Fixed Points of anti-attracting maps and Eigenforms on Fractals

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Received , revised , accepted
Published online

Key words Fixed point theorems, fractals, Dirichlet forms
MSC (2010) 47H10, 31C25, 28A80

An important problem in analysis on fractals is the existence of a self-similar energy on finitely ramified fractals. The self-similar energies are constructed in terms of eigenforms, that is, eigenvectors of a special nonlinear operator. Previous results by C. Sabot and V. Metz give conditions for the existence of an eigenform. In this paper, I prove this type of result in a different way. The proof given in this paper is based on a general fixed-point theorem for anti-attracting maps on a convex set.

1 Introduction

The subject of this paper is analysis on fractals. Much of analysis on fractals is built based on the notion of an energy. Therefore, an important problem is the construction of self-similar Dirichlet forms on fractals, i.e. energies. In this paper, we investigate the finitely ramified fractals. This means more or less that the intersection of each pair of copies of the fractal is a finite set. Examples of finitely ramified fractals are the Sierpinski Gasket (or Sierpinski triangle) and its generalizations, the Vicsek Set and the Lindstrøm Snowflake, while a well-known example of infinitely ramified fractal is the Sierpinski Carpet.

The Gasket can be constructed in the following way. Take an equilateral triangle T_0 (0-step of the construction), next divide T_0 into 4 copies of it and remove the central one; call T_1 the set so obtained (1-step of the construction); next use the same process on every of the remaining three triangles and call this set T_2, and so on. The Gasket will be the intersection of all T_n. The Vicsek set and the Carpet are obtained taking a square as the 0-step of the construction. In Figures 1 and 2 the 0-step and 1-step in the construction of the Gasket are depicted. In Figures 3 and 4 the Vicsek Set and the Carpet are depicted (1-step of the construction). We see in Figures 2 and 3 that in the Gasket and in the Vicsek Set, the intersection of two copies of the fractal either is empty or consists of a singleton, while in the Carpet, depicted in Figure 4, the intersection of some two copies of the fractal consists of a line-segment.

An important and general class of finitely ramified fractals is that of the P.C.F. self-similar sets, introduced by J. Kigami in [2] and a general theory with many examples can be found in [3]. In this paper, we consider a subclass of the class of the P.C.F. self-similar sets, with a very mild additional requirement, which is described in Section 2. This is the same setting as in other papers of mine e.g., [6] and is essentially the same setting as in [1], and in other papers ([9], [8]). We require that every point in the initial set is a fixed point of one of the contractions defining the fractal. Moreover, we require that the fractal is connected. All fractals considered in this paper are subsets of a metric space X *generated by a finite set* Ψ *of maps from* X *to* X, in the sense that the fractal $\mathcal{F}$ is the unique non-empty compact subset of X such that

$$\mathcal{F} = \bigcup_{\psi\in\Psi} \psi(\mathcal{F}). \tag{1.1}$$

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On such a class of fractals, the basic tool used to construct a Dirichlet form is a self-similar *discrete* Dirichlet form defined on a special finite subset $V^{(0)}$ of the fractal. This subset is a sort of boundary of the fractal. For example, in the Gasket, we have $V^{(0)} = \{P_1, P_2, P_3\}$ and in the Vicsek Set we have $V^{(0)} = \{P_1, P_2, P_3, P_4\}$ (see Figures 2 and 3). Such self-similar discrete Dirichlet forms are the *eigenforms*, i.e., the eigenvectors of a

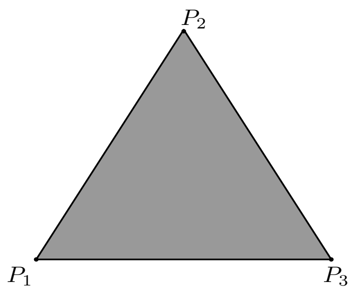

Figure 1. The Gasket, 0-step

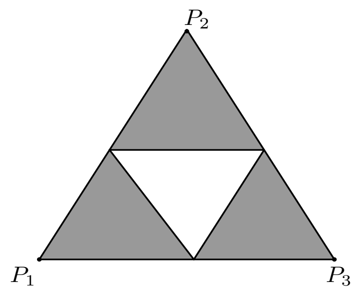

Figure 2. The Gasket, 1-step

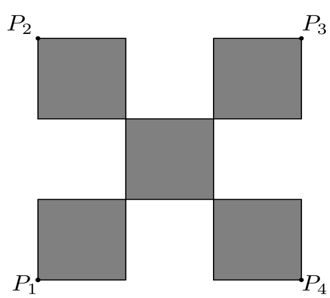

Figure 3. The Vicsek Set

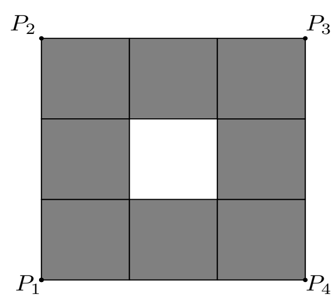

Figure 4. The Carpet

special nonlinear operator Λ_r called *renormalization operator*, which depends on a set of positive *weights* r_i placed on the cells of the fractal.

Thus, an important problem in analysis on fractals is whether on a given P.C.F. self-similar set there exists an eigenform with prescribed weights. Another important problem is whether such an eigenform is unique up to a multiplicative constant (it can be easily verified that a positive multiple of an eigenform is an eigenform as well). In this paper, we will consider only the problem of the existence of an eigenform.

In some specific cases (e.g., the Gasket) an explicit eigenform can be given. The first result of existence of an eigenform on a relatively general class of fractals was given by T. Lindstrøm in [4], where it was proved that there exists an eigenform with all weights equal to 1 on the nested fractals, a class of fractals with good properties of symmetry including for example, the Lindstrøm snowflake. C. Sabot in [8] proved a rather general criterion for the existence of an eigenform, and V. Metz in [5] improved the results in [8]. In fact, he removed an additional requirement present in the paper of Sabot and also, considered more general classes of fractals than those considered in [8] and in the present paper. Moreover, the result given in [5] provides a condition for the existence of an eigenform, which is in some sense almost necessary and sufficient although it does not cover some usual cases (see Remark 3.3). More details on the existence results of Sabot and Metz are given in Section 3.

In this paper, I prove essentially the same existence result as that in [5], here described in Section 3, Theorem 3.2. Note that in [5], other criteria are also given, but apparently less appropriate in the context considered here. Strictly speaking, the result given here is slightly weaker than the result of Metz here described in Theorem 3.2. However, I think that, in the practical cases, the ranges of application of the two results are probably essentially the same. Moreover, the condition "almost" necessary and sufficient for the existence here mentioned in Remark 3.3 can be also deduced from the result given in this paper. See Remark 6.9 for the details.

The idea behind the proof given here is entirely different. Note that it avoids almost completely the use of the theory of Hilbert's projective metric and also the use of the notion of P-parts is reduced and limited to the last theorem (Theorem 6.8) and a preliminary lemma. Instead, the proof is based on a natural and general principle.

This principle is that a map from the (non-empty) interior of a compact and convex set A in $\mathbb{R}^n$ into itself has a fixed point if it is *anti-attracting*. To illustrate it, in this situation call repulsing a map ϕ from A into itself if it has the following property: for every $\overline{x} \in \partial A$, ϕ sends a suitable neighborhood of $\overline{x}$ toward the interior of A. The name repulsing appears to be appropriate since every point of the boundary is repulsing for ϕ. Then, it is well-known that a repulsing map ϕ has a fixed point on A. We say that the map is anti-attracting if, more generally, for every $\overline{x} \in \partial A$, it sends a suitable neighborhood of $\overline{x}$ in a direction which is not opposite to a given element $\widetilde{x}$ chosen in the interior of A. The previous, of course, are informal definitions.

Moreover, I will use a stronger notion of anti-attracting map, namely, I will require that the map is anti-attracting with respect to a point $\widetilde{x}$, which is not constant but depends continuously on $\overline{x}$. In the original version of this paper, I have used only the versions of the fixed point theorems with constant $\widetilde{x}$. The proof of the final result was definitely simpler than in the present version. However, after the referee asked for a better comparison with the previous result, I realized that in that version, the result was substantially weaker than that of Metz.

The precise definitions and theorems (Theorem 4.4 and Theorem 4.5) are discussed in Section 4. Since at least the versions with constant $\widetilde{x}$ of such theorems, especially Theorem 4.4, are simple consequences of Brouwer's fixed point Theorem, we can reasonably conjecture that at least some version of them could be already known, but since I do not know any reference to them, I prove them in this paper. In the proof of the existence result described in this paper, which, as previously told, is based on a natural and general result, we cannot however avoid a technical point. Namely, in Lemma 6.3 we use a rather technical result proved in [8]. Such a type of result, is in any case, a key-point also in the proofs in [8] and in [5].

Note that in [8] and in [5] also sufficient conditions for non-existence of an eigenform as well as the uniqueness of the eigenforms are discussed. These problems are not discussed in this paper, apart from in Section 3, which is a Section devoted to previous results. Finally, note that many examples of fractals are known where for some prescribed weights there exist no eigenforms. A natural conjecture was considered in literature, namely whether on every fractal (of the type considered in this paper) there exists an eigenform *for a suitable set of weights*.

Such a problem has been solved in [6] and in [7] by two different points of view. More precisely, in [7] an example of a fractal is shown where for every set of weights there exist no eigenforms on the fractal. However, in [6] a weak form of the conjecture is proved. Namely, it is proved that there exists always an eigenform with suitable weights but with respect to a suitable set of maps generating the fractal, which is not necessarily the original set of maps (as Ψ in (1.1)).

2 Notation

In this Section, we introduce the notation, based on that of [6]. This type of construction was firstly considered in [1]. A notion similar to that of a fractal triple was discussed first in [2], Appendix A, and called an *ancestor*.

First, we define the general fractal setting. The basic notion is that of *fractal triple*. By this, we mean a triple $\big(V^{(0)}, V^{(1)}, \Psi\big)$, where $V^{(0)}$ is a finite set with $N \geq 2$ elements, $V^{(1)}$ is a finite set and Ψ is a finite set of one-to-one maps from $V^{(0)}$ into $V^{(1)}$ satisfying $V^{(1)} = \bigcup_{\psi\in\Psi} \psi\big(V^{(0)}\big)$. Put

$$V^{(0)} = \{P_1, ..., P_N\}.$$

We require that

a) for each $1 \leq j \leq N$, there exists a (unique) function $\psi_j \in \Psi$ such that $\psi_j(P_j) = P_j$, and $\Psi = \{\psi_1, ..., \psi_k\}$, with $k \geq N$;

b) $P_j \notin \psi_i\big(V^{(0)}\big)$ when $i \neq j$ (in other words, if $\psi_i(P_h) = P_j$ with $i \in \{1, ..., k\}$, $j, h \in \{1, ..., N\}$, then $i = j = h$);

c) all pairs of points in $V^{(1)}$ can be connected by a path every edge of which is contained in a set of the form $\psi_i(V^{(0)})$; in other words for every $Q, Q' \in V^{(1)}$ there exists a sequence of points $Q_0, ..., Q_n \in V^{(1)}$ such that $Q_0 = Q$, $Q_n = Q'$ and for every $h = 1, ..., n$ there exists $i_h = 1, ..., k$ such that $Q_{h-1}, Q_h \in \psi_{i_h}(V^{(0)})$.

Note that $V^{(0)} \subseteq V^{(1)}$. As discussed in Introduction, $V^{(0)}$ is seen as a sort of boundary of the fractal. By definition, a *1-cell* (or simply a *cell*) is a set of the form $V_i := \psi_i\big(V^{(0)}\big)$ with $i = 1, ..., k$. The points P_j, $j = 1, ..., N$, will be called *vertices*. Let

$$J = J\big(V^{(0)}\big) = \big\{\{j_1, j_2\} : j_1, j_2 \in \{1, ..., N\}, j_1 \neq j_2\big\}.$$

Based on a fractal triple, we can construct in a standard way a (unique) finitely ramified fractal, more precisely a P.C.F. self-similar set. See, for example, [2], Appendix A, for the details of such a construction.

Next, we define the Dirichlet forms on $V^{(0)}$, invariant with respect to an additive constant. Namely, denote by $\mathcal{D}\big(V^{(0)}\big)$, or simply $\mathcal{D}$, the set of functionals E from $\mathbb{R}^{V^{(0)}}$ into $\mathbb{R}$ of the form

$$E(u) = \sum_{\{j_1,j_2\}\in J} E_{\{j_1,j_2\}}\big(u(P_{j_1}) - u(P_{j_2})\big)^2,$$

where $E_{\{j_1,j_2\}} \geq 0$. The numbers $E_{\{j_1,j_2\}}$ will be called *coefficients* of E. Denote by $\widetilde{\mathcal{D}}\big(V^{(0)}\big)$, or simply $\widetilde{\mathcal{D}}$, the set of the irreducible Dirichlet forms, i.e.

$$\widetilde{\mathcal{D}} = \{E \in \mathcal{D} : E(u) = 0 \text{ if and only if } u \text{ is constant}\}.$$

We remark that, in particular, if $E \in \mathcal{D}$ and all coefficients of E are strictly positive, then $E \in \widetilde{\mathcal{D}}$. However, there are forms in $\widetilde{\mathcal{D}}$ that have some coefficients equal to 0. More precisely, if $E \in \mathcal{D}$, then $E \in \widetilde{\mathcal{D}}$ if and only if the graph $\mathcal{G}(E)$ defined on $V^{(0)}$ as

$$\mathcal{G}(E) := \big\{\{P_{j_1}, P_{j_2}\} : E_{\{j_1,j_2\}} > 0\big\},$$

is connected. This means that for every $P_j, P_{j'} \in V^{(0)}$ there exists a sequence $j_0 = j, j_1, ..., j_n = j'$ such that $E_{\{j_{h-1},j_h\}} > 0$ for every $h = 1, ..., n$.

Note that a form $E \in \mathcal{D}$ is uniquely determined by its coefficients. Thus, we can identify $E \in \mathcal{D}$ with the set of its coefficients $E_{\{j_1,j_2\}}$ in $\mathbb{R}^J$. In fact,

$$E_{\{j_1,j_2\}} = \frac{1}{4}\Big(E\big(\chi_{\{P_{j_1}\}} - \chi_{\{P_{j_2}\}}\big) - E\big(\chi_{\{P_{j_1}\}} + \chi_{\{P_{j_2}\}}\big)\Big).$$

Accordingly, we will equip $\mathcal{D}$ with the euclidean metric in $\mathbb{R}^J$. We will also use the following convention:

$$E \le E' \iff E(u) \le E'(u) \quad \forall\, u \in \mathbb{R}^{V^{(0)}},$$

$$E \preceq E' \iff E_d \le E'_d \quad \forall d \in J.$$

Note that $E \preceq E'$ implies $E \le E'$ but the converse does not hold. The following lemma is standard. I merely sketch the proof.

Lemma 2.1 *If $E_1, E_2 \in \widetilde{\mathcal{D}}$ there exist positive constants c, c' such that $cE_1 \le E_2 \le c' E_1$.*

Proof. Let

$$S := \{u \in \mathbb{R}^{V^{(0)}} : u(P_1) = 0, ||u|| = 1\}.$$

The ratio $\frac{E_2}{E_1}$ attains its minimum c and its maximum c' over S. Thus, for every non-constant $u \in \mathbb{R}^{V^{(0)}}$, putting $\widetilde{u} := \dfrac{u - u(P_1)}{||u - u(P_1)||}$, we have $\widetilde{u} \in S$, thus

$$\frac{E_2(u)}{E_1(u)} = \frac{E_2(\widetilde{u})}{E_1(\widetilde{u})} \in [c, c'].$$

□

Next, we recall the definition of the *renormalization operator* Λ_r. For every $r \in W :=]0, +\infty[^k$, every $E \in \mathcal{D}$ and every $v \in \mathbb{R}^{V^{(1)}}$, define

$$S_{1,r}(E)(v) = \sum_{i=1}^{k} r_i\, E(v \circ \psi_i).$$

Here, an element r of W can be written as $(r_1, ..., r_k)$ and the number $r_i > 0$ is called the *weight* placed on the cell V_i. Note that $S_{1,r}(E)$ is a sort of sum of E on all cells. It is easy to see that $S_{1,r}(E)$ is a Dirichlet form on $\mathbb{R}^{V^{(1)}}$. Now, for $u \in \mathbb{R}^{V^{(0)}}$ let

$$\mathcal{L}(u) = \left\{v \in \mathbb{R}^{V^{(1)}} : v = u \text{ on } V^{(0)}\right\},$$

and let us define $\Lambda_r(E)(u)$ for $u \in \mathbb{R}^{V^{(0)}}$ as

$$\Lambda_r(E)(u) = \inf \{S_{1,r}(E)(v) : v \in \mathcal{L}(u)\}.$$

The form $\Lambda_r(E)$ is called the *restriction* of $S_{1,r}(E)$ on $V^{(0)}$. Note that Λ_r maps $\mathcal{D}$ into $\mathcal{D}$ and $\widetilde{\mathcal{D}}$ into $\widetilde{\mathcal{D}}$.

If $r \in W$, we say that $E \in \widetilde{\mathcal{D}}$ is an *r–eigenform* (with eigenvalue ρ) if there exists $\rho > 0$ such that

$$\Lambda_r(E) = \rho\, E. \tag{2.1}$$

We say that E is an r-degenerate eigenform (with eigenvalue ρ) if $E \in \widetilde{\mathcal{D}} \setminus \mathcal{D}$ satisfies (2.1). The following lemma is standard and can be easily proved

Lemma 2.2 *i) The map $(E, u) \mapsto E(u)$ from $\mathcal{D} \times \mathbb{R}^{V^{(0)}}$ to $\mathbb{R}$ is continuous.*
ii) The map $(r, E) \mapsto \Lambda_r(E)$ from $W \times \mathcal{D}$ to $\mathcal{D}$ is continuous.

Proof. (i) is trivial, and for ii) see for example [6], Lemma 3.1. □

The aim of this paper will be to give sufficient conditions for the existence of an r-eigenform.

3 Previous Results

We here adopt some notations of [5], or light variants of them. If $E \in \mathcal{D}$, we denote by $\ker(E)$ the set $E^{-1}(0)$. Moreover, we denote by $\mathcal{D}_E$ the set $\{E' \in \mathcal{D} : \ker(E') = \ker(E)\}$. The sets of the form $\mathcal{D}_E$, $E \in \mathcal{D}$ will be called *P-parts*. Note that $\mathcal{D}_E$ is the unique P-part containing E. We will put $\ker(\mathcal{D}_E) = \ker(E)$. Of course, $\widetilde{\mathcal{D}}$ and $\{0\}$ are P-parts. We will call $\widetilde{\mathcal{D}}$ ands $\{0\}$ the *trivial* P-parts. Note that in [8], instead of P-parts, the equivalent notion of G-relation is used. Here G stands for a group, which in our case is the trivial group.

We observe that $\mathcal{D}_E$ only depends on the set graph $\mathcal{G}(E)$ defined in Section 2. Therefore, there are only finitely many P-parts.

Given a P-part P, the P-part $\mathcal{D}_{\Lambda_r(E)}$ is independent of $E \in P$ and $r \in W$ and will be denoted by $\Lambda(P)$. We will say that a P-part P is Λ-invariant if $\Lambda(P) = P$. Note that the trivial P-parts are Λ-invariant. Given two P-parts P_1 and P_2 we put $P_1 \preceq P_2$ if $\ker(P_1) \subseteq \ker P_2$.

We will now describe the existence results of Sabot and of Metz. We will reformulate them using our notation. In the present context some statements can be simplified. We need to introduce a notion due to Sabot (see [8]) whose aim is to approximate Λ_r near $\overline{E} \in \mathcal{D} \setminus \widetilde{\mathcal{D}}$ by minimizing along functions in $\ker(\overline{E})$. Namely, we define $\Lambda_{r,\overline{E}}(E) : \ker(\overline{E}) \to \mathbb{R}$ as

$$\Lambda_{r,\overline{E}}(E)(u) = \inf \big\{ S_{1,r}\big(E\big)(v) : v \in \mathcal{L}(u), v \circ \psi_i \in \ker(\overline{E})\ \forall\, i = 1, ..., k \big\} \quad \forall\, u \in \ker(\overline{E}).$$

Of course, $\Lambda_{r,\overline{E}}(E)$ is independent of $\overline{E}$ in a given P-part. We are now ready to state the existence result of Sabot $\big($ii) of Theorem 5.1 in [8]$\big)$.

Theorem 3.1 *Suppose the nontrivial* Λ*-invariant* P*-parts are mutually incomparable (with respect to the order* $\preceq$*). Suppose for every non-trivial* Λ*-invariant* P*-part P there exist* $\overline{E} \in P$ *and* $E \in \widetilde{\mathcal{D}}$ *such that*

$$\inf \Big\{ \frac{\Lambda_{r,\overline{E}}(E)(u)}{E(u)} : u \in \textit{ker}(\overline{E}), u \textit{ non-constant} \Big\} > \sup \Big\{ \frac{\Lambda_r(\overline{E})(u)}{\overline{E}(u)} : u \in \mathbb{R}^{V^{(0)}} \setminus \textit{ker}(\overline{E}) \Big\}.$$

Then there exists an r*-eigenform (and it is unique up to a multiplicative constant).*

Now, I will describe the existence result of Metz. In [5], in fact there is a general but not easily verifiable existence criterion (Theorem 25 there) and later more practical criteria are deduced. I will describe the criterion which is the most appropriate in our setting, that is Corollary 28 combined with Theorem 25 in [5] and called Sabot's test there.

Theorem 3.2 *Suppose that for every non-trivial* Λ*-invariant* P*-part containing a degenerate* r*-eigenform* $\overline{E}$ *with eigenvalue* $\rho > 0$ $\Lambda_{r,\overline{E}}$ *has an eigenvector* $E \in \mathcal{D}$ *with eigenvalue strictly greater than* ρ*. Then there exists an* r*-eigenform (and it is unique up to a multiplicative constant).*

Note that in the statement of Theorem 3.2, we could avoid the use of the P-parts, and could instead use only the degenerate r-eigenforms. However, referring to the P-parts clarifies better that we have to check only finitely many possibilities. Also, it is important to note that the eigenvalue does not depend on the degenerate r-eigenform in the given P-part (see [5], Corollary 13), so that in Theorem 3.2 the statement does not depend on the degenerate r-eigenform $\overline{E}$.

Remark 3.3 I here explain what are the main differences between the result of Metz and that of Sabot. First of all, in Theorem 3.2 the hypothesis that the nontrivial Λ-invariant P-parts are mutually incomparable is removed. Moreover, in the result by Metz, we are reduced to check only the P-parts containing a degenerate r-eigenform. As observed in [5], Remark 34 and subsequent comments, this provides a condition for the existence of an r-eigenform which is "almost" necessary and sufficient. Namely, enumerate the nontrivial Λ-invariant P-parts containing a degenerate r-eigenform as $P_1, ..., P_s$. For every $i = 1, ..., s$, let $\overline{E}_i$ be a degenerate r-eigenform with eigenvalue ρ_i. Let

$$\gamma_i = \sup_{E \in \widetilde{\mathcal{D}}} \inf \Big\{ \frac{\Lambda_{r,\overline{E}_i}(E)(u)}{E(u)} : u \in \ker(\overline{E}_i), u \text{ non-constant} \Big\}.$$

Then it follows from Theorem 3.2 that if

$$\max\{\rho_i : i = 1, ..., s\} < \min\{\gamma_i : i = 1, ..., s\},$$

then there exists an r-eigenform. On the other hand, if

$$\max\{\rho_i : i = 1, ..., s\} > \min\{\gamma_i : i = 1, ..., s\},$$

then there exist no r-eigenforms. The proof of the latter statement proposed in [5], Remark 34, is not clear to me. However, as mentioned there, it is also proved in the part i) of the result of Sabot (Theorem 5.1 in [8]).

In some sense, this condition is analogous to the fact that in the one-dimensional dynamics, a fixed point $\overline{x}$ of a function f is attracting if $|f'(\overline{x})| < 1$ and repelling if $|f'(\overline{x})| > 1$. Note however that this condition does not work in some usual fractals, e.g., when we have existence but not uniqueness.

4 The fixed point Theorems.

In this Section, we give two fixed point Theorems, useful for the following. They are simple variants of the Brouwer fixed point Theorem. The first concerns maps from a convex and compact set *not necessarily into itself* but such that any point x of the boundary is mapped not on the half-line with end-point at x and opposite to an interior point depending on x. The second theorem is a variant of the first but for *open* convex sets. First, recall some notation. An affine subset of $\mathbb{R}^n$ is a set in $\mathbb{R}^n$ of the form $X + a$ where X is a linear subspace of $\mathbb{R}^n$ and $a \in \mathbb{R}^n$. Now, let Z be an affine set in $\mathbb{R}^n$, and let $v, w \in Z$. Let

$$]v, w[:= \{v + t(w - v) : t \in]0, 1[\},$$

$$[v, w[:=]v, w[\cup \{v\}, \quad]v, w] :=]v, w[\cup \{w\}, \quad [v, w] :=]v, w[\cup \{v, w\}.$$

In the following, if Z is an affine subset of $\mathbb{R}^n$, every topological notion on Z will be meant to be with respect to the topology on Z inherited from the euclidean topology on $\mathbb{R}^n$. For example, if $A \subseteq Z$, we will denote by $\text{int}(A)$ the interior of A with respect to such a topology. The following lemma is standard and can be easily proved.

Lemma 4.1 *Suppose Z is an affine subset of $\mathbb{R}^n$ and A is a convex subset of Z. Then*
i) If $v \in A$ and $w \in \text{int}(A)$, we have $]v, w[\subseteq \text{int}(A)$.
ii) $\text{int}(A)$ is convex.

Let $\widetilde{x}$, x be points of the affine subset Z of $\mathbb{R}^n$. Then, we define

$$\text{Ext}_{\widetilde{x}}(x) = \big\{\widetilde{x} + t(x - \widetilde{x}) : t > 1\big\}.$$

Lemma 4.2 *Let K be a compact convex subset of the affine subset Z of $\mathbb{R}^n$. Then*
i) For every $\widetilde{x} \in \text{int}(K)$ and every $x \in Z \setminus \{\widetilde{x}\}$ there exists a unique $y = p_{\widetilde{x}}(x) \in \partial K$ of the form $y = \widetilde{x} + t(x - \widetilde{x})$, $t > 0$.
ii) The map $(\widetilde{x}, x) \mapsto p_{\widetilde{x}}(x)$ from $\big\{(\widetilde{x}, x) \in \text{int}(K) \times Z : x \neq \widetilde{x}\big\}$ into ∂K is continuous.
iii) Let $\widetilde{x} \in \text{int}(K)$. If $x \in \partial K$, then $p_{\widetilde{x}}(x) = x$;
if $x \in K \setminus \{\widetilde{x}\}$, then $x \in [p_{\widetilde{x}}(x), \widetilde{x}]$;
if $x \in Z \setminus \text{int}(K)$, then $p_{\widetilde{x}}(x) \in [x, \widetilde{x}]$.
iv) If $\widetilde{x} \in \text{int}(K)$, $x \in K \setminus \{\widetilde{x}\}$ and $x_1 \in \text{Ext}_{\widetilde{x}}(x) \cap K$, then $x_1 \in [x, p_{\widetilde{x}}(x)]$.
v) If $\widetilde{x} \in \text{int}(K)$ and $x \in Z \setminus K$, then $y = p_{\widetilde{x}}(x)$ is the unique point in ∂K satisfying $x \in \text{Ext}_{\widetilde{x}}(y) \cup \{y\}$.

Proof. (Sketch) i) Clearly, $H := \partial K \cap \{\widetilde{x} + t(x - \widetilde{x}) : t > 0\}$ is non-empty by connectedness. In fact, the point $\widetilde{x} + t(x - \widetilde{x})$ belongs to $\text{int}(K)$ for $t = 0$, and moreover, in view of the boundedness of K, it lies in $Z \setminus K$ for sufficiently large t. Moreover, H cannot contain two different points by Lemma 4.1. ii) This follows at once from the uniqueness of the point defining $p_{\widetilde{x}}(x)$. iii) It is easy to see that in the definition of $p_{\widetilde{x}}(x)$ we have $t = 1$ if $x \in \partial K$, $t \geq 1$ if $x \in K \setminus \{\widetilde{x}\}$, $t \leq 1$ if $x \in Z \setminus \text{int}(K)$. iv) and v) are trivial. □

In order to prove the main results of this Section, we recall a standard version of Tietze's Theorem where the functions take values in a convex set in $\mathbb{R}^n$. Stronger versions of this theorem are known but the present suffices for our aims. We will use Lemma 4.3 also in Section 6.

Lemma 4.3 *If f is a continuous function from a non-empty compact subset Y of a metric space X with values in a convex set C in $\mathbb{R}^n$, then there exists a continuous function $\widetilde{f} : X \to C$ that extends f.*

P r o o f. Let $f = (f_1, ..., f_n)$. Since every f_i is continuous with values in $\mathbb{R}$, the usual form of Tietze's Theorem provides continuous functions $\widehat{f_i} : X \to \mathbb{R}$ extending f_i. Thus we find a continuous function $\widehat{f} : X \to \mathbb{R}^n$ extending f. Now, let π be the projection of $\mathbb{R}^n$ over $\operatorname{co}\big(f(Y)\big)$. Then the function $\widetilde{f}$ defined as $\widetilde{f} = \pi \circ \widehat{f}$ satisfies the Lemma. □

Theorem 4.4 *Let Z be an affine subset of $\mathbb{R}^n$, let K be a non-empty compact convex subset of Z. Let θ be a continuous map from ∂K to* $\operatorname{int}(K)$ *and let $\phi : K \to Z$ be a continuous map such that $\phi(x) \notin \operatorname{Ext}_{\theta(x)}(x)$ for every $x \in \partial K$. Then ϕ has a fixed point on K.*

P r o o f. In view of Lemma 4.3, we can extend θ to a continuous function, which we call θ as well, from Z to $\operatorname{int}(K)$. Define $\widetilde{\phi} : K \to K$ by

$$\widetilde{\phi}(y) = \begin{cases} p_{\theta(y)}\big(\phi(y)\big) & \text{if } \phi(y) \in Z \setminus K \\ \phi(y) & \text{if } \phi(y) \in K \end{cases}.$$

Since $p_{\theta(y)}\big(\phi(y)\big) = \phi(y)$ when $\phi(y) \in \partial K$, then $\widetilde{\phi}$ is continuous on all of K with values in K and amounts to ϕ on $\phi^{-1}(K)$. Thus it has a fixed point $\overline{x} \in K$. We claim that $\phi(\overline{x}) = \overline{x}$. In fact, if $\overline{x} \in \partial K$ and $\phi(\overline{x}) \neq \overline{x} = \widetilde{\phi}(\overline{x})$, by the definition of $\widetilde{\phi}$ we have $\overline{x} = p_{\theta(\overline{x})}\big(\phi(\overline{x})\big)$ and $\phi(\overline{x}) \in Z \setminus K$. Thus, by Lemma 4.2 v) we have $\phi(\overline{x}) \in \operatorname{Ext}_{\theta(\overline{x})}(\overline{x})$, contrary to our assumption. Thus $\overline{x} \in \operatorname{int}(K)$, and consequently $\phi(\overline{x}) \in K$. In fact, in the opposite case, $\overline{x} = p_{\theta(\overline{x})}\big(\phi(\overline{x})\big) \in \partial K$. Therefore $\overline{x} = \widetilde{\phi}(\overline{x}) = \phi(\overline{x})$. □

Let Z and K be as in Theorem 4.4, let ϕ be a continuous map from $\operatorname{int}(K)$ into itself, and let θ be a continuous map from ∂K to $\operatorname{int}(K)$. We say that $\overline{x} \in \partial K$ is *anti-attracting for* (ϕ, θ) if there exists a neighborhood $U_{\overline{x}}$ of $\overline{x}$ in Z such that for every $x \in U_{\overline{x}} \cap \operatorname{int}(K)$ and every $x' \in U_{\overline{x}} \cap \partial K$ we have $\phi(x) \notin \operatorname{Ext}_{\theta(x')}(x)$. We say that ϕ is θ-anti-attracting if every $\overline{x} \in \partial K$ is anti-attracting for (ϕ, θ).

Theorem 4.5 *Let Z and K be as in Theorem 4.4. Let θ be a continuous map from ∂K into* $\operatorname{int}(K)$*, and let ϕ be a θ-anti-attracting map from* $\operatorname{int}(K)$ *into itself. Then ϕ has a fixed point on* $\operatorname{int}(K)$.

P r o o f. For every $\overline{x} \in \partial K$, let $U_{\overline{x}}$ be a neighborhood of $\overline{x}$ in Z as in the definition of an anti-attracting point. We can and do assume that $U_{\overline{x}}$ is open and, moreover, its closure has the same property, namely

$$\Big(x \in \overline{U_{\overline{x}}} \cap \operatorname{int}(K), x' \in \overline{U_{\overline{x}}} \cap \partial K\Big) \Rightarrow \phi(x) \notin \operatorname{Ext}_{\theta(x')}(x). \tag{4.1}$$

Moreover, we can extend θ continuously on $K \setminus \{\widetilde{x}\}$ where $\widetilde{x}$ is an arbitrary point in $\operatorname{int}(K)$ by putting $\theta(x) = \theta\big(p_{\widetilde{x}}(x)\big)$. By continuity we can choose $U_{\overline{x}}$ such that the following variant of (4.1) holds

$$x \in \overline{U_{\overline{x}}} \cap \operatorname{int}(K) \Rightarrow \Big(x \notin \{\widetilde{x}\} \cup \theta(\partial K),\ \phi(x) \notin \operatorname{Ext}_{\theta(x)}(x)\Big). \tag{4.2}$$

By compactness, there exist $\overline{x}_1, ..., \overline{x}_m \in \partial K$ such that

$$U := \bigcup_{i=1}^{m} U_{\overline{x}_i} \supseteq \partial K.$$

Let $\widetilde{K} := co\big(K \setminus U\big)$. Note that, in view of Lemma 4.1 ii), we have

$$\widetilde{K} \subseteq \operatorname{int}(K). \tag{4.3}$$

We also have

$$\partial \widetilde{K} \subseteq \overline{U}. \tag{4.4}$$

In fact, in the opposite case, there exists $x \in \partial \widetilde{K}$ such that

$$x \in \operatorname{int}(K) \setminus \overline{U} \subseteq K \setminus U \subseteq \widetilde{K},$$

and since int$(K) \setminus \overline{U}$ is open in Z, then $x \notin \partial\widetilde{K}$, a contradiction, thus (4.4) holds. By (4.3) and (4.4), for every $x \in \partial\widetilde{K}$ we have $x \in \overline{U_{\overline{x}_i}} \cap \text{int}(K)$ for some $i = 1, ..., m$, thus by (4.1)

$$\phi(x) \notin \text{Ext}_{\theta(x)}(x).$$

Next, $\widetilde{x} \notin \overline{U}$ by (4.2). Therefore, $\widetilde{x} \in K \setminus U \subseteq \widetilde{K}$, but in view of (4.4), $\widetilde{x} \notin \partial\widetilde{K}$, thus $\widetilde{x} \in \text{int}(\widetilde{K})$. In particular, the compact and convex set $\widetilde{K}$ is non-empty.

Moreover, if $x \in \partial\widetilde{K} \subseteq K \setminus \{\widetilde{x}\}$, then $\theta(x) \in \text{int}(\widetilde{K})$. In fact, on one hand, $\theta(x) \in K$ by hypothesis and $\theta(x) \notin U$ by (4.2) thus $\theta(x) \in \widetilde{K}$ by definition of $\widetilde{K}$, on the other $\theta(x) \notin \partial\widetilde{K}$ by (4.2) and (4.4). The map ϕ from $\widetilde{K}$ into Z thus satisfies all hypotheses of Theorem 4.4, thus ϕ has a fixed point on $\widetilde{K} \subseteq \text{int}(K)$. □

5 Anti-attracting forms on Fractals.

In this Section, we investigate the notions of Section 4 in the setting of forms in $\mathcal{D}$. Recall that a form in $\mathcal{D}$ can be seen as an element of $\mathbb{R}^J$. So, we define specific sets in $\mathbb{R}^J$ which will play the roles of Z and K in Section 4. Moreover, we will investigate the notion of an anti-attracting form with respect to a map obtained normalizing Λ_r. Let us define

$$\widetilde{L}(x) := \sum_{d \in J} x_d \quad \forall\, x \in \mathbb{R}^J,$$

$$|x| := \sum_{d \in J} |x_d| \quad \forall\, x \in \mathbb{R}^J,$$

$$Z := \left\{x \in \mathbb{R}^J : \widetilde{L}(x) = 1\right\},$$

$$\mathcal{D}_{\mathcal{N}} =: \left\{E \in \mathcal{D} : |E| = 1\right\} = \left\{E \in Z : E_d \geq 0 \quad \forall\, d \in J\right\}.$$

So, Z is an affine set in $\mathbb{R}^J$ and $\mathcal{D}_{\mathcal{N}}$ is a non-empty compact and convex subset of Z. Note that

$$\max_{d \in J} E_d \geq \widetilde{m} := \frac{1}{\#(J)} \quad \forall\, E \in Z. \tag{5.1}$$

We easily characterize int$(\mathcal{D}_{\mathcal{N}})$. In fact we have int$(\mathcal{D}_{\mathcal{N}}) = \mathcal{D}_{\mathcal{N}}^{(1)}$ where

$$\mathcal{D}_{\mathcal{N}}^{(1)} := \{E \in \mathcal{D}_{\mathcal{N}} : E_d > 0\, \forall\, d \in J\} \subseteq \widetilde{\mathcal{D}}.$$

We next want to study the map $\widetilde{\Lambda^*}_r$ defined as

$$\widetilde{\Lambda^*}_r(E) := \frac{\Lambda_r(E)}{\left|\Lambda_r(E)\right|}.$$

As it is known that if $E \in \widetilde{\mathcal{D}}$ satisfies $E_d > 0$ for every $d \in J$, so does $\Lambda_r(E)$ (see, for example, [8], Prop. 1.15), then $\widetilde{\Lambda^*}_r$ maps continuously $\mathcal{D}_{\mathcal{N}}^{(1)}$ into itself. However, in general $\widetilde{\Lambda^*}_r$ cannot be extended continuously on all of $\mathcal{D}_{\mathcal{N}}$. In fact, we could have $\Lambda_r(E) = 0$ for some $E \in \mathcal{D} \setminus \widetilde{\mathcal{D}}$. We so need a nice decomposition of $\partial\mathcal{D}_{\mathcal{N}}$. Let

$$\mathcal{D}_{\mathcal{N}}^{(2)} := \mathcal{D}_{\mathcal{N}} \cap \widetilde{\mathcal{D}} \setminus \mathcal{D}_{\mathcal{N}}^{(1)},$$

$$\mathcal{D}_{\mathcal{N}}^{(3)} = \{E \in \mathcal{D}_{\mathcal{N}} \setminus \widetilde{\mathcal{D}} : \Lambda_r(E) \neq 0\},$$

$$\mathcal{D}_{\mathcal{N}}^{(4)} = \{E \in \mathcal{D}_{\mathcal{N}} \setminus \widetilde{\mathcal{D}} : \Lambda_r(E) = 0\},$$

where $r \in W$. In fact, it can be proved that the formula $\Lambda_r(E) = 0$ is independent of $r \in W$, but this is not important for our considerations since we fix a given $r \in W$. We easily have

$$\partial\mathcal{D}_{\mathcal{N}} = \mathcal{D}_{\mathcal{N}}^{(2)} \cup \mathcal{D}_{\mathcal{N}}^{(3)} \cup \mathcal{D}_{\mathcal{N}}^{(4)}.$$

Claerly, $\widetilde{\Lambda^*}_r$ maps continuously $\mathcal{D}_{\mathcal{N}}^{(1)} \cup \mathcal{D}_{\mathcal{N}}^{(2)} \cup \mathcal{D}_{\mathcal{N}}^{(3)}$ into $\mathcal{D}_{\mathcal{N}}$. Note also that, when $E \in \mathcal{D}_{\mathcal{N}}^{(1)} \cup \mathcal{D}_{\mathcal{N}}^{(2)} \cup \mathcal{D}_{\mathcal{N}}^{(3)}$, then E is a (possibly degenerate) r-eigenform if and only if it is a fixed point of $\widetilde{\Lambda^*}_r$.

We are going to prove that every point $E \in \partial\mathcal{D}_{\mathcal{N}}$ which is not an r-degenerate eigenform is anti-attracting for $\widetilde{\Lambda^*}_r$. We need the following lemma, which is well-known, but however, I will prove it.

Lemma 5.1 *If $E, E' \in \widetilde{\mathcal{D}}$ and $0 < \rho < \rho'$ we cannot have $\Lambda_r(E) \le \rho E$ and $\Lambda_r(E') \ge \rho' E'$.*

P r o o f. By contradiction, if $\Lambda_r(E) \le \rho E$ and $\Lambda_r(E') \ge \rho' E'$, using an inductive argument we obtain

$$\Lambda_r{}^n(E) \le \rho^n E, \quad \Lambda_r{}^n(E') \ge \rho'^n E' \ \ \forall n \in \mathbb{N}.$$

However, in view of Lemma 2.1, there exist positive c and c' such that $cE' \ge E \ge c'E'$ which implies, $\Lambda_r{}^n(E) \ge c'\Lambda_r{}^n(E')$ for every positive integer n, so that

$$\rho^n E \ge \Lambda_r{}^n(E) \ge c'\Lambda_r{}^n(E') \ge c'\rho'^n E' \ge \frac{c'}{c}\rho'^n E$$

and, since $0 < \rho < \rho'$, this cannot hold for large n. □

Now, for every form $\widetilde{E} \in \mathrm{int}(\mathcal{D}_{\mathcal{N}}) = \mathcal{D}_{\mathcal{N}}^{(1)}$, according to the notation of the previous section, put

$$\mathrm{Ext}_{\widetilde{E}}(E) = \left\{\widetilde{E} + t(E - \widetilde{E}) : t > 1\right\} \ \forall E \in Z \setminus \{\widetilde{E}\}.$$

Here, $\mathcal{D}_{\mathcal{N}}$ plays the role of K in Section 4. Next, define $p_{\widetilde{E}} : Z \setminus \{\widetilde{E}\} \to \partial\mathcal{D}_{\mathcal{N}}$ as in the previous section.

Lemma 5.2 *Let $r \in W$ and let θ be a continuous map from $\partial\mathcal{D}_{\mathcal{N}}$ to $\mathcal{D}_{\mathcal{N}}^{(1)}$. Then every $\overline{E} \in \mathcal{D}_{\mathcal{N}}^{(4)}$ is anti-attracting for $(\widetilde{\Lambda^*}_r, \theta)$.*

P r o o f. First, prove that there exists a neighborhood U of $\overline{E}$ such that

$$p_{\theta(E')}(E)_d \le 2E_d \quad \forall E \in U \cap \mathcal{D}_{\mathcal{N}}, \ \forall E' \in U \cap \partial\mathcal{D}_{\mathcal{N}}, \ \forall d \in J. \tag{5.2}$$

Note that by Lemma 4.2 iii), if $E \in \mathcal{D}_{\mathcal{N}} \setminus \{\theta(E')\}$ we have $E \in [p_{\theta(E')}(E), \theta(E')]$, thus

$$E_d \in [p_{\theta(E')}(E)_d, \theta(E')_d] \quad \forall d \in J. \tag{5.3}$$

If $d \in J$ satisfies $\overline{E}_d = 0$, since $\theta(\overline{E}) \in \mathcal{D}_{\mathcal{N}}^{(1)}$, thus $\theta(\overline{E})_d > 0$, by continuity we have $E_d < \theta(E')_d$, thus by (5.3) we have $p_{\theta(E')}(E)_d \le E_d$, and a fortiori (5.2). On the other hand, by continuity we have

$$E_d \underset{E \to \overline{E}}{\longrightarrow} \overline{E}_d, \quad p_{\theta(E')}(E)_d \underset{(E,E') \to (\overline{E}, \overline{E})}{\longrightarrow} p_{\theta(\overline{E})}(\overline{E})_d = \overline{E}_d \quad \forall d \in J.$$

We have used the fact that, since $\overline{E} \in \mathcal{D}_{\mathcal{N}}^{(4)} \subseteq \partial\mathcal{D}_{\mathcal{N}}$, by Lemma 4.2 iii) we have $p_{\theta(\overline{E})}(\overline{E}) = \overline{E}$. Thus (5.2) holds for a suitable U, also for d such that $\overline{E}_d > 0$, and (5.2) is proved. We now prove by contradiction that, possibly restricting U, given $E \in U \cap \mathcal{D}_{\mathcal{N}}^{(1)}$, $E' \in U \cap \partial\mathcal{D}_{\mathcal{N}}$ we have

$$\widetilde{\Lambda^*}_r(E) \notin \mathrm{Ext}_{\theta(E')}(E). \tag{5.4}$$

By Lemma 4.2 iv) we have $\widehat{E}_d \in [E_d, p_{\theta(E')}(E)_d]$ if $\widehat{E} \in \mathrm{Ext}_{\theta(E')}(E) \cap \mathcal{D}_{\mathcal{N}}$, for every $d \in J$. Thus, by (5.2), if (5.4) does not hold we have

$$\widetilde{\Lambda^*}_r(E)_d \le 2E_d \quad \forall d \in J. \tag{5.5}$$

Take a positive ε which we will specify later. Since $\overline{E} \in \mathcal{D}_{\mathcal{N}}^{(4)}$, by definition we can choose U such that

$$\big(\Lambda_r(E)\big)_d < \varepsilon \quad \forall E \in U \cap \mathcal{D}_{\mathcal{N}}^{(1)}, \quad \forall d \in J. \tag{5.6}$$

For such E, by the definition of $\widetilde{\Lambda^*}_r$ we have $\alpha\widetilde{\Lambda^*}_r(E) = \Lambda_r(E)$ for some $\alpha > 0$ (depending on E). Thus, by (5.1) and (5.6), for some $\overline{d} \in J$ we have

$$\alpha\widetilde{m} \leq \alpha\big(\widetilde{\Lambda^*}_r(E)\big)_{\overline{d}} = \big(\Lambda_r(E)\big)_{\overline{d}} < \varepsilon.$$

It follows that $\alpha < \frac{\varepsilon}{\widetilde{m}}$. Hence, in view of (5.5) we have

$$\big(\Lambda_r(E)\big)_d = \alpha\big(\widetilde{\Lambda^*}_r(E)\big)_d \leq 2\frac{\varepsilon}{\widetilde{m}}E_d \quad \forall\, d \in J.$$

Thus, we have $\Lambda_r(E) \preceq \dfrac{2\varepsilon}{\widetilde{m}}E$, hence $\Lambda_r(E) \leq \dfrac{2\varepsilon}{\widetilde{m}}E$. Fix $\widetilde{E} \in \mathcal{D}_{\mathcal{N}}^{(1)}$ and take $c > 0$ so that $\Lambda_r(\theta(E')) \geq c\widetilde{E}$ (see Lemma 2.1). Thus, if we choose ε so that $\dfrac{2\varepsilon}{\widetilde{m}} < c$, we have contradicted Lemma 5.1. Such a contradiction shows that (5.4) holds and the Lemma is proved. □

Lemma 5.3 *Let $r \in W$ and let θ be a continuous map from $\partial\mathcal{D}_{\mathcal{N}}$ to $\mathcal{D}_{\mathcal{N}}^{(1)}$. Then every $\overline{E} \in \mathcal{D}_{\mathcal{N}}^{(2)} \cup \mathcal{D}_{\mathcal{N}}^{(3)}$ such that $\widetilde{\Lambda^*}_r(\overline{E}) \neq \overline{E}$ is anti-attracting for $(\widetilde{\Lambda^*}_r, \theta)$.*

Proof. Since $p_{\theta(\overline{E})}(\overline{E}) = \overline{E}$ we have $\widetilde{\Lambda^*}_r(\overline{E}) \notin [\overline{E}, p_{\theta(\overline{E})}(\overline{E})] = \{\overline{E}\}$. By continuity, there exists a neighborhood U of $\overline{E}$ such that for every $E \in U \cap \mathcal{D}_{\mathcal{N}}^{(1)}$ and every $E' \in U \cap \partial\mathcal{D}_{\mathcal{N}}$ we have $\widetilde{\Lambda^*}_r(E) \notin [E, p_{\theta(E')}(E)]$, thus, by Lemma 4.2 iv), $\widetilde{\Lambda^*}_r(E) \notin \mathrm{Ext}_{\theta(E')}(E)$. □

6 The Final Results.

In view of Lemmas 5.2 and 5.3, we can use Theorem 4.5, provided that also every degenerate r-eigenform in $\partial\mathcal{D}_{\mathcal{N}}$ is anti-attracting for $(\widetilde{\Lambda^*}_r, \theta)$. However, this does not necessarily hold, but depends on the r-eigenform. More precisely, we have to study carefully the local behavior of Λ_r near a degenerate r-eigenform. Recall that if $\overline{E} \in \mathcal{D} \setminus \widetilde{\mathcal{D}}$, $\overline{E} \neq 0$, then ker$(\overline{E})$ strictly contains the set of the constant functions. We say that the degenerate r-eigenform $\overline{E} \in \mathcal{D}_{\mathcal{N}}^{(3)}$ with eigenvalue ρ is $\widetilde{E}$*-repulsing* if

$$\exists\, \rho' > \rho:\ \Lambda_{r,\overline{E}}(\widetilde{E})(u) \geq \rho'\widetilde{E}(u)\ \forall\, u \in \mathrm{ker}(\overline{E}). \tag{6.1}$$

Recall that $\Lambda_{r,\overline{E}}(\widetilde{E})$ has been defined in Section 3. We also say that the degenerate r-eigenform is repulsing if it is $\widetilde{E}$-repulsing for some $\widetilde{E} \in \mathcal{D}_{\mathcal{N}}^{(1)}$. We now prove that if θ is a continuous map from $\partial\mathcal{D}_{\mathcal{N}}$ to $\mathcal{D}_{\mathcal{N}}^{(1)}$ then every degenerate r-eigenform $\overline{E} \in \mathcal{D}_{\mathcal{N}}^{(3)}$, $\theta(\overline{E})$-repulsing, is anti-attracting for $(\widetilde{\Lambda^*}_r, \theta)$. We need some preliminary lemmas. Here, we consider $\overline{E} \in \mathcal{D}_{\mathcal{N}}^{(3)}$ fixed in the following lemmas.

Lemma 6.1 *If $E, \widetilde{E} \in \mathcal{D}_{\mathcal{N}}^{(1)}$, then the ratio $\dfrac{E}{\widetilde{E}}$ attains a minimum $\eta_{E,\widetilde{E}}$ on the set of all non-constant functions in ker$(\overline{E})$.*

Proof. The proof is similar to that of Lemma 2.1. □

Lemma 6.2 *If $E, \widetilde{E} \in \mathcal{D}_{\mathcal{N}}^{(1)}$, we have*
i) $\eta_{E,\widetilde{E}} > 0$.
ii) $E(u) \geq \eta_{E,\widetilde{E}}\widetilde{E}(u) \quad \forall\, u \in$ ker$(\overline{E})$.
iii) There exists $\overline{u}_{E,\widetilde{E}} \in$ ker$(\overline{E}) \cap S$ such that $E(\overline{u}_{E,\widetilde{E}}) = \eta_{E,\widetilde{E}}\widetilde{E}(\overline{u}_{E,\widetilde{E}})$, where S is the set defined in Lemma 2.1.

Proof. Trivial. □

The following Lemma is the most technical point in this paper, where we use a previous result of Sabot (a similar result was proved later by Metz in [5]) whose proof is rather long.

Lemma 6.3 *Let $r \in W$ and let $\widetilde{E} \in \mathcal{D}^{(1)}_{\mathcal{N}}$. If $\overline{E} \in \mathcal{D}^{(3)}_{\mathcal{N}}$ and $\widetilde{\Lambda^*}_r(\overline{E}) = \overline{E}$, then for every $\alpha < 1$ there exists a neighborhood U of $\overline{E}$ such that*

$$\Lambda_r(E)(u) \geq \alpha \eta_{E,\widetilde{E}} \Lambda_{r,\overline{E}}(\widetilde{E})(u) \quad \forall\, E \in U \cap \mathcal{D}^{(1)}_{\mathcal{N}}\ \forall u \in ker(\overline{E}).$$

Proof. This is a consequence of the arguments in [8]. For example, by [8], Prop. 4.23 (see also Prop. 23 in [5]) there exists a neighborhood U of $\overline{E}$ such that for every $E \in U \cap \mathcal{D}^{(1)}_{\mathcal{N}}$ and every $u \in \ker(\overline{E})$ we have $\Lambda_r(E)(u) \geq \alpha \Lambda_{r,\overline{E}}(E)(u)$. The Lemma follows from Lemma 6.2 ii) and the definition of $\Lambda_{r,\overline{E}}(E)$. □

Lemma 6.4 *Let $r \in W$ and let θ be a continuous map from $\partial\mathcal{D}_{\mathcal{N}}$ to $\mathcal{D}^{(1)}_{\mathcal{N}}$. Suppose a degenerate r-eigenform $\overline{E} \in \mathcal{D}^{(3)}_{\mathcal{N}}$ is $\theta(\overline{E})$-repulsing. Then $\overline{E}$ is anti-attracting for $(\widetilde{\Lambda^*}_r, \theta)$.*

Proof. By Lemma 6.3 and (6.1), given $\alpha \in]0,1[$, we find a neighborhood U of $\overline{E}$ such that, if $E \in U \cap \mathcal{D}^{(1)}_{\mathcal{N}}$ and $u \in \ker(\overline{E})$ we have

$$\Lambda_r(E)(u) \geq \rho' \alpha \eta_{E,\theta(\overline{E})} \theta(\overline{E})(u). \tag{6.2}$$

On the other hand, since the ratio $\frac{\theta(\overline{E})}{\theta(E')}$ attains all values on the compact set S defined in Lemma 2.1, it can be easily seen, using Lemma 2.2 i) and the continuity of θ, that, given $\sigma > 0$, possibly restricting U, if $E' \in U$ we have

$$(1-\sigma)\theta(E')(u) \leq \theta(\overline{E})(u) \leq (1+\sigma)\theta(E')(u) \quad \forall\, u \in \mathbb{R}^{V^{(0)}}.$$

It follows that $\eta_{E,\theta(\overline{E})} \geq \frac{1}{1+\sigma}\eta_{E,\theta(E')}$. Thus, in view of (6.2), we have

$$\Lambda_r(E)(u) \geq \rho'' \alpha \eta_{E,\theta(E')} \theta(E')(u) \quad \forall\, u \in \ker(\overline{E}) \tag{6.3}$$

for every $E \in U \cap \mathcal{D}^{(1)}_{\mathcal{N}}$ and every $E' \in U \cap \partial\mathcal{D}_{\mathcal{N}}$, where $\rho'' = \frac{1-\sigma}{1+\sigma}\rho' > \rho$ for sufficiently small $\sigma > 0$.

Next, note that $\Lambda_r(\overline{E}) = \rho\overline{E}$, hence $|\Lambda_r(\overline{E})| = \rho|\overline{E}| = \rho$. Thus, for $u \in \ker(\overline{E})$, in view of (6.3) we have

$$\begin{aligned}&\widetilde{\Lambda^*}_r(E)(u)\\ &= \frac{\Lambda_r(E)(u)}{|\Lambda_r(\overline{E})|}\frac{|\Lambda_r(\overline{E})|}{|\Lambda_r(E)|}\\ &\geq \frac{\rho''}{\rho}\alpha\eta_{E,\theta(E')}\frac{|\Lambda_r(\overline{E})|}{|\Lambda_r(E)|}\theta(E')(u).\end{aligned}$$

Also, possibly restricting U, by the continuity of Λ_r we can assume that $|\Lambda_r(\overline{E})| \geq \alpha|\Lambda_r(E)|$. Hence,

$$\widetilde{\Lambda^*}_r(E)(u) \geq \alpha^2 \frac{\rho''}{\rho}\eta_{E,\theta(E')}\theta(E')(u).$$

Since $\rho'' > \rho$ we can choose α such that $\alpha^2\frac{\rho''}{\rho} > 1$. Thus if $u \in \ker(\overline{E})$ is non-constant we have

$$\widetilde{\Lambda^*}_r(E)(u) > \eta_{E,\theta(E')}\theta(E')(u). \tag{6.4}$$

It follows that $\overline{E}$ is anti-attracting for $(\widetilde{\Lambda^*}_r, \theta)$ as $\widetilde{\Lambda^*}_r(E) \notin \mathrm{Ext}_{\theta(E')}(E)$ for every $E \in U \cap \mathcal{D}^{(1)}_{\mathcal{N}}$ and every $E' \in U \cap \partial\mathcal{D}_{\mathcal{N}}$. In fact, in the opposite case, for some $E \in U \cap \mathcal{D}^{(1)}_{\mathcal{N}}$ and some $E' \in U \cap \partial\mathcal{D}_{\mathcal{N}}$, by Lemma 4.2 iv), we have

$$\widetilde{\Lambda^*}_r(E) \in [E, p_{\theta(E')}(E)]. \tag{6.5}$$

On the other hand, by Lemma 4.2 iii) we have

$$E \in [p_{\theta(E')}(E), \theta(E')]. \tag{6.6}$$

Next, possibly restricting U, in view of Lemma 2.2 i), since the set S defined in Lemma 2.1 is compact, we can assume that for every $E \in U \cap \mathcal{D}_{\mathcal{N}}$ and every $u \in S$ we have

$$|E(u) - \overline{E}(u)| < \delta, \quad \delta := \min\big\{\theta(E'')(u) : E'' \in \partial\mathcal{D}_{\mathcal{N}}, u \in S\big\}. \tag{6.7}$$

Note that the minimum defining δ does exist since $\partial\mathcal{D}_{\mathcal{N}}$ and S are compact and the map $(E'', u) \mapsto \theta(E'')(u)$ is continuous by Lemma 2.2 i) and the continuity of θ. On the other hand, we have $\delta > 0$ as θ takes values in $\mathcal{D}_{\mathcal{N}}^{(1)}$ and every $u \in S$ is non-constant.

Thus, if we fix $E \in U \cap \mathcal{D}_{\mathcal{N}}^{(1)}$ and $E' \in U \cap \partial\mathcal{D}_{\mathcal{N}}$ and put $\overline{u} := \overline{u}_{E,\theta(E')}$, since $\overline{u} \in \ker(\overline{E}) \cap S$, in view of (6.7) we have $E(\overline{u}) < \theta(E')(\overline{u})$. Thus, by (6.6) we have $p_{\theta(E')}(E)(\overline{u}) \leq E(\overline{u})$, thus, by (6.5) and Lemma 6.2 iii) we have $\widetilde{\Lambda^*}_r(E)(\overline{u}) \leq E(\overline{u}) = \eta_{E,\theta(E')}\theta(E')(\overline{u})$. This contradicts (6.4) and the Lemma is proved. □

Theorem 6.5 *Suppose that there exists a continuous map $\theta : \partial\mathcal{D}_{\mathcal{N}} \to \mathcal{D}_{\mathcal{N}}^{(1)}$ such that every degenerate r-eigenform $\overline{E} \in \mathcal{D}_{\mathcal{N}}^{(3)}$ is $\theta(\overline{E})$-repulsing. Then there exists an r-eigenform.*

Proof. Suppose there exist no r-eigenforms, in particular $\widetilde{\Lambda^*}_r(E) \neq E$ for every $E \in \mathcal{D}_{\mathcal{N}}^{(2)}$. By Lemmas 5.2, 5.3 and 6.4, the hypothesis of Theorem 4.5 is satisfied with $K = \mathcal{D}_{\mathcal{N}}$, and $\phi = \widetilde{\Lambda^*}_r$. Thus, there exists $\overline{E} \in \mathcal{D}_{\mathcal{N}}^{(1)}$ such that $\widetilde{\Lambda^*}_r(\overline{E}) = \overline{E}$, hence $\overline{E}$ is an r-eigenform. □

We now want to prove that, in the hypothesis of previous theorem, we can remove the continuity of θ. In other words, if every r-degenerate eigenform $\overline{E}$ in $\mathcal{D}_{\mathcal{N}}^{(3)}$ is repulsing, that is repulsing with respect to some $\theta(\overline{E})$, we can choose such a function θ to be continuous. We need two simple lemmas.

Lemma 6.6 *We have $\Lambda_{r,\overline{E}}(\sum_{h=1}^{s} a_h E_h) \geq \sum_{h=1}^{s} a_h \Lambda_{r,\overline{E}}(E_h)$ whenever $a_h \geq 0$ and $E_h \in \widetilde{\mathcal{D}}$.*

Proof. This is a standard and simple fact (see [5], Proposition 2 ii) for the analogous statement on Λ_r). □

Lemma 6.7 *i) $\mathcal{D}_{\mathcal{N}}^{(3)} \cup \mathcal{D}_{\mathcal{N}}^{(4)}$ is closed.*
ii) If P is a P-part, then $\overline{P} \subseteq \bigcup\big\{P' : P'$ is a P-part$, P \preceq P'\big\}$.

Proof. i) If $E_h \in \mathcal{D}_{\mathcal{N}}^{(3)} \cup \mathcal{D}_{\mathcal{N}}^{(4)}$ and $E_h \underset{h\to+\infty}{\longrightarrow} E$, then $E \in \mathcal{D}_{\mathcal{N}}^{(3)} \cup \mathcal{D}_{\mathcal{N}}^{(4)}$. In fact, in the opposite case, we have $E \in \widetilde{\mathcal{D}}$, but then there exists $\delta > 0$ such that $E(u) \geq \delta$ for every $u \in S$, where S is as usual, the set defined in Lemma 2.1. By Lemma 2.2 i) and the compactness of S we have, for large h, $E_h(u) > 0$ for every $u \in S$. This easily implies $E_h(u) > 0$ for every non-constant $u \in \mathbb{R}^{V^{(0)}}$, thus $E_h \in \widetilde{\mathcal{D}}$, a contradiction. ii) Clearly, the kernel of the limit of a sequence in P contains $\ker(P)$. □

Theorem 6.8 *Suppose that every degenerate r-eigenform $\overline{E} \in \mathcal{D}_{\mathcal{N}}^{(3)}$ is repulsing. Then there exists a continuous map $\theta : \partial\mathcal{D}_{\mathcal{N}} \to \mathcal{D}_{\mathcal{N}}^{(1)}$ such that every degenerate r-eigenform $\overline{E} \in \mathcal{D}_{\mathcal{N}}^{(3)}$ is $\theta(\overline{E})$-repulsing, thus there exists an r-eigenform.*

Proof. Since on every P-part the eigenvalue of possible degenerate r-eigenforms does not depend on the r-eigenform as observed in Section 3, and we have finitely many P-parts, the degenerate r-eigenforms in $\mathcal{D}_{\mathcal{N}}^{(3)}$ have only finitely many eigenvalues. Call them $\rho_i > 0, i = 1, ..., n$. Let

$$B_i = \big\{E \in \mathcal{D}_{\mathcal{N}}^{(3)} : \Lambda_r(E) = \rho_i E\big\}.$$

Also using Lemma 6.7 i), we easily see that the sets B_i are closed and mutually disjoint. Now, when P is a P-part put $B_{i,P} = B_i \cap P$, and, if P' is another P-part, put $B_{i,P} \preceq B_{i,P'}$ if $P \preceq P'$. We define

$$\mathcal{B}_i = \big\{B_{i,P} : P \text{ P-part}, B_{i,P} \neq \varnothing\big\}.$$

We enumerate the P-parts that intersect B_i as P_h, $h = 1, ..., s_i$, and put $B_{i,h} := B_{i,P_{i,h}}$, thus the sets $B_{i,h}$, $h = 1, ..., s_i$, are the elements of $\mathcal{B}_i$. We order the sets $P_{i,h}$ so that the sets $B_{i,h}$, $h = 1, ..., s_i'$, with $1 \leq s_i' \leq s_i$, are the maximal elements of $\mathcal{B}_i$ (with respect to the order $\preceq$). Note that, by Lemma 6.7 ii), the sets $B_{i,h}$, $h = 1, ..., s'_i$, are closed. Let

$$B'_i = \bigcup_{h=1}^{s'_i} B_{i,h},$$

thus B'_i is closed.

By hypothesis, for every $B_{i,h}$, $h = 1, ..., s'_i$ there exists $E_{i,h} \in \mathcal{D}^{(1)}_{\mathcal{N}}$ such that an arbitrarily fixed $\overline{E} \in B_{i,h}$ is $E_{i,h}$-repulsing. But, by the definition of $\Lambda_{r,\overline{E}}$ this implies that *every* $\overline{E} \in B_{i,h}$ is $E_{i,h}$-repulsing. Put $\theta = E_{i,h}$ on $B_{i,h}$. Now, let

$$L_{i,h} = \bigcup \big\{ B_{i,h'} : h' = 1, ..., s_i, B_{i,h'} \not\preceq B_{i,h} \big\}.$$

By Lemma 6.7 ii) again, $L_{i,h}$ is closed. Since $B_{i,h}$ and $L_{i,h}$ are disjoint closed subset of B_i, by Tietze's Theorem (in its usual form) there exists a continuous function $\alpha_{i,h} : B_i \to \mathbb{R}$ that attains the value 1 on $B_{i,h}$, the value 0 on $L_{i,h}$, and is strictly between 0 and 1 elsewhere. In particular, $\alpha_{i,h}$ attains the value 0 on the sets of the form $B_{i,h'}$, $h' = 1, ..., s'_i$, $h' \neq h$, as they are subsets of $L_{i,h}$ by definition. We now extend θ on B_i putting

$$\theta(E) = \sum_{h=1}^{s'_i} \beta_{i,h}(E) E_{i,h}, \quad \beta_{i,h}(E) := \frac{\alpha_{i,h}(E)}{\sum_{h=1}^{s'_i} \alpha_{i,h}(E)}.$$

Note that, given $E \in B_i$, then $E \in B_{i,h'}$ for some $h' = 1, ..., s_i$, and there exists $h = 1, ..., s'_i$ such that $B_{i,h'} \preceq B_{i,h}$, therefore $\alpha_{i,h}(E) > 0$, thus the function $\sum_{h=1}^{s'_i} \alpha_{i,h}$ is positive on B_i. Also, observe that, since $\mathcal{D}^{(1)}_{\mathcal{N}}$ is convex, then $\theta(E) \in \mathcal{D}^{(1)}_{\mathcal{N}}$. Since $\alpha_{i,h'}$ amounts to 1 on $B_{i,h'}$ and to 0 on the other $B_{i,h}$, the function so defined in fact extends θ.

Next, suppose $E \in B_i$ so that in the definition of $\theta(E)$ we can restrict the sum to the h's that satisfy Ker$(P_{i,h})$ $\supseteq$Ker(E), as we have $\alpha_{i,h}(E) = 0$ for the other h's. Let $\overline{E}_{i,h} \in B_{i,h}$, so that by definition we have $\Lambda_{r,E} \geq \Lambda_{r,\overline{E}_{i,h}}$. Thus, also using Lemma 6.6, for every $u \in \ker E \subseteq \ker \overline{E}_{i,h}$, we have

$$\begin{aligned}
\Lambda_{r,E}\big(\theta(E)\big)(u) &\geq \sum_{h=1}^{s'_i} \beta_{i,h}(E) \Lambda_{r,E}(E_{i,h})(u) \\
&\geq \sum_{h=1}^{s'_i} \beta_{i,h}(E) \Lambda_{r,\overline{E}_{i,h}}(E_{i,h})(u) \\
&\geq \sum_{h=1}^{s'_i} \beta_{i,h}(E) \rho'_{i,h} E_{i,h}(u) \\
&\geq \min_h \{\rho'_{i,h}\} \sum_{h=1}^{s'_i} \beta_{i,h}(E) E_{i,h}(u) = \min_h \{\rho'_{i,h}\} \theta(E)(u),
\end{aligned}$$

where $\rho'_{i,h}$ are numbers greater than ρ_i. These numbers exist since $\overline{E}_{i,h}$ is $E_{i,h}$-repulsing. In sum, E is $\theta(E)$-repulsing and $\theta : B_i \to \mathcal{D}^{(1)}_{\mathcal{N}}$ is continuous. Since the sets B_i are mutually disjoint closed subsets of the compact set $\partial\mathcal{D}_{\mathcal{N}}$ and θ is a continuous function from $\bigcup_{i=1}^{n} B_i$ with values in the convex set $\mathcal{D}^{(1)}_{\mathcal{N}}$, using Lemma 4.3, θ can be extended continuously to all of $\partial\mathcal{D}_{\mathcal{N}}$ with values in $\mathcal{D}^{(1)}_{\mathcal{N}}$. □

Remark 6.9 We now compare Theorem 6.8 with Theorem 3.2. First, note that when Theorem 6.8 is applicable also Theorem 3.2 is so. To prove this, we use a variant of an argument in [5], namely Lemma 8 there. Here we use similar considerations but replacing Λ_r with $\Lambda_{r,\overline{E}}$. Suppose the hypothesis of Theorem 6.8 is satisfied, that is, for every degenerate r-eigenform $\overline{E} \in \mathcal{D}^{(3)}_{\mathcal{N}}$ with eigenvalue ρ

$$\exists \widetilde{E} \in \mathcal{D}^{(1)}_{\mathcal{N}}, \exists \rho' > \rho : \Lambda_{r,\overline{E}}(\widetilde{E})(u) \geq \rho' \widetilde{E}(u) \; \forall \, u \in \ker(\overline{E}). \tag{6.8}$$

We now want to prove that the hypothesis of Theorem 3.2 is satisfied. Let $\overline{E}$ be a nonzero degenerate r-eigenform with eigenvalue $\rho > 0$. It immediately follows that $\frac{\overline{E}}{|\overline{E}|} \in \mathcal{D}^{(3)}_{\mathcal{N}}$, thus $\overline{E}$ satisfies (6.8). Note in fact, that by definition we have $\Lambda_{r,\overline{E}} = \Lambda_{r,\frac{\overline{E}}{|\overline{E}|}}$. Also, note that we can associate to the P-part containing $\overline{E}$ a partition $\{W_1, ..W_s\}$ of $V^{(0)}$ such that $\ker(\overline{E})$ is the set of the functions in $\mathbb{R}^{V^{(0)}}$ that are constant on every W_i. This fact is well-known. The graph $\mathcal{G}(\overline{E})$ defined in Section 2 is disconnected as $\overline{E} \notin \widetilde{\mathcal{D}}$ and the sets $W_1, ..., W_s$ are the components of $\mathcal{G}(\overline{E})$. Define $\mathcal{D}(\overline{E})$ to be the set of the restrictions of $E \in \mathcal{D}$ to $\ker(\overline{E})$. Note that the coefficients $E_{\{j_1,j_2\}}$ of an element E of $\mathcal{D}(\overline{E})$ are not unique, since different sets of coefficients can give the same function on $\ker(\overline{E})$. However, we can find a canonical set of coefficients. Select $P_{j(h)} \in W_h$ and put

$$E(u) = \sum_{h,h'=1}^{s} E_{j(h),j(h')} \big(u(P_{j(h)}) - u(P_{j(h')})\big)^2 \tag{6.9}$$

with $E_{j(h),j(h')} \geq 0$ [1]. Moreover, put $\mathcal{D}_{\mathcal{N}}(\overline{E}) = \big\{E \in \mathcal{D}(\overline{E}) : |E| = 1\big\}$. Here, $|E|$ is defined as usual using the coefficients of E given by (6.9). Next, define $\widetilde{\Lambda^*}_{r,\overline{E}}(E) = \frac{\Lambda_{r,\overline{E}}(E)}{|\Lambda_{r,\overline{E}}(E)|}$. Consider now the set

$$A := \big\{E \in \mathcal{D}_{\mathcal{N}}(\overline{E}) : \Lambda_{r,\overline{E}}(E)(u) \geq \rho' E(u) \; \forall \, u \in \ker(\overline{E})\big\}.$$

Then A is non-empty by (6.8), convex and compact. Moreover, $\widetilde{\Lambda^*}_{r,\overline{E}}$ maps continuously A into itself, thus it has a fixed point on A. As a consequence $\Lambda_{r,\overline{E}}$ has an eigenvector in $\mathcal{D}_{\mathcal{N}}$ with eigenvalue greater than ρ, thus satisfies the hypothesis of Theorem 3.2.

On the other hand, Theorem 6.8 is (at least apparently) slightly weaker since in the hypothesis of Theorem 3.2 it is not clear that we find a function $\widetilde{E} \in \mathcal{D}^{(1)}_{\mathcal{N}}$ satisfying (6.1). In fact, this would be trivial if the eigenvector E given in Theorem 3.2 belongs to $\widetilde{\mathcal{D}}$, since we can approximate it with forms in $\mathcal{D}^{(1)}_{\mathcal{N}}$. In general this approximation does not work to obtain (6.1) for $u \in \ker(E)$.

However, I think that in the usual cases where Theorem 3.2 works so also does Theorem 6.8. In particular, note that the condition given in Remark 3.3 can be deduced from Theorem 6.8. I also suspect that refining the argument in Theorem 6.8 we could obtain the exact statement of Theorem 3.2.

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[1] In [8] and in [5], instead of selecting points in W_i, the authors have considered forms on the set of W_i. Here, for simplicity, I have preferred to avoid the use of equivalence classes.