An algebraic condition for the Bisognano-Wichmann property

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The Bisognano-Wichmann property for local, Poincaré covariant nets of standard subspaces is discussed. We present a sufficient algebraic condition on the covariant representation ensuring Bisognano-Wichmann and Duality properties without further assumptions on the net. Our *"modularity"* condition holds for direct integrals of scalar massive and masselss representations. We conclude that in these cases the Bisognano-Wichmann property is much weaker than the Split property. Furthermore, we present a class of massive modular covariant nets not satisfying the Bisognano-Wichmann property.

Keywords: Algebraic quantum field theory; Bisognano-Wichmann property; modular covariance; standard subspaces.

1. Introduction

The Tomita-Takesaki theory for von Neumann algebras has a crucial role in the study of geometric properties of Quantum Field Theories. Bisognano and Wichmann showed that, in Wightman theories, modular groups associated to wedge regions have a geometrical interpretation: they implement pure Lorentz transformations.[1,2]

Let $\mathbb{R}^{1+3} \supset \mathcal{O} \mapsto \mathcal{A}(\mathcal{O}) \subset \mathcal{B}(\mathcal{H})$ be a local Poincaré covariant net of von Neumann algebras on a fixed Hilbert space $\mathcal{H}$, from certain open, connected bounded regions of the Minkowski spacetime $\mathbb{R}^{1+3}$. Let U be a positive energy Poincaré representation acting covariantly on $\mathcal{A}$ and let Ω be the unique (up to a phase) normalized U-invariant vector, namely the vacuum vector. Ω is supposed to be cyclic for all the local algebras. The **Bisognano-Wichmann** property (B-W) is stated as follows:

$$U(\Lambda_W(2\pi t)) = \Delta^{-it}_{\mathcal{A}(W),\Omega}, \tag{1}$$

for any wedge shaped region W, where $t \mapsto \Lambda_W(t)$ is the one parameter group of boosts associated to the wedge W and $\Delta^{it}_{\mathcal{A}(W),\Omega}$ is the modular group associated to $\mathcal{A}(W) = (\cup_{\mathcal{O}\subset W}\mathcal{A}(\mathcal{O}))''$ w.r.t. Ω. The Bisognano-Wichmann property strictly links the covariant representation of the Poincaré group to the net: in a precise sense it is enclosed in the net once we choose the vacuum state and the algebra of the observables.

It is a basic fact in many aspects of QFT: it holds in massive theories[14], it implies the Spin-Statistics canonical relations[9] and the essential Duality property. References on Duality and Bisognano-Wichmann properties can be found in Refs. 4, 5, 15, 7, 14, 6. In general, counter-examples have a pathological nature. For

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instance, theories having an infinite number of particles in the same mass multiplet do not need to satisfy the B-W property (cf. Sect. 5).

The Bisognano-Wichmann property cares only about the modular structure of the algebraic model which is contained in the standard subspace structure of the net. Standard subspace techniques give a fruitful approach to QFT. For instance, they are useful to model first quantization nets, giving a canonical structure to free fields (cf. Brunetti-Guido-Longo construction in Ref. 3), and they have a keyrole in finding localization properties of Infinite Spin particles, cf. Ref. 12.

One should expect an algebraic description of the B-W property. We ask when a (unitary, postive energy) Poincaré representation U is *modular*, i.e. any net of standard subspaces it acts covariantly on satisfies the B-W property, in particular U is implemented by modular operators. This is expected, at least, for irreducible Poincaré, positive energy representations. We present an *algebraic condition* ensuring the modularity of a representation. It holds at least for scalar Poincaré representations in any spacetime $\mathbb{R}^{1+s}$, with $s \geq 3$.

This analysis is useful to compute counterexamples. We describe a massive, modular covariant net, not respecting the B-W property. The massless case is contained in Ref. 12. Furthermore, an outlook on the relation between modular covariance and Split property on generalized free fields is given. Details about this analysis can be found in Ref. 13.

2. One particle net

A linear, real, closed subspace H of a complex Hilbert space $\mathcal{H}$ is called *cyclic* if $H + iH$ is dense in $\mathcal{H}$, *separating* if $H \cap iH = \{0\}$ and *standard* if it is cyclic and separating. Let H be a standard subspace, the Tomita operator S_H is the closure, of the densely defined anti-linear involution

$$H + iH \ni \xi + i\eta \mapsto \xi - i\eta \in H + iH, \qquad \xi, \eta \in H.$$

Its polar decomposition $S_H = J_H \Delta_H^{1/2}$ defines the positive selfadjoint *modular operator* Δ_H and the anti-unitary *modular conjugation* J_H. Δ_H is invertible and

$$J_H \Delta_H J_H = \Delta_H^{-1}. \tag{2}$$

Standard subspaces are in 1-1 correspondence with densely defined antilinear involutions which are in 1-1 correspondence with pairs of an antiunitary, involutions and selfadjoint operators satisfying (2). Here is a basic Lemma on standard subspaces.

Lemma 2.1. [10,11] *Let $H, K \subset \mathcal{H}$ be standard subspaces and $U \in \mathcal{U}(\mathcal{H})$ be a unitary operator on $\mathcal{H}$ s.t. $UH = K$. Then $U\Delta_H U^* = \Delta_K$ and $UJ_H U^* = J_K$.*

Consider the (proper, orthocronous) Poincaré group $\mathcal{P}_+^\uparrow$ of the 3+1 dimensional Minkowski spacetime $\mathbb{R}^{3+1}$. $\mathcal{P}_+^\uparrow$ is the semidirect product of 4-translations and the connected component of the identity of the Lorentz group denoted by $\mathcal{L}_+^\uparrow$. A wedge shaped region W in $\mathbb{R}^{3+1}$ is an open region of the form gW_1 where $g \in \mathcal{P}_+^\uparrow$ and

$W_1 = \{p \in \mathbb{R}^{3+1} : |p_0| < p_1\}$. W' denotes the causal complement of W w.r.t. the causal structure of $\mathbb{R}^{3+1}$. The set of wedges is denoted by $\mathcal{W}$. $\mathcal{W}_0$ denotes the subset of wedges of the form gW_1 where $g \in \mathcal{L}_+^\uparrow$. For every wedge $W \in \mathcal{W}$ there exists a unique 1-parameter group of Poincaré boosts $t \mapsto \Lambda_W$ preserving W, i.e. $\Lambda_W(t)W = W$ for every $t \in \mathbb{R}$.

Let U be a unitary, positive energy representation (p.e.r.) of the Poincaré group $\mathcal{P}_+^\uparrow$ on an Hilbert space $\mathcal{H}$. We shall call a U-covariant net of standard subspaces on wedges a map

$$H : \mathcal{W} \ni W \longmapsto H(W) \subset \mathcal{H},$$

associating to every wedge in $\mathbb{R}^{1+3}$ a closed real linear subspace of $\mathcal{H}$, s.t.:

1. **Isotony:** if $W_1, W_2 \in \mathcal{W}$, $W_1 \subset W_2$ then $H(W_1) \subset H(W_2)$
2. **Poincaré Covariance:** $\forall W \in \mathcal{W}$, $g \in \mathcal{P}_+^\uparrow$: $U(g)H(W) = H(gW)$.
3. **Reeh-Schlieder Property:** $\forall W \in \mathcal{W}$, $H(W)$ is a cyclic subspace of $\mathcal{H}$.
4. **Locality:** $\forall W_1, W_2 \in \mathcal{W}$ s.t. $W_1 \subset W_2'$: $H(W_1) \subset H(W_2)'$.

We shall indicate with (U, H) a Poincaré covariant net of standard subspaces satisfying 1-4[†]. The **Bisognano-Wichmann Property** in this setting is stated as follows:

$$U(\Lambda_W(2\pi t)) = \Delta_{H(W)}^{-it}, \qquad \forall t \in \mathbb{R}.$$

Duality property states that $H(W) = H(W')'$.

The proper Poincaré group $\mathcal{P}_+$ is generated by $\mathcal{P}_+^\uparrow$ and the space and time reflection Θ. An irreducible unitary representation of the Poincaré group always extends to an (anti-)unitary representation of $\mathcal{P}_+$ (i.e. unitary on $\mathcal{P}_+^\uparrow$, anti-unitary on $\theta\mathcal{P}_+^\uparrow$) except for finite helicity representations which have to be coupled. The following theorem is a consequence of the Brunetti-Guido-Longo construction.

Theorem 2.1. [3] *There is a 1-1 correspondence between:*

a. (Anti-)unitary positive energy representation of $\mathcal{P}_+$.
b. Local nets of standard subspaces satisfying 1-4 and B-W Property.

In view of Theorem 2.1, our question becomes: which families of representations preserve the 1-1 correspondence when the B-W property is not assumed?

3. A modularity condition for the Bisognano-Wichmann Property

For every $W \in \mathcal{W}_0$, we shall call G_W^0 the set $\{g \in \mathcal{L}_+^\uparrow : gW = W\}$. G_W denotes the Poincaré subgroup generated by the translation group and G_W^0. By transitivity

[†]It is possible to generalize the assumptions to fermionic representations assuming twisted-locality on H.[12] The forthcoming analysis continues to hold.

of $\mathcal{P}_+^\uparrow$ on wedges, G_W^0 and G_W can be defined for any $W \in \mathcal{W}$. The strongly continuous map

$$Z_{H(W)} : \mathbb{R} \ni t \mapsto \Delta_{H(W)}^{it} U(\Lambda_W(2\pi t)) \in \mathcal{U}(\mathcal{H})$$

is the proper quotient between the modular group of a wedge subspace and the unitary implementation of the one parameter boost transformations associated to W. Indeed, it has to be identity map if the B-W property holds.

The following proposition is a consequence of $\mathcal{P}_+^\uparrow$-covariance and Lemma 2.1.

Proposition 3.1. *Let (U, H) be a Poincaré covariant net of standard subspaces. Then, for every $W \in \mathcal{W}$, $t \mapsto Z_{H(W)}(t)$ defines a one-parameter group and $Z_{H(W)}(t) \in U(G_W)'$, $\forall t \in \mathbb{R}$.*

Here is stated our algebraic condition, sufficient to modularity.

Modularity condition: let $r_W \in \mathcal{P}_+^\uparrow$, s.t. $r_W W = W'$, then

$$U(r_W) \in U(G_W)''. \tag{M}$$

Theorem 3.1. *Let (U, H) be a Poincaré covariant net of standard subspaces. Let $W \in \mathcal{W}$, and $r_W \in \mathcal{P}_+^\uparrow$ be s.t. $r_W W = W'$. If $Z_{H(W)}$ commutes with $U(r_W)$, then the Bisognano-Wichmann and Duality properties hold.*

The proof goes as follows. By hypotheses and covariance we have $Z_{H(W)} = Z_{H(W')}$. As $Z_{H(W)}$ is a one parameter group of automorphisms of $H(W)$, by the KMS condition, Duality and B-W properties follow.

It is a corollary that representations satisfying (M) are modular.

Corollary 3.1. *With the assumptions of Theorem 3.1, if condition* (M) *holds on U, then Bisognano-Wichmann and Duality properties hold on H.*

The above property easily extends to direct integrals and multiples of representations as the following proposition shows.

Proposition 3.2. *Let U and $\{U_x\}_{x\in X}$ be p.e.r of $\widetilde{\mathcal{P}}_+^\uparrow$ satisfying (M).*

Let $\mathcal{K}$ be an Hilbert space, (M) holds for $U \otimes 1_{\mathcal{K}} \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$.

Let (X, μ) be a standard measure space. Assume that $U_x|_{G_W}$ and $U_y|_{G_W}$ are disjoint for μ-a.e. $x \neq y$ in X. Then $U = \int_X U_x d\mu(x)$ satisfies (M).

4. The scalar case

As we have announced, the above analysis holds for scalar representations.

Proposition 4.1. *Let U be a unitary irreducible scalar representation of the Poicaré group. Then U satisfies the modularity condition* (M).

The proof relies on two facts. Translations of a scalar representation generate a maximal abelian sub-algebra (MASA) of $\mathcal{B}(\mathcal{H})$ and r_W preserves G_W^0 orbits of the m-mass hyperboloid (a.e. orbits in the massless case).

Theorem 4.1. *Let* $U = \int_{[0,+\infty)} U_m d\mu(m)$ *where* $\{U_m\}$ *are (finite or infinite) multiples of the scalar representation of mass* m. *Then* U *satisfies* (M), *hence Duality and Bisognano-Wichmann properties hold.*

It is a consequence of Theorem 4.1 and Proposition 3.2.

This analysis holds in any Minkowski spacetime $\mathbb{R}^{s+1}$ with $s \geq 3$.

5. Counterexamples and Outlook

Consider $U_{m,s}$ m-mass, and s-spin, unitary, irreducible representation of the Poincaré group $\mathcal{P}_+^\uparrow$. Let $W \mapsto H(W)$ be the canonical net associated to $U_{m,s}$. Let V be a unitary, non trivial, representation of $\mathcal{L}_+^\uparrow$ on an Hilbert space $\mathcal{K}$, which is real, i.e. there exists a standard subspace $K \subset \mathcal{K}$ preserved by V and s.t. $\Delta_K = 1$ and $J_K K = K$. We can define the following new net of standard subspaces (cf. Ref.12 for standardness),

$$K \otimes H : \mathcal{W} \ni W \longmapsto K \otimes H(W) \subset \mathcal{K} \otimes \mathcal{H}.$$

Two Poincaré representations act covariantly on $K \otimes H$:

$$U_I(a,\Lambda) \equiv 1 \otimes U_{m,s}(a,\Lambda) \qquad \text{and} \qquad U_V(a,\Lambda) \equiv V(\Lambda) \otimes U_{m,s}(a,\Lambda),$$

where $\Lambda \in \mathcal{L}_+^\uparrow$, $a \in \mathbb{R}^{3+1}$. U_I is implemented by $K \otimes H$ modular operators and Bisognano-Wichmann property holds w.r.t. U_I. Note that when $s = 0$, U_I satisfies the modularity condition (M), hence the B-W property also follows by Corollary 3.1. It is obvious by construction that U_V does not satisfy B-W property on $K \otimes H$ except when V is the trivial representation. Notice that V could be chosen as a representation of $\mathrm{SL}(2,\mathbb{C})$ (i.e. the universal covering of $\mathcal{L}_+^\uparrow$) and thus a wrong Spin-Statistics relation would follow for U_V. U_V decomposes into a direct sum of infinitely many inequivalent representations of mass m, in particular infinitely many spins have finite multiplicity.[13] It is interesting to look for natural counterexamples to modular covariance, in the class of representations excluded by this discussion, if they exist. We expect that a modularity condition could be established on a finite direct sum of factorial Poincaré representations.

We briefly give an outlook on the relation between B-W and Split property, a strong statistical independence property in QFT defined by Doplicher and Longo in Ref. 8. An inclusion of von Neumann algebras $(N \subset M, \Omega)$ is said to be *Split* if there exists an intermediate type I factor F $(N \subset F \subset M)$. Analogously we define an inclusion of standard subspaces $K \subset H$ *Split* iff their second quantization von Neumann algebras give a Split inclusion. Our analysis suggests that the Bisognano-Wichmann property is *much weaker* than the Split property. Indeed, consider a net (U, H) be a Poincaré covariant net, assume that U is a direct integral of scalar representations. The Split property on the U-canonical net requires that U shall be purely atomic on masses, concentrated on isolated points and for each mass there can only be a finite multiple of the scalar representation.[8,13] The disintegration satisfies the modularity condition (M) and the modularity of U follows by Corollary 3.1.

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