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Minimal surfaces and conservation laws for bidimensional structures

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Abstract

We discuss conservation laws for thin structures which could be modelled as a material minimal surface, i.e. a surface with zero mean curvatures. The models of an elastic membrane and micropolar (six-parameter) shell undergoing finite deformations are considered. We show that for a minimal surface it is possible to formulate a conservation law similar to three-dimensional nonlinear elasticity. It brings us a path-independent J-integral which could be used in mechanics of fracture. So the class of minimal surfaces extends significantly a possible geometry of two-dimensional structures which possess conservation laws.

Keywords

conservation law, minimal surface, membrane, micropolar shell, finite deformations

1 Introduction

The conservation laws play a central role in continuum physics. Indeed, it is worth to mention conservation laws of mass, energy, momentum, moment of momentum [1;2]. In addition to these classic conservation laws it is possible to establish other, trivial or non-trivial, conservation laws [2–5]. Let us briefly recall the definition of a conservation law. Let a problem under consideration be described through a set of functions of many variables $u_i = u_i(x_j)$, $i = 1, \dots, m$, $j = 1, \dots, n$, which satisfy a system of partial

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differential equations (PDEs)

$$l_p\left(x_j, u_i, \frac{\partial u_i}{\partial x_j}, \dots\right) = 0, \quad p = 1, \dots, k. \tag{1}$$

Let $\vec{P} = (P_1, \dots, P_n) \in \mathbb{R}^n$ be a vector-valued function with components

$$P_q = P_q\left(x_j, u_i, \frac{\partial u_i}{\partial x_j}\right), \quad q = 1, ..., n.$$

Then, if the following equation

$$\mathrm{div}\vec{P} \equiv \sum_{i=1}^{n} \frac{\partial P_i}{\partial x_i} = 0, \tag{2}$$

holds true for any solution of (1), it is called a *conservation law*.

For derivation of conservation laws in the three-dimensional (3D) elasticity one can apply various techniques including Noether's theorem and its extensions such as the Bessel–Hagen and the neutral action methods, see [5–9]. After Noether it is known that conservation laws are closely related to invariance properties of a total energy functional that called also variational symmetries. For example, a homogeneity, i.e. local invariance with respect to infinitesimal translations, results in conservation law for the Eshelby tensor [2;4;5], which brings us well-known path-independent J-integral and some other invariant integrals. Conservation laws are widely used in mechanics of fracture, theory of stress-induced phase transitions and for description of other inhomogeneity in solids [2;3;5].

Instead, in the case of two-dimensional (2D) structures such as shells, one faces a problem of homogeneity as a shell is an inhomogeneous 2D medium, since its geometry is point-dependent, in general. The 3D conservation laws could be transformed for plane geometry, i.e. for plates. See results for first-order shear-deformable linear plates [10;11], linear second-order plate theory [12], von Kármán plates [13]. As a result, unlike to plate theory conservation laws for shells were established for particular geometries, such as spherical, cylindrical or shells of revolution, see [5;14;15]. Path independent integrals were introduced for cylindrical shells and shells of revolution within the Sanders–Koiter variant of linear shell theory and nonlinear membrane theory in [16]. Conservation laws are also known for linear shallow shell model with applications to cracked cylindrical shell [17], and nonlinear shallow shell models [18] including Marguerre-von Kármán theory [19]. In fact, the concept of shallow shell inherits plane geometry from plates.

The aim of this paper is to discuss new conservation laws for 2D structures which could be modelled using a minimal surface as a base surface carrying physical properties of the structure. The principal property of a minimal surface is zero mean curvature, see [20–22] for basic properties of the minimal surfaces. Recently, some structures based on minimal surface geometry were proposed for advanced composites, see e.g. [23–26]. Let us note that one can easily meet minimal surfaces in natures, for example, as seashells [27;28].

The paper is organised as follows. First, in Section 2, we briefly recall necessary information from differential geometry including the surface divergence theorems. In Section 3 we discuss the kinematics of a material surface considering membrane theory[29] and enriched (Cosserat-like) surfaces. The latter model has straightforward relation to micropolar shells[30;31] called also six-parameter shell model[32]. It could be treated as 2D Cosserat continuum, i.e. a 2D medium with translational and rotational degrees of freedom, and with surface stresses and surface couple stresses. In Sections 4 and 5 we introduced Eshelby tensors for these models and present the corresponding conservation laws and invariant integrals. Modeling stress-induced phase transformations in micropolar shells the 2D Eshelby tensor was introduced in[33], whereas its relation to the 3D counterpart was discussed in[34]. Recently, the 2D Eshelby tensor was also used for modelling of adhesion of thin structures[35]. In Section 6 we briefly discuss 3D-to-2D reduction as an alternative way of derivation of 2D conservation laws.

In what follows we almost always use the direct (index-free) tensor calculus as in[36;37].

2 Preliminaries

First, let us briefly introduce some formulae of differential geometry. Let $\Sigma \in \mathbb{R}^3$ be a smooth enough surface with a boundary $\Gamma = \partial\Sigma$. Σ could be parameterized with a position vector given as a function of two surface coordinate s^1 and s^2:

$$\mathbf{X} = \mathbf{X}(s^1, s^2) = X_1(s^1, s^2)\mathbf{i}_1 + X_2(s^1, s^2)\mathbf{i}_2 + X_3(s^1, s^2)\mathbf{i}_3, \tag{3}$$

where X_j and $\mathbf{i}_j$ are Cartesian coordinates and corresponding unit base vectors, respectively, see Fig. 1. We introduce the surface nabla-operator ∇ and the natural and reciprocal base vectors as follows

$$\nabla = \mathbf{E}^\alpha \frac{\partial}{\partial s^\alpha}, \quad \mathbf{E}^\alpha \cdot \mathbf{E}_\beta = \delta^\alpha_\beta, \quad \mathbf{E}^\alpha \cdot \mathbf{N} = 0, \quad \mathbf{E}_\beta = \frac{\partial \mathbf{X}}{\partial s^\beta}, \quad \alpha, \beta = 1, 2,$$

where δ^α_β is the Kronecker symbol, $\mathbf{N} = \dfrac{\mathbf{E}^1 \times \mathbf{E}^2}{|\mathbf{E}^1 \times \mathbf{E}^2|}$ is the unit normal to Σ, "$\cdot$" and "$\times$" denote dot and cross products, respectively. Hereinafter Greek indices takes values 1 and 2, whereas Latin indices will take values 1, 2, and 3, and Einstein's summation rule is used.

For any differentiable surface field $\mathbf{T}$ we introduce the surface divergence theorem (the Gauss–Ostrogradsky theorem)[29;37]

$$\iint\limits_{\Sigma} (\nabla \cdot \mathbf{T} + H\mathbf{N} \cdot \mathbf{T})\, d\Sigma = \oint\limits_{\Gamma} \boldsymbol{\nu} \cdot \mathbf{T}\, ds. \tag{4}$$

Here $H \equiv -\frac{1}{2}\nabla \cdot \mathbf{N}$ is the mean curvature of Σ, $\boldsymbol{\nu}$ is the external unit normal to $\Gamma = \partial\Sigma$ such that $\boldsymbol{\nu} \cdot \mathbf{N} = 0$. Let us note that $\mathbf{T}$ could be a vector-valued or tensor-valued surface field of any order. There are other forms of the surface divergence

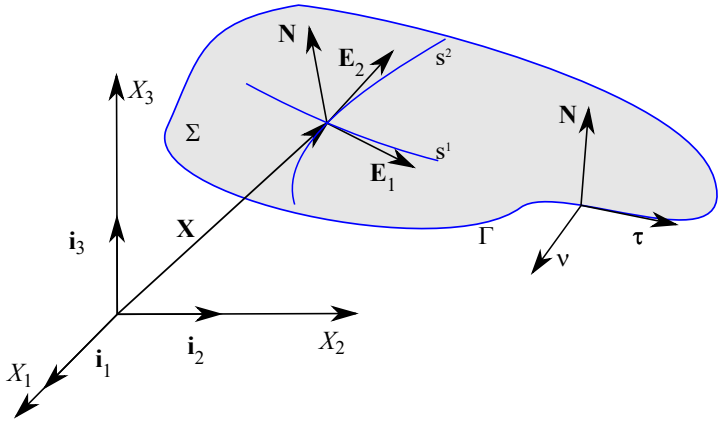

Figure 1. Surface with parametrization.

theorem

$$\iint_{\Sigma} (\nabla \mathbf{T} + H\boldsymbol{N} \otimes \mathbf{T})\, d\Sigma = \oint_{\Gamma} \boldsymbol{\nu} \otimes \mathbf{T}\, ds, \tag{5}$$

$$\iint_{\Sigma} (\nabla \times \mathbf{T} + H\mathbf{N} \times \mathbf{T})\, d\Sigma = \oint_{\Gamma} \boldsymbol{\nu} \times \mathbf{T}\, ds, \tag{6}$$

$$\iint_{\Sigma} \nabla \times (\mathbf{N} \otimes \mathbf{T})\, d\Sigma = \oint_{\Gamma} \boldsymbol{\tau} \otimes \mathbf{T}\, ds. \tag{7}$$

In (5) and (7) "$\otimes$" is the dyadic product, $\boldsymbol{\tau}$ denotes the unit vector tangent to Γ, $\boldsymbol{\tau} \times \boldsymbol{\nu} = \mathbf{N}$, see Fig. 1.

Obviously, the form of surface divergence theorems (4)–(6) differs from its 3D counterparts due to presence of terms related to the mean curvature. If the mean curvature of Σ vanishes, that is if

$$H = 0, \tag{8}$$

then there is no such difference. A surface whose mean curvature is zero at any point is called a *minimal surface*. Basic properties of the minimal surfaces can be found in [20–22].

3 Kinematics of a material surface

In what follows we utilize the concept of a material surface [29]. In the theory of plates and shells it is also called the direct approach [38]. We introduce a deformation of a material surface $\mathcal{S}$ as a differentiable mapping from a reference placement κ into a current placement χ. Let Σ and σ be surfaces describing $\mathcal{S}$ in κ and χ, respectively. Within Lagrangian description we introduce a displacement vector $\mathbf{u}$ of a point $z \in \mathcal{S}$ with coordinates s^1 and s^2 defined on Σ as follows

$$\mathbf{u} = \mathbf{u}(s^1, s^2) = \mathbf{x} - \mathbf{X}, \tag{9}$$

where $\mathbf{x} = \mathbf{x}(s^1, s^2)$ and $\mathbf{X} = \mathbf{X}(s^1, s^2)$ are position vectors of z in κ and χ, respectively, see Fig. 2.

In order to describe deformations of kinematically enriched (Cosserat-like) material surfaces, in addition to position vector of $z \in \mathcal{S}$ we consider two triples of unit orthogonal vectors called directors. So we have two triples $\{\mathbf{D}_k\}$ and $\{\mathbf{d}_k\}$, $k = 1, 2, 3$, defined in reference and current placements, respectively. Using these triples we introduce an orthogonal tensor

$$\mathbf{Q} = \mathbf{Q}(s^1, s^2) = \mathbf{D}_k \otimes \mathbf{d}_k \tag{10}$$

as a complementary kinematical descriptor, see[30;37] for more details. As a result, for enriched material surface we have two kinematical descriptors $\mathbf{u}$ (or $\mathbf{x}$) and $\mathbf{Q}$ which could be treated as translational and rotational degrees of freedom used in the theory of shells[30–32].

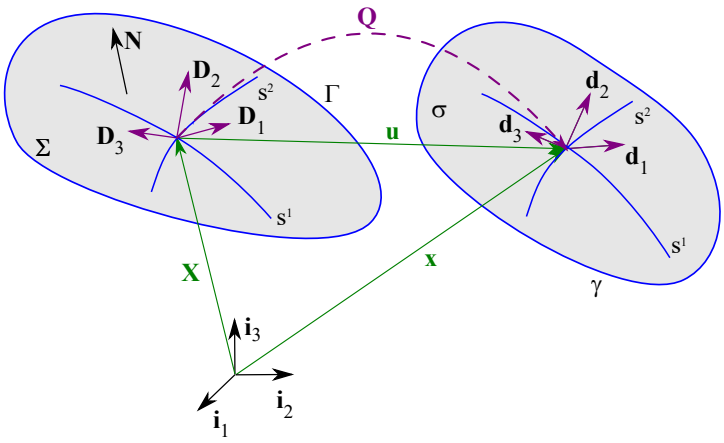

Figure 2. Deformation of a material surface $\mathcal{S}$.

4 Eshelby tensor and conservation laws: elastic membrane

In order to discuss a derivation of conservation laws for thin structures modeled using the minimal surface property, first let us study a simple case, that is an elastic membrane.

4.1 Finite deformations

For a hyperelastic membrane there exists a surface strain energy W. In what follows we restrict ourselves to homogeneous membranes, so W does not depend on $\mathbf{X} \in \Sigma$. So it is a function of the surface deformation gradient $\mathbf{F} = \nabla\mathbf{x}$:

$$W = W(\mathbf{F}). \tag{11}$$

Applying to (11) the material frame-indifference principle[39] we came to the dependence

$$W = W(\mathbf{C}), \tag{12}$$

where $\mathbf{C} = \mathbf{F} \cdot \mathbf{F}^T$ is the surface Cauchy-Green strain measure[29]. Note that for simplicity we keep in (12) the same notation for the energy function.

Neglecting surface forces we have the following Lagrangian equilibrium equation

$$\nabla \cdot \mathbf{P} = \mathbf{0}, \tag{13}$$

where $\mathbf{P}$ is the surface first Piola-Kirchhoff stress tensor. It is given by the formulae

$$\mathbf{P} = \frac{\partial W}{\partial \mathbf{F}} = \mathbf{S} \cdot \mathbf{F}, \quad \mathbf{S} = 2\frac{\partial W}{\partial \mathbf{C}}.$$

Here $\mathbf{S}$ is the surface second Piola-Kirchhoff stress tensor. Note that $\mathbf{N} \cdot \mathbf{P} = \mathbf{0}$. Eqs. (11)–(13) constitute a 2D counterpart of governing equations of the 3D nonlinear elasticity. So an elastic membrane model could be treated as a 2D Cauchy continuum, or as a 2D simple medium in sense of Noll[39;40].

For an elastic membrane the Eshelby tensor is defined as follows

$$\mathbf{B}_m = W\mathbf{A} - \mathbf{P} \cdot \mathbf{F}^T = W\mathbf{A} - \mathbf{S} \cdot \mathbf{C}, \tag{14}$$

where $\mathbf{A} = \mathbf{I} - \mathbf{N} \otimes \mathbf{N}$ and $\mathbf{I}$ is the 3D unit tensor. So, by definition $\mathbf{N} \cdot \mathbf{B}_m = \mathbf{0}$. $\mathbf{B}_m$ has also the following property

$$\nabla \cdot \mathbf{B}_m = 2HW\mathbf{N}. \tag{15}$$

Indeed, using the identities

$$\nabla \cdot \mathbf{A} = \nabla \cdot (\mathbf{I} - \mathbf{N} \otimes \mathbf{N}) = -(\nabla \cdot \mathbf{N})\mathbf{N} = 2H\mathbf{N}, \tag{16}$$

$$\nabla W = \mathbf{E}^\beta \frac{\partial W}{\partial s^\beta} = \nabla\mathbf{F} : \mathbf{P}, \tag{17}$$

we have that

$$\begin{aligned}\nabla \cdot \mathbf{B}_m =& \nabla \cdot (W\mathbf{A} - \mathbf{P} \cdot \mathbf{F}^T)\\ =& (\nabla W) \cdot \mathbf{A} + W\nabla \cdot \mathbf{A} - (\nabla \cdot \mathbf{P}) \cdot \mathbf{F}^T - \mathbf{E}^\alpha \cdot \mathbf{P} \cdot \frac{\partial \mathbf{F}^T}{\partial s^\alpha}\\ =& \nabla\mathbf{F} : \mathbf{P} + 2HW\mathbf{N} - \mathbf{P} : \nabla\mathbf{F}^T = 2HW\mathbf{N}.\end{aligned}$$

Here ":" stands for the double dot product. For dyads and triads of vectors it could be defined as follows

$$\begin{aligned}(\mathbf{a} \otimes \mathbf{b}) : (\mathbf{c} \otimes \mathbf{d}) &= (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}),\\ (\mathbf{a} \otimes \mathbf{b}) : (\mathbf{c} \otimes \mathbf{d} \otimes \mathbf{e}) &= (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d})\mathbf{e},\\ (\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c}) : (\mathbf{d} \otimes \mathbf{e}) &= (\mathbf{b} \cdot \mathbf{d})(\mathbf{c} \cdot \mathbf{e})\mathbf{a},\end{aligned}$$

and by linearity could be extended for tensors of any order.

Using the surface divergence theorem (4) for $\mathbf{B}_m$ we get the identity

$$\iint\limits_{\Sigma_\Upsilon} 2HW\mathbf{N}\, d\Sigma = \oint\limits_{\Upsilon} \boldsymbol{\nu} \cdot \mathbf{B}_m\, ds, \tag{18}$$

for any part $\Sigma_\Upsilon \subset \Sigma$ with the boundary Υ. Eq. (18) shows that the right side of (18) does not constitute a path-independent integral, in general.

Instead, for a minimal surface $H = 0$ and we get the conservation law

$$\nabla \cdot \mathbf{B}_m = \mathbf{0}, \tag{19}$$

and with (4) we came to the path-independent J-integral

$$J_m \equiv \oint\limits_\Upsilon \boldsymbol{\nu} \cdot \mathbf{B}_m \, ds = \mathbf{0} \tag{20}$$

Using (13) and (19) we can derive another useful integral identity. First, we have two relations

$$\nabla \cdot (\mathbf{B}_m \cdot \mathbf{X}) = \operatorname{tr} \mathbf{B}_m, \quad \operatorname{tr} \mathbf{B}_m = 2W - \mathbf{P} : \mathbf{F}. \tag{21}$$

Then, we can see that

$$\oint\limits_\Upsilon \boldsymbol{\nu} \cdot \mathbf{B}_m \cdot \mathbf{X} \, ds = \iint\limits_{\Sigma_\Upsilon} \nabla \cdot (\mathbf{B}_m \cdot \mathbf{X}) \, d\Sigma = 2 \iint\limits_{\Sigma_\Upsilon} W \, d\Sigma - \iint\limits_{\Sigma_\Upsilon} \mathbf{P} : \mathbf{F} \, d\Sigma. \tag{22}$$

Here $\Sigma_\Upsilon \subset \Sigma$ is an area bounded by Υ. The last integral in (22) could be transformed as follows

$$\iint\limits_{\Sigma_\Upsilon} \mathbf{P} : \mathbf{F} \, d\Sigma = - \iint\limits_{\Sigma_\Upsilon} \nabla \cdot (\mathbf{P} \cdot \mathbf{x}) \, d\Sigma + \oint\limits_\Upsilon \boldsymbol{\nu} \cdot \mathbf{P} \cdot \mathbf{x} \, ds = \oint\limits_\Upsilon \boldsymbol{\nu} \cdot \mathbf{P} \cdot \mathbf{x} \, ds. \tag{23}$$

As a result, Eq. (22) takes the form

$$\oint\limits_\Upsilon \boldsymbol{\nu} \cdot \mathbf{B}_m \cdot \mathbf{X} \, ds = 2 \iint\limits_{\Sigma_\Upsilon} W \, d\Sigma - \oint\limits_\Upsilon \boldsymbol{\nu} \cdot \mathbf{P} \cdot \mathbf{x} \, ds. \tag{24}$$

So we came to

$$\oint\limits_\Upsilon \boldsymbol{\nu} \cdot \left[\mathbf{B}_m \cdot \mathbf{X} + \mathbf{P} \cdot \mathbf{x} \right] \, ds = 2 \iint\limits_{\Sigma_\Upsilon} W \, d\Sigma. \tag{25}$$

4.2 Infinitesimal deformations

Surface integral in (25) could be transformed into a contour one only for very particular cases such as small deformations. Let us consider it in more detail. For infinitesimal deformations W takes the form

$$W = W(\boldsymbol{\varepsilon}), \quad \boldsymbol{\varepsilon} = \frac{1}{2} \left(\nabla \mathbf{u} \cdot \mathbf{A} + \mathbf{A} \cdot (\nabla \mathbf{u})^T \right), \tag{26}$$

where $\boldsymbol{\varepsilon}$ is a linear surface strain tensor. Equations of equilibrium transform into

$$\nabla \cdot \boldsymbol{\sigma} = \mathbf{0}, \quad \boldsymbol{\sigma} = \frac{\partial W}{\partial \boldsymbol{\varepsilon}}, \tag{27}$$

where $\boldsymbol{\sigma}$ is the symmetric surface stress tensor. The Eshelby tensor is modified as follows

$$\mathbf{B}_{sm} = W\mathbf{A} - \boldsymbol{\sigma} \cdot (\nabla \mathbf{u})^T. \tag{28}$$

Repeating derivations (21)–(24) we came to the identity

$$\oint_{\Upsilon} \boldsymbol{\nu} \cdot [\mathbf{B}_{sm} \cdot \mathbf{X} + \boldsymbol{\sigma} \cdot \mathbf{u}] \; ds = 2 \iint_{\Sigma_\Upsilon} W \, d\Sigma. \tag{29}$$

For a linear membrane $W = \frac{1}{2}\boldsymbol{\sigma} : \boldsymbol{\varepsilon}$ and $\iint_{\Sigma_\Upsilon} W \, d\Sigma$ could be represented as a contour integral. Indeed, we have

$$\begin{aligned} \iint_{\Sigma_\Upsilon} W \, d\Sigma =& \frac{1}{2} \iint_{\Sigma_\Upsilon} \boldsymbol{\sigma} : \boldsymbol{\varepsilon} \, d\Sigma = \frac{1}{2} \iint_{\Sigma_\Upsilon} \boldsymbol{\sigma} : \nabla \mathbf{u} \, d\Sigma \\ =& -\frac{1}{2} \iint_{\Sigma_\Upsilon} (\nabla \cdot \boldsymbol{\sigma}) \cdot \mathbf{u} \, d\Sigma + \frac{1}{2} \oint_{\Upsilon} \boldsymbol{\nu} \cdot \boldsymbol{\sigma} \cdot \; ds \\ =& \frac{1}{2} \oint_{\Upsilon} \boldsymbol{\nu} \cdot \boldsymbol{\sigma} \cdot \; ds. \end{aligned} \tag{30}$$

So instead of (29) we get the formula

$$\oint_{\Upsilon} \boldsymbol{\nu} \cdot \mathbf{B}_{sm} \cdot \mathbf{X} \, ds = 0. \tag{31}$$

As in the case of plane stress state[5], in the theory of linear membranes path-independent integral (31) could be called M-integral.

A M-integral-type identity could be also derived for a power-law constitutive relation, that is for W given by

$$W = \frac{1}{2m} \left(\boldsymbol{\varepsilon} : \mathbf{C} : \boldsymbol{\varepsilon} \right)^m, \tag{32}$$

where $\mathbf{C}$ and m are material parameters, and a fourth-order tensor $\mathbf{C}$ has the same symmetry properties as in the case of linear plane stress elasticity. So we have that

$$\boldsymbol{\sigma} = \left(\boldsymbol{\varepsilon} : \mathbf{C} : \boldsymbol{\varepsilon} \right)^{m-1} \mathbf{C} : \boldsymbol{\varepsilon}, \quad W = \frac{1}{2m} \boldsymbol{\sigma} : \boldsymbol{\varepsilon},$$

and instead of (31) we came to another M-integral

$$\oint_{\Upsilon} \boldsymbol{\nu} \cdot \left[\mathbf{B}_{sm} \cdot \mathbf{X} + \frac{m-1}{m} \boldsymbol{\sigma} \cdot \mathbf{u} \right] \; ds = 0. \tag{33}$$

Constitutive equation (32) could be useful for modelling of some hardening phenomena in inelastic materials. In fact, in plasticity they used power law-type constitutive equations like $\sigma = K\varepsilon^n$, where K is a strength coefficient and n is an exponent, see, e.g., [41] p. 94. Power-law constitutive relations such as Norton's law are also widely used the theory of plasticity and creep, see, [42;43] and the references therein. For example, J- and M-integrals for power-law materials were used in [44–46] in order to estimate a stress concentration in vicinity of crack tips.

5 Eshelby tensor and conservation laws: micropolar (six-parameter) shell

As an example of more complex 2D model we consider micropolar or six-parameter shells[30–32]. Within the model we have an extended kinematics wich includes two kinematically independent fields of translations and rotations.

5.1 Finite deformations

For a hyperelastic micropolar (six-parameter) shell, a surface strain energy density U depends on two surface strain measures $\mathbf{E}$ and $\mathbf{K}$[30;37]

$$U = U(\mathbf{E}, \mathbf{K}), \tag{34}$$

where

$$\mathbf{E} = \mathbf{F} \cdot \mathbf{Q}^T - \mathbf{A}, \quad \mathbf{K} = \frac{1}{2}\mathbf{E}^\alpha \otimes \left(\frac{\partial \mathbf{Q}}{\partial s^\alpha} \cdot \mathbf{Q}^T \right)_\times. \tag{35}$$

Here $\mathbf{T}_\times$ is the vectorial invariant of a second-order tensor $\mathbf{T}$, see, e.g.,[37]. For a dyad of two vectors $\mathbf{a}$ and $\mathbf{b}$ we have $(\mathbf{a} \otimes \mathbf{b}) = \mathbf{a} \times \mathbf{b}$. In what follows we consider only homogeneous shells that is shells whose strain energy density depends only on the strain measures $\mathbf{E}$ and $\mathbf{K}$.

Without surface forces and couples the Lagrangian equations of statics takes the form

$$\nabla \cdot \mathbf{T} = \mathbf{0}, \quad \nabla \cdot \mathbf{M} + \left[\mathbf{F}^T \cdot \mathbf{T}\right]_\times = \mathbf{0}, \tag{36}$$

where

$$\mathbf{T} = \frac{\partial U}{\partial \mathbf{F}} = \mathbf{S}_1 \cdot \mathbf{Q}, \quad \mathbf{M} = \mathbf{S}_2 \cdot \mathbf{Q}, \quad \mathbf{S}_1 = \frac{\partial U}{\partial \mathbf{E}}, \quad \mathbf{S}_2 = \frac{\partial U}{\partial \mathbf{K}}. \tag{37}$$

Here $\mathbf{T}$ and $\mathbf{M}_\varkappa$ are the surface stress and couple stress tensors of the first Piola–Kirchhoff type, whereas the stress measures $\mathbf{S}_1$ and $\mathbf{S}_2$ are the referential stress and couple stress tensors similar to the respective second Piola–Kirchhoff stress tensors of 3D nonlinear elasticity.

Within the six-parameter shell model the Eshelby tensor $\mathbf{B}$ was introduced in[33] for description of stress-induced phase transitions. More precisely, using $\mathbf{B}$ the thermodynamic compatibility condition along a phase interface was formulated. $\mathbf{B}$ is defined as follows

$$\mathbf{B} = U\mathbf{A} - \mathbf{T} \cdot \mathbf{F}^T - \mathbf{M} \cdot \mathbf{Q}^T \cdot \mathbf{K}^T, \tag{38}$$

or as

$$\mathbf{B} = U\mathbf{A} - \mathbf{S}_1 \cdot \mathbf{E}^T - \mathbf{S}_2 \cdot \mathbf{K}^T, \tag{39}$$

In[34] it was shown that under some conditions $\mathbf{B}$ could be obtained from its 3D counterpart using the through-the-thickness integration similar to derivation of stress resultants[32].

For $\mathbf{B}$ we have the identity

$$\nabla \cdot \mathbf{B} = 2HU\mathbf{N}, \tag{40}$$

which could be proven similarly to (15). For brevity we omit awkward calculations here.

As a result, we came to the integral identity

$$\iint\limits_{\Sigma_\Upsilon} 2HU\mathbf{N}\,d\Sigma = \oint\limits_{\Upsilon} \boldsymbol{\nu}\cdot\mathbf{B}\,ds, \tag{41}$$

Again, for a minimal surface we get the conservation law and J-integral

$$\nabla\cdot\mathbf{B}=\mathbf{0}, \quad J\equiv\oint\limits_{\Upsilon}\boldsymbol{\nu}\cdot\mathbf{B}\,ds=\mathbf{0}. \tag{42}$$

This conservation law is the 2D counterpart of the 3D one derived for nonlinear micropolar continua in [47;48] with the use of Noether's theorem.

5.2 Small deformations

In the case of small deformations we can provide a similar study of conservation laws. For small rotations instead of the microrotation tensor $\mathbf{Q}$ one can use the infinitesimal vector $\boldsymbol{\phi}$ since $\mathbf{Q}$ can be approximated as follows [30;49]

$$\mathbf{Q}\approx\mathbf{I}+\boldsymbol{\phi}\times\mathbf{I}.$$

As a result, $\mathbf{E}$ and $\mathbf{K}$ could be replaced by the linear strain measures $\mathbf{e}$ and $\mathbf{k}$

$$\mathbf{e}=\nabla\mathbf{u}-\mathbf{I}\times\boldsymbol{\phi}, \quad \mathbf{k}=\nabla\boldsymbol{\phi}.$$

Equilibrium equations take the form

$$\nabla\cdot\mathbf{T}=\mathbf{0}, \quad \nabla\cdot\mathbf{M}+\mathbf{T}_\times=\mathbf{0} \tag{43}$$

with surface stress $\mathbf{T}$ and couple stress $\mathbf{M}$ tensors. The latter relate to a surface strain energy density through the formulae

$$\mathbf{T}=\frac{\partial U}{\partial\mathbf{e}}, \quad \mathbf{M}=\frac{\partial U}{\partial\mathbf{k}}, \quad U=U(\mathbf{e},\mathbf{k}). \tag{44}$$

The Eshelby tensor $\mathbf{B}_s$ has the form

$$\mathbf{B}_s=U\mathbf{A}-\mathbf{T}\cdot(\nabla\mathbf{u})^T-\mathbf{M}\cdot(\nabla\boldsymbol{\phi})^T. \tag{45}$$

Let us note that (45) is symmetrized with respect to translations and rotations. Indeed, here we face a full symmetry under replacements

$$\mathbf{u}\leftrightarrows\boldsymbol{\phi}, \quad \mathbf{T}\leftrightarrows\mathbf{M}.$$

We can again prove the identity

$$\nabla\cdot\mathbf{B}_s=2HU\mathbf{N}. \tag{46}$$

Unlike the case of finite deformations, here the calculations are more simple. Indeed, similar to (17) we have

$$\nabla U = \nabla \mathbf{e} : \mathbf{T} + \nabla \mathbf{k} : \mathbf{M} = \nabla\nabla \mathbf{u} : \mathbf{T} + \nabla\nabla \boldsymbol{\phi} : \mathbf{M} - \nabla(\mathbf{I} \times \boldsymbol{\phi}) : \mathbf{T},$$

and we came to a series of identities

$$\begin{aligned}\nabla \cdot \mathbf{B}_s =& \nabla U + 2HU\mathbf{N} - (\nabla \cdot \mathbf{T}) \cdot (\nabla \mathbf{u})^T - (\nabla \cdot \mathbf{M}) \cdot (\nabla \boldsymbol{\phi})^T \\ & - \mathbf{T} : \nabla(\nabla \mathbf{u})^T - \mathbf{M} : \nabla(\nabla \boldsymbol{\phi})^T \\ =& \nabla\nabla \mathbf{u} : \mathbf{T} - \mathbf{T} : \nabla(\nabla \mathbf{u})^T + \nabla\nabla \boldsymbol{\phi} : \mathbf{M} - \mathbf{M} : \nabla(\nabla \boldsymbol{\phi})^T \\ & - \nabla(\mathbf{I} \times \boldsymbol{\phi}) : \mathbf{T} - \mathbf{T}_\times \cdot (\nabla \boldsymbol{\phi})^T + 2HU\mathbf{N} = 2HU\mathbf{N},\end{aligned}$$

that results in (46). Thus, for a minimal surface we get the conservation law and corresponding J-integral for small deformations

$$\nabla \cdot \mathbf{B}_s = \mathbf{0}, \quad J \equiv \oint_\Upsilon \boldsymbol{\nu} \cdot \mathbf{B}_s \, ds = \mathbf{0}. \tag{47}$$

This conservation law is similar to 3D analogous in linear micropolar elasticity, see, e.g.,[50]. Unlike the case of elastic membrane, as for micropolar 3D solids[50] M-integral for micropolar shell cannot be derived, in general. It could be possible for a particular class of constitutive equations with symmetric stress resultant tensor, $\mathbf{T}_\times = \mathbf{0}$, or for decoupled relations with an energy in the form $U = U(\boldsymbol{\varepsilon}, \mathbf{k})$, see[51] for a discussion on this class of 3D constitutive relations.

6 On 3D-to-2D reduction of conservation laws

We have discussed so-called direct approach to bidimensional structures. Within the approach the basic governing equations, i.e. equations of equilibrium and constitutive equations, are formulated as for a 2D continuum. As a result, additional non-trivial conservation laws could be derived using these 2D governing equations as was demonstrated above. In other words, derived 2D conservation laws are *exact* consequence of 2D equilibrium and constitutive equations.

An alternative way could be a 3D-to-2D reduction also applied to 3D conservation laws. Any 3D-to-2D reduction results in 2D equations, so one could also apply it to 3D conservation laws. Let us note that any reduction procedure results in an *approximate* representation of a 3D state through its 2D counterpart. Using such an approach one should be aware of the following:

– obtained conservation law could be reduction-dependent, i.e. could depend on the chosen reduction procedure. Indeed, in the literature there are known various shell models which have different conservative laws, in general;

– reduction of a conservation law could result in an identity which is not a 2D conservation law according to definition (1) and (2).

Let us discuss this matter in more detail using the through-the-thickness procedure. This 3D-to-2D reduction leads to the nonlinear resultants six-parameter shell theory[32]. For finite deformations of a nonlinear elastic solids we have the Lagrangian equitation

of equilibrium and the conservation law for the Eshelby tensor $\mathbf{b}$ in the form[2]

$$\mathrm{Div}\,\boldsymbol{\Sigma} = \mathbf{0}, \tag{48}$$

$$\mathrm{Div}\,\mathbf{b} = \mathbf{0}, \qquad \mathbf{b} = V\mathbf{I} - \boldsymbol{\Sigma}\cdot\mathbf{G}^T, \tag{49}$$

where $\boldsymbol{\Sigma}$ is the first Piola–Kirchhoff stress tensor, $\mathbf{G} = \mathrm{Grad}\,\mathbf{r}$ is the deformation gradient, V is a strain energy function, $\mathbf{r}$ is a 3D position vector in a current placement, Div and Grad are Lagrangian divergence and gradient operators, respectively.

Following[32;37] we consider deformations of a shell-like body $\mathcal{B}$ as a differentiable invertible mapping from a reference placement into a current one. Let $V = \{(s^1, s^2, \zeta) : (s^1, s^2) \in \Sigma, \zeta \in [-h^-, h^+]\}$ be a volume of $\mathcal{B}$ in the reference placement, where Σ is a referential base surface, $h = h^- + h^+$ is the shell total thickness, see Fig. 3. So the position vector of a given point z of $\mathcal{B}$ in the reference placement is given by

$$\mathbf{R} = \mathbf{X}(s^1, s^2) + \zeta\mathbf{N}.$$

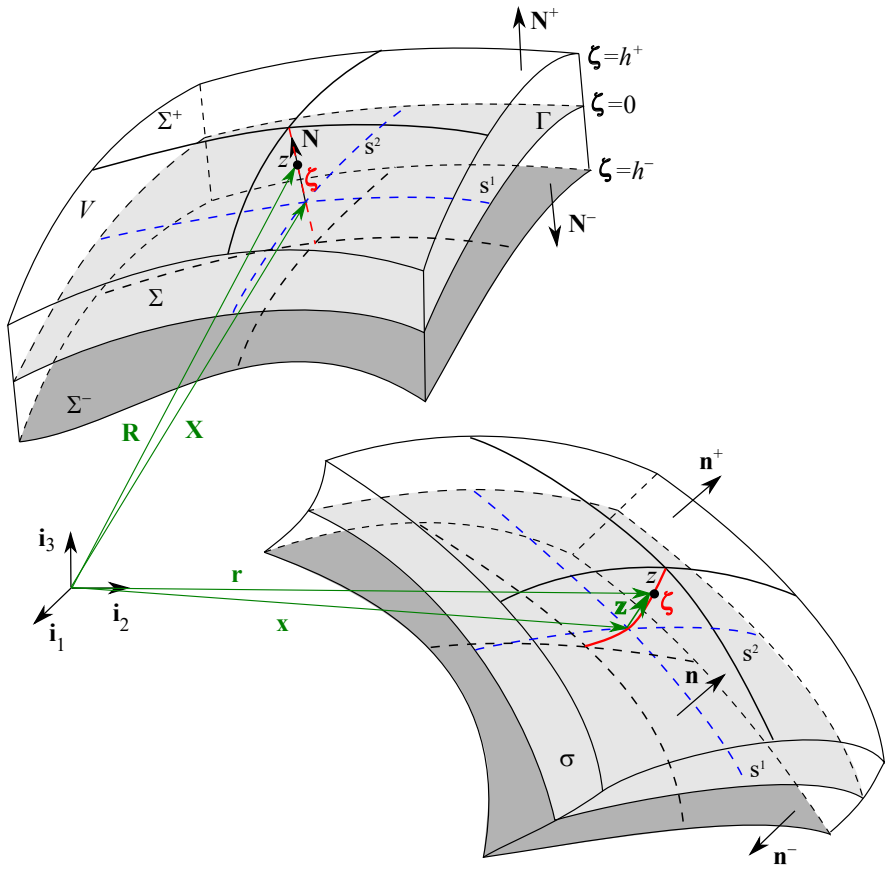

Figure 3. Deformation of a shell-like body $\mathcal{B}$.

In a current placement z could be represented through its position vector

$$\mathbf{r} = \mathbf{x}(s^1, s^2) + \mathbf{z}(s^1, s^2, \zeta),$$

where $\mathbf{z} = \mathbf{r} - \mathbf{x}$ is called the base reference deviation vector[32].

Integrating (48) through the thickness we came to $(43)_1$ with $\mathbf{T}$ defined as follows[37]

$$\mathbf{T} = \int_{-h^-}^{h^+} (\mathbf{A} - \zeta\mathbf{H})^{-1} \cdot \boldsymbol{\Sigma}\, \mu\, d\zeta, \tag{50}$$

where $\mathbf{H} = -\nabla\mathbf{N}$ is the curvature tensor of Σ and μ is the scale factor defined by the formulae

$$dV = \mu\, d\zeta\, d\Sigma, \quad \mu \equiv \det(\mathbf{A} - \zeta\mathbf{H}) = 1 - 2H\zeta + K\zeta^2,$$

$K \equiv \det \mathbf{H}$ is the Gaussian curvature of Σ. Note that here we have used the assumption

$$\mathbf{N} \cdot \boldsymbol{\Sigma}\Big|_{\Sigma^\pm} = \mathbf{0}, \tag{51}$$

i.e. we assumed that faces Σ^- and Σ^+ are free.

To derive $(43)_2$ we cross-multiply (48) by $\mathbf{z}$ from the left and integrate the result through the thickness. We get $(43)_2$ with $\mathbf{M}$ defined as

$$\mathbf{M} = \int_{-h^-}^{h^+} (\mathbf{A} - \zeta\mathbf{H})^{-1} \cdot \boldsymbol{\Sigma} \times \mathbf{z}\, \mu\, d\zeta. \tag{52}$$

So (43) are exact consequence of the 3D equilibrium equations.

Similarly, the 2D strain energy density U could be introduced as follows

$$U = \int_{-h^-}^{h^+} V\, \mu\, d\zeta. \tag{53}$$

Obviously, within this 3D-to-2D reduction some part of an energy stored in $\mathcal{B}$ could be lost, see[32] for more details.

Let us now repeat the same integration technique to (49). Similarly, we came to

$$\nabla \cdot \widehat{\mathbf{B}} + \mu^+ V^+ \mathbf{N}^+ + \mu^- V^- \mathbf{N}^- = \mathbf{0}, \tag{54}$$

$$\widehat{\mathbf{B}} = \int_{-h^-}^{h^+} (\mathbf{A} - \zeta\mathbf{H})^{-1} \cdot \mathbf{b}\, \mu\, d\zeta, \tag{55}$$

where we have used (51), $\mu^\pm$ and $V^\pm$ are the values of μ and V taken on $\Sigma^\pm$, respectively, i.e. at $\zeta = \pm h^\pm$, and $\mathbf{N}^-$ and $\mathbf{N}^+$ are unit outward normals to $\Sigma^\pm$, see Fig. 3. Obviously, $\widehat{\mathbf{B}}$ does not coincide nor with $\mathbf{B}_s$ neither with $\mathbf{B}$. Moreover, Eq. (54)

does not constitute a conservation law, in general. So one has to apply additional assumptions of kinematical and/or smallness type to get a conservation law.

As an example, let us transform (54) and (55) to the case of an elastic nonlinear membrane. First, we restrict ourselves to a symmetric case $h^+ = h^- = h/2$. In addition we assume that $\mathbf{G}$ does not depend on ζ or that such dependence is negligible. So for $\mathbf{G}$ we use an approximation $\mathbf{G} = \mathbf{F} + \mathbf{N} \otimes \mathbf{n}$, where $\mathbf{n}$ is a normal to σ. As a result, V does not depend on ζ. For a thin enough structure we also assume that $\mathbf{N}^{\pm} = \pm\mathbf{N}$. As a result, we get the formulae

$$\mu^{\pm} = 1 \mp Hh + \frac{1}{4}Kh^2, \quad \mu^+ V^+ \mathbf{N}^+ + \mu^- V^- \mathbf{N}^- = 2HhV\mathbf{N}.$$

With these assumptions $\widehat{\mathbf{B}}$ transforms into

$$\widehat{\mathbf{B}} = hV\mathbf{A} - \int\limits_{-h/2}^{h/2} \mathbf{A} \cdot \boldsymbol{\Sigma}\, d\zeta \cdot \mathbf{F}^T. \tag{56}$$

Thus, introducing W and $\mathbf{P}$ as follows

$$W = hV, \quad \mathbf{P} = \int\limits_{-h/2}^{h/2} \mathbf{A} \cdot \boldsymbol{\Sigma}\, d\zeta$$

we came to (15) with $\mathbf{B}_m = \widehat{\mathbf{B}}$. For derivation of 2D equations of elastic membranes we also refer to[32;52].

Transformation of (54) and (55) to the case of six-parameter (micropolar) shells can be provided similarly. It requires additional and more complex kinematical assumptions, see Eq. (28) in[34] for normal components of $\mathbf{b}$ and $\mathbf{B}$.

7 Conclusions

We have discussed a few conservation laws for thin-walled structures, i.e. elastic membranes and six-parameter shells, modelled using material minimal surface. Let us underline that minimal surfaces significantly extended a class of geometry of shells and membranes for which it is possible to introduce such conservation laws. Using the property of a minimal surface (8) we have demonstrated that conservation laws for two-dimensional systems are similar to the case of 3D nonlinear elasticity. With conservation laws one can derive invariant (path-independent) integrals such as J-integrals. The latter could be useful in mechanics of fracture, for example, for estimation of stress concentration in the vicinity of geometrical singularities such as holes, crack tips, notches, and rigid inclusions. Moreover, they could be related to the energy release rate for quasistatically propagating defects in thin structures.

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Declaration of Conflicting Interests

The author declares that he has no conflict of interest.

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