview session was carried out in consideration of the guidelines of the New South Wales Department of Education and the policies of the specific schools. The setting involved a quiet room in a general area where the participant and researcher remained visible to passing staff of the school throughout the interview. The interviews were audio-taped and conducted within normal Mathematics lesson time in consultation with staff and students. On completion, the interviews were transcribed and combined with written responses from the students to form the data set for the investigation.

The interview proforma is included in Appendix B and the interview process is described briefly below in regard to Study 1. Each session, approximately one hour in length, began with the student being asked to list and sketch an acute-angled scalene triangle, and then to recall, list, and draw other triangle types. This process was designed to help clarify that all triangle types including both angle and side classifications were to be the focus of the activity. The relationships identified by the students were presented in diagrammatical form to illustrate the links between the different triangles. This diagram was then used as a catalyst for discussion concerning the reasons for the existence of these links. The student list was then supplemented with any of the triangles missing from the list of possibilities. If necessary, a second tree diagram was designed by the student and discussion with diagrammatical stimulus followed. In the case of a second tree diagram being completed, the student was requested to return to the original tree diagram to add the newly identified triangles. A third discussion then took place concerning the design of the tree diagram.

This format was chosen as it enabled the students to work with familiar recalled information, supplemented information, individual tree designs, and discussion involving prompts and probes. The continual revisiting of the same relationships on different tree diagram designs provided a vehicle for extracting further information and expanding on information already provided.

The second section of the interview relating to triangles involved the student's understanding of the relationships among triangle properties. In brief, each student was provided with a collection of triangle property cards (Appendix C) and was asked to choose all cards that belonged to the equilateral triangle. The student was then given the problem of leaving clues for that particular figure which would be interpreted correctly as that figure by a friend. The student was asked to provide as many combinations as possible. During the task the student was asked to provide reasons for his/her choices and to elaborate on his/her responses. This task and questioning sequence was then repeated for the right isosceles triangle. A similar sequence was followed for Study 2 in regards to quadrilaterals.

Study 3 involved two parts, namely, a quantitative analysis of all data collected applying Rasch, and secondly, a longitudinal qualitative analysis presented in the form of four case studies. The interview followed the same procedures as Study 1 and Study 2. The research design is summarised in Figure 3.2.

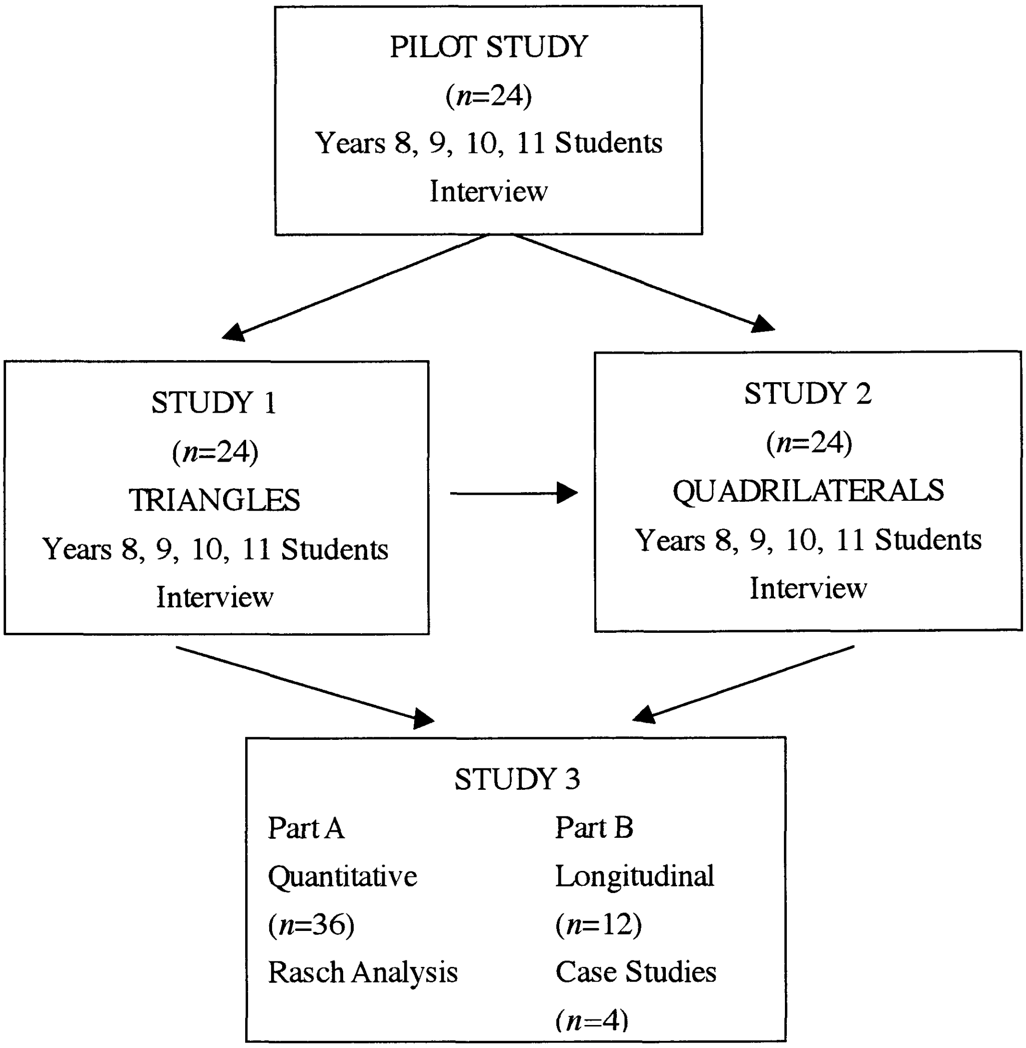


Figure 3.2 Overview of the research design

METHODOLOGICAL ISSUES

This section considers the research design from a theoretical perspective in terms of the selection of the research instrument, and issues considered throughout the development and implementation of the interviews with participants. This analysis provides a justification for the appropriateness of interviews as the data-collecting tool, and issues relating to the longitudinal data collection.

Justification of Interview Design

Burns (1990, p. 10) stated that "qualitative methods are concerned with processes, rather than consequences, with organic wholeness rather than independent variables, and with meanings rather than behavioural statistics. Interest is directed towards context-bound conclusions that could potentially point the way to new policies and educational decisions, rather than towards scientific generalisations that may be of little use at the coal-face." It is in this light that the qualitative methodology chosen for this research study required careful consideration. The appropriateness of the interview as the chosen research tool for the in-depth study of students' understandings of geometrical relationships is emphasised in the quote below:

> The interview, when coupled with an adequate schedule of pretested worth, is a potent and indispensable research tool, yielding data that no other research tool can yield. It is adaptable, capable of being used with all kinds of respondents in many kinds of research, and uniquely suited to exploration in depth.
>
> (Kerlinger, 1986, p. 446)

Distinguishing characteristics of the qualitative interview are summarised by Sarantakos (1998) as the following:

- They use open-ended questions only.
- They are predominantly single interviews, questioning one person at a time.
- The question structure is not fixed or rigid, allowing change of question order, even the addition of new questions where necessary.
- They offer interviewers more freedom in presenting the questions, changing wording and order, and adjusting the interview so that it meets the goals of the study.

(Sarantakos, 1998. pp. 255–256)

Often the interview is compared with the written survey or questionnaire as an alternative research tool. Wiersma (1991) described the following three advantages of the interview over the use of a questionnaire:

1. if the interview is granted there is no problem with non-response;
2. the interview provides opportunity for in-depth probing and elaboration and clarification of items, if necessary; and,
3. interviews can be used with individuals from whom data cannot otherwise be obtained.

(Wiersma, 1991, p. 190)

These general descriptions provide a broad outline of the components of the interview as a qualitative research tool, however, each characteristic described above requires decisions to be made by the researcher based upon the research questions involved. It is essential for the researcher to consider both advantages and disadvantages in order to justify the adopted methodology. These issues are discussed below and include, interview type, response rate, open-ended items, response effect, tasks as catalysts for promoting discussion, interviewer effect and bias.

Interview type

One of the most discussed advantages of the interview as a research tool is flexibility in terms of participant observation and repeating questions, and probing for further information (Burns, 1990). It must also be recognised that such flexibility may also cause difficulties with coding responses. This disadvantage, however, can be minimised as the interviewer has control over the questioning sequence. The type of interview protocol chosen determines this control. This study utilised a combination of methods within the one interview. These were, a semi-structured interview (sometimes referred to as structure or dilemma interview), and the clinical interview. The semi-structured interview as defined by Sarantakos (1998, p. 251) involves an interview guide where the "order of questions are relatively firmly set, but there is freedom to add supplementary questions. In a typical case, the interviewer presents the interviewees with a story (or stories) containing a decision problem (dilemma) which they must solve; they must also justify the suggested solutions." Depending on the research focus, "both the solutions as well as the reasons offered by the interviewees are considered and evaluated." Clinical interviews as defined by Romberg (1992, p. 56) "begin in a manner similar to structured interviews, but the sequence of questions varies with each respondent, depending on prior answers."

An advantage of the semi-structured interview is that all participants are asked the same questions in the same order, thus eliminating fluctuations in the data that result from inconsistencies in when and how questions are asked. This process assists in the ease in which the responses can be summarised and analysed. The unstructured interview is described as involving general questions where the respondent is free to answer in their own words. While allowing for more complete answers, "they are often more difficult to code and analyse" (Bordons & Abbott, 1991, p. 198). Bordons and Abbott (1991, p. 198) stated that there are "advantages from each method by combining them in one interview. For example, begin with the structured format by asking prepared questions; later in the interview switch to an unstructured format" (p. 198).

Response rate

Once the participant has consented to participate in the interview and has presented for the interview, the response rate to the interview questions is high (Burns, 1990). One disadvantage is the limited number of participants, however, the qualitative data collected require fewer participants due to the richness of these data.

Open-ended items

Cohen and Manion (1994, p. 277) defined three types of interview questions, namely, fixed-alternative items, open-ended items, and scale items. This study utilised open-ended items, which are defined as "those that supply a frame of reference for respondents' answers, but put a minimum restraint on the answers and their expression" (p. 277). While the subject of the question is determined by the nature of the problem, restrictions are not placed upon the content or the manner of the interviewers' reply, thus "facilitating a richness and intensity of response" (Burns, 1990, p. 285).

Cohen and Manion (1994), and Burns (1990) described a number of advantages in regards to open-ended items, including flexibility. In this case such items enable the interviewer to probe for greater depth of responses; they provide a means for clarification of any misunderstandings; they enable the interviewer to seek optimal responses from the participants; they assist in encouraging co-operation and establishing rapport; and they allow the interviewer to make a truer and clearer assessment of what the participant believes.

The utilisation of open-ended items in this study required careful consideration of the appropriate use and timing of prompts and probes. Probes, which can be characterised as directed or undirected, were used consistently throughout the interview to elicit optimum responses and clarifications. Prompts, however, were utilised towards the end of each student's investigation of tasks. Responses were noted to be prompted or unprompted. Examples of probes and prompts include:

Directed Probe	Can you tell me more about the link that you have made between the isosceles triangle and the equilateral triangle?
Undirected Probe	Can you tell me some more about the links that you have made?
Prompt	You have not made a link between your parallelogram and the square. What would you think if I placed a link between these shapes?

In general, "probes are employed when a partial, irrelevant or inaccurate response is given, when the respondent has difficulty in answering a question or remains silent. In intensive interviewing, probing helps to encourage the respondent to talk and to direct the discussion towards the objectives of the study without causing bias or distortion" (Sarantakos, 1998, p. 263). Thus, probes are described as neutral statements which do not affect the respondent's direction of thinking, but assist in directing a more complete response, based upon the respondent's thinking. The funnel approach to questioning as described by Kerlinger (1986) assisted in the preparation of the interview protocol as it involves the notion of starting with a broad question and narrowing down the focus progressively to the important specific points. Thus the prompts and probes were organised in a manner where the set of questions was directed towards getting information on a single important topic or a single set of related topics.

Tasks as catalysts for promoting discussion

Traditionally, researchers present the participant with particular problems, and then analyse the participant's responses to the presented problem. Practical difficulties, that may arise from this process include:

- participants may have attempted the problems before
- participants may not understand the complexity of the question
- different participants can often give the same response, but a variety of thinking processes could underlie the same response.

As a consequence of the potential difficulties flagged by Collis (1992), the focus of the research instrument for this study concerned the reasons behind students' actions. They were given the opportunity to return to the four stimulus tasks on two or three occasions, within the same interview, to elicit a deeper response. Each task also maintained a focus for questioning, and a focal point, which remained individual to each participant. Often it appeared that for optimal responses, the students needed to 'warm up.' Hence the decision was made that the items would act as a catalyst for discussion concerning the relationships among figures and their properties. In addition, the interview proforma was designed in a manner that allowed each student to return to the same task on a couple of occasions as the student became more familiar with the ideas.

Interviewer effect and bias

It is important to note that the interviewer can influence the outcome of the study in both positive and negative ways. It is essential for the interviewer to consider and either eradicate, or at least control, sources of error and bias. These sources may include:

- quality of the interviewer
- misconduct
- presentation
- expectations
- probing
- interviewer effect

While the first five points do not require defining, interviewer effect concerns "variations in use of interview techniques, including tone of voice and the inconsistent use of probes" which may reduce standardisation (Burns, 1990, p. 303). Bordons and Abbott (1981) provided two suggestions for combatting such problems: extensive training in interview techniques to enable consistency in the manner in which questions are asked to each participant without accentuation of particular words, phrases, or questions; and the running of a pilot study where the interview procedure is trialed on a small sample. It is essential for the researcher to maintain an awareness of the effect of interviewer bias (Sarantakos, 1998) and to make all efforts to limit the amount of bias as much as possible. Reducing such bias has the effect of enhancing validity and reliability in terms of consistency and timing.

Response effect

The response effect is concerned with participants providing inaccurate or dishonest information. This effect may be minimised through the use of open-ended questions and non-reaction to certain responses consistently across each interview. While such interview techniques, as applied in this study, are relatively time-consuming, the literacy barrier is broken down, the motivation of the respondent is high, there is greater rapport due to face-to-face interaction, and, most importantly, the participants are not made to feel 'on the spot' (Burns, 1990).

In summary, "the interview is a form of questioning characterised by the fact that it employs verbal questioning as its principal technique of data collection" (Sarantakos, 1998, p. 246). Although interviews are considered a part of everyday life, the interview as a research instrument is different, in regard to preparation, construction and execution: "First, because it is prepared and executed in a scientific way, second, because it is controlled for the researcher to avoid bias and distortion, and thirdly, due to the fact that it is related to a specific research question and a specific purpose" (p. 246).

Rationale for Longitudinal Perspective

One of the most significant advantages of a longitudinal perspective, often referred to as developmental research (Cohen & Manion, 1994), is the positive contribution to cross-sectional findings. As "most researchers have a tendency to interpret cross-sectional differences as representing developmental changes or trends similar to those in longitudinal studies of development, such interpretation must be made with great care since there may be other factors affecting performance differences between groups" (Drew, Hardman, & Hart, 1985, p. 209). Hence, the disadvantages of the cross-sectional sample are reduced by the advantages of the longitudinal design, and vice versa.

The longitudinal design is described by Drew et al. (1985), as possessing an "attractive quality." It allows developmental changes to be seen clearly. The development of behaviour can be witnessed. This advantage may make the longitudinal design worth the rather large investment of time it takes to collect the data" (Drew et al., 1985, pp. 177–178). Thus, when used in conjunction with data providing a 'snap-shot' of a cross-sectional sample from which hierarchical patterns emerge, a longitudinal perspective can provide validation.

One commonly described disadvantage of longitudinal research is the "unscheduled sample attrition with the passage of time" (Goldstein, 1979, p. 3). The effects of sample mortality can be eliminated by selecting a cross-section of participants (rather than following a sample over time). Considering these factors, Goldstein (p. 17) stated "the most efficient sampling scheme will have a longitudinal element" where there is consideration of the "most efficient mixture of longitudinal and cross-sectional elements." Threats to internal reliability also require necessary consideration due to repeated interventions. In regard to this concern, Meltzoff (1998, p. 209) stated that "if the aim of the study is to examine time passage as an experimental variable, then maturation, in that sense, is the topic under study and is not a threat to internal reliability."

In the light of these issues, a longitudinal perspective is valuable to research concerning an investigation of cognitive growth. Characteristically, developmental research is "concerned both to describe what the present relationships are among variables in a given situation and to account for changes occurring in those relationships as a function of time" (Cohen & Manion, 1994, p. 67) and involves data collection of "the same group of people by taking successive measurements on some variable over an extended period of time" (Meltzoff, 1998, p. 284).

DATA ANALYSIS PLAN

> In educational research, theory serves a synthesising function, combining ideas and individual bits of empirical information into a set of constructs that provides for deeper understanding, broader meaning, and wider applicability.
> (Wiersma, 1991, p. 19)

The design of the study, as described earlier, involves the implementation of interviews as the research instrument incorporating a longitudinal perspective. As highlighted in the quote above by Wiersma (1991) the utilisation of the SOLO model to assist in the analysis of the data provides an effective framework for gaining deeper insights within the context of the van Hiele Theory. These data were analysed using both qualitative and quantitative methods. The following section details the methodologies applied when analysing the collected responses.

Qualitative Analysis

Coding has been defined by Cohen and Manion (1994, p. 286) as "the translation of question responses and respondent information to specific categories for the purpose of analysis." They described two methods as appropriate for analysing open-ended responses:

1. The interviewer may precode her interview schedule so that while an interviewee is responding freely, the interviewer is assigning the content of her responses, or parts of it, to predetermined coding categories. Classifications of this kind may be developed through pilot studies. (p. 286)

2. Data may be post-coded. Having recorded the interviewee's response, either by summarising it during or after the interview itself, or verbatim by tape-recorder, the researcher may subject it to content analysis. (p. 286)

The qualitative analysis of this study involved post-coding of responses in four stages:

Stage 1: Task analysis

Stage 2: Responses categorised according to 'likeness' elements based upon the task analysis

Stage 3: Groupings of responses viewed in terms of the SOLO model

Stage 4: Coder reliability check

The qualitative analysis components of the main study, namely, Study 1, Study 2, and the qualitative longitudinal component of Study 3, consisted of post-coding procedures. The process involved a description of the range of responses in terms of their groupings, and the utilisation of the SOLO model, as described in Chapter 2, to code the grouped students' responses into levels. To enable this grouping, it was necessary to undergo a task analysis, which entailed a consideration of all possible plausible elements of a response. The task analysis, which is described in detail in Chapters 4 and 5, and Appendices F and G, was based on information gained from previous research focused upon students' understandings of geometrical concepts as they relate to triangles and quadrilaterals (Burger & Shaughnessy, 1996; de Villiers, 1993; Mayberry, 1981; Pegg & Davey, 1991; Usiskin, 1982) and the New South Wales Mathematics Syllabus. The elements of the task analysis then provided a means for determining similarities between responses that enabled response groupings, which could be characterised by type. In the light of the complexity of response type, these groupings were placed into levels based upon the demonstrated degree of understanding of the relationships among figures and their properties. The hierarchical groups of responses could then be considered within the levels and modes provided by the SOLO model, to ascertain a more detailed characterisation of the identified developmental progression.

On completion of the identification of SOLO response categories for each section of Study 1 and Study 2, the researcher checked the consistency of this coding procedure. To establish interrater and intrarater reliability a procedure was undertaken which is detailed in Appendix D. The results of this qualitative analysis appear in Chapters 4 and 5.

The qualitative longitudinal component of Study 3, which was analysed in the same manner as Study 1 and Study 2, provided the opportunity to investigate students' responses to the task through further probing and clarification. The longitudinal perspective enabled further clarification of similarities and differences noted within the components of the interview concerning relationships among figures and properties within the contexts of triangles and quadrilaterals. Of particular importance is the manner in which second intervention responses shed light upon the developmental component of different categories of responses. The nature of the responses over the two-year period is discussed in Chapter 7.

Quantitative Analysis

A synthesis of the data concerning relationships among figures, and relationships among properties, was possible with the application of the Rasch Analysis. The quantitative results were able to support the developmental paths evident in the qualitative analysis. The analysis gives a degree of difficulty ranking for the items through the comparison of students' success rates, and produces fit statistics that facilitate the identification of poorly discriminating items. The term 'partial credit' represents a general type of data which "comes from an observation format which requires the prior identification of several ordered levels of performance on each item and thereby awards partial credit for partial success on items" (Masters, 1982, p. 150). This discrimination offers a more precise estimate of a student's ability rather than allocating a pass or fail score.

The data chosen for the quantitative analysis in the present study included the total 36 interviews undertaken across the seven tasks. This procedure enabled a comparison of item difficulty inclusive of Studies 1, 2 and 3. In addition, the analysis was chosen due to its potential to ascertain the appropriateness of the tasks in the light of the research themes. A detailed description of the quantitative analysis and the results that emerged are presented in Chapter 6.

EVALUATION OF THE DESIGN

The success of the interview as a research instrument "depends to a large extent on the ability of the instrument, the interviewers, or both to solicit unbiased and valid responses from the respondents" (Lin, 1976, p. 220). "While reliability is concerned with the replicability of scientific findings, validity is concerned with the accuracy of scientific findings" (LeCompte & Goetz, 1982, p. 31). It is essential to consider the perceived strengths and weaknesses of the research design and analysis plan in terms of validity and reliability from an internal and external perspective, as discussed below. In addition, issues concerning ethics are considered in the light of the methodology used in this study.

Validity in Research Design and Analysis

Essentially, "validity involves two concepts simultaneously: the extent to which the results can be accurately interpreted and the extent to which the results can be generalised to populations and conditions. The former concept is called internal validity, and the latter is called external validity" (Wiersma, 1991, p. 4). The evidence of validity is often too difficult to "demonstrate because it refers to the degree in which the evidence being gathered is directly related to the phenomenon of interest" (Romberg, 1992, p. 58).

Validity can be interpreted as whether the questions asked look as if they are measuring what they claim to measure. The cause of invalidity is bias which is defined as a systematic or persistent tendency to make errors in the same direction, that is, to overstate or understate the true value of an attribute" (Cohen & Manion, 1994, p. 281).

The sources of bias are often directly related to the characteristics of the interviewer, respondents, and the content of the questions. Cohen and Manion (1994) described these as:

> the attitudes and opinions of the interviewer; a tendency for the interviewer to see the respondent in her own image; a tendency for the interviewer to seek answers that support her preconceived notions; misperceptions on the part of the interviewer of what the respondent is saying; and misunderstandings on the part of the respondent of what is being asked.
>
> (Cohen & Manion, 1994, p. 282)

Cohen and Manion (1994) summed up various writers' suggestions for reducing bias as:

> careful formulation of questions so that the meaning is crystal clear; thorough training procedures so that an interviewer is more aware of the possible problems; probability sampling of the respondents; and sometimes by matching the interviewer characteristics with those of the sample being interviewed.
>
> (Cohen & Manion, 1994, p. 282)

Internal validity

"Internal validity refers to the extent to which scientific observations and measurement are authentic representations of some reality" (LeCompte & Goetz, 1982, p. 32). Various factors affect the internal validity of a study. The first of these factors, ***history and maturation***, refers to the element of change likely to occur throughout the study. Hence, this concerns the comparison of individuals who have participated at different times throughout the study. 'History' refers to "changes that occur in the overall social scene," while 'maturation' refers to "changes that involve progressive development in individuals." To reduce the effects of history in the present study, the interviews were undertaken over a one-week period for each of the schools involved. In addition, students from the same cohort in a particular school were interviewed during the same day. Maturation, however, is a significant factor in this study and was an important component of Study 3, which added a longitudinal perspective to investigate students' understandings of class inclusion concepts in Geometry. This longitudinal perspective was achieved by a second intervention of a selection of student's two years after the initial intervention.

Observer Effects concerns the possible threat to validity imposed by an observer upon the natural setting. In the present study, while the interviewer was not familiar to the participants, a familiar setting was maintained. The interviews were scheduled at times suitable to the participants and within their own school environment. The nature of the interview tasks also took into consideration observer effects as the participant had the opportunity to return to their responses and add to and/or complement their own responses as necessary.

Mortality refers to the possible difficulties associated with changes to the research sample as a result of losses or gains of participants. While the initial intervention was not affected by mortality, this was a possible threat to the second intervention, which required base line data as collected in the first intervention. Due to the teacher selection of suitable students, mortality was not an issue in regard to Study 1 and Study 2. While mortality was a possible threat to the second round of interviews to be carried out upon half the sample, all students were able and willing to participate in the follow-up interviews.

Spurious conclusions concerns the potential threat of "observing relationships where there may be none or assuming nonrelationships where they may be obscured by an artefact of instrumentation or treatment" (LeCompte & Goetz, 1982, p. 49). To address this issue, and hence ensure that conclusions drawn from the data were derived from accurate interpretations, two different contexts (triangles and quadrilaterals) were utilised. The responses collected via interview procedures included justifications and the opportunity to return to the same task on several occasions. The responses were initially coded characteristically in the light of the task analysis. The categories of responses were then coded using the SOLO model, which assisted the interpretation of the hierarchical nature of the response groupings. In addition, the longitudinal perspective was utilised to validate the developmental pathway evident in students' understandings of class inclusion concepts in Geometry. A quantitative analysis, using the Rasch partial credit modelling process, further validated the developmental growth that emerged.

While it is necessary to consider possible threats to the valid interpretation of results, possible threats to the generalisability must also be considered. The following discussion outlines the procedures adopted in this study to minimise the potential threat to external validity.

External validity

The following issues address threats to the degree to which the representations drawn may be compared legitimately across groups (LeCompte & Goetz, 1982). In consideration of threats to external validity, three factors as described by LeCompte and Goetz (1982) are discussed below. These include setting effects, history effects, and construct effects.

To reduce the potential threat to generalisability of the results of this study, a number of features were adopted in the design, described as *setting effects*. Firstly, the sample of students each matched a criterion based upon mathematical ability. Each student was enrolled in a New South Wales Secondary School and was taught within the structure of the same Mathematics Syllabus. The cross-sectional sample was also derived from two different schools with gender balance. Hence, a similar sample of students could be easily identified in other contexts. Secondly, due to the application of the SOLO model as described in Chapter 2, an analytical tool was provided which allowed the researcher to focus upon structure of the response in addition to the content described in the task analysis.

As indicated above, *history effects*, in regards to the background of the participants of the sample, must be known and acknowledged. The research sample for this study was chosen from those students considered to be in the top 30% of their cohort in the light of previous geometrical experience. Past research (Pegg & Davey, 1991) concerning the likelihood of student understanding of class inclusion notions in Geometry has indicated that few students reach Level 3 (van Hiele) thinking while in Year 8 of secondary school. Capable students therefore have greater potential to reach this level. As a consequence, the decision was made to use an equivalent cohort within Years 8, 9, 10, and 11. This cross section will also aid in the identification of developmental growth patterns that may emerge with a longitudinal perspective.

Construct effects refer to "the extent to which abstract terms, generalisations, or meanings are shared across times, settings, and populations" (LeCompte & Goetz, 1982,

p. 53). An interesting feature of this study was the flexibility of the student tasks in maintaining suitability for each participant. The study included semi-structured interview tasks and interview formats, which allowed each participant the opportunity to work with a familiar, yet non-routine setting. The interview structure was assessed by University lecturers and Mathematics teachers in secondary schools, and piloted on students of varying ages. The contexts chosen were familiar to each participant as they are described as content within the New South Wales Mathematics Syllabus. The second intervention interviews also provided a means for validating the findings associated with the initial intervention.

As discussed above, through the consideration of potential threats to validity, a number of procedures have been adopted in the present study to balance possible effects on internal validity and external validity.

Reliability in Research Design and Analysis

Reliability of the study concerns the accuracy of the findings "in the light of the way in which it was gathered" (Romberg, 1992, p. 58). Synonyms for reliability that were described by Wiersma (1991, p. 7) include "dependability, stability, consistency, predictability, and accuracy. Reliability refers to the consistency of the research and the extent to which studies can be replicated".

Internal reliability

There are a number of procedures, as described by Wiersma (1991) and LeCompte and Goetz (1982), in regards to internal reliability. Each of these refers to "the extent that data collection, analysis, and interpretation are consistent given the same conditions" (Wiersma, 1991, p. 7). A lack of internal reliability results in the data becoming a "function of who collects them rather than what actually happened" (p. 7). Essentially, the research instrument should always be administered under standard, well controlled, and similar conditions. The following discussion refers to "procedures that maximise the degree to which other researchers, given a set of previously generated constructs, would match them with data in the same way as the original researcher" (LeCompte & Goetz, 1982, p. 32). These constraints include low-inference descriptors, multiple researchers and peer examination, and mechanically recorded data.

Low inference descriptors refer to the precise and descriptive accounts of findings, which allow for the accurate presentation of evidence. This presentation should provide the reader with means to reject or accept the findings based upon the richness of the material presented. The range of data presented in this study includes students' verbal responses to interviews in transcript form, summaries of student diagrams, and a written record of student actions during the interview concerning property card selection. Studies which provide the reader with multiple examples of the data collected are considered to be most credible.

While *multiple researchers* were not present during the interviews, the design included the involvement of an experienced researcher in this field during the coding procedures. In addition, intrarater reliability was evident in the present study where a single researcher consistently coded the data on different occasions. Interrater reliability was also a feature of the design where the initial researcher and a second researcher congruently coded the responses. This is procedure is described in more detail in Appendix D.

One method of improving internal reliability concerns the utilisation of *mechanical recording equipment* such as audio and visual recorders. While such items may be perceived as a threat to validity, the present study required tape recording of each interview to maintain internal reliability. This was essential for the concise depiction of student responses both at the coding level, and for future readers. Mechanical recorders also facilitated an environment where the researcher could focus upon the interview schedule and participant, without having to make written records of verbal responses during the intervention.

A number of considerations are mentioned above which address the need to implement strategies to maximise consistency in relation to the interpretation of data. To follow are the necessary issues concerning replicability of the study which require careful consideration.

External reliability

This refers to "the issue of whether or not independent researchers can replicate studies in the same or similar settings" (Wiersma, 1991, p. 7). Thus the conditions and procedures of the study must be adequately described to enable independent researchers to "discover the same phenomena or generate the same constructs in the same or similar settings" (LeCompte & Goetz, 1982, p. 32). Factors affecting external reliability have been discussed above. They include the role played by the interviewer, the need to characterise participants, the context of the data collection, the need to discuss the structure behind the analysis procedures, the inclusion of intrarater reliability, and clear descriptions of methodologies utilised.

Kitwood (1977, cited in Cohen & Manion, 1994, p. 282) drew attention to the conflict concerning traditional views of validity and reliability and stated that "where increased reliability of the interview is brought about by greater control of its elements, this is achieved … at the cost of reduced validity." The distinctively human element of the interview is necessary to its 'validity.' The more the interviewer becomes rational, calculating, and detached, the less likely the interview is to be perceived as a friendly transaction, and the more calculated the response also is likely to be.

Ethical Considerations

The design of the study, as discussed above, required careful consideration of the balance between validity and reliability. While methodologies are employed which strive for the equilibrium between these issues, ethical considerations are also of importance. The centrality of maintaining a positive ethical stance is highlighted in the following quote:

> Research and the pursuit of truth are vital functions in higher education institutions. Central to the long established principles that guide research are the maintenance of high ethical standards, and validity and accuracy in the reporting of data.
>
> (University of New England, accessed 31st Aug 1999)

An application based on the consideration of issues such as informed consent, access and acceptance, and privacy, was approved by the University of New England's Ethics Committee which strictly adheres to the National Health and Medical Research Council (NHMRC) *Statement of Human Experimentation and Supplementary Notes*. Ethical issues relating to the study are discussed below.

Informed consent

Diener and Crandall (1978, cited in Cohen & Manion, 1994, p. 350) defined informed consent as "the procedures in which individuals choose whether to participate in an investigation after being informed of facts that would be likely to influence their decisions." Ethical treatment of participants mandates that "they be informed of the nature, purpose, and requirements of your study, and be given the opportunity to decline participation" (Bordons & Abbott, 1991, p. 122). Before commencing research, the free consent of the participants was obtained. This involved the circulation of a Plain Language Statement / Consent Form (Appendix E) which provided the guardian and participant with sufficient information about the purpose, methods, demands, risks, inconveniences and discomforts of the study at his or her level of comprehension. The participants were also informed in writing of the voluntary nature of their participation. At the beginning of each interview session, the participants were reminded that they were free at any time to withdraw consent to further participation and that they would not be disadvantaged if they chose to do so. At any time during the interview the participant had the opportunity to ask questions. Time was allocated at the completion of each interview to enable debriefing.

Access and acceptance

This consideration involves the access to, and the acceptance of, the institution or organisation in which the research instrument is administered. Access and acceptance needs to be addressed before consent letters are distributed to the participants. Accordingly, an official letter briefly outlining the structure and aims of the study, with particular emphasis on the level of involvement required at the school level, was given in person to the Principal of each of the schools involved. The Principal was given time to read the relevant information and ask any questions in regards to queries or extra information required. Both Principals then took the information to the next staff meeting where the decision was made in regards to the school's participation in the study. Finally, a meeting then took place with the Mathematics Head Teacher at each of the schools to discuss selection of the sample, suitable interviewing areas in line with ethical protocols, and appropriate interview times for each party involved.

Privacy

Privacy in the research context has been considered by Diener and Crandall (1978, cited in Cohen & Manion, 1994, p. 365) as a consideration of three perspectives, namely, "sensitivity of the information being given, the setting being observed, and dissemination of information." A consideration of each of these perspectives in the light of this study yields the following results. The content of the interview was not potentially threatening or of a personal nature. Interview techniques were employed which created a positive environment with no room for student failure, because of open-ended tasks and questions. To enable dissemination of the findings while maintaining privacy for both participants and schools, pseudonyms were used to protect anonymity, and confidentiality was maintained.

In summary, "with respect to data subjects, researchers should be conscious of their intrusive potential, and should seek to minimise any intrusion; the confidentiality of data must be respected and protected by positive measures; and data subjects should be told the purposes of the research and should have adequate opportunity to withhold their cooperation" (Burgess, 1989, p. 13). In the light of this, ethical issues of importance to this study include, informed consent, access and acceptance, and privacy. The nature of these ethical issues require consideration before, during, and after the research study.

CONCLUSION

This chapter provided an outline of the research methodology devised to investigate students' understandings of class inclusion concepts in Geometry. This outline was presented in five sections that considered a range of factors that affected the chosen research design. The first section described the contextual setting of the research and detailed the geographical and mathematical background of the students within the sample. This section was followed by an overview of the design, which included the process taken and implementation procedures. The third section comprised a discussion of the theoretical issues pertinent to the design and interview tasks in relation to the five research themes. The fourth section outlined the methodologies associated with the analysis of data from both qualitative and quantitative aspects. The final section discussed the validity and reliability of the design, inclusive of ethical considerations.

The following chapters present the data analysis resulting from the consideration and application of the methodological aspects outlined in this chapter. They begin with the presentation of results concerning students' understandings of the relationships among triangle figures, and among triangle properties. In the light of these findings, students' understandings of the relationships among figures, and relationships among properties are explored in Chapter 5 within the context of quadrilaterals. Chapter 6 comprises a quantitative synthesis concerning students' understandings of the relationships among figures and their properties using a partial credit model. The data analysis involves a longitudinal perspective in the form of four case studies, which are presented in Chapter 7.

CHAPTER FOUR

TRIANGLE RESULTS

The purpose of this chapter is to present the results of research directed at triangles. In particular, the focus of the results is on Research Theme 1:

To investigate students' understandings of class inclusion concepts concerned with different types of triangles.

1.1 What are the characteristics of students' understandings demonstrated in a classification task of seven different triangles?

1.2 Is there evidence of some developmental pattern in the different responses to a classification task of seven different triangles?

1.3 Does the SOLO model offer a framework to explain the identified categories of responses concerning students' understandings of relationships among different triangles?

1.4 Can students' demonstrated understandings of relationships between triangle properties be categorised into identifiable groups according to similar characteristics?

1.5 Is there evidence of some developmental pattern in the different responses to a task requiring the utilisation of relationships among triangle properties?

1.6 Does the SOLO model offer a framework to explain the identified categories of responses concerning students' understandings of relationships among triangle properties?

This chapter is divided into three main sections, namely, Relationships Among Triangles, Relationships Among Triangle Properties, and Conclusion.

RELATIONSHIPS AMONG TRIANGLES

The following discussion, directed at relationships among triangles, is concerned with students' understandings of links that exist between different triangle types demonstrated through a classification activity. The most sophisticated level of understanding is the application of class inclusion notions. The discussion is divided into five sub-sections. These are: Background, which includes the background information and the analysis procedures; Results, involving detailed descriptions of coded responses; Summary, which ties together and compares the groups of responses; Theoretical Perspective, which considers the results in the light of the SOLO model; and, Conclusion, which addresses the findings in relation to the research questions.

Background

The methodology of this section of the research was discussed previously in Chapter 3. The interview began with students' recalling and drawing all possible triangle types after being directed to write down and draw an acute-angled scalene triangle. This diagram was then used as a catalyst for discussion concerning the reasons for the existence of links among the seven triangles. Opportunities were provided for students to supplement the list where necessary. This allowed for further discussion if triangles were missing from the original list, and if necessary, a second and even a third tree diagram was designed by the student. The purpose of this section of the study was to have students identify and justify relationships among seven different triangle types, namely, acute scalene, obtuse scalene, right scalene, acute isosceles, obtuse isosceles, right isosceles, and equilateral. The format of the interview is contained in Table 4.1 and includes student tasks and the questioning focus common to each interview. This provided a set structure with room for individual dialogues between the interviewer and student where necessary.

Table 4.1 Interview format for the task concerning relationships among triangles

Triangle Relationships	
(i) Int:	I would like you to write a list of all the triangle names you can think of. Begin with acute-angled scalene. Draw each triangle.
(ii) Int:	Design a tree diagram which links the different triangles. Draw a sketch to link each type. (discussion follows concerning the reasons for links and/or lack of links) (the following three points are addressed if required)
(iii) Int:	There are some triangles that we can add to this list. (provide triangles not recalled) Draw a sketch of each new triangle.
(iv) Int:	Design a second tree diagram incorporating all the triangles on the list. (discussion follows concerning the reasons for links and/or lack of links)
(v) Int:	Return now to your first map. I would like you to add the new triangles to your original tree. (discussion follows concerning the reasons for links and/or lack of links)

This format was chosen as it enabled the students to work with familiar recalled information, supplemented information, and individual tree designs. The continual revisiting back to the same relationships on different tree diagram designs provided a vehicle for extracting further information and expanding on information already provided. Prior to the interviews taking place, a task analysis was carried out which included all expected elements of a response. The task analysis of the activity concerning students' understanding of the relationships among triangles is contained in Appendix F. This task analysis assisted in the identification of characteristically similar groups of responses.

Results

The responses were grouped according to the identification of the relationships formed among the seven triangle types, and the reasons for the existence of these relationships. There were nine types of responses identified and a description of each follow. The descriptions include a diagrammatical summary developed from the student maps and transcript excerpts. This presentation provides a visual summary of the information presented and is not meant to mirror the true diagrams drawn by the students. The reasons for the links that appear on each diagram are consistent with the language used by the students within that group of responses.

The nine groups are coded A to I, and are described below in an order which represents an increased sophistication. In some areas, categories within types were found. This occurs when variations were identified while major characterisations of the group of responses remained similar.

Type A

This type of response indicates the use of a single similar feature or property to relate triangles together. The groupings of triangles exist according to whether they contain acute angles, unequal sides, or equal sides. Only one feature or property is used in each grouping. The groupings change according to the property or feature that is the focus, and a student may create several different groupings. There was only one response coded as Type A.

Narelle's response is summarised in Figure 4.1 below.

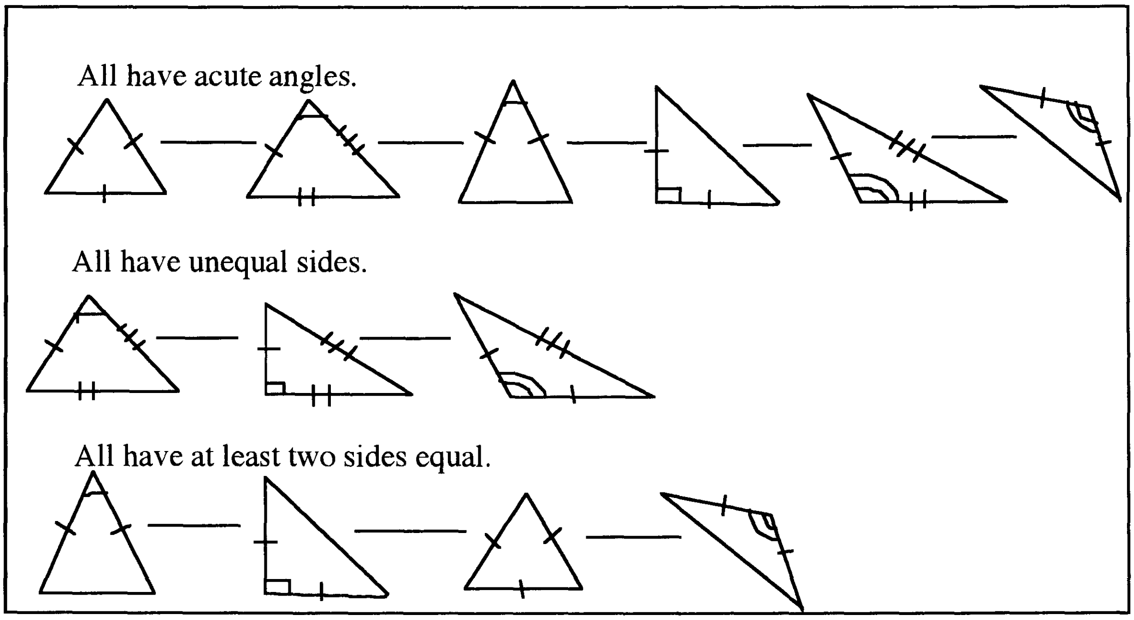


Figure 4.1 Narelle's triangle groupings summary

The dialogue below centres upon Narelle's justifications for her groupings. In Narelle's case the groupings were based on acute angles, unequal sides, and having at least two sides equal. The transcripts illustrate how the groupings formed, and how they changed.

Int: *What is that link there for?*
Narelle: *Because it has got acute angles.*
Int: *Anything else?*
Narelle: *Um no.* (pause) *There I have put all the unequal sides together.*
Int: *No more links there?*
Narelle: *No ...*
Int: *What have you done up on this top row?*
Narelle: *They all have three sides and they all have at least one angle that is an acute angle.*
Int: *What have you done on the next row?*
Narelle: *Um they all have uneven sides.*
Int: *And here?*
Narelle: *They all have at least two sides the same.*

Narelle provided links across the groups only when she was prompted to do so.

Int: *Could I link across these lines here? I mean could I link the right-angled isosceles with the right-angled scalene?*
Narelle: *Yes.*
Int: *How come?*
Narelle: *Because they have both got right angles.*
Int: *Any other reason that you could make those links?*
Narelle: *Um these are acute these two here* (points to acute angles on the right isosceles and right scalene) *and that is all.*

Overall, a Type A response forms groups of triangles based on a single similar feature. The groupings change as frequently as the identifying characteristic changes without the formation of dominant groupings. Different types of triangles can belong to a number of groups.

Type B

In these responses the seven triangles are formed into three distinct classes based on a single similar property. Examples of a similar property identified within the class include side, angle, or symmetrical properties. There are four responses coded within this category. The scalene, equilateral, and isosceles classes of triangles not only exist due to an identifiable property, but the property also excludes links among classes because of the differences identified.

Category 1

These responses included the formation of mutually exclusive classes of triangles based on a similar property. Each of these classes represented a unit, with a specific name to encapsulate the group. The three classes of triangles identified were scalene, isosceles and equilateral. Category 1 contains two responses (Peter and Arthur) and comprises three triangle-type classes only, namely, scalene, isosceles and equilateral. This is summarised diagrammatically in Figure 4.2.

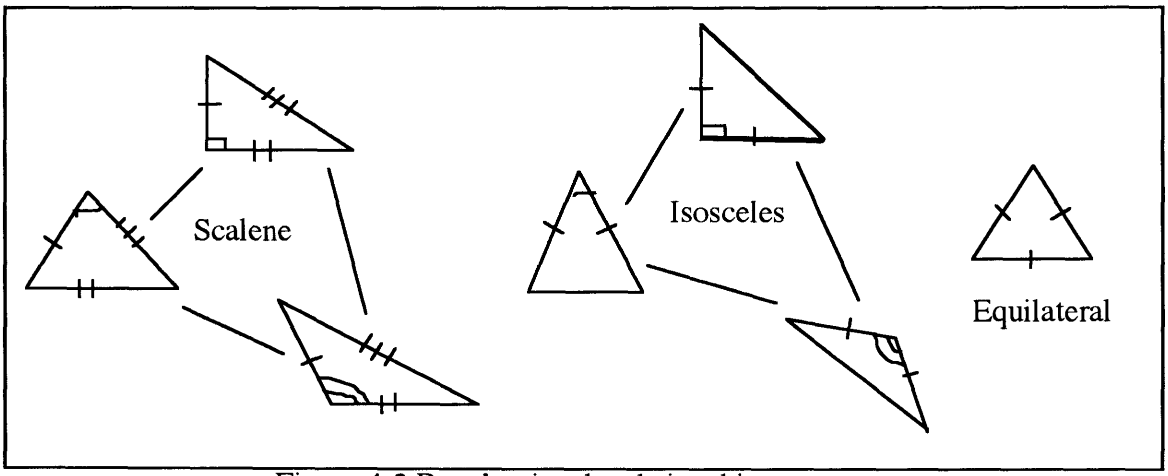


Figure 4.2 Peter's triangle relationships summary

When discussing the relationships that exist among the triangles, both respondents utilised the meaning carried by the name of each class. Peter used the names of each class consistently after describing the similarity in terms of equality of sides with the equilateral triangle having three equal sides.

Int: *What about your link between here and here?* (obtuse isosceles and acute isosceles)
Peter: *Because they are both isosceles.*

Arthur described his classes of triangles as a progression moving from three sides equal, to two sides equal, and, finally, no sides equal.

Arthur: *Because it is the next one down because it has three and that has two and that has none so they are just in order.*

The structure of the group of triangles is not inclusive of subsets as the differences identified over-ride the similarity noted, e.g., "the equilateral triangle and the acute isosceles triangle are acute, but they can't link because that has all sides equal and that has two sides equal." None of the responses in this category made a link between the class of isosceles and the equilateral triangle. When questioned as to the appropriateness of such a link, the equilateral triangle was described as containing three equal sides thus being unlike all other triangles.

Peter: *The equilateral is really on its own but an equilateral is also acute.*
Int: *So could I put a line between the equilateral and the isosceles?*
Peter: *Um no. They are a completely different shape.*

No angle-type links were made across classes in this category, as the classes remained separate identities based on a single similar property (side length). When adding the right-angled scalene triangle into the tree diagram already containing a right isosceles triangle and two other scalene triangles, only Arthur connected it to the scalene class of triangles.

Arthur: *I would simply add these ones in here. The isosceles with the isosceles and the right-angled scalene with the scalenes. These are all different here.*

Category 2

These responses are similar to a Type B Category 1 response with the addition of a right-angle link. This category contains two responses (Ellen and Tracy) and they are summarised in Figure 4.3.

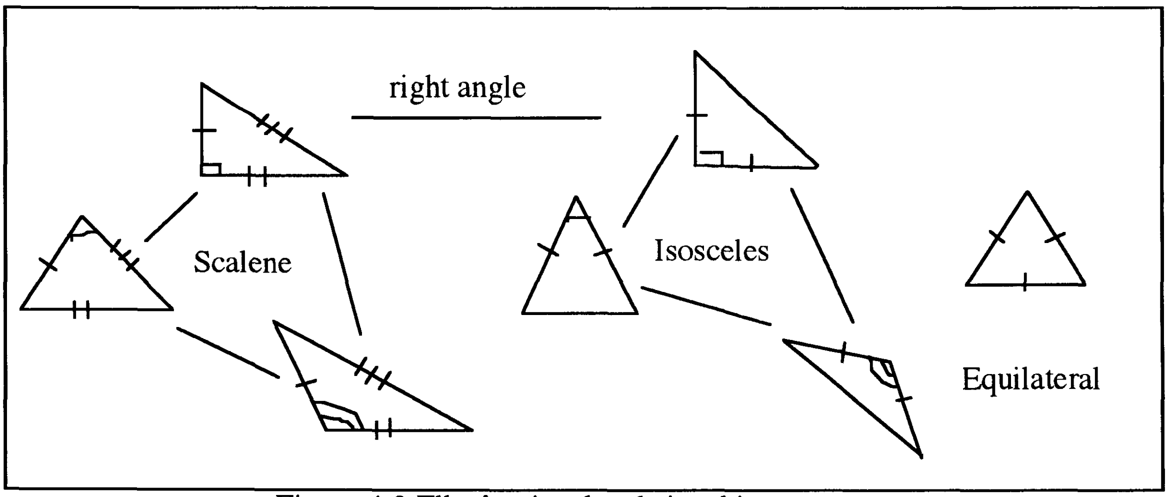


Figure 4.3 Ellen's triangle relationships summary

Ellen's discussion described the relationships clearly in terms of classes based on a single similar property. The equilateral triangle has three sides equal, the isosceles triangles have two sides equal, and the third class was described by name only as scalene triangles.

Ellen: *I have put the isosceles together.*
Int: *Why can they be linked?*
Ellen: *Because they have both got two sides the same.*
Ellen: *I have put these two scalenes together.*
Int: *Does it matter that one is acute and one is obtuse?*
Ellen: *Well they are still both scalene triangles.*

When prompted to make a link between the equilateral triangle and the isosceles triangles Ellen stated:

Ellen: *That one* (equilateral) *there won't link with any of them. It can only link because it has got three sides.*

Thus the differences identified did not allow a link to be made. When asked again in regards to her second tree diagram she replied:

Ellen: *It has got all equal sides and it is the only one.*

Tracy's reasons for not linking the equilateral to any other triangle also reinforced the dominating element of the differences noted, and the formation of classes based on a single similar property.

Int: *Can the equilateral triangle link anywhere?*
Tracy: *No not really because those ones have only got two sides that are equal and that has three and doesn't have a right angle.*
Int: *Can you explain the diagram?*
Tracy: *Well those three are all connected to each other because they are all isosceles and that one is connected to the scalene-right angled one because they are both right angles and all those three are scalene triangles.*
Int: *What about the equilateral?*
Tracy: *I can't see how it can fit in with the other ones.*

Ellen's response is also characterised by links across classes based on right angles. When prompted to make more links, Ellen declined to make further links on the basis of angle-types across established triangle-type classes.

Ellen: *That is the right-angled scalene and that is the right-angled isosceles there.*
Int: *No other links that you could add?*
Ellen: *No.*

Links across classes based on right-angle links are illustrated in both Tracy's and Ellen's responses. Both students described confidently the link between the right isosceles triangle and the right scalene triangle. While these students incorporated angle-type links as well as a single property link, they were unable to utilise this additional link to construct a relationship between the equilateral triangle and the isosceles class of triangles.

In summary, the Type B responses formed three distinct classes of triangles, which were justified on the basis of a single property. Each class was recognised by a name that encapsulated the known single property of the group of triangles. Links across classes were not identified as any differences noted prevented relationships forming across groups. In the better Category 2 responses angle-type links were identified, such as right-angle links, however, these could not be utilised to form significant links across the equilateral and isosceles classes of triangles.

Type C

These responses are similar to a Type B response in that the responses clearly identify three individual classes of triangles. The difference lies in the justifications as they are based upon two or more properties. In addition, these responses include three angle-type links, being right-angled, obtuse-angled, and acute-angled. The links across classes were made according to angle-types with the exception of the equilateral triangle. A link based on acute angles was not made from the isosceles triangles to the equilateral triangle as the equilateral triangle was described specifically as having three angles equal, and, therefore, it was not seen to be possible to link the classes for any reason.

This category contains four responses (Megan, Alice, Brendan, and Andrew) and the relationships identified are summarised in Figure 4.4.

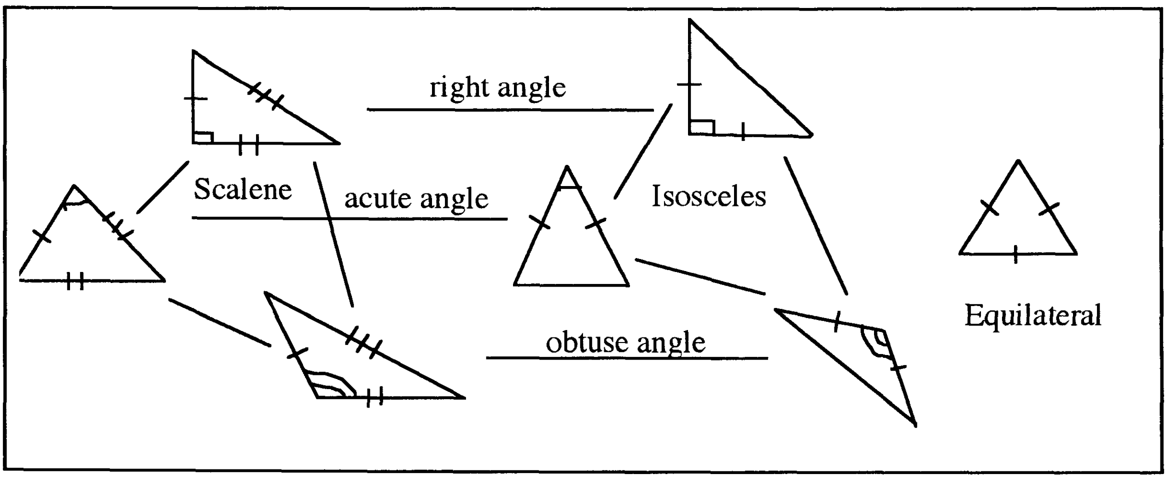


Figure 4.4 Megan's triangle relationships summary

Megan's justifications for classes of shapes were based on the following similar properties:

(i) the equilateral has three sides and/or three angles equal;

(ii) the isosceles class of triangles has two sides equal, two angles equal, and/or one line of symmetry; and,

(iii) the scalene triangles have no sides equal and/or no angles equal.

The reasons for these relationships are clear in Megan's discussion of the classes. For example:

Int: *So can you tell me some more about these links?*
Megan: *These link together because they are all isosceles triangles. They all have two equal angles and equal sides. These ones here link together because they are all scalene and they have no equal angles and no equal sides.*

Megan confidently made connections across classes due to angle-type links but refused to make the link between the isosceles triangles and the equilateral triangle. The following interview excerpt illustrates this.

Megan: *The right-angled scalene is also connected to the right-angled isosceles as they are both the same.*
Int: *What is the same?*
Megan: *Well they are both right-angled. This is connected to the acute-angled isosceles and this is connected to the obtuse-angled. The equilateral triangle is on its own.*
Int: *Do you think that the isosceles and the equilateral belong together?*
Megan: *Oh no not really.*
Int: *Why don't they?*
Megan: *Because that one there can be different. The only thing that they have in common is that they have some equal sides and equal angles. An isosceles only has two equal sides and two equal angles where as an equilateral has three equal sides and three equal angles. With these scalene triangles each side can be all different lengths and I think that sometimes they can be right-angled and the same as the isosceles.*
Megan: *Um this one is a right-angled scalene um I don't want that link between the equilateral and the isosceles. Those isosceles are linked because they both have a line of symmetry.*

In summary, Type C responses all contained three separate triangle-type classes, namely, scalene triangles, isosceles triangles, and the equilateral triangle. The classes are mutually exclusive and were formed based on reference to two or more similar properties. The name of each class carries meaning, and a triangle must contain all the characteristics of the class, and no others, to belong to that class. For example typical Type C reasoning for an equilateral triangle not being related to an isosceles triangle is that two sides and two angles must be equal, not three sides equal and three angles equal. None of the responses in Type C made a link to the equilateral triangle for any reason beyond the fact that all triangles contain three sides and three angles.

Type D

These responses are similar to a Type C description with the addition of a tentative link made between the equilateral triangle and the isosceles class of triangles. The similarities are noted, but the differences identified do not allow the link to be made. Three responses (Suzanne, Scott, and Louise) were coded within this category and these are summarised in Figure 4.5. The dotted line represents the tentative link between the isosceles class of triangles and the equilateral triangle.

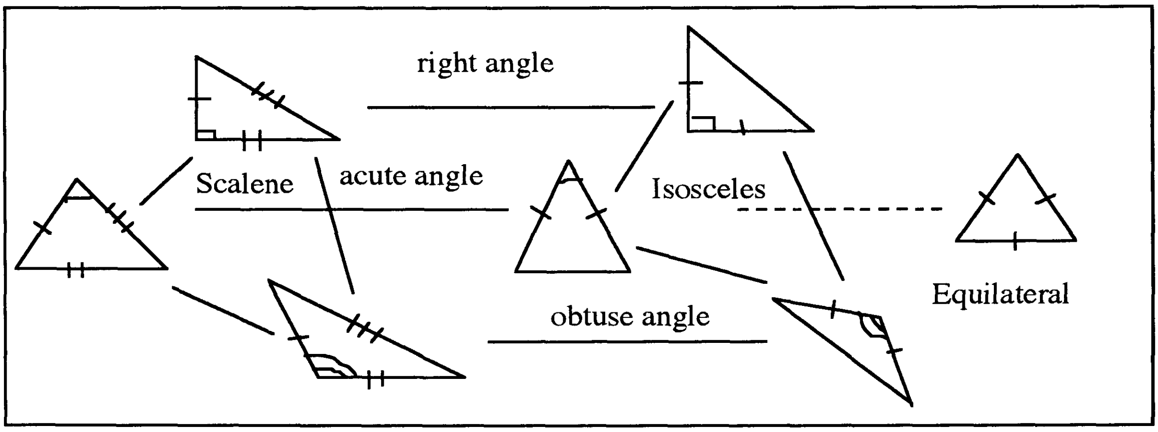


Figure 4.5 Suzanne's triangle relationships summary

The discussion below concerns the appropriateness of constructing a link between the equilateral triangle and the isosceles class of triangles. The response indicated the awareness that the equilateral triangle also has two sides equal and/or two angles equal, however, a link is not made based on the equilateral triangle containing three sides equal and/or three angles equal. Suzanne's discussion concerning the tree diagram described the formation of three triangle-type classes based on similar properties, and the relationships based on angle-type links. The difference in these responses was made evident by the indecisive comments concerning a possible link between the equilateral triangle and the isosceles triangles.

Suzanne: *So I can link up the scalenes because they have no sides that are equal. Um the equilateral um it can be linked to there, it is not really linked it is sort of tacked on the end.*
Int: *Do you think that link there exists?*
Suzanne: *Well I can't see how it can link. I should cross that one out.*
Int: *So why are these others here linked?*
Suzanne: *They all have two sides that are equal.*
Int: *Anything else?*
Suzanne: *These bottom angles are equal.*
Int: *What about these?*
Suzanne: *Um they are all scalene and none of the angles are equal.*
Int: *Could you make any other links here?*
Suzanne: *Um yeah these two are right-angled triangles and these two have acute and these two have obtuse.*

When questioned again about the appropriateness of a link between the equilateral triangle and the isosceles class of triangles, Suzanne described both a similarity and the differences between the classes, and ended with an inability to make a decision concerning the relationship.

Int: *Now you still have the equilateral linked on there. What do you think about that?*
Suzanne: *Um since these are both 45 degrees there and no they are all 60 degrees on that one um these turn out to be equal um I don't know maybe two angles are equal on both of them.*

Scott's comments concerning the link between the equilateral triangle and the isosceles class of triangles noted the similarities also, but the fact that all the angles are the same on the equilateral triangle did not allow the link to be made. As a result, Scott decided not to make the link between the classes.

Scott: *This one has got three sides the same and this one has got two.*
Int: *Well how can they relate together?*
Scott: *Because they have both got two sides the same and one of them has to have the other and it can have two angles the same. Oh no they can't that can have 60 and all the same.*
Int: *So do you think that they can be linked together for angle reasons?*
Scott: *No they can't.*

Overall, the responses in this group contained the three triangle-type classes and angle-type relationships across classes with the exception of the equilateral triangle. The distinguishing feature of the three responses is that the similarities between the equilateral triangle and the isosceles triangles are noted, but the differences identified precluded the link from being made.

Type E

This group of responses describe a definite relationship between the equilateral and isosceles triangle classes based on similar properties. Eight responses were coded as Type E and could further be divided into four categories. Most significantly, these responses include the three triangle-type classes and a description of a connection between the isosceles class of triangles and the equilateral triangle.

Category 1

Each of the responses in this category included three classes of triangles (equilateral, isosceles, and scalene) that are based on similar properties with the addition of a link between the isosceles triangles and the equilateral triangle. The relationship is based on both classes containing two equal sides and/or two equal angles. The equilateral triangle and isosceles class of triangles are related on the grounds of similar properties, but the equilateral triangle is not yet described as a subset of the isosceles class of triangles. There were three responses (Cameron, Jason, and Jenny) in this category, and, as illustrated in Figure 4.6, no angle-type links were made across the three triangle classes.

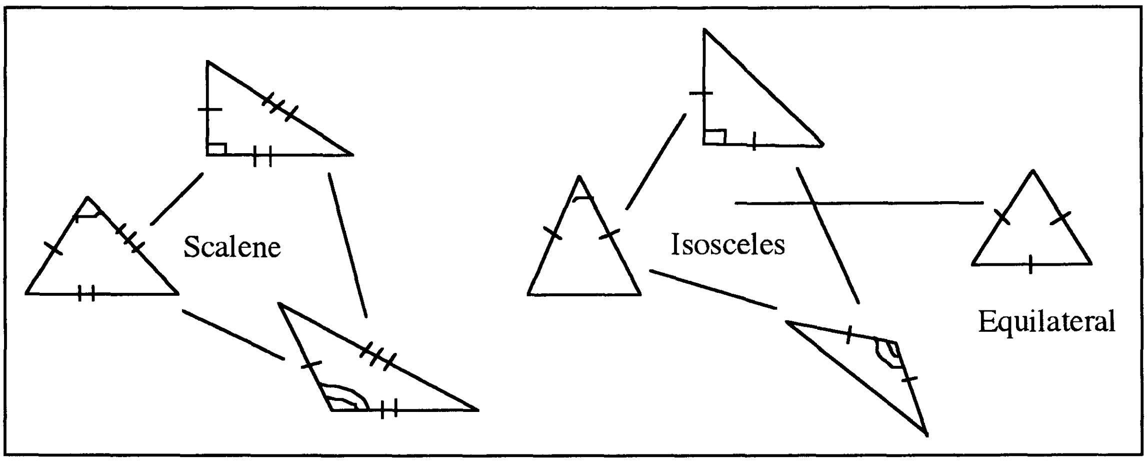


Figure 4.6 Cameron's triangle relationships summary

When a probe is directed at the link made between the equilateral triangle and the isosceles triangles, Cameron stated that the connection is based on the existence of similar properties, namely, two sides equal and two angles equal. Cameron also defined the differences between the classes, but unlike a Type C or Type D response, this did not dominate the similarities between the classes. The relationship was defined clearly and there was no hesitation when making the link between the classes:

Int: *Can you tell me about the link you have made between the equilateral and the isosceles?*

Cameron: *The equilaterals link because they have both got two sides that are equal and two angles that are equal but the isosceles have one side and angle that is not going to be the same as the rest.*

Int: *And this one?*

Cameron: *This one is changing all the angles and the sides so that none of them are going to be the same as an equilateral.*

Category 2

This category contains responses that define three triangle-type classes based on similar properties with a relationship between the isosceles triangles and the equilateral triangle. The difference between Category 2 and Category 1 lies in the discussion of the connections, as these responses also acknowledged the angle-type links in discussion but did not allow them to cross established triangle-type classes. There were three responses within this category (Michael, Allan, and Kathy). Hence links were not drawn between the scalene and isosceles classes, but the angle-type similarities were noted. Therefore, the diagrammatical summary of relationships is the same as Figure 4.6.

Kathy's relationship between the isosceles class of triangles and the equilateral triangle is described below. The relationship exists as both classes contain two equal sides:

Int: *And what triangle do you think links to that one?* (isosceles)
Kathy: *Probably the equilateral.*
Int: *Can you tell me why they link together?*
Kathy: *Because the equilateral has got two sides the same.*

When Kathy was asked to explain why she linked the acute scalene triangle and the acute isosceles triangle she stated:

Kathy: *Um they are both scalene and that one is two or more and that one is none equal so I shouldn't have the link in there.*

Michael's explanation of the triangle relationship illustrates the inability, due to the differences in the classes, to make links across the isosceles and scalene triangle classes based on similar angle types:

Int: *Can you make any other links across them?*
Michael: *Yes probably but I can't think.*
Int: *Could you link it to that one there?* (right isosceles to right scalene)
Michael: *Well I don't see how I can when that one has three different sides.*
Int: *So why can't you link to that one?*
Michael: *Because that one doesn't have all different sides.*

The similarities between the isosceles and equilateral classes of triangles formed a significant link where the similarities dominated the differences:

Int: *Where is your equilateral?*
Michael: *It is not anywhere because I forgot about it. I can put it here.*
Int: *So how can you link it to the isosceles?*
Michael: *Because it has got two sides the same length and these have two sides the same.*

Category 3

This response is similar to a Type E Category 1 response with the addition of right-angle links. This category contains one response (Frances). Figure 4.7 illustrates the three triangle-type classes, the right-angle link, and the relationship between the isosceles class of triangles and equilateral triangle based on similar properties.

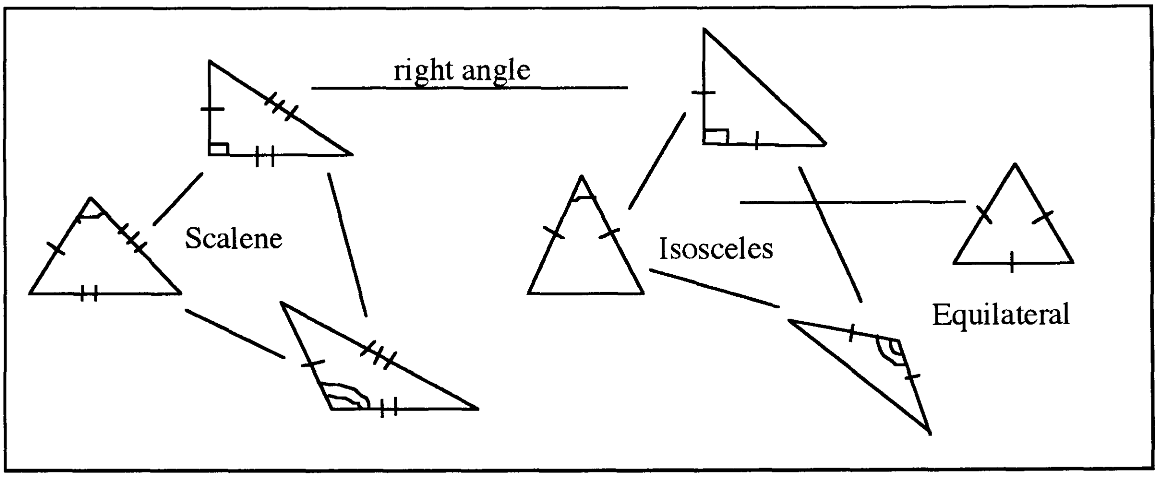


Figure 4.7 Frances' triangle relationships summary

Frances described the relationship between the isosceles class and equilateral class as existing due to both classes containing two equal sides:

Frances: *Well they are both scalene and that one there to that.* (isosceles and equilateral)
Int: *What reasons have you got for why they do link?*
Frances: *Well they have all got two equal sides.*
Int: *For what reason do you think that they don't link?*
Frances: *Um that those ones have got all different angles.*

When prompted to make a link between the acute-angled scalene triangle and acute-angled isosceles triangle, Frances described them as being too different, but was confident in making the link across classes based on right angles:

Int: *What is the reason for that?* (right isosceles and right scalene)
Frances: *Um because they have a right angle ...*
Int: *Could I link the acute-angled isosceles with the acute-angled scalene?*
Frances: *You could because they are both acute-angled but not really.*
Int: *Would you be happy linking those two together?*
Frances: *Um no, probably not, they are too different.*

Category 4

This category includes the three triangle-type classes with a relationship between the isosceles class of triangles and the equilateral triangle based on similar properties. Links across triangle-type classes, namely, right-angled, acute-angled and obtuse-angled links, are described also. There was one response (Beth) in this category, and the relationships identified are represented diagrammatically in Figure 4.8.

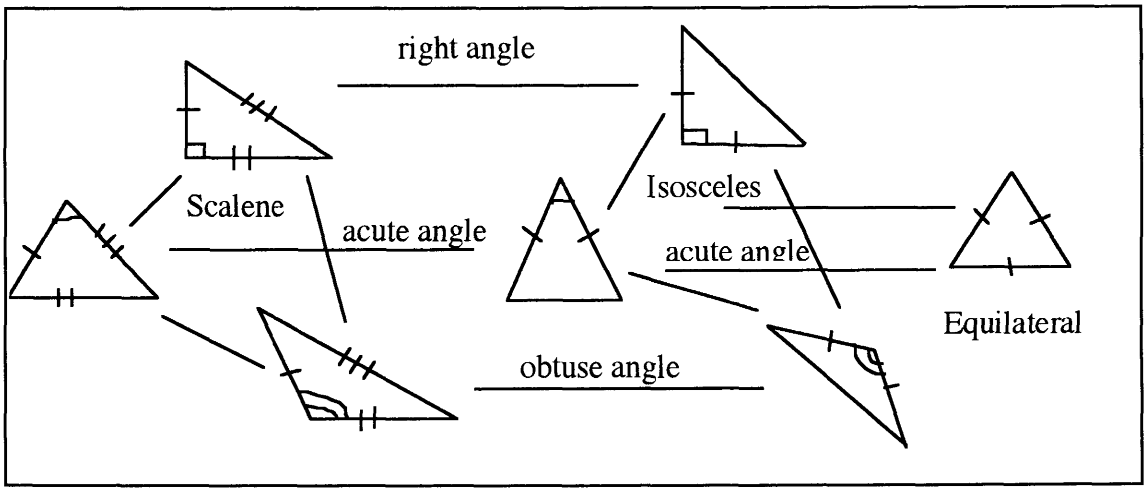


Figure 4.8 Beth's triangle relationships summary

Beth described the triangle-type classes in terms of similar properties, in this case, on the grounds of equal sides. The link between the equilateral triangle and the isosceles class of triangles exists on the basis of both having two sides the same length. Beth also acknowledged that the equilateral triangle is acute, and linked the equilateral triangle with the acute isosceles triangle for this reason:

Int: *Why are those two linked together?*
Beth: *Because they are both scalene triangles.*
Int: *And what does that mean?*
Beth: *That all the sides are different lengths. Oh that one there is supposed to be obtuse.*
Int: *How can the isosceles be linked to your scalene triangle?*
Beth: *Because they are both obtuse I suppose.*
Int: *And why do they link to your equilateral?*
Beth: *Because they are all acute.*
Int: *Any other reason why the equilateral links to the isosceles?*
Beth: *Um because it has got two of the sides the same on both of them. I can link that with that because they are all acute-angled.*

In summary, the Type E responses made a relationship between the equilateral triangle and the isosceles class of triangles. The relationship between these classes is based on similar properties. At no stage was the equilateral triangle described as a subset of the isosceles class of triangles but a relationship exists between the two classes.

The four categories differ from each other through the gradual addition of angle-type links across the scalene and isosceles classes, and the equilateral triangle. Category 1 contains no angle-type links. Category 2 acknowledges the existence of angle-type links in discussion but does not allow them to cross the established triangle-type groupings. Category 3 contains right-angle links across triangle-type classes only. Finally, Category 4 adds relationships based on right-angled, acute-angled, and obtuse-angled similarities across the three triangle-type groupings.

Type F

This group of responses includes statements concerning the equilateral triangle's relationship to the isosceles class of triangles as not based on similar properties only, but due to the equilateral triangle being a form of an isosceles triangle. The single response (Nathan) suggested that the equilateral triangle may be a subset of the isosceles triangle but the idea is not fully accepted.

Nathan's response is characteristic of a Type E Category 4 coding with one differing element. The distinguishing feature that separates this response from the Type E responses was Nathan's suggestion that the equilateral triangle may be a subset of the isosceles triangle class. This is represented diagrammatically by the dotted boundary in Figure 4.9.

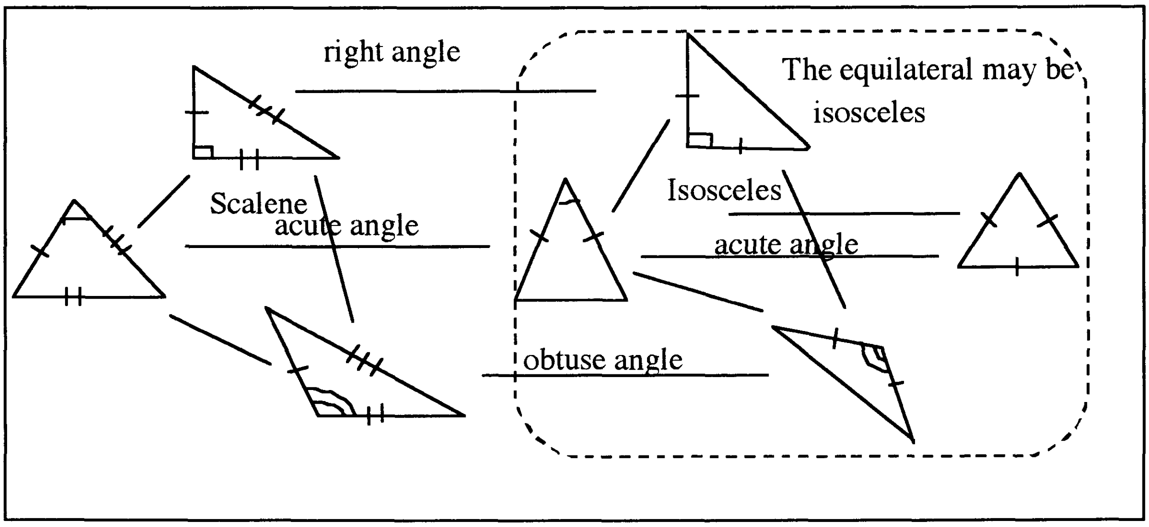


Figure 4.9 Nathan's triangle relationships summary

When Nathan was asked to provide a reason for the relationship between the acute-angled isosceles triangle and the equilateral triangle he began by explaining that both triangles have acute angles. Nathan stopped before continuing and went on to explain that both classes of triangles do have two equal sides. When prompted to place the equilateral triangle within the isosceles class of triangles Nathan restated the similarity between them but was hesitant to accept the equilateral triangle as a subset of the isosceles class of triangles:

Int: *Now why do you have those two linked?*
Nathan: *Because they both have acute angles. Um* (pause) *that is because they have all got acute angles.*
Int: *Any other reason why the equilateral and the isosceles link?*
Nathan: *Because on those two, the sides are equal like that. Yeah they are both isosceles I think.*
Int: *So do you think the equilateral is isosceles?*
Nathan: *Yes I think so. It has got two equal sides like an isosceles.*

In summary, the Type F responses included tentative statements concerning the possibility of the equilateral triangle being a member of the isosceles class of triangles. While this notion was raised within the response, the decision was made not to include the equilateral triangle within the isosceles class of triangles. The relationship between these classes of triangles remained on the basis of similar properties.

Type G

The Type G response accepted the notion of the equilateral triangle as a subset of the isosceles class of triangles, however, the student was unable to provide justification for the decision. The three triangle-type classes were distinct with a clear statement concerning the inclusion of the equilateral triangle in the isosceles class of triangles. There is one response (Dianne) coded within Type G.

Although the notion of class inclusion was made clear, Dianne was not able to provide justification. No angle-type links were made with a concentration on relationships based on triangle-types. Figure 4.10 includes a dotted line around the isosceles triangles as the inclusion of the equilateral triangle in the isosceles group is accepted, but the reasons for this relationship are not evident.

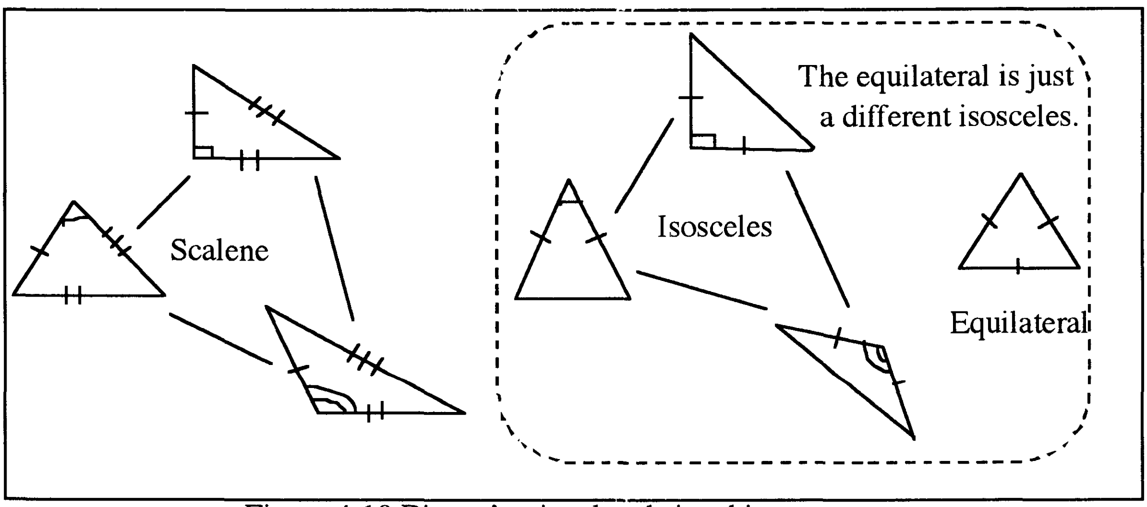


Figure 4.10 Dianne's triangle relationships summary

It is clear from Dianne's discussion that the relationship between the equilateral triangle and the isosceles class of triangles is based on the notion of subsets. Despite this, Dianne did not provide an explanation for this relationship based on properties and did not provide a justification for the inclusion of the equilateral triangle within the isosceles class:

Dianne: *Well these are linked because they no equal sides.*
Int: *Any other reason why they relate?*
Dianne: *They have no equal angles on them.*

Dianne: *Um these are linked because these are isosceles but that is just a different isosceles.*

Dianne's response also illustrates an inability to make angle-type links across the scalene and isosceles classes of triangles. Dianne acknowledged the similarities but the differences between the classes did not allow the link to be made:

Dianne: *These are the isosceles here and you would probably have the right-angled in the middle and the acute and obtuse on either side.* (pause) *You would have your equilateral there and the scalenes there. Do you think I can link that?*
Int: *Why?*
Dianne: *Because it is an acute-angled isosceles and the acute-angled scalene. That one can't go to that.*
Int: *Why not?*
Dianne: *Because they are scalene and that is completely different.*

Overall, the Type G response included a description of the equilateral triangle as a subset of the isosceles class of triangles. The response described the three triangle-type classes of triangles and a relationship that exists between the isosceles class of triangles and the equilateral triangle. Although containing fewer relationships, the student confidently and without prompting provided the information that the equilateral triangle belongs to the isosceles class of triangles. Dianne's response does lack clarity in regard to the justification of this relationship.

Type H

A Type H response describes the isosceles class of triangles as containing a subset, namely, the equilateral triangle. There was one response coded as Type H (David). The equilateral triangle was identified as a form of the isosceles triangle, and the student was able to justify the equilateral triangle's existence within this class. Also, he could argue why an equilateral triangle is an isosceles triangle, but an isosceles triangle is not necessarily an equilateral triangle. The diagram developed from the information given by David is contained in Figure 4.11.

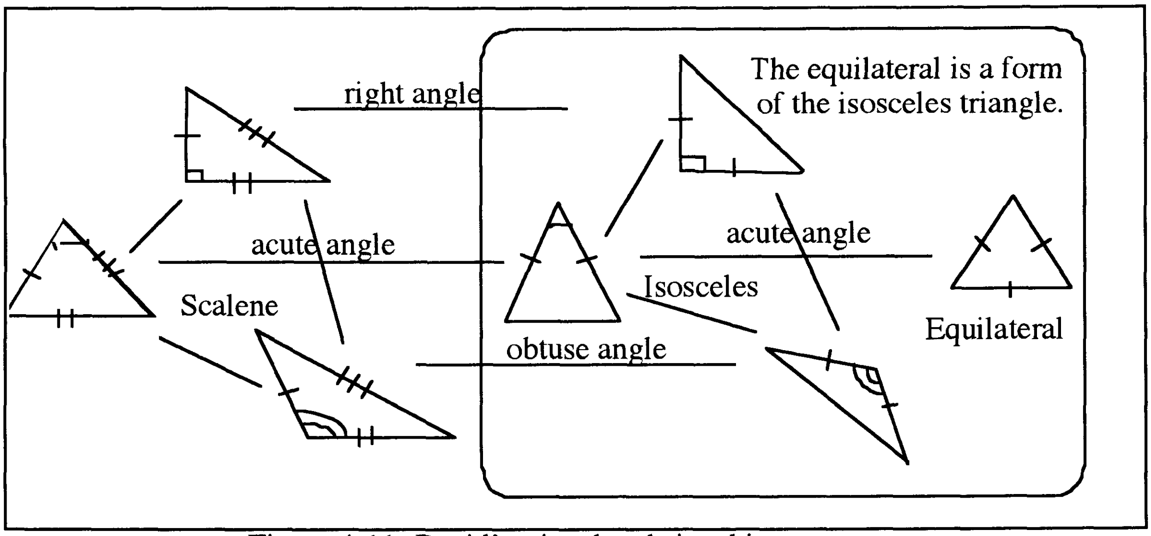


Figure 4.11. David's triangle relationships summary

David's explanation of the tree diagram illustrates the clearly defined relationships that exist among the classes of triangles. David included the notion of class inclusion as an important feature of his tree design and was able to justify this on the basis of properties:

David: *Um, all the triangles begin the tree. Then I differentiate between the side length with two or more equal sides and sides are not equal.*

Int: *So you have ended up with the equilateral and the isosceles on the same branch. Do you see those two triangles linking?*

David: *In that they have equal sides and equal angles. You could say that the equilateral triangle is a form of the isosceles triangle in that it does have two equal angles and two equal sides.*

David acknowledged the angle-type relationships in his discussion and explained that it is another method of grouping the triangles. David went on to explain that an important element of the design was to make sure that the equilateral triangle branches from the isosceles triangle branch:

David: *Um the first step and the easiest way to differentiate them was by side length and that is to divide them up into three different areas and then um the only other way you could differentiate them was by angle size. Are the angles equal to 90 degrees, less than 90 degrees or greater than 90 degrees because there wasn't a case where a triangle had all three characteristics.*

Int: *Do you think doing it this way shows how the triangles relate together?*

David: *Yes but I think I should have had the equilateral coming off the isosceles branch because it does have two equal sides where as a scalene could not be seen as an isosceles where as an equilateral can be seen as isosceles...*

When making a comparison between the tree diagram based on angle-type classes and the tree diagram based on triangle-type classes, David preferred the diagram that illustrates the notion of class inclusion between the isosceles class of triangles and equilateral triangle. This concept had become a dominating feature of the relationships among the different triangles.

David: *I have done much the same thing but I think that it works better in that it includes the equilateral and the isosceles on the same branch.*

In summary, the Type H response included the equilateral triangle as a subset of the isosceles class of triangles based on similar properties. The equilateral triangle is clearly identified as a form of the isosceles triangle. The student was able to justify this class inclusion notion and argue that the converse is not true.

Type I

This type of response also makes explicit that the equilateral triangle is a subset of the isosceles class of triangles. The difference lies in the acquisition of further conditions upon these subsets. There was one response coded as Type I (Adam), and the relationships are summarised in Figure 4.12.

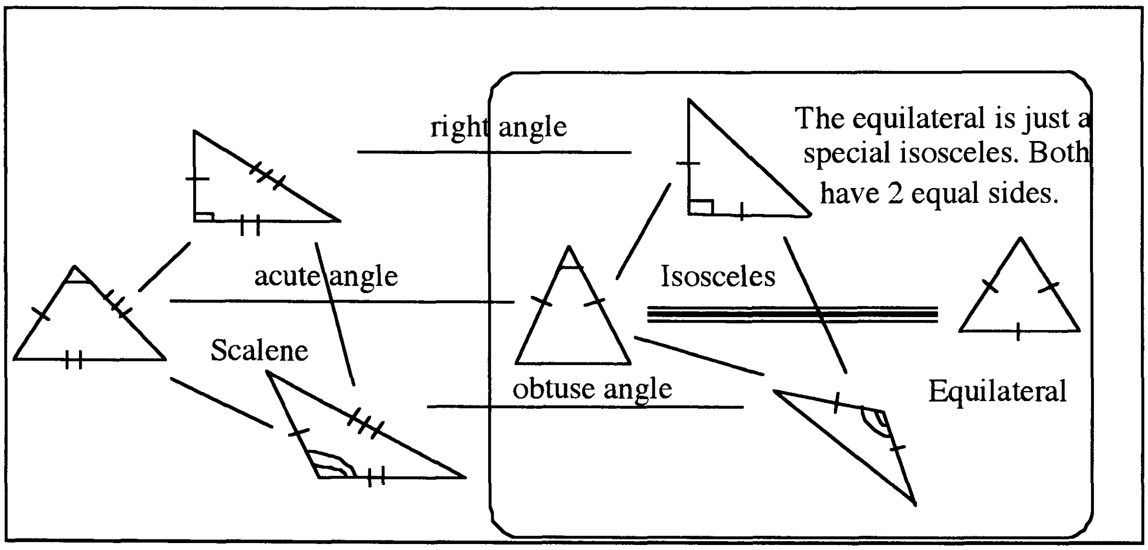


Figure 4.12 Adam's triangle relationships summary

Adam explained that the equilateral is a "special isosceles" based on properties, namely, two equal sides:

Adam: *The equilateral can be linked to the isosceles.*
Int: *How come?*
Adam: *Because these triangles both have two equal sides and the equilateral is just a special isosceles. Some right-angled triangles can be isosceles.*

Adam developed the relationships further, as the link between the acute-angled isosceles triangle and the equilateral triangle became significant. A greater link was acknowledged between these two triangles, based on the equilateral triangle's existence within the isosceles class of triangles as well as the acute-angled triangle class:

Int: *What is the main reason these are linked together?*

Adam: *They* (equilateral) *have the same things that these* (isosceles) *have.* (Link drawn between acute-angled isosceles triangle and equilateral triangle)
Int: *Do you see a link to all those?*
Adam: *No just the acute ones. Because 60 is less than 90.*

Figure 4.12, developed as a summary of Adam's response, illustrates the difference between the Type I and Type H response. Adam acknowledged the significance of the relationship between the acute-angled isosceles triangle and the equilateral triangle. Adam stated that the equilateral triangle is a subset of the isosceles class of triangles, and made a major link to the acute-angled isosceles triangle when considering angle-type relationships.

The Type I response tied together the relationships that exist due to angle-type links, triangle-type classes, and the notion of class inclusion. Each of these relationships was justified on the basis of properties. The subset formed has acquired further conditions. In summary, Type I differs from a Type H response due to the inclusion of a significant link with justification between the acute-angled isosceles triangle and the equilateral triangle.

Summary

This section considers the features of the task analysis and those features identified as characteristic within response categories. The students' responses were coded into nine types. These can be summarised as:

Type A — A single similar feature is identified to link triangles. Triangles appear in more than one group depending upon the identifying feature for each group.

Type B — Scalene, isosceles and equilateral classes of triangles are formed and they are characterised by name and related by a single similar property. No link is made between the isosceles and equilateral classes. The two categories separate responses on the addition of angle-type links across the isosceles and scalene classes.

Type C — Scalene, isosceles and equilateral classes of triangles are formed and they are characterised by name and related by two or more similar properties. Angle-type links are also identified among the scalene and isosceles classes of triangles only.

Type D — Three triangle-type classes are formed with angle-type links across classes. Similar properties are noted between the isosceles and equilateral triangles but differences described do not allow the link to be made.

Type E Relationships are formed across the equilateral and isosceles classes of triangles based on similar properties, but the equilateral triangle is not regarded as a subset of the isosceles class. The four categories within this type separate the responses on the addition of angle-type links across the three classes of triangles.

Type F These responses are similar to a Type E response with indecisive statements concerning the possible inclusion of the equilateral triangle within the isosceles class of triangles; however, this is not fully accepted.

Type G The equilateral triangle is accepted as a subset of the isosceles class of triangles; however, this notion of class inclusion is not justified.

Type H The equilateral triangle is a subset of the isosceles class of triangles with justification based on properties.

Type I These responses are similar to Type H but the subsets formed acquire further conditions. The relationship between the acute-angled isosceles and the equilateral triangle becomes significant and can be justified.

Overall, the comparison between the task analysis (Appendix F) and the nine groups of coded responses illustrated a pattern of growth between the Type A response, where links are based on the spontaneous identification of a single similar feature, to a culmination in responses typical of Type I, where a generalisation is formed based on the interrelationship of the triangle-type and angle-type networks of relations. Responses characteristic of elements described in the task analysis were identified, along with Type E and Type H responses which were not described within the task analysis. One group, Type D, was identified as transitional responses, as they are characterised by the previous response group with the addition of tentative relationships characteristic of those within the following response type. In summary, the response groups appear to form a developmental sequence leading to the identification and justification of relationships based upon the notion of class inclusion. The following section applies a general framework, the SOLO model, to assist in the interpretation of the identified categories of responses.

Theoretical Perspective

The previous section provided a description of the nine groups of coded responses in relation to the task analysis. Through this comparison, additional groups were identified, as well as those that corresponded to the elements of the task analysis. In this section, the

SOLO model is applied to the response categories to create a framework based upon the structure of the response groupings.

In the light of the SOLO model, the majority of the coded groups of responses fall within the concrete symbolic mode. Most of the responses include the expression of concepts which are drawn from the context of the student's experienced world. In Geometry, this means that the focus of the student's reasoning is primarily on properties that can be triggered by reference to a diagram. The remaining responses fall within the formal mode as a real world reference is no longer required and the focus has shifted to relationships connecting properties, which form the bases of generalisations.

The Type A response is characteristic of a relational response within the first cycle of the concrete symbolic mode, and incorporates links based upon a single similar feature or property, drawn upon visual cues, without the formation of a workable unit of triangle types. Types B, C, D, and E are characteristic of responses within the second cycle of the concrete symbolic mode and include the three levels, namely, unistructural, multistructural, and relational responses, with one transitional group. The final groups of responses, Types F, G, H, and I are classified as in the formal mode. The development begins with Type F where the notion of class inclusion is considered although not utilised. Type G responses are characterised by the utilisation of class inclusion without justification. Type H demonstrates consistency in the utilisation and justification of class inclusion notions as an integrating feature of the response. The final group of responses, Type I, is classified as within the second cycle of the formal mode. The response goes further in fine-tuning the nature of the class inclusion notion by addressing additional conditions than were the·case in the previous category. A detailed summary of the response codings in terms of their SOLO classification is contained in Table 4.2.

Table 4.2 The SOLO model and relationships among triangles

Level	Type	Description
R_1 (CS)	A	Links among triangles based on a single similar feature. Hence, students focus on a single quantifiable aspect. Triangle groupings form spontaneously as a result of the identification of the linking feature with a reliance on visual cues. Lack of consistency is evident in groupings. Due to the spontaneous nature of the links, classes of triangles have not formed generic categories. The description of the linking feature is sometimes not applicable to all triangles of the group.

U_2 (CS)	B	The responses indicate that the described single property combines to form three classes of triangles, namely, scalene, equilateral, and isosceles. A class represents an identifiable unit. Inconsistency is evident due to the inability to consider the equilateral triangle in the class of isosceles triangles. The responses include links based on negative instances, such as 'has two sides equal and one different,' which excludes the formation of subsets. Some responses include angle-type links across the isosceles and scalene classes of triangles.
M_2 (CS)	C	These responses incorporate two or more properties to form three distinct classes of triangles, being equilateral, isosceles, and scalene. The focus upon negative instances precludes a link between the isosceles and equilateral triangles. Angle-type links are evident to a varying degree within responses across the isosceles and scalene classes of triangles.
TRANS M_2/R_2 (CS)	D	These responses are similar to the multistructural response above, with the addition of a tentative link between the equilateral triangle and the isosceles class of triangles. With prompting, a connection between these triangles based on a similar property is suggested, but not fully accepted. Reference is made to a connection between the equilateral and isosceles triangles, but the differences described hinder the formation of a relationship.
R_2 (CS)	E	These responses are characterised by the existence of relationships between the equilateral triangle and isosceles class of triangles, based on similar properties. The relating properties of the isosceles class of triangles are described as being applicable to the equilateral triangle. Unlike the unistructural and multistructural responses where the differences dominated the similarities, the descriptions of each class enable a connection across groups. Responses vary according to the addition of angle-type links.
U_1 (F)	F	These responses include tentative statements concerning the equilateral triangle as a subset of the isosceles class of triangles. When prompted, these responses describe the notion of class inclusion as a possibility but without acceptance. Hence, the student becomes disoriented in the task, although s/he still identifies relationships between classes.
M_1 (F)	G	The response includes an acceptance of the equilateral triangle as a subset of the isosceles class of triangles. While this notion of class inclusion is utilised in an unprompted situation, the notion cannot be justified adequately.

R_1 (F)	H	These responses include the notion of class inclusion as the integrating feature of the described relationships. The equilateral triangle is considered a subset of the isosceles class of triangles, thus illustrating consistency in terms of the described relating features of the triangle classes. The scalene, equilateral, and isosceles classes of triangles each maintain a workable identity, which takes into consideration the network of relationships based upon the properties of each class.
U_2 (F)	I	This response re-establishes the class inclusion concept identified in the previous level by narrowing the focus precluding inappropriate examples of isosceles triangles. The equilateral triangle is described as a subset of the isosceles class of triangles with a significant relationship to the acute isosceles triangle.

Overall, the levels and modes identified in the types of responses have assisted in a deeper interpretation of the response categories. Through the application of the SOLO model, a framework emerged which highlighted a pattern of growth in student understanding leading to the acquisition of the skill of class inclusion. While the first cycle responses in the concrete symbolic mode included links based on similar features, it was not until the second cycle in the concrete symbolic mode that the classes of triangles emerged and became a workable identity. Movement through the levels of the second cycle indicates growth, which moves from a focus on each class as separate identities, to relationships based on similar properties among classes. The ability to incorporate subsets of different triangles within the class of isosceles triangles did not appear until the first cycle of the formal mode. The second cycle of the formal mode brought further conditions to the subsets incorporating the equilateral triangle in the class of isosceles triangles. Hence, by considering the structure of the responses, a hierarchical framework has emerged which sheds light on the development of student understanding of the relationships among triangles.

RELATIONSHIPS AMONG TRIANGLE PROPERTIES

Section 2 of triangle results investigates the Relationships Among Triangle Properties. This section is divided into five sub-sections that are similar to Section 1, i.e., Background, Equilateral Triangle Results, Isosceles Triangle Results, Summary and Theoretical Perspective.

Background

Issues relating to the methodology of this section of the research were discussed in Chapter 3. This part of Study 1 focused upon students' understanding of the relationships among triangle properties. The triangles chosen for this task were the equilateral triangle and right isosceles triangle. Each of these was treated separately in the interview. At the beginning of this section of the interview the student was shown a selection of twelve cards. These are referred to as 'characteristic cards' and are listed in Table 4.3.

Table 4.3 Triangle characteristic cards

3 SIDES
3 ANGLES
3 SIDES EQUAL
3 ANGLES EQUAL
HAS RIGHT ANGLE
1 AXIS OF SYMMETRY
NO AXES OF SYMMETRY
3 AXES OF SYMMETRY
HAS OBTUSE ANGLE
HAS ACUTE ANGLES
2 ANGLES EQUAL
2 SIDES EQUAL

The student was then instructed to choose all cards that belonged to the named triangle, beginning with equilateral triangle. This provided a focus for the activity and a means for the student to work within their known domain. After selection of all known characteristic cards, the student was placed into the context of leaving clues for a friend to identify the figure. This enabled a minimum description/definition of the particular triangle depending upon the known relationships. The students were probed for justifications for the chosen combinations, and then they were asked to provide and justify as many combinations as possible. The same procedure was repeated for the right isosceles triangle.

The purpose of this section of the study was to have the students complete a task which focused upon known relationships among triangle properties. The students were provided with a focus for discussion, which involved properties of the triangles known by the individual student. Through discussion of the clue combinations, a vehicle was provided which initiated discussion concerning triangle property relationships. While a summary of the interview is contained in Table 4.4, which includes student tasks and questioning focus common to each interview, the structure allowed for individual dialogue incorporating prompts and probes where necessary.

Table 4.4 Interview format for the task concerning relationships among triangle properties

Triangle Property Relationships	
(i) Int:	We are going to look closely at a few triangles. I have placed some cards in front of you with triangle characteristics on them. I would like you to begin by choosing the cards which belong to the equilateral triangle. (selection made) Look carefully to make sure that you have included all the cards which belong to that triangle.
(ii) Int:	Suppose you wanted to leave some clues for a friend. Do you think that your friend would need to see all these properties to know that you are thinking about an equilateral triangle? What combination could you leave? (discussion follows concerning reasons for cards included in the combination and those that have been removed) Do you think it could be made simpler? (discussion follows concerning reason for the simplification or inability to make simpler)
(iii) Int:	Let's put all the cards back. I would like you to make a different set of clues for your friend. (point (ii) repeated until student has provided all known combinations).
(iv)	First three steps repeated for the right isosceles triangle.

Equilateral Triangle Results

Context

It is necessary to consider the working domain of the 24 students in regards to known properties of the equilateral triangle before categorising the responses on the basis of understanding relationships among properties. The initial selection, which set the context of the task provided, enabled the responses to be placed into groups according to the selected characteristic cards. These groupings assisted in providing information concerning the properties that were known by the students. The optimum response comprised the following cards:

- 3 SIDES
- 3 ANGLES
- 3 SIDES EQUAL
- 3 ANGLES EQUAL
- 3 AXES OF SYMMETRY
- 2 SIDES EQUAL
- 2 ANGLES EQUAL
- 1 AXIS OF SYMMETRY
- HAS ACUTE ANGLES

The 24 responses are divided into the following three groups.

Group 1

There were six responses (Allan, Beth, David, Michael, Nathan, and Suzanne) which consisted of the optimum list of property cards listed above as belonging to the equilateral triangle. These students were familiar with all properties belonging to the equilateral triangle (when prompted by the cards). Suzanne's response included a hesitation to include properties of an inclusive nature in her selection, namely, two sides equal, two angles equal, and one axis of symmetry.

Group 2

This group comprised of seventeen responses (Adam, Alice, Andrew, Arthur, Brendan, Cameron, Dianne, Ellen, Frances, Jason, Jenny, Kathy, Louise, Megan, Narelle, Scott and Tracy) which did not include the following cards:

2 SIDES EQUAL
2 ANGLES EQUAL
1 AXIS OF SYMMETRY

These seventeen students thus indicated a lack of understanding of the inclusive nature of 'three equal sides,' 'three equal angles,' and 'three axes of symmetry.'

Group 3

One student (Peter) provided a similar selection to those described above in Group 1; however, Peter did not include 'three sides,' or 'three axes of symmetry.'

Task analysis

As with the investigation concerning relationships among triangles, it was necessary to consider a task analysis for this section of the study. The initial selection task was designed to focus students on the new activity. The observations of interest to this task relate to discussion concerning relationships among triangle properties and justifications for the identified relationships. All possible elements of a response concern the following relationships among the properties of the equilateral triangle:

1. The relationship between three sides and three angles.
2. The relationship between equality of sides and equality of angles.
3. The relationship between symmetry and equality of sides.
4. The relationship between symmetry and equality of angles.
5. The interrelationships between equality of sides, equality of angles, and symmetry.

Of interest to this analysis are the combinations of relationships utilised, and the variety of justifications provided within the responses. While a wide range of minimum descriptions/definitions exists for the two triangles, the focus is upon the justification of these combinations in terms of property relationships.

Response categories

Taking into consideration the property relationships identified within each response and reasons given for the utilisation of these relationships, it was possible to divide the 24 responses for each triangle into five groups. A description of each of the response types follows. Each description includes a diagrammatical triangle properties relationships summary, interview excerpts which illustrate identified relationships and reasons for these, and examples of the language characteristic of the response type. The diagrammatical summaries include the following symbols as described in Table 4.5 below.

Table 4.5 Key to diagrammatical summary symbols

SYMBOLS	MEANING
s	side property
a	angle property
sym	symmetry property
○	property is known by the student and is included in initial selection
●	property is utilised by the student when providing effective minimum descriptions/definitions
	relationship between properties is tentative or is utilised in a single direction (although the minimisation is based upon the relationship between two properties, combinations include one direction only)
	relationship between properties is known, utilised, and is justified bi-directionally (relationship between the two properties is the focus of the minimisation, and combinations formed include both directions of the relationship)

Through the consideration of the property relationships described in the task analysis, it was possible to divide the 24 responses for each of the triangle property tasks into a number of characterised types. Each response indicated that the relationship between number of sides and number of angles was known and utilised, and therefore is not incorporated into the analysis.

Type A (Equ)

This type of response indicated a strong reliance on specific visual or traced examples of the triangle type from which the properties were determined. While side, angle, and symmetrical properties were chosen from the list provided, in each case the student first drew or traced the figure. There were three responses coded as Type A (Alice, Ellen, and Jenny).

The student was aware that these properties were relevant to the figure, however, there was no indication of known relationships between the properties. This type of response is characterised by the figure determining the properties; thus, the response is driven by the notion that if the property belongs to the shape it is required in the description. The relevant chosen properties are recognised from the figure individually. A typical Type A response is summarised below in Figure 4. 13.

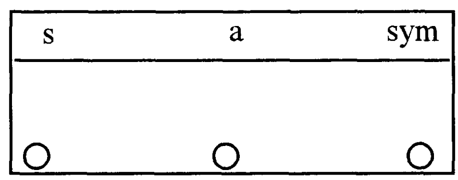


Figure 4. 13 Type A equilateral triangle properties relationships summary

As described earlier, the open dots above imply that the property was included in the initial selection of cards, however, it was not used in the minimisation of properties. Both Ellen and Alice, while making their initial selection of property cards, found it necessary to draw an example of the equilateral triangle, and constantly refer back to it in attempts to allocate properties. Ellen provided the following as a minimum set of cards to depict the equilateral triangle: 3 sides equal, 3 angles equal, and 3 axes of symmetry together (minimum combination 1). When prompted to remove 3 sides equal or 3 angles equal, Ellen was adamant that if the property card is not included, it will not be known by the other properties given. Symmetry was not identified as important and was removed in the final combination.

MINIMUM CONBINATION (1)

3 SIDES EQUAL   3 ANGLES EQUAL   3 AXES OF SYMMETRY

Int: *What if I took out that the three sides are equal?*
Ellen: *They wouldn't know then.*
Int: *So you need to keep it there?*

Ellen: *Yes.*

MINIMUM COMBINATION (2)

[3 SIDES EQUAL] [3 ANGLES EQUAL] [HAS ACUTE ANGLES]

Int: *What if I took out three angles equal?*
Ellen: *No it has to stay there because you need to know that.*

Alice began with an identical minimum combination to Ellen (minimum combination 1). When prompted to remove [3 sides equal] she agreed and carried out the task (minimum combination 2). Alice considered that the equilateral would still be depicted, however, her justification is not consistent with the combinations left.

MINIMUM COMBINATION (1)

[3 SIDES EQUAL] [3 ANGLES EQUAL] [3 AXES OF SYMMETRY]

Int: *What if I took out three sides are equal?*
Alice: *Oh yeah because you would still have three axes of symmetry and the angles.*

MINIMUM COMBINATION (2)

[3 ANGLES EQUAL] [3 AXES OF SYMMETRY]

MINIMUM COMBINATION (3)

[3 SIDES EQUAL] [3 AXES OF SYMMETRY]

Overall, the Type A responses have no identified relationships among triangle properties. The response includes a reliance on an image of the equilateral triangle from which the property cards are chosen from the selection of cards provided. Visual examples of the figure are required to determine each property.

Type B (Equ)

The Type B responses also incorporated no relationships among the properties of the equilateral triangle. The difference in this group of responses is that the students did not rely on overt visual examples when choosing relevant properties. The student provided more than one minimum description. Each included one property, which was used in isolation, as a necessary indicator of the equilateral triangle. There are eight responses coded as Type B (Andrew, Arthur, Dianne, Frances, Kathy, Narelle, Nathan and Suzanne).

There are no perceived relationships between the equilateral triangle properties. The property combinations chosen in the minimisations are justified in terms of significant indicators of the triangle, rather than property-to-property relationships. Figure 4.14

details a variety of examples identified within Type B and is followed by student dialogue, which illustrates the justifications for the combination chosen.

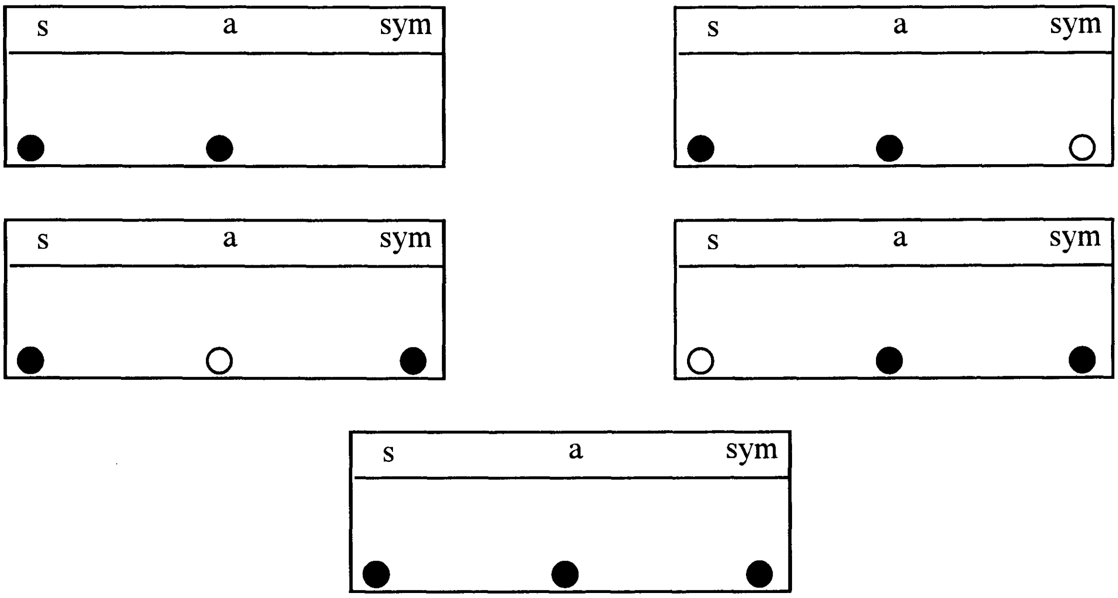


Figure 4.14 Type B equilateral triangle properties relationships summary

The characteristics of the Type B response are reflected in Nathan's attempt at providing minimum descriptions/definitions for the equilateral triangle and his justifications that follow.

MINIMUM COMBINATION (1)

3 SIDES EQUAL

Nathan: *I could just use that one.*
Int *So why can you just use that one?*
Nathan: *There is no other triangle that has just three sides equal.*

MINIMUM COMBINATION (2)

3 ANGLES EQUAL

Int: *So why can you have either of those?*
Nathan: *Because um either it makes it unique.*
Int: *Could you have another set of clues?*
Nathan: *You could have three axes of symmetry.*
Int: *Why would that work?*
Nathan: *Because it is the only triangle with three axes of symmetry.*

It is evident that Nathan's minimum combinations are based upon the knowledge of three sides equal, three angles equal, and three axes of symmetry as unique properties of the equilateral triangle. While these are considered in isolation of one another, Nathan is able to provide more than one minimum combination. The unique properties of the equilateral triangle are linked directly back to the shape, as opposed to a link between the properties.

Overall, the Type B responses are characterised by the utilisation of more than one property, where each property is used in isolation to signify the equilateral triangle. While multiple minimum combinations are provided which utilise three equal sides, or three equal angles, or three axes of symmetry, they are not linked in any manner. The individual properties of the equilateral triangle are considered as unique to that particular triangle, and are treated in isolation.

Type C(Equ)

This group of responses is similar to the Type B response, with the addition of a single link between two properties, which is utilised in one direction at any one time. While these responses are characterised by the inclusion of one or more isolated property signifiers of the equilateral triangle, the students' justification was also based upon a tentative connection between two properties. There are two responses coded as Type C (Brendan and Louise).

While one property is described as relating to another, the relationship has not become a workable unit. Hence the student was unable to include a second combination based upon the link. Figure 4.15 illustrates this link as described by the students.

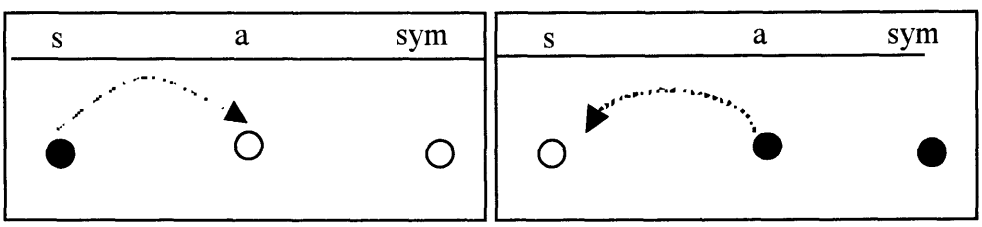


Figure 4.15 Type C equilateral triangle properties relationships summary

Louise began by using [3 axes of symmetry] within her minimum combinations of the equilateral triangle as this property is considered to be unique to the equilateral triangle. Her justifications for this included no relationship to side or angle properties. Louise's second attempt included [3 angles equal]. When asked to justify her reasons for removing [3 sides equal], Louise responded that "if it has got, um, three angles that are equal, it has got three sides that are equal." As in Dianne's response, when prompted to find another combination, she was unable to utilise the relationship in the reverse direction.

MINIMUM COMBINATION (1)
[3 SIDES] [3 AXES OF SYMMETRY]

Int: *Why is that enough?*
Louise: *Well the only triangle that has three axes of symmetry is the equilateral.*

MINIMUM COMBINATION (2)
[3 ANGLES EQUAL]

Louise: *Just three angles. If the angles are equal it is an equilateral.*

Int: *So why don't you need to have that the three sides are equal?*
Louise: *Because if it has got um three angles that are equal it has three sides that are equal.*
Int: *Can you find another combination of cards, which denote the equilateral triangle?*
Louise: *No, that would be it.*

In general, this group of responses included one or more significant property signifiers of the shape, with justifications suggesting the existence of a link to another property. Hence, in this category, not only did the students utilise unique properties of the equilateral triangle in isolation, one property is connected to another. However, the link did not formed a workable unit to enable spontaneous utilisation in both directions. Responses within this category demonstrated a link based upon an ordering of properties, thus precluding the spontaneous utilisation of the property relationship in both directions.

Type D (Equ)

This group of responses made explicit reference to a single relationship between two properties as the basis for the minimum property combinations formed. While a third property is known, it does not link to any other property. The third property is either chosen and not utilised when providing a minimum combination, or is incorporated within minimum descriptions/definitions as an isolated signifier of the particular triangle. There are nine responses coded as Type D (Adam, Beth, Cameron, David, Jason, Michael, Peter, Scott and Tracy).

These responses included justifications and property combinations which utilise one bi-directional relationship between two properties. A summary of this type of response appears in Figure 4.16.

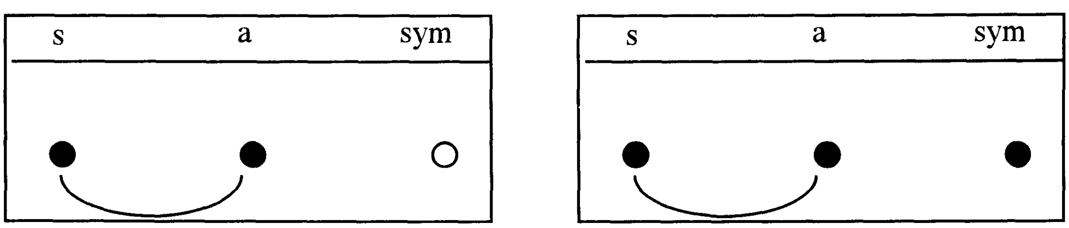


Figure 4.16 Type D equilateral triangle properties relationships summary

It is evident in both Peter's and Michael's justifications below that their chosen combinations for the equilateral triangle are based upon the bi-directional relationship between 'three sides equal' and 'three angles equal.' At no stage did Peter attempt to use 'three axes of symmetry' within his minimum property descriptions.

MINIMUM COMBINATION (1)

3 ANGLES | 3 SIDES EQUAL | HAS ACUTE ANGLES

Peter:	*That is about all that you would need.*
Int:	*How come I can remove three angles equal?*
Peter:	*Well if there were three sides equal then the angles would be equal as well. I suppose I could have removed that one and left that one it doesn't really make that much of a difference.*

MINIMUM COMBINATION (2)
3 ANGLES   3 ANGLES EQUAL

Int:	*Can you come up with any other combinations?*
Peter:	*No, I don't think so.*

Michael's response below illustrates that the relationship between equality of angles and equality of sides forms the basis of his justifications. 'Three axes of symmetry' is chosen as relevant to the equilateral triangle, but is not utilised in minimum property descriptions/definitions.

MINIMUM COMBINATION (1)
3 ANGLES EQUAL

Int:	*Why is that enough on its own?*
Michael:	*Um because if has equal sides it has to have equal angles.*

MINIMUM COMBINATION (2)
3 SIDES EQUAL

Int:	*Another one?*
Michael:	*Um* (pause) *no.*
Int:	*What if I tried to use just the three axes of symmetry on its own?*
Michael:	*Yeah.*
Int:	*Do you think that would work?*
Michael:	*Um probably. I don't know.*
Int:	*Would you like to use it in a combination?*
Michael:	*No.*

In summary, the Type D responses included the utilisation of one property relationship that is applied to the task as a workable unit. Hence, the response incorporates property combinations, which are focused upon the relationship that exists between two properties. The properties within this relationship are considered to work together. The property relationship forms the equilateral triangle, as opposed to signifying the triangle. Thus, unlike Types A, B, and C, the relationship connecting the properties determines the figure.

Type E (Equ)

This group of responses is characterised by minimum property combinations which incorporate two or more relationships among known properties. The minimum descriptions/definitions are formed on the basis of a combination of relationships between equality of sides and angles, equality of sides and symmetry, equality of angles and symmetry. These responses included property combinations, which have formed separate workable units. While more than one relationship exists among the properties, these are treated in isolation. Hence, while more than one relationship is present, they have not formed a network of relationships. There were two responses coded as Type E (Megan and Allan). The summary diagram contained in Figure 4.17 illustrates two isolated relationships between two pairs of properties.

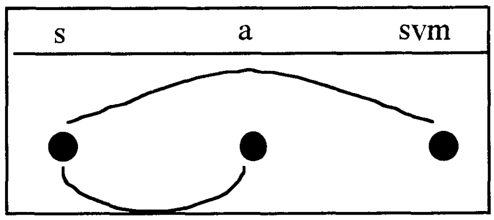


Figure 4. 17 Type E equilateral triangle properties relationships summary

It is evident in the excerpt below that Allan is focused upon two isolated relationships, these being: between three sides equal and three angles equal, and three sides equal and three axes of symmetry. Allan does not indicate in his response the utilisation of the relationship between angle and symmetrical properties.

MINIMUM COMBINATION (1)

| 3 SIDES EQUAL |

Int *Now why would you only need that?*

Allan: *Because by having three sides equal it is going to have three angles equal anyway because of the three sides, um and three sides equal will mean that it has got three axes of symmetry and three sides are equal well it has got to have three acute angles because it has got to be 180 divided by three and two angles will be equal for the same reason as three.*

MINIMUM COMBINATION (2)

| 3 ANGLES EQUAL |

MINIMUM COMBINATION (3)

| 3 SIDES | | 3 AXES OF SYMMETRY |

Allan's combinations demonstrate his ability to utilise 'three angles equal,' 'three sides equal,' and 'three axes of symmetry.' The justification provided is focused upon the relationships between four paired properties.

The Type E responses are characterised by the focus upon more than one relationship between pairs of properties. While more than one link may exist from one property to other properties, these are not interrelated. Hence, each relationship between two properties is treated in isolation. While more than one relationship has formed and has the potential to form a network, an overview does not exist and therefore has not become a workable unit.

Right Isosceles Triangle Results

Context

The optimum response expected in the initial selection of properties associated with the right isosceles triangle includes the following list of seven properties.

3 SIDES

3 ANGLES

2 SIDES EQUAL

2 ANGLES EQUAL

1 AXIS OF SYMMETRY

RIGHT ANGLE

HAS ACUTE ANGLES

The sample of 24 students was divided into three groups according to their working domain. These are described below.

Group 1

The first group comprises responses which include all properties of the right isosceles triangle. There are seventeen students within this group.

Group 2

The cards chosen by students in this group correctly belong to the right isosceles triangle, but the list is not complete. There are six responses in this group. One response (Ellen) included six of the cards and did not include 'two angles equal.' Three responses (Narelle, Suzanne, Adam) included six of the cards and did not include 'has acute angles. One response (Andrew) included five of the cards and did not include 'has acute angles' and 'one axis of symmetry.' One student (Peter) chose five of the cards and did not include 'three sides' and 'two sides equal.'

Group 3

There is only one response (Jason) categorised as Group 3. Jason's selection included five of the cards described in the optimum response, but Jason incorrectly included 'three axes of symmetry,' and did not include 'has acute angles.'

Task analysis

All possible elements of a response concerning the relationships among the properties of the right isosceles triangle include:

1. The relationship between three sides and three angles.
2. The relationship between the right angle and acute angles.
3. The relationship between two equal sides and two equal angles.
4. The relationship between one axis of symmetry and two equal sides.
5. The relationship between one axis of symmetry and two equal angles.
6. The interrelationships between two equal sides, two equal angles, and one axis of symmetry.

Similar to the equilateral triangle results, of interest to this analysis are the combinations of relationships utilised and the variety of justifications provided within the responses. The same diagrammatical symbols are utilised in the following discussion as were incorporated in the discussion of equilateral property relationships. The right angle property is not included in the diagram as the right angle characteristic card was included in all minimum combinations.

Type A (Rt isos)

These responses are characterised by the utilisation of one known property to provide a minimum description/definition of the right isosceles triangle. The ultimate reference point is the right isosceles triangle, which is considered to have a single property, which is unique to this type. There are two responses within this group (Ellen and Peter).

While the single minimum description provided initially reflects a possible understanding of property relationships, when the students were probed to justify the chosen combinations this understanding was not evident. Instead, the other known properties are not considered to be adequate signifiers and have no relationship to the property chosen. Figure 4.18 illustrates a typical Type A response.

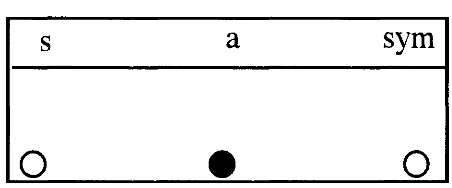


Figure 4.18 Type A right isosceles triangle properties relationships summary

Peter began the task by describing the properties of the right-angled isosceles triangle as he chose the appropriate cards.

Peter: *It has got one axis of symmetry and that is because it is right angled and it is even on all sides of it. It has two equal angles and it will have to have acute angles if it has a right angle. It has three angles because it is a triangle and it has a right angle because it is right angled.*

MINIMUM COMBINATION (1)

3 ANGLES | HAS RIGHT ANGLE | 2 ANGLES EQUAL

Peter: *I don't need the axis of symmetry.*
Int: *Why can you take that one out?*
Peter: *I just don't think that you would really need to know that. It is not really necessary.*
Int: *What about two angles equal?*
Peter: *Oh yeah that is pretty necessary if it is isosceles.*
Int: *Is there any other combination that you could use?*
Peter: *No that is the only way.*

The card combination and dialogue above emphasise that although the side and symmetrical properties are known, Peter identifies 'two equal angles' as the only significant signifier of the isosceles triangle. Both 'one axis of symmetry' and 'two equal sides' are known to belong to the isosceles triangle, but do not signify this triangle.

In general, the Type A responses are characterised by the recognition of one property, which is a necessary signifier of a particular triangle type. Other properties of the triangle are known in terms of belonging to that triangle, but are not employed as signifiers of the triangle.

Type B (Rt isos)

The Type B responses are characterised by the utilisation of more than one property to signify the right isosceles triangle. This group is comprised of six responses (Alice, Brendan, Michael, Narelle, Nathan and Suzanne). These responses indicate that there is more than one perceived significant indicator for the triangle type. Figure 4.19 below illustrates that both 'two sides equal' and 'two angles equal' can be used in isolation to signify the isosceles triangle.

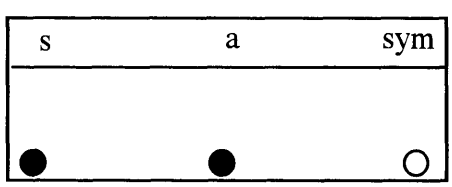


Figure 4.19 Type B right isosceles triangle properties relationships summary

Nathan's justifications below illustrate the perceived notion of important and unimportant characteristics of the right isosceles triangle. While both 'two sides equal' and 'two angles equal' are utilised to provide more than one minimum description/definition they are not related to each other.

MINIMUM COMBINATION (1)
HAS RIGHT ANGLE | 2 SIDES EQUAL

Int: *Why can you just have those two?*
Nathan: *Because there is only two right angled triangles. Isosceles and scalene.*
Int: *Now you have taken out these. How do you know they still have all these things when you only have these two?*
Nathan: *Um because a triangle, oh well it has all those things because they are not all important things to find out it is that sort of triangle.*

MINIMUM COMBINATION (2)
HAS RIGHT ANGLE | 2 ANGLES EQUAL

This notion was emphasised by Brendan as he removed 1 AXIS OF SYMMETRY and provided the following justification.

Brendan: *Yes, and the axis of symmetry. You don't have to know that.*
Int: *Why not?*
Brendan: *Because when you work it out from these ones well the axis of symmetry is just a feature of it. It is not like a description um a major description, there is enough information here to do it.*

Alice provided the following combination of cards. When probed to justify her selection, and the removal of 2 SIDES EQUAL, Alice related the 1 AXIS OF SYMMETRY to the isosceles triangle only, hence, there is no relationship to other properties of the shape.

MINIMUM COMBINATION (1)
3 SIDES | HAS RIGHT ANGLE | 1 AXIS OF SYMMETRY

Alice: *Well I would need the sides, the angle and the axis of symmetry.*
Int: *You haven't included anything about the two sides equal?*
Alice: *I don't have to because it has got the axis of symmetry.*
Int: *And what does that tell me?*
Alice: *That it is a type of isosceles.*

While Suzanne's response is characteristic of a Type C, as reference is made in regards to the link between the axis of symmetry and other properties, however, this required further prompting for it to be utilised in the combinations devised.

Suzanne: *You would know that it has an axis of symmetry with the two sides equal and the two angles equal.*
Int: *You do need both of those?*
Suzanne: *Yes.*
Int: *What if I took out the two sides are equal?*
Suzanne: *Um um yeah you would still know.*

The Type B responses are characterised by a focus upon more than one property, in isolation, which signifies the right isosceles triangle. While the response incorporates more than one property combination they are not related to one another. Instead, the properties are specifically related to the triangle type. Hence, a selection of properties are utilised as unique indicators of the right isosceles triangle.

Type C(Rt isos)

The Type C responses make explicit reference to one link between two properties; however, this link has not formed a workable unit that can be spontaneously utilised in both directions. There are five responses (Adam, Dianne, Louise, Beth, and Kathy) in this group. Each of these responses focuses upon a link between two properties when justifying the minimum combinations provided. These responses are characterised by an ordering between two properties. A typical Type C response to the right isosceles task is summarised in Figure 4.20 below.

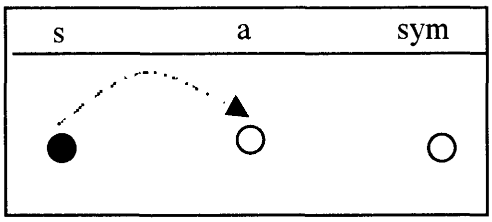


Figure 4.20 Type C right isosceles triangle properties relationships summary

The nature of the link is highlighted in the following excerpt. Dianne's response indicates that all three properties are known. The 'one axis of symmetry' is included but is not justified. Of special interest is the removal of 'two angles equal' and the justification that follows. Dianne is confident that this card is not necessary, as "if there are going to be two sides equal there are going to be two angles equal" but when probed to provide another combination the relationship was not provided.

MINIMUM COMBINATION (1)

3 ANGLES | HAS RIGHT ANGLE

1 AXIS OF SYMMETRY | 2 SIDES EQUAL

Dianne: *Um, I would use the axis of symmetry and if you have two sides equal then you have two angles equal and I would need the right angle. Do they know it is a triangle?*

Int: *No.*

Dianne: *Well then I could use either three sides or three angles.*

Int: *Now why didn't I need those other ones?*

Dianne: *Well obviously if there are going to be two sides equal there are going to be two angles equal.*

Int: *What about the acute angles?*

Dianne: *Well if you have got a right angle that is 90 then the other two have got to be less then 90.*
Int: *Can you come up with another combination of cards?*
Dianne: (considers each card) *No I think that is it.*

In summary, the Type C responses are characterised by a single link between two properties of the right isosceles triangle. The focus of justifications is upon the link between the two properties, as opposed to the ultimate reference point being the figure itself. An ordering between two properties takes place but this has not formed a workable unit, hence precluding multiple minimum combinations based upon the relationship between two properties. While a third property may be known and utilised in chosen combinations, this property is viewed as a significant indicator of the triangle and is not related to the other properties.

Type D (Rt isos)

The Type D responses are characterised by a focus upon a single relationship between two properties. These differ from Type C, as the relationship is a workable unit from which multiple minimum combinations are provided. Hence, the focus of relationships is no longer limited between property and figure, but is based upon property to property links. There were eight responses (Andrew, Arthur, Cameron, David, Jason, Megan, Scott, and Tracy) coded as Type D. Responses typical to this group appear in Figure 4.21 below.

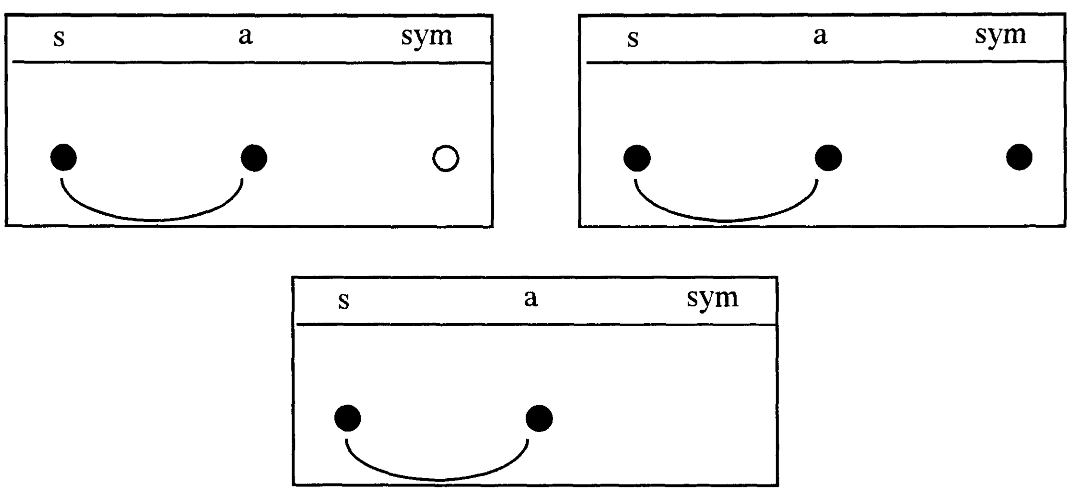


Figure 4.21 Type D right isosceles triangle property relationships summary

The combinations and justifications below, given by Arthur, illustrate the spontaneous focus upon the relationship between two sides equal and two angles equal. Both properties are utilised within different minimum combinations due to the bi-directional relationship between the two properties. While 'has acute angles' remains with 'has right angle' in the first minimum combination, acute angles is later removed.

MINIMUM COMBINATION (1)

[3 SIDES] [HAS RIGHT ANGLE]

[HAS ACUTE ANGLES] [2 ANGLES EQUAL]

Arthur: *I could take that out because if there are two angles equal there are two sides equal.*
Int: *Do you think that you still need all these?*
Arthur: *Well do they know it is a triangle?*
Int: *No.*
Int: *Could I take out has acute angles?*
Arthur: *Um Yeah.*
Int: *How come?*
Arthur: *Because if that is a right angle and the other two angles have to equal 90 then it obviously would have to be acute.*

MINIMUM COMBINATION (2)

[3 SIDES] or [3 ANGLES]

[HAS RIGHT ANGLE]

[2 ANGLES EQUAL] or [2 SIDES EQUAL]

Arthur: *Well you could have either of those three sides and three angles, and either of two angles equal and two sides equal.*

The explicitness of the bi-directional relationship between side and angle properties is also highlighted in Tracy's reponse below.

MINIMUM COMBINATION (1)

[3 SIDES] [HAS RIGHT ANGLE] [2 SIDES EQUAL]

Int: *How come I can take out two angles are equal?*
Tracy: *Because if those two sides are equal then two of the angles are equal.*
Int: *What about your axis of symmetry?*
Tracy: *Well to be a right-angled isosceles you would need the two equal sides.*

MINIMUM COMBINATION (2)

[3 ANGLES] [HAS RIGHT ANGLE] [2 SIDES EQUAL]

Tracy: *Well you could have three angles instead of three sides or you could take out two sides equal and put in two angles equal.*

Overall, the Type D responses include spontaneous reference to the bi-directional relationship between two properties of the right isosceles triangle. The focus of the response is the link between two properties, as opposed to the link between a property and the figure. While the relationships between the two properties are together related to the right isosceles triangle, it is the link between the properties themselves that is the basis for more than one minimum combination.

Type E (Rt isos)

The Type E responses are similar to those in Type D as they include a focus upon a single bi-directional relationship between two properties. However, they differ with the addition of one or more property relationships which are utilised in a single direction. There are two responses (Jenny and Allan) coded as Type E. Examples of the typical responses identified in this group appear in Figure 4.22. The diagrammatical summaries below illustrate the relationship, which is utilised in both directions, and those that are utilised in a single direction.

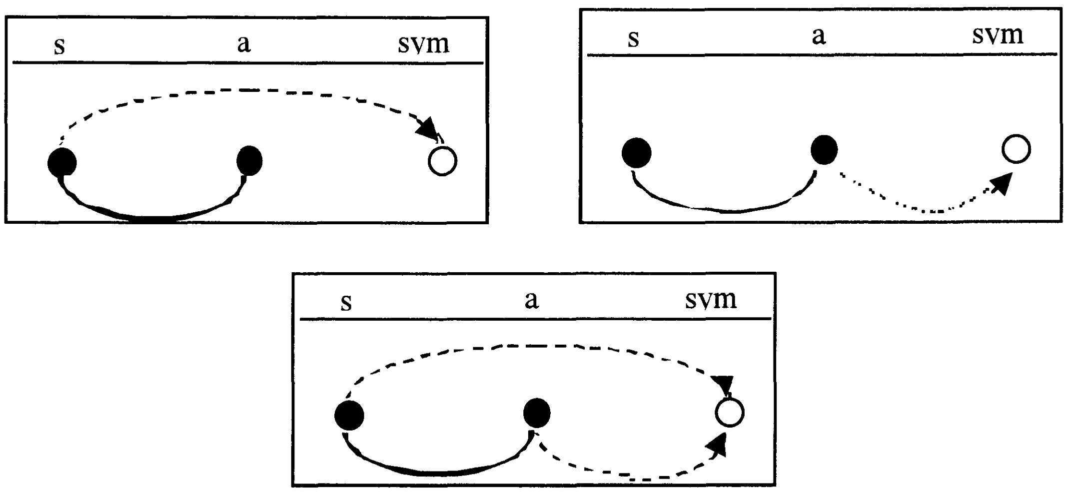


Figure 4.22 Type E right isosceles triangle properties relationships summary

Allan's response below indicates clearly the utilisation of the relationship between side and angle properties. While Allan mentions that if the triangle has two sides equal it will also have an axis of symmetry, this relationship is utilised in a single direction and he is unable to use the property of symmetry within his minimum combinations.

MINIMUM COMBINATION (1)

3 SIDES | HAS RIGHT ANGLE | 2 SIDES EQUAL

Int:	*Now why can acute angles go?*
Allan:	*Um no I am just rearranging it.* (further removals) *I could take out that because if it has got a right angle it has to have other acute angles.*
Int:	*What about three angles?*
Allan:	*Well if it has three sides it has got to have three angles.*
Int:	*And the axis of symmetry?*
Allan:	*Well by having the right angle with the two equal sides um it has got an axis of symmetry.*

MINIMUM COMBINATION (2)

3 ANGLES | HAS RIGHT ANGLE | 2 ANGLES EQUAL

Allan:	*I suppose I can get rid of the two sides equal.*
Int:	*How come?*
Allan:	*Because if you have got two angles equal and a right angle then those two sides have got to be equal.*
Int:	*Are there any other combinations that you could use?*

Allan: No.

This type of response is also reflected in Jenny's justification for her chosen combinations.

Int: *How come you didn't need two sides equal?*
Jenny: *Because of the two angles equal. If you have them they will be the same.*
Int: *Why can you take out has one axis of symmetry?*
Jenny: *Well actually you can because if you have two angles there it would have that.*

The Type E group of responses is characteristic of the Type D responses with the addition of other property relationships which are utilised in a single direction. The justifications are based upon the relationships which exist between properties, rather than being dominant signifiers of particular triangle types.

Type F (Rt isos)

The Type F responses utilise all relationships that exist between the known properties. The minimum combinations chosen are based upon the bi-directional relationships that exist between side, angle, and symmetrical properties. There is only one response (Frances) coded as Type G. All properties are utilised and justified through their relationship with other properties of the particular triangle. The relationships described and justified within this group of responses appear in Figure 4.23.

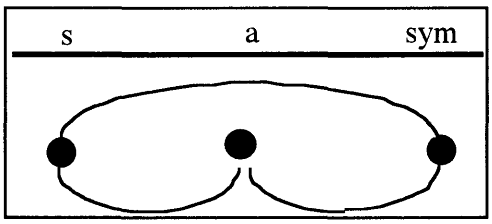


Figure 4.23 Type F right isosceles triangle properties relationship summary

It is evident in the excerpt below, given by Frances, that the focus is upon the interrelationships between 'two sides equal,' 'two angles equal,' and 'one axis of symmetry.' Each of these properties is discussed in terms of its relationship with the others, as opposed to being described as separate links. Although Frances' language is not succinct, for example "two bottom ones" for angles, and "you fold it" for symmetry, Frances spontaneously utilises the relationships between these properties, and no prompting is required.

MINIMUM COMBINATION (1)

3 SIDES | HAS RIGHT ANGLE | 2 SIDES EQUAL

Int: *How come two angles equal can go?*
Frances: *Because being isosceles like those two are the same then those two bottom ones are the same, and you have the axis of symmetry.*

Int:	*What if I took out has acute angles?*
Frances:	*No you would need that, um no because it is a right angle I don't need that.*
Int:	*Can you come up with another combination?*

MINIMUM COMBINATION (2)

3 ANGLES | HAS RIGHT ANGLE | 2 ANGLES EQUAL

MINIMUM COMBINATION (3)

3 ANGLES | HAS RIGHT ANGLE | 1 AXIS OF SYMMETRY

Int:	*Why is that one enough when you haven't mentioned anything about two sides or two angles equal?*
Frances:	*Um because it means that you can fold it and then those two sides will be equal and those two angles.*

In summary, this group of responses utilises the bi-directional relationships between more than one property. Within the context of triangles, which incorporates only three such relationships, all three are utilised: the relationship between equality of sides and equality of angles, equality of sides and symmetry, and equality of angles and symmetry. The response focuses upon an overview of the property relationships in a given justification. Hence, the basis of the response concerns the interrelationships between more than one property.

Summary

This section considered the similarities between the response groupings for both triangle property relationship tasks. Based upon the identified similarities and differences between the response groupings a set of groupings was formed which combines both the equilateral triangle and right isosceles triangle tasks. This combined set of response groupings is considered through a comparison of relationships identified in the task analysis, with those features identified as characteristic within the categorised responses.

The groups of responses concerning property relationships of the equilateral triangle and the right isosceles triangle are summarised separately below.

Equilateral triangle

The five groups of responses concerning the relationships among equilateral properties can be summarised as:

Type A (Equ): These responses incorporate a strong reliance on visual cues and specific examples from which the properties of the equilateral triangle are drawn.

The ultimate point of reference is the figure from which a single unique property is identified.

Type B (Equ): This type of response involves more than one property, but the properties are used in isolation as necessary indicators of the equilateral triangle. No links appear to exist between the known properties. More than one property is identified as unique to the equilateral triangle. Multiple minimum property descriptions are provided.

Type C (Equ): While the Type C responses also incorporate more than one property to signify the equilateral triangle, reference is also made to a link between two of these properties. While the response indicates a focus upon this link, it is verbose and is utilised in a single direction. While one property relates to another, the link is not balanced. Instead, an ordering of properties is evident.

Type D (Equ): This response makes explicit reference to a single relationship between two properties. The balanced relationship between the pair of known properties has formed a workable unit and is the focus of the response. The point of reference is a single relationship between two properties of the equilateral triangle.

Type E (Equ): Similar to the Type D response, this type of response focuses upon more than one relationship between pairs of properties. Each of these property relationships is considered in isolation.

Right isosceles triangle

The six groups of responses concerning the relationships among right isosceles properties can be summarised as:

Type A (Rt Isos): While combinations of side, angle, and symmetrical properties are known by the student, this type of response is characterised by the utilisation of a single property perceived as a unique indicator for the right isosceles triangle. A single minimum description is provided.

Type B (Rt Isos): This type of response is similar to the Type A responses, however, more than one property is considered to be a unique signifier of the particular triangle type. Multiple minimum property descriptions are provided.

Type C (Rt Isos): The Type C responses make reference to a single relationship between a pair of properties. While mention is made concerning this link, it remains verbose and is not readily available to the student in both directions.

Type D (Rt Isos): Type D responses are characterised by the spontaneous explicit focus upon the existing relationship between two properties. Hence, the properties are no longer viewed in isolation and related to figure only, but instead, the property to property relationship determines the figure.

Type E (Rt Isos): This type of response is similar to the Type D response with the addition of one or more property to property relationships, which are utilised in a single direction. Hence, while one relationship remains a reversible unit, this reversibility is not maintained across additional property links.

Type F (Rt Isos): This type of response focuses upon the interrelationships between the identified property links. The links between pairs of properties are not viewed in isolation. Instead, the focus of the response is considered in terms of the relationship they have with other pairs of properties. An overview of the relationships that exist among the properties is the central focus.

Combined property relationships response groups

Table 4.6 below indicates the student responses categorised within each group for the equilateral triangle and the right isosceles triangle. The third column places the groups of coded responses into a combined coding based upon the comparison between the equilateral triangle response groups and the right isosceles triangle response groups. These are discussed below in terms of the eight combined groups.

Table 4.6 Combined triangle properties response groupings

Equilateral	Right Isosceles	Combined Coding
A (Equ)		A (Comb)
	A (Rt Isos)	B (Comb)
B (Equ)	B (Rt Isos)	C (Comb)
C (Equ)	C (Rt Isos)	D (Comb)
D (Equ)	D (Rt Isos)	E (Comb)
	E (Rt Isos)	F (Comb)
E (Equ)		G (Comb)
	F (Rt Isos)	H (Comb)

In the task analysis four possible relationships among triangle properties where identified, namely;

1. The relationship between equality of sides and equality of angles.
2. The relationship between symmetry and equality of sides.
3. The relationship between symmetry and equality of angles.
4. The interrelationships between equality of sides, equality of angles, and symmetry.

When considering the relationships identified in the task analysis in the light of the eight response categories, there is no defined order in which the property relationships appear. There is, however, a pattern of progression evident which results in an understanding of one property relationship, i.e. (1), (2), or (3) of the task analysis. This progression culminates in an understanding of the interrelationships between the three identified relationships of the task analysis.

Overall, when considering a comparison between the task analysis and combined triangle properties response groupings, there is an illustrated pattern of growth between Type A (Comb) and Type H (Comb). The earlier responses, Type A (Comb), Type B (Comb), and Type C (Comb), utilise no relationships as described in the task analysis. These responses are characterised by a reliance on visual cues, which reduces in dependence from Type A (Comb) to Type C (Comb). The Type A (Comb) responses include all known properties when students were asked to provide a minimum description and will minimise only when prompted. The Type B (Comb) responses identify a single property as a unique signifier of a particular triangle, while the Type C (Comb) responses incorporate more than one property as unique signifiers.

The Type D (Comb) responses make reference to one of the relationships described in the task analysis. However, this relationship is not utilised effectively and is discussed in one direction, and is therefore characterised as a link. An ordering exists between two properties within the Type D responses.

Types E (Comb), F (Comb), G (Comb) incorporate relationships as described in the task analysis. There is a progression from utilising a single link between two properties in one direction, to the utilisation of the relationship as a workable unit when providing multiple combinations, as evident in Type D (Comb) and Type E (Comb). Type G (Comb) responses are characterised by more than one relationship. The Type F (Comb) response is transitional as it incorporates elements of the Type D response with the addition of links between other properties. The final response, Type H (Comb) is characterised by the utilisation of the interrelationships between relationships described in the task analysis. In summary, there appears to be an identifiable developmental progression between the response types, which leads to an understanding of the interrelationships among properties. To assist in the interpretation of the identified groups of responses, the following section utilises the SOLO framework.

Theoretical Perspective

The previous section outlined the eight response categories in the light of the task analysis. Through this comparison elements of the task analysis were evident in some of the groups; however, earlier groups included no relationships as described in the task analysis, while the final group comprised the integration of elements of the task analysis. Additionally, there were various transition groups. This section applies the SOLO model to the response categories to assist in the creation of a developmental framework based upon the structure of the coded response groupings.

Four of the eight response groups fall within the concrete symbolic mode when considered in the light of the SOLO model. That is, the responses are based upon conceptual reference to the context of their experienced world, and often have diagrammatical cues. The Type A (Comb) response is characteristic of the concrete symbolic mode when the student is prompted; however, there is considerable support from the ikonic mode as reasoning is focused upon examples of the figure.

Responses coded as Type A (Comb) are coded as relational within the first cycle of the concrete symbolic mode, as there is also a reliance on specific examples from which a list of properties may be generated and on a demonstrated need for visual cues. The following three response groups focus upon the properties and/or relationships between them at varying levels of complexity. Types B, C, and D (Comb) are characteristic of unistructural, multistructural, and relational levels, respectively, in the second cycle of the concrete symbolic mode. Hence, there is an elaboration of the perceived significance of a property.

Types E, F, G, and H (Comb) are characteristic of the formal mode as there is no longer a need for a real world referent. Instead, the focus is upon the relationships that exist among the properties. Types E, F, and G (Comb) are characteristic of the first cycle of the formal mode, namely, unistructural, a transitional group, and multistructural. The final group of responses, Type H (Comb), falls into the second cycle of the formal mode and is coded as unistructural. This group of responses focuses upon the network of relationships among the properties and justifies minimisations on the basis of property interrelationships. Table 4.7 below outlines a detailed description of each response group's SOLO classification.

Table 4.7 The SOLO model and relationships among triangle properties

Level	Type	Description
R_1 (CS)	A (Comb)	The properties are perceived as features that are determined by the figure. All known properties are included in the description as they are all considered to belong to the figure. There is a reliance on specific examples of the particular triangle type to generate the list of properties. Minimisation does not occur; however, when prompted respondents are able to reduce the combination effectively.
U_2 (CS)	B (Comb)	This response indicates the recognition of one property as a unique and necessary signifier for a particular triangle type. Hence, only one minimum description/definition is identified within the response. The figure is the main point of reference, and determines the property. Minimisation is based upon the uniqueness of a single property to a particular triangle.
M_2 (CS)	C (Comb)	More than one property is identified as unique to a particular triangle type. More than one minimum description is provided, but, no links exist between the properties. The properties utilised are determined by the figure and perceived as significant signifiers. The response involves a series of isolated closures.
R_2 (CS)	D (Comb)	A link between two properties is evident in the justification of minimisations; however, it remains verbose, tentative, and is not incorporated readily in both directions. Hence, the link has not formed a workable unit. Ordering exists between two properties.
U_1 (F)	E (Comb)	The focus of the minimisations provided is the single link between two properties. This single relationship has formed a readily available unit. The response indicates a perception that the property relationships determine the figure, as opposed to being determined by the figure or belonging to the figure. The justification for the minimisations is succinctly described in terms of the relationship between two properties. This relationship is perceived as a workable identity which forms the basis of minimum descriptions.

U_1/M_1 (F)	F (Comb)	These responses are similar to the unistructural response above, with the addition of links between other properties, which are not utilised in both directions. While one relationship between two properties is readily available, other property links are introduced. The additional links are limited as they are only applied in one direction. Inconsistency exists with additional property links made.
M_1 (F)	G (Comb)	Response is based upon the links existing between more than one pair of properties; however, they are perceived to be in isolation. The links remain separate identities, and, more than one link is focused upon in the response.
U_2 (F)	H (Comb)	These responses include the consideration of the network of relationships among the triangle properties. Responses are focused upon the interrelationships between the properties. There is a consistency in justifications for each minimisation based upon the relationships between each of the properties.

Overall, the application of the SOLO model has provided a deeper interpretation of the response categories associated with the development of relationships between triangle properties. Through the identification of levels and modes within the response categories a framework emerged which has highlighted a pathway leading to an understanding of the interrelationships among triangle properties. The early responses, relational in the first cycle of the concrete symbolic mode, included reliance upon visual cues of specific examples of particular triangles and the properties were generated from them.

In the second cycle of the concrete symbolic mode the properties were perceived as unique signifiers of triangle types. More than one minimum description was evident in the multistructural responses (CS), however, this was based upon multiple properties as unique signifiers of a particular triangle type. The second cycle responses of the concrete symbolic mode culminate in a focus upon a single link between two properties. This results in an ordering of properties. The ordering may be carried out in more than one way. Nevertheless, the link has not yet formed a workable unit.

It was not until the first cycle of the formal mode that there was a shift in focus from the triangle type determining the property, to a focus upon the property relationships determining the figure. Movement through the levels of the first cycle (formal mode) indicate growth which moves from a focus upon a single relationship that exists between two properties, i.e., students use as a unit the linking of property A with property B and vice versa, to a focus on more than one relationship among properties of a given triangle type. It is envisaged that a relational response in the first cycle of the formal mode would be characterised by a focus upon interrelationships among properties, which are not spontaneously utilised or which require prompting. The second cycle of the formal mode was entered when the response included a focus upon the interrelationships of relationships among property to property relationships in a spontaneous manner. This consideration of the structure of response categories in light of the SOLO model has resulted in the emergence of a hierarchical framework. This framework sheds light on the development of student understanding of relationships among triangle properties.

CONCLUSION

The utilisation of student-designed tree diagrams as a catalyst to unpack student understanding was particularly successful. By providing students with the opportunity to revisit ideas on up to three occasions, in addition to prompting and probing by the interviewer, a clear picture of how the students in the sample saw the relationships among triangles was identified. In particular, the six research questions posed at the beginning of the chapter were able to be addressed formally.

Research Question 1.1 stated, *What are the characteristics of students' understandings demonstrated in a classification task of seven different triangles?* There was evidence of characteristically different thinking underpinning the links made by different students. For example, in this particular sample, the lower level responses included spontaneous links on the basis of a single similar feature. Other responses involved the type of triangle taking on significance, with each class characterised by its properties and known by name. However, the classes of triangles remained in isolation. Another group of responses described the individual classes of triangles and linked the equilateral triangle to the isosceles class of triangles on the basis of similar properties. The responses also included those that described the equilateral triangle as a subset of the isosceles class of triangles, and the final category placed further conditions on the joining of the two triangle groups.

An important implication of the identification of different types of responses which all incorporate connections among triangles was highlighted by the justification, and in some cases the absence, of the links. While links were made between the same triangles by different students, the nature of the links varied. For example, when considering the connection between the equilateral triangle and the isosceles class of triangles they ranged from: a link based on one aspect only, therefore, resulting in inconsistencies within the group; to responses which incorporated relationships based on properties which culminate to form classes, but whose closed definition of the class make the link between the equilateral triangle and isosceles class of triangles not possible; and finally, to those that incorporated the notion of class inclusion with conditions when describing the relationship.

Research Question 1.2 stated, *Is there evidence of some developmental pattern in the different responses to a classification task of seven different triangles?* A developmental path was identified as earlier notions were subsumed by later ideas. For example, the simple aspects originally used to identify a particular class of triangles, which are independent and isolated, were later used to relate the classes together, and, finally, used to justify the inclusion of different types of triangles. In addition, transitional responses were noted under probing where students were striving for a more comprehensive response.

The observed developmental path leading to the notion of class inclusion has an important implication when considered in the light of past literature. This pattern of growth illustrates that class inclusion is a gradually acquired skill, and not a skill that develops suddenly once all properties of each class reach a totality as previously implied.

Research Question 1.3 stated, *Does the SOLO model offer a framework to explain the identified categories of responses concerning students' understandings of relationships among triangles?* The SOLO model when considered from a multiple cycle perspective provided a structure in which the hierarchy could be embedded. For example, the responses were identified in the concrete symbolic mode and in the formal mode. Due to the range of responses, it was possible to identify a number of levels. Although only one group of responses fell into the first cycle of the concrete symbolic mode, each level of the second cycle (CS) was identified. In addition, the first cycle of the formal mode was identified, and a unistructural response of the second cycle in the formal mode was also evident.

The significance of this successful application of the SOLO framework is that it extends previous research. In the past, Pegg and Davey (1989) were unable to distinguish between responses coded as R_1(CS) and U_2(CS) as both levels encompass knowledge, at any one time, of a single property. This research has shown that while the property is known at R_1, and can be used as a sorting device, it is not a signifier of that class of shapes, as is the case with a U_2 response. Also significantly, the structure of the responses highlights the inconsistency evident in the concrete symbolic mode until the student reaches a relational level within the second cycle. This is particularly evident in those responses that indicate an inability to relate the equilateral triangle to the isosceles class of triangles on the basis of similar properties. It is not until the formal mode is entered that students are able to place the equilateral triangle in the isosceles class of triangles and utilise the description of the isosceles triangle being applicable to the equilateral triangle. The significance of closure is made evident when considering those students unable to consider class inclusion as a possibility, when their later responses are more open enabling this notion to be a consideration.

The triangle property task, within the context of property card selection and minimisations, provided an effective means for eliciting information regarding students' understanding of property relationships. The initial selection made by the student placed the task within the working domain of each student. The opportunity for optimum responses was available, as the student was able to return to the selection on a number of occasions. The property cards provided an avenue for revisiting selections, while justifications under prompting and probing unpacked the types of property relationships understood by the student.

Research Question 1.4 stated, *Can students' demonstrated understandings of relationships between triangle properties be categorised into identifiable groups according to similar characteristics?* There was evidence of characteristically different thinking associated with students' understanding of the relationships among properties. In this particular sample, the lower level responses included no relationships among properties. At the lower levels, properties could be generated from specific examples of triangles types and all known properties were included in their given minimum descriptions, while one of these groups perceived one property as a necessary indicator. Another group of responses made explicit reference to one mono-directional relationship between two properties, while other groups of responses utilised one or more bi-directional property relationships. The final group of responses based their minimisations on the interrelationships among property relationships.

Research Question 1.5 stated, *Was there evidence of some developmental pattern in the different responses to a task requiring the utilisation of relationships among triangle properties?* A developmental path was identified due to the observed change in perception of the role of properties and the relationships between them. The first level of understanding, as evident in this sample, perceived a property as a feature of specific examples of triangle types—the property is seen to be generated from the shape. This perception is subsumed by another, where a single property is perceived as a unique signifier of a particular triangle type. The property can be used in isolation when minimising; however, it is not linked to another property. It is not until the property is perceived as determining the particular triangle that a progression to utilising and justifying descriptions and definitions on the basis of property to property relationships occurs, and finally, justifications based upon the interrelatedness of properties among triangles.

Research Question 1.6 stated, *Does the SOLO model offer a framework to explain the identified categories of responses concerning students' understandings of relationships among triangle properties*? The application of the SOLO model in regards to modes and cycles of levels, provided a means for interpreting the hierarchical structure evident in the groups of responses. For example, while all the responses fell into the concrete-symbolic mode and the formal mode, the earliest level of response was characteristic of first cycle (CS). Of the remaining concrete-symbolic responses, a number of second cycle levels were identified. In addition, identification of first level responses in the formal mode was made, and the significance of entering the second cycle of the formal mode was made evident.

Having established this broad framework for triangles, the next chapter explores the case for Quadrilaterals. In particular, the focus is on the following classes of shapes, namely, square, rectangle, parallelogram, rhombus, kite, and trapezium. The property relationships are considered in the context of the square, parallelogram, and rhombus.