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Ellipticity of gradient poroelasticity

Victor A. Eremeyev[a,b,c,*]

[a] *University of Cagliari, Via Marengo, 2, 09123 Cagliari, Italy*
[b] *Gdańsk University of Technology, ul. Gabriela Narutowicza 11/12, 80-233 Gdańsk, Poland*
[c] *National Research Lobachevsky State University of Nizhni Novgorod, pr. Gagarin, 23, 603022 Russia*

Abstract

We discuss the ellipticity properties of an enhanced model of poroelastic continua called dilatational strain gradient elasticity. Within the theory there exists a deformation energy density given as a function of strains and gradient of dilatation. We show that the equilibrium equations are elliptic in the sense of Douglis–Nirenberg. These conditions are more general than the ordinary and strong ellipticity but keep almost all necessary properties of equilibrium equations. In particular, the loss of the ellipticity could be considered as a criterion of a strain localization or material instability.

Keywords: Douglis–Nirenberg ellipticity, strong ellipticity, strain gradient elasticity, poroelasticity, dilatational strain gradient elasticity

1. Introduction

Partial differential equations (PDEs) constitute a basis of physics and mechanics of solids and fluids. Considering systems of PDEs we usually distinguish hyperbolic, parabolic and elliptic systems. The latter almost relate to statics or to quasistatics. Among definitions of elliptic systems of PDEs one can find ordinary ellipticity or Petrowsky ellipticity (Petrowsky, 1939), strong ellipticity (Vishik, 1951; Nirenberg, 1955), Douglas–Nirenberg ellipticity (Douglis & Nirenberg, 1955), or even more general definitions (Volevich, 1965), see also Agranovich (1997). From the mathematical point of

*Corresponding author
Email address: eremeyev.victor@gmail.com; victor.eremeev@unica.it (Victor A. Eremeyev)

view ellipticity brings regularity of solutions, solvability and well-posedness of corresponding boundary-value problems. From the physical point of view, a violation of ellipticity may result in a certain material instability such as a strain localization, folding, and appearance of multiple solutions, it may also prevent wave propagation in certain points or in some directions. In particular, Hill (1962) and Rice (1976) considered loss of ellipticity as a criterion for detection of strain localization and transition to a plastic regime of deformation, see also Bigoni (2012); Staber et al. (2021) and the references therein. So the analysis of ellipticity conditions brings an essential a priori information about a solution of a problem under consideration and a possible material response. Moreover, even for finite deformations ellipticity conditions take a form of algebraic problem that is more simple, in general.

Within the classic nonlinear elasticity ellipticity conditions were analyzed in many works, summarized in (Lurie, 1990; Ogden, 1997; Truesdell & Noll, 2004). It was shown how the strong ellipticity and its weak form called Hadamard's inequality relate to infinitesimal stability. In particular, infinitesimal stability implies Hadamard's inequality. So the latter can serve as a necessary condition of stability and a violation of Hadamard's inequality can indicate possible instabilities. The converse statement, i.e. the strong ellipticity results in stability for a particular affine class of deformations and for Dirichlet's boundary conditions. Strong ellipticity was studied also for so-called implicit constitutive relations in Mai & Walton (2015a,b).

For the enhanced models of continua such as micropolar and strain gradient media, the connection of ellipticity with strain localization phenomena and material instabilities is similar to the case of simple materials, in general. Localization of deformations in micropolar elastoplastic solids with connections to the loss of ellipticity was studied by De Borst (1991); De Borst et al. (1993); De Borst & Muhlhaus (1992); Dietsche et al. (1993); Tejchman & Bauer (1996), see also more recent papers by Hasanyan & Waas (2018); Russo et al. (2020) and the references therein. Ellipticity and its relation to waves propagation and instabilities in micropolar solids was analysed by Eremeyev (2005); Lakes (2018, 2021); Soldatos et al. (2021); Passarella et al. (2011). Another model of continua related to strain localization is based on strain gradient approach, see e.g. Muhlhaus & Aifantis (1991); Fleck & Hutchinson (2001, 1993). Ellipticity within the strain gradient elasticity was considered by Eremeyev (2021); Eremeyev & Reccia (2022); Eremeyev & Lazar (2022).

Considering strain gradient media it is worth to mention the couple-stress theory introduced by Koiter (1964); Toupin (1962); Mindlin & Tiersten (1962) as a possible simplest version of the strain gradient models. With its modified version by Yang et al. (2002) it was used for modelling materials and thin-walled structures at the micro- and nanoscales, see e.g. recent papers by Zhang & Liu (2020); Farajpour et al. (2020); Dastjerdi et al. (2020); Malikan et al. (2020); Malikan & Eremeyev (2023); Nobili & Volpini (2021); Dehrouyeh-Semnani & Mostafaei (2021); Dehrouyeh-Semnani (2021); Dastjerdi et al. (2021); Shahmohammadi et al. (2023), and the reviews by Ghayesh & Farajpour (2019); Kong (2021). The couple-stress theory could be also considered as a micropolar medium with constraint rotations, see Nowacki (1970); Eremeyev et al. (2013). Within the couple-stress theory the loss of ellipticity and related material instabilities were analysed by Gourgiotis & Bigoni (2016a,b, 2017); Bigoni & Gourgiotis (2016). In particular, in (Gourgiotis & Bigoni, 2016a) it was remarked that the principal part of the symbol of the operator is degenerate, so that the system of PDEs in couple-stress elasticity is not elliptic in the standard sense. Nonetheless, by considering a modified equivalent couple-stress operator adding to the governing operator an additional fourth-order operator as a sort of null Lagrangian, ellipticity conditions may still be defined. These procedure involved not only the fourth-order part of the couple stress operator but also the second-order part of the operator. It is closely related to the fact that P-waves are not dispersive in couple stress theory, but S-waves are.. For an isotropic and orthotropic materials the modified conditions of ellipticity were given in (Gourgiotis & Bigoni, 2016a,b), where one can see that loss of ellipticity results in folding of the material. Similar observation on non-ellipticity was made by Eremeyev et al. (2023), where another transformation towards elliptic formulation was done.

The aim of this paper is to discuss ellipticity conditions for the dilatational strain gradient elasticity (Eremeyev et al., 2021; Lurie et al., 2021). In the case of small deformations this model complements the couple-stress theory by Mindlin & Tiersten (1962) to the gradient complete Toupin–Mindlin strain gradient elasticity (Toupin, 1962; Mindlin & Eshel, 1968; Mindlin, 1964). The model can be applied to pressure sensitive materials such as considered in the poroelasticity by Nunziato & Cowin (1979); Cowin & Nunziato (1983); Coussy (2004). In this case a possible violation of ellipticity may model pressure-induced phase changes in porous solids or other localization phenomena. As an example, one can mention materials with voids and related analysis given by Chiriţă & Ghiba (2010). Let us also note that the discussed model belongs to the class of constitutive equations with scalar microstructure Capriz (1989); Eringen (1999). Among of such media it is worth to mention two-phase mixtures (Clayton, 2022) and other models of porous media discussed in (Sciarra et al., 2008; Liu et al., 2021; Rajagopal, 2021; Zheng et al., 2022; Kazemian et al., 2022; Zhou et al., 2023; Ma et al., 2022).

The reminder of the paper is organized as follows. In Section 2 we briefly recall the Douglis–Nirenberg ellipticity definition as in (Douglis & Nirenberg, 1955). The main content of the paper is given in Section 3. Here we introduce the governing equations of the dilatational strain gradient elasticity and show that the linearized equations does not form an elliptic in ordinary sense. Nevertheless, we can show that another form of equilibrium equations is elliptic in the Douglis–Nirenberg sense. This form is similar to one used for linearized Navier-Stokes equations of incompressible fluids which also form a Douglis–Nirenberg elliptic system (Volevich, 1965).

2. Douglis–Nirenberg ellipticity

Let us recall the definition of the Douglis–Nirenberg ellipticity. Let $\mathbf{w} = (w_1(\mathbf{X}), w_2(\mathbf{X}), \ldots, w_N(\mathbf{X}))$ be a vector of unknown functions, whereas $\mathbf{b} = (b_1(\mathbf{X}), b_2(\mathbf{X}), \ldots, b_N(\mathbf{X}))$ be a vector of given functions. For $\mathbf{w}(\mathbf{X})$ we consider the following system of linear differential equations

$$\mathcal{A}(\mathbf{X}, D)\mathbf{w} = \mathbf{b}, \tag{1}$$

or in the component form

$$\sum_{k=1}^{N} \mathcal{A}_{mk}(\mathbf{X}, D) w_k = b_m, \quad m = 1, \ldots, N. \tag{2}$$

Hereinafter we have used the following standard notations: $\mathbf{X} = (X_1, \ldots, X_n)$ is a position vector, X_p, $p = 1, \ldots, n$, are Cartesian coordinates. Moreover, $\mathcal{A}_{mk}(\mathbf{X}, D)$ is a linear differential operator of order α_{mk}

$$\mathcal{A}_{mk}(\mathbf{X}, D) = \sum_{|\alpha| \le \alpha_{mk}} a_{mk}^{(\alpha)}(\mathbf{X}) D^{\alpha}, \tag{3}$$

where $D = (D_1, \ldots, D_n)$, $D_p = -i\partial/\partial x_p$, $i = \sqrt{-1}$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multiindex, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $\alpha_p \ge 0$ are integers, $p = 1, \ldots, n$. In addition we assume that $\mathcal{A}_{mk} = 0$ if $\alpha_{mk} < 0$.

Following Douglis & Nirenberg (1955), we assume that $\alpha_{mk} = s_m + t_k$, where s_p and t_p are some integers. We introduce the principal symbol of (1) by the formula

$$A_0(\mathbf{X}, \boldsymbol{\xi}) = \det \mathbb{A}(\mathbf{X}, \boldsymbol{\xi}), \quad \mathbb{A}_{mk} = \sum_{|\alpha|=s_m+t_k} a_{mk}^{(\alpha)}(\mathbf{X})\boldsymbol{\xi}^{\alpha}, \quad \boldsymbol{\xi} \in \mathbb{R}^n. \qquad (4)$$

The Douglis–Nirenberg ellipticity at the point $\mathbf{X}$ means that

$$A_0(\mathbf{X}, \boldsymbol{\xi}) \neq 0, \quad \forall \;\; \boldsymbol{\xi} \in \mathbb{R}^n, \quad \boldsymbol{\xi} \neq \mathbf{0}. \qquad (5)$$

Within the Douglis–Nirenberg ellipticity one explicitly assumed that each equation and each dependent variable in (2) can have different orders of differentiation. Note that if $s_p = 0$ and $t_p = t$ we have the simplest case of ordinary ellipticity. Petrowsky considered also more general case with $s_p = 0$ and different t_p. Strong ellipticity conditions involves ordinary ellipticity.

3. Dilatational strain gradient elasticity

3.1. Governing equations

Following Eremeyev et al. (2021) let us briefly introduce the basic equations of the dilatational strain gradient elasticity. A deformation of an elastic solid body can be modelled as an invertible differentiable mapping

$$\mathbf{x} = \mathbf{x}(\mathbf{X}),$$

where $\mathbf{x}$ and $\mathbf{X}$ are position vectors in a reference and current placement, respectively. Within the model there exists a strain energy density introduced as a function of deformation gradient $\mathbf{F}$ and the gradient of its determinant J, i.e. the gradient of volume change,

$$W = W(\mathbf{F}, \mathbf{k}), \quad \mathbf{F} = \nabla \mathbf{x}, \quad \mathbf{k} = \nabla J, \quad J = \det \mathbf{F}, \qquad (6)$$

where ∇ is the referential nabla-operator.

The Lagrangian equilibrium equations take the form (Eremeyev et al., 2021)

$$\nabla \cdot \mathbf{P} - \nabla \cdot \left[(\nabla \cdot \mathbf{m}) J \mathbf{F}^{-T} \right] + \rho \mathbf{f} = \mathbf{0}, \qquad (7)$$

where $\mathbf{P}$ and $\mathbf{m}$ are the first Piola–Kirchhoff stress tensor and the first Piola–Kirchhoff double force vector, "$\cdot$" stands for the dot product, ρ is a mass density in the reference placement, and $\mathbf{f}$ is a mass force vector. $\mathbf{P}$ and $\mathbf{m}$ are given by formulae

$$\mathbf{P} = \frac{\partial W}{\partial \mathbf{F}}, \quad \mathbf{m} = \frac{\partial W}{\partial \mathbf{k}}.$$

The case of small deformations was also studied by Eremeyev et al. (2021); Lurie et al. (2021). For an isotropic solid the strain energy density has the form

$$W = \frac{1}{2}\lambda e^2 + \mu \boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} + \frac{1}{2}\beta \mathbf{k}\cdot\mathbf{k}, \tag{8}$$

where

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T), \quad \mathbf{u} = \mathbf{x} - \mathbf{X},$$
$$e = \operatorname{tr}\boldsymbol{\varepsilon} = \nabla\cdot\mathbf{u}, \quad \mathbf{k} = \nabla e = \nabla\nabla\cdot\mathbf{u},$$

λ and μ are Lamé moduli, β is an additional elastic modulus related to gradient of dilatation, and ":" denotes the double dot product. The stress tensor and the double stress vector transform to

$$\mathbf{P} = \lambda e\mathbf{I} + 2\mu\boldsymbol{\varepsilon}, \quad \mathbf{m} = \beta\mathbf{k}, \tag{9}$$

whereas the equilibrium equation (7) takes the form

$$\mu\Delta\mathbf{u} + (\mu+\lambda)\nabla\nabla\cdot\mathbf{u} - \beta\Delta\nabla\nabla\cdot\mathbf{u} + \rho\mathbf{f} = \mathbf{0}, \quad \Delta = \nabla\cdot\nabla. \tag{10}$$

Hereinafter $\mathbf{I}$ is the 3D unit tensor.

3.2. Loss of ordinary ellipticity

The considered model is a particular case of the general strain gradient elasticity introduced by (Toupin, 1962; Mindlin & Eshel, 1968; Mindlin, 1964), see also Bertram & Forest (2020); Bertram (2023). So ordinary and strong ellipticity of (7) can be studied within general framework as in (Mareno & Healey, 2006; Eremeyev, 2021; Eremeyev & Lazar, 2022). Considering this model in the case of small deformations it was noted by Eremeyev et al. (2023) that the equilibrium equations in displacements (10) does constitute neither ordinary elliptic nor strongly elliptic system as the principal symbol is degenerated. The same conclusion is valid for finite deformations. Indeed, the principal symbol of (7) has the form of a dyad

$$\mathbb{A}(\boldsymbol{\xi}) = J^2\boldsymbol{\xi}\cdot\frac{\partial^2 W}{\partial\mathbf{k}\partial\mathbf{k}}\cdot\boldsymbol{\xi}\,(\boldsymbol{\xi}\cdot\mathbf{F}^{-T})\otimes(\boldsymbol{\xi}\cdot\mathbf{F}^{-T}), \tag{11}$$

where $\otimes$ is the dyadic product. Obviously, here $\det \mathbb{A}(\boldsymbol{\xi}) = 0$ and the conditions of ordinary ellipticity is violated. Since ordinary ellipticity is a necessary condition of the strong ellipticity, the latter is also violated.

3.3. Douglis–Nirenberg ellipticity

In order to bring ellipticity properties to the equilibrium equations we use a certain correspondence between the dilatational strain gradient elasticity and the poroelasticity by Nunziato & Cowin (1979). We reformulate the equilibrium equations as follows. First, we introduce a new scalar variable φ as an additional kinematical descriptor, "porosity" in the sense of the nonlinear poroelastticty. So a strain energy density takes the form

$$W = W(\mathbf{F}, \nabla\varphi).$$

Treating φ as independent field subjected to the constarint

$$\varphi = J \equiv \det \mathbf{F}, \tag{12}$$

we come to the following system of equations

$$\nabla \cdot \mathbf{P} - \nabla \cdot (\gamma J \mathbf{F}^{-T}) + \rho \mathbf{f} = \mathbf{0}, \quad \mathbf{P} = \frac{\partial W}{\partial \mathbf{F}}, \tag{13}$$

$$\nabla \cdot \mathbf{m} - \gamma = 0, \quad \mathbf{m} = \frac{\partial W}{\partial \nabla\varphi}. \tag{14}$$

Here γ is a Lagrange multiplier related to (12). Excluding it from (13) and (14) we get again (7). Instead, we consider (13), (14), and (12) as a system of PDEs with respect to $\mathbf{w} = (\mathbf{u}, \varphi, \gamma)$. As this system consists of PDEs of different order, it cannot be treated using standard ellipticity definition. On the other hand, the Douglis–Nirenberg ellipticity works and brings the following inequality

$$\det \mathbb{A}(\boldsymbol{\xi}) \neq 0, \quad \mathbb{A}(\boldsymbol{\xi}) = \begin{pmatrix} \mathbf{Q}(\boldsymbol{\xi}) & \mathbf{0} & -i\boldsymbol{\xi} \cdot J\mathbf{F}^{-T} \\ 0 & \boldsymbol{\xi} \cdot \dfrac{\partial^2 W}{\partial \mathbf{k} \partial \mathbf{k}} \cdot \boldsymbol{\xi} & 0 \\ i\boldsymbol{\xi} \cdot J\mathbf{F}^{-T} & 0 & 0 \end{pmatrix}, \tag{15}$$

where $\mathbf{Q}(\boldsymbol{\xi})$ is the classic acoustic tensor given by the formulae

$$Q_{ij} = C_{minj}\xi_m\xi_n, \quad \mathbf{C} = \frac{\partial^2 W}{\partial \mathbf{F} \partial \mathbf{F}}.$$

Here we used the following set of integers s_p and t_p, $p = 1, \ldots, 5$:

$$t_1 = 3, \quad t_2 = 3, \quad t_3 = 3, \quad t_4 = 3, \quad t_5 = 2,$$

$$s_1 = -1, \quad s_2 = -1, \quad s_3 = -1, \quad s_4 = -1, \quad s_5 = -2.$$

With another technique similar, but not the same, constraints were obtained by Zee & Sternberg (1983) for incompressible materials .

What is remarkable is that the Douglis–Nirenberg ellipticity condition (15) includes also the classic ellipticity condition, i.e. the condition of non-singularity of the acoustic tensor. This is an essential difference from the strong ellipticity conditions which do not imply such constraints, see Eremeyev (2021); Eremeyev & Lazar (2022).

In order to clarify the Douglis–Nirenberg ellipticity condition let us study the case of small deformations in more details. Now system (13), (14), and (12) take the form

$$\mu\Delta\mathbf{u} + (\lambda+\mu)\nabla\nabla\cdot\mathbf{u} - \nabla\gamma + \rho\mathbf{f} = \mathbf{0}, \tag{16}$$

$$\beta\Delta\varphi - \gamma = 0, \tag{17}$$

$$\varphi - \nabla\cdot\mathbf{u} = 0. \tag{18}$$

The corresponding symbolic representation of the differential operator $\mathcal{A}(\mathbf{X}, D)$ in (1) is given by

$$\begin{pmatrix} -\mu\boldsymbol{\xi}\cdot\boldsymbol{\xi}\mathbf{I} - (\lambda+\mu)\boldsymbol{\xi}\otimes\boldsymbol{\xi} & \mathbf{0} & -i\boldsymbol{\xi} \\ 0 & -\beta\boldsymbol{\xi}\cdot\boldsymbol{\xi} & -1 \\ i\boldsymbol{\xi} & 1 & 0 \end{pmatrix}.$$

As a result, the principal symbol introduced in (4) has the form

$$A_0(\mathbf{X}, \boldsymbol{\xi}) = \det \mathbb{A}(\boldsymbol{\xi}), \tag{19}$$

$$\mathbb{A}(\boldsymbol{\xi}) =$$
$$\begin{pmatrix} -\mu\xi^2 - (\lambda+\mu)\xi_1^2 & -(\lambda+\mu)\xi_1\xi_2 & -(\lambda+\mu)\xi_1\xi_3 & 0 & -i\xi_1 \\ -(\lambda+\mu)\xi_2\xi_1 & -\mu\xi^2 - (\lambda+\mu)\xi_2^2 & -(\lambda+\mu)\xi_2\xi_3 & 0 & -i\xi_2 \\ -(\lambda+\mu)\xi_3\xi_1 & -(\lambda+\mu)\xi_3\xi_2 & -\mu\xi^2 - (\lambda+\mu)\xi_3^2 & 0 & -i\xi_3 \\ 0 & 0 & 0 & -\beta\xi^2 & 0 \\ i\xi_1 & i\xi_2 & i\xi_3 & 0 & 0 \end{pmatrix},$$

where $\xi^2 = \boldsymbol{\xi} \cdot \boldsymbol{\xi}$. Here we have the formula

$$\det \mathbb{A}(\boldsymbol{\xi}) = \beta\mu^2\xi^8. \tag{20}$$

As a result, the the Douglis–Nirenberg ellipticity conditions take the form of two inequalities

$$\beta \neq 0, \quad \mu \neq 0. \tag{21}$$

We can see that the ellipticity conditions include constraints for first-order and higher order elastic moduli.

These inequalities could be also obtained if one decompose the displacements using the Helmholtz decomposition $\mathbf{u} = \nabla\Phi + \nabla \times \boldsymbol{\Psi}$, $\nabla \cdot \boldsymbol{\Psi} = 0$, where Φ and $\boldsymbol{\Psi}$ are potentials. For the latter we have two equations

$$(\lambda + 2\mu)\Delta\Phi - \beta\Delta^2\Phi + f = 0,$$
$$\mu\Delta\boldsymbol{\Psi} + \mathbf{p} = \mathbf{0},$$

where we also used the Helmholtz decomposition of the mass force $\rho\mathbf{f} = \nabla f + \nabla \times \mathbf{p}$. Obviously, both equations are elliptic if and only if (21) are fulfilled.

4. Conclusions

We demonstrated that the dilatational strain gradient elasticity belongs to the class of elliptic systems in the Douglis–Nirenberg sense. So the general theory of elliptic systems could be applied to these models of continua. Let also note that unlike the ordinary ellipticity the Douglis–Nirenberg ellipticity is invariant under change of variables, so it could be more useful for various transformations of the systems under considerations. Similar results one can expect for other models with additional scalar degree of freedom. In addition we demonstrated that the provided conditions of ellipticity inherited the ones from the simple materials. In other words they includes inequalities for low- and high-order elastic moduli, whereas the standard ellipticity requires constrains for higher order elastic moduli, see e.g. Eremeyev & Lazar (2022). The approach based on the Douglis–Nirenberg definition could be also useful for other models of elasticity, such as ones with implicit or incremental constitutive relations Rajagopal (2007); Rajagopal & Srinivasa (2007).

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