

The impact of mathematics teachers' professional competence on instructional quality and students' mathematics learning outcomes

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Teacher quality is a critical factor that influences instructional quality and student learning outcomes. Recently, the authors have proposed broadened views of teacher competence that include dispositions, such as knowledge, and more situation-specific aspects, such as noticing, and span from the dispositions of teachers to their performance in the classroom. Mathematics-education researchers have enriched the conceptualization of teacher competence, developed new measurements, and explored effective ways to facilitate the development of teacher competence. Many empirical studies have been conducted in this field to investigate the impact of teacher professional competence on instructional quality and student mathematics learning outcomes. With this review, we intend to provide a synopsis of the state-of-the-art in this topic and outline new research perspectives.

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Introduction

Over the past few decades, teacher quality has been widely accepted as a factor that influences teachers' instructional quality and students' learning outcomes [1–3]. However, the majority of previous empirical studies in mathematics education adopted an exclusively cognitive perspective to investigate the associations between professional knowledge,

such as mathematics content knowledge (MCK) and mathematical pedagogical content knowledge (MPCK); teaching behaviors in the classroom; and students' mathematical achievement [3,4]. Recently, this cognitive perspective has been challenged by researchers such as Depaepe et al. [4], who argued that the relationship between the three constructs is "far from simple and straightforward" [p. 181]. More situation-specific aspects that mediate the relationship between teachers' cognitive aspects of knowledge and their teaching behaviors have been proposed as additional elements of teacher competence, with reference to the discourse on teacher noticing [5,6]. Currently, educational researchers widely accept that teachers' professional competence should include both cognitive facets, such as knowledge, and situation-specific facets, such as teacher noticing, which in turn includes facets such as perception, interpretation, and decision-making [6].

In the past few years, there have been increased efforts to expand the conceptualization of teachers' professional competence, develop new measurement methods, and explore how the growth of teachers' competence can be facilitated. Based on the broadened view of teacher competence, empirical studies have investigated the ways in which various components of mathematics teachers' professional competences jointly influence teachers' instructional quality and students' mathematics learning.

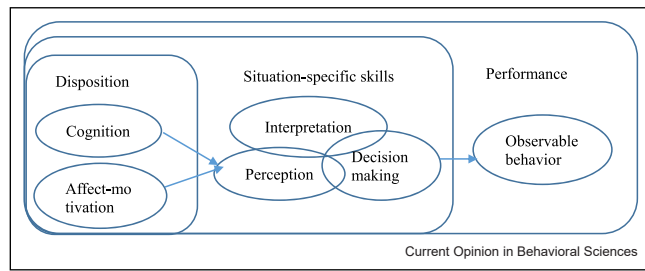
The aim of this review is to provide a synopsis of the current knowledge in this area and develop suggestions for future research. More specifically, we focus on those empirical studies that include larger samples of pre- or in-service mathematics teachers and that have been published in influential international mathematics education and teacher-education journals within the last five years, during which time this discourse has been strongly enriched by new conceptualizations [5]. Although the review focuses on mathematics education, the reviewed papers stem from a general discussion on teacher education.

Theoretical conceptualization of teachers' professional competence and empirical evidence

Theoretical conceptualization of teachers' professional competence

In the last decade, mathematics teachers' professional competence has been conceptualized based on the

Figure 1



Model of teachers' professional competence ([5], p. 7).

integration of cognitive and situated perspectives. The most influential framework to date was proposed by Blömeke, Gustafsson, and Shavelson [5] (see Figure 1). Teachers' professional competences are conceptualized as a continuum from teachers' individual dispositions, such as professional knowledge and beliefs, to performance. Situation-specific cognitive skills, such as perception, interpretation, and decision-making, serve as mediators between disposition and performance, as their enactment refers strongly to teaching situations and brings them therefore closer to teaching practice as dispositions.

Since the publication of this framework, several authors have attempted to enrich the constructs of teacher knowledge and teaching from similar perspectives. Schoenfeld [7] reframed the concept of teacher knowledge by distinguishing between "small k knowledge", which denotes an individual's documental knowledge, and "big K Knowledge", which includes teachers' perceptions, inclinations, and orientations as well as understanding and related proficiencies. The latter are constructs similar to noticing or situation-specific skills.

van Es and Sherin [8] recently expanded their original theoretical framework of teacher noticing. The first facet of teacher noticing — attending — was modified to include not only identification of noteworthy features of classroom situations but also disregard of some features. The second facet — interpretation — was enriched by a line of inquiry going beyond the original description of interpretation as using one's knowledge and experiences to make sense of observed events. Finally, a more theory-guided third facet — shaping — was proposed. This facet includes the construction of interactions and contexts that provide access to additional information.

Recent publications have discussed the cultural nature of teacher noticing. For example, Louie [9] clearly pointed out that teacher noticing is "socially and culturally constructed" [p. 61]. From a similar perspective,

van Es, Hand, Agarwal, and Sandoval [10••] proposed a multidimensional framework of noticing for equity that "captures the stretch and expanse of teachers' attention and sense making of the local, sociocultural, and historical aspects of mathematics classrooms" [p. 114]. *Stretch* in the framework captures the relation of what teachers notice to their own and their students' past and futures, while *expanse* reflects "the breadth and range of what teachers identify as noteworthy" [p. 115] in teaching.

In connection with these developments, research has challenged the linear conceptualization of teachers' professional competence, according to which teacher dispositions influence situation-specific skills and teaching behavior. For example, Santagata and Yeh [11] revised the model developed by Blömeke et al. [5], hypothesizing a bidirectional relationship between teachers' dispositional factors and their situation-specific skills. The authors [11] argued that "changes in competence would not be possible if teachers did not deliberately attend to and interpret practice and make decisions that create new knowledge and new beliefs" [p. 163].

Empirical evidence for the construct of teachers' professional competence

In addition to theoretically discussing the construct of mathematics teachers' professional competence, in the past few years, researchers have performed empirical investigations of the construct of teachers' competence and its dimensionality. First, in terms of the relationship between MCK and MPCK, after a test and analysis of 373 Cypriot primary school teachers' MCK and MPCK on fractions, Agathangelou and Charalambous [12] found that MCK could be considered a prerequisite of MPCK. However, in another study by Charalambous, Hill, Chin, and McGinn [13] that included 200 fourth- and fifth-grade teachers in the United States, the competence components of advanced common content knowledge and knowledge for mathematics teaching could not be separated. In contrast, when studying 602 American high school teachers' subject matter knowledge for teaching geometry, Ko and Herbst [14] found that the aspects of understanding students' work and choosing givens for a problem were distinguishable.

Referring to the three main widely accepted components of teacher noticing, namely attending, interpreting, and deciding, based on a sample of 213 preservice teachers in the United States, Thomas et al. [15] reported that the three components of teacher noticing were weakly correlated with each other. In addition, the coherence of the quality of the performance across the components could be fostered by a thematic interrelatedness of the noticing situations.

In addition, efforts have also been made to explore the dimensionality of mathematics teachers' knowledge (e.g. MCK and MPCK) and the relation to noticing. For example, Copur-Gencturk and Tolar found that mathematics teachers' content knowledge, pedagogical content knowledge, and content-specific noticing skills were separate components of teachers' professional competence [16].

Newly developed measures for mathematics teachers' professional competence

In the last decade, researchers have developed new measurement methods and instruments to investigate the broadened construct of teachers' professional competence. For example, Santagata and Sandholtz [17] developed a classroom video-analysis test to examine preservice mathematics teachers' teaching competence. This test measures teachers' usable knowledge for teaching mathematics by assessing teachers' analysis of classroom events shown in a series of video clips. Similarly, Kersting, Smith, and Vezino [19] developed a novel measurement instrument based on video clips of authentic classroom instruction to examine teachers' moment-to-moment noticing as knowledge-filtered perception.

Formats beyond video clips or video vignettes have also been proposed to measure teacher noticing. For example, Phelps-Gregory and Spitzer [18] used lesson narratives to examine preservice teachers' ability to analyze evidence of student thinking.

Recently, new means and ways have been developed to investigate teachers' situation-specific competence. For example, one study used 360-degree videos, finding that preservice teachers who viewed these videos attended to more students' actions than their peers who viewed standard videos [20]. Another study used a virtual-reality classroom method to investigate teachers' noticing [21].

The development of mathematics teachers' professional competence

Several studies have focused on characteristics of the development of mathematics teachers' professional competence, influencing factors, and effective promotion strategies. Researchers have identified various achievement levels of the three facets of teacher noticing. Attending/perceiving is considered the easiest activity, especially for preservice teachers. It is more challenging to make reasonable interpretations, and even more difficult to respond adequately or develop reasonable decisions. The responding phase of noticing seems to be the most difficult competence component for teachers to develop [22,23].

To investigate the development of teacher noticing, cross-sectional studies based on the expert–novice

paradigm examined differences in noticing skills between expert and novice teachers. Stahnke and Blömeke [24] investigated 39 novice and expert teachers' perception, interpretation, and decision-making skills regarding classroom management. They found that, compared with novice teachers, expert teachers interpreted more and suggested more alternative courses of action. Similarly, in a cross-sectional study conducted in China, Yang et al. [25•] compared the noticing skills of preservice teachers, early career teachers, and experienced mathematics teachers. The findings suggest linear growth of teacher noticing among the three teacher cohorts. In contrast, in a cross-sectional study with preservice teachers at a master's level, early career teachers, and experienced teachers, Bastian et al. [26] identified a considerable increase in professional noticing between master's students and practicing teachers and a slight decrease in professional noticing between early career teachers and experienced teachers. Findings obtained from the perspective of the expert–novice paradigm imply that teaching experience influences the development of teacher noticing, although teaching experience is not equivalent to expertise [27].

Researchers have studied possible ways to facilitate the development of teachers' professional competence in general and teacher noticing in particular [28–30]. For example, Lee [28•] examined the effects of a technology-aided intervention to facilitate the growth of preservice mathematics teachers' noticing skills. The pre- and post-intervention results suggest significant improvements in teachers' noticing skills of attending, interpreting, and responding.

The use of video is a specific way to foster teachers' noticing skills. For example, van Es et al. [30] investigated the development of preservice mathematics teachers' noticing within a mathematics pedagogy course that included video-based elements. They identified variations in preservice mathematics teachers' noticing, especially concerning the relation between student thinking, teaching practice, and mathematical content showing the potential of video-based elements in noticing courses.

Some studies have examined effective ways to develop teachers' competence to improve teaching quality based on innovative teaching methods, such as differentiated instruction or increased classroom discourse [31,32]. For example, video-based professional-development programs have used, among other methods, a discourse visualization tool called the classroom discourse analyzer to support participating teachers in reflecting on classroom discourse [32]. It was found in the study that the intervention teachers significantly increased the use of productive talk in teaching.

The relationship between teachers' professional competence, instructional quality, and student learning outcomes

This section first analyzes the relation between teachers' professional competence and instructional quality and then discusses student learning outcomes.

The relationship between teachers' professional competence and instructional quality

In the last few decades, teachers' competence has been widely accepted as a crucial factor that significantly influences teachers' teaching practice or overall instructional quality even though the results are not entirely consistent [3,33•–36]. Baier et al. [34] identified a significant moderate relationship between in-service mathematics teachers' pedagogical/psychological knowledge and their students' reported instructional quality for aspects of teaching such as learning support. However, constructivist beliefs were not found to be significant predictors of instructional quality. Yurekli, Stein, and Correnti [35] found that although in-service primary and secondary school teachers had positive beliefs about promoting conceptual understanding for mathematics teaching, they did not always implement this practice in their classrooms. In addition, Backfisch et al. [36] found that mathematics teachers' pedagogical content knowledge, not their MCK, was significantly correlated with aspects of instructional quality, such as cognitive activation, instructional support, and the quality of technology exploitation.

Otumfuor and Carr [37] analyzed subject-based instructional quality, investigating the influence of teachers' spatial skills on their geometry instruction. Their findings suggest that teachers with better spatial skills are more likely to use representational gestures and show better content and pedagogical knowledge during instruction.

Other studies investigated the relationship between teachers' knowledge and instructional quality while considering classroom management, student support, and cognitive activation [34,38]. For example, Blömeke et al. [38] included aspects of teachers' professional competence, such as MCK, teacher noticing related to the aspect of mathematics instruction, classroom-management skills, and beliefs. Using latent-profile analysis, they identified four profiles of mathematics teachers' professional competence associated with different types of instructional quality.

The relationship between instructional quality/practice and student learning outcomes

Using different research designs and perspectives, researchers have examined how instructional quality or practice influences students' mathematics learning outcomes. One common way to conduct such investigations

is to use video analysis [39–41]. For example, using the influential Mathematical Quality of Instruction observation instrument, Lynch, Chin, and Blazar [40] investigated the relationship between instructional quality and students' mathematics achievement based on 298 elementary mathematics teachers. The authors identified a generally strong relationship between instructional quality and student achievement, although this relationship differed by school district. Similarly, based on the analysis of the effects of the levels of teacher responsiveness to students' mathematics thinking to students' learning, Bishop [42] identified a significant positive relationship between teachers' responsiveness and students' learning.

The second common method of investigation is to use a questionnaire survey to collect data from students and teachers in order to evaluate teachers' teaching behaviors or instructional quality [43–45], although it is worth noting that self-reports face limitations due to their subjective criteria. One example of a study using survey data is that of Westphal et al. [43], who asked students to rate teachers' diagnostic skills and classroom management. The authors found that if teachers demonstrated better diagnostic skills and classroom-management skills, students experienced more enjoyment and less anxiety and boredom [43]. Similarly, in Li et al.'s [44] study, students reported that the quality of cognitive activation was positively related to their mathematics performance, with mathematics self-efficacy acting as a vital mediator.

In addition, experimental or longitudinal studies have also been carried out to investigate the effect of teaching practice or instructional quality on students' mathematics learning. For example, the experimental study by Kutnick et al. [46] examined how specific teaching strategies influence students' mathematics learning outcomes, such as the effectiveness of group work on primary school students' mathematics achievement in a Confucian cultural context in Hong Kong. They found that enhanced mathematical achievement was supported by improved peer-based communication skills. Dimosthenous, Kyriakides, and Panayiotou [47••] empirically examined the longitudinal effects of teachers' instructional quality/practice on students' mathematics learning outcomes and found that quality of teaching, such as aspects of questioning techniques, had both short- and long-term effects on students' mathematics achievement.

The relationship between teachers' professional competence and student learning outcomes

The quality of mathematics teachers is commonly accepted as a decisive factor affecting students' learning outcomes. Recently, studies have investigated the relationship between mathematics teachers' MPCK and

students' learning outcomes [48,49], between MCK and students' performance [50], and between teachers' MCK and MPCK and students' achievement [13]. In general, these studies identified a significant positive association between teachers' knowledge and students' achievement in mathematics. In particular, Tchoshanov et al. [50] found that different cognitive types of teachers' MCK (e.g. knowing basic facts and procedures or understanding of concepts or mathematical models and generalizations) contribute differently to students' mathematics learning. Significant correlations could be identified between the first two cognitive types of teachers' knowledge, but not for the last one.

Gabriele et al. [51] investigated the association between elementary mathematics teachers' judgment accuracy, confidence, calibration accuracy, and students' mathematics achievement. Teachers' calibration accuracy, but not their task-specific judgment accuracy, significantly predicted students' mathematics achievement.

Links among teachers' competence, instructional quality, and mathematics learning

The relationship between teachers' professional competence, instructional quality, and student learning outcomes has been recently examined with the help of new perspectives in the discourse [2,52–55]. For example, studies conducted by Hill et al. [52,53] investigated the connections between teachers' knowledge of students or mathematical knowledge, instructional quality, and student-achievement outcomes. Although they identified significant correlations among teacher knowledge, instruction, and achievement, they also found that such correlations were heavily conflated with differences among students, classrooms, schools, and school districts. They concluded that teacher knowledge must be proximal to the instructional domain to allow for identification of such associations [52,53].

König et al. [54••] examined the links between pedagogical competence, including general pedagogical knowledge (GPK), situation-specific classroom-management expertise (CME), instructional quality (measured by a classroom observation-rating instrument), and mathematics achievement among large samples comprising data from 59 teachers and 1220 students in Germany. It was found that both GPK and CME predict instructional quality, but only a direct statistical effect on students' mathematical progress could be identified, there were no indirect statistical effects via instructional quality.

Very recently, Blömeke et al. [56••] investigated the relationship between cognitive aspects of mathematics teachers' competence, such as MCK and MPCK, and situated aspects of these teachers' competence, such as perception, interpretation, and decision-making,

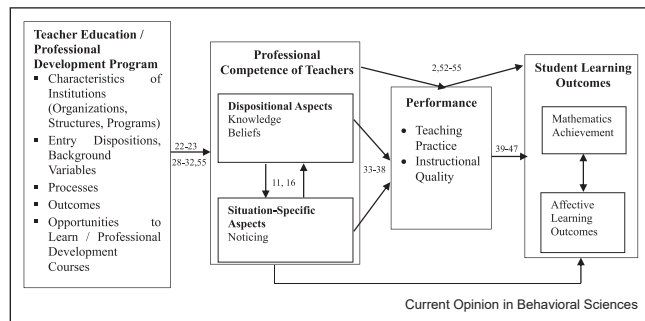
instructional quality (measured by classroom observations), and students' mathematics learning progression. This relationship was examined with large-scale samples comprising 3496 students from 154 classrooms with 89 teachers in Germany. The situated aspects of competence and instructional quality played a mediating role between the cognitive aspects of competence and students' mathematics learning progression. Teachers' knowledge had no direct effects on students' progress, which emphasizes the important role of teachers' situation-specific skills.

It has been argued that attending professional-development programs is an influential factor affecting the development of teachers' professional competence and students' mathematics learning outcomes. A few recent empirical studies examined the relationship between professional-development programs, teachers' professional competence, instructional quality, and student learning outcomes, and reported somewhat inconsistent findings [32,55,57,58••]. Jacob, Hill, and Corey [55] reported findings from a three-year evaluation of a well-developed mathematics professional program that focused on the improvement of teachers' knowledge of mathematics, understanding of students' mathematics learning, effective use of formative assessment, and classroom instructional strategies. They found that the training positively impacted teachers' mathematics knowledge for teaching, but had no effects on instructional quality and student outcomes. In contrast, Chen et al. [32] reported that after attending a video-based professional-development program, the 24 intervention teachers significantly increased their use of productive talk. Furthermore, students of the intervention teachers demonstrated significantly higher mathematics achievement scores than students in the control group [32].

Conclusions, limitations, and recommendations

This review shows that a wealth of studies in the last decade, particularly during the last few years, have investigated mathematics teachers' professional competence in general and teacher noticing in particular. A great number of studies have investigated how teachers' professional competence impacts instructional quality and students' learning. As depicted in Figure 2, researchers have made new theoretical advancements in the past few years concerning methods of investigating situation-specific skills (e.g. noticing), aspects of the effectiveness of teachers' professional-development programs, the characteristics of teachers' professional competence, instructional quality, and students' mathematics learning. Figure 2 shows the central components of teachers' competence — disposition and situation-specific skills — and their impact on teachers' performance. Teaching practice and instructional quality

Figure 2



Relationships between professional development, teacher competence, performance, and student learning outcomes.

Adapted from the model developed by Blömeke et al. [5], with reference to the model developed by Kaiser and König [59].

serve as mediating factors that influence teachers' performance. The whole impact chain can be modeled in four columns showing that teacher education and professional-development programs influence dispositions, situation-specific skills, and the performance of the teachers in the classroom, which in turn impacts students' learning gains, mathematical achievements, and affective learning outcomes. The arrows in Figure 2 display the identified relations and influences between these factors of teacher competence, their bases, and outcomes with the numbers at the arrows showing the studies reporting the referring results. The diagram shows quite clearly the mediating role of instructional quality between teacher competence and student learning gains.

Despite the strong progress that has been made in this field, our literature survey revealed some limitations of the current discourse that must be overcome in the future. Only a few empirical studies have employed a broadened view of teachers' professional competence that integrates cognitive and situated perspectives to investigate the link between teachers' professional competence, instructional quality, and student achievement. Thus, very few studies have examined the whole impact chain, from teachers' competence, including knowledge and situation-specific skills, to students' learning gains, including mediating variables such as instructional quality. Furthermore, in regard to student learning outcomes, a majority of studies focused on students' cognitive achievements. Very few studies considered students' affective development, which is of high importance for students' learning. In addition, the majority of current studies in this field included elementary school mathematics teachers, and few studies investigated mathematics teachers at the secondary level. Finally, the majority of studies have been

conducted in the so-called Western context, especially in the United States and Germany, only a few studies have been conducted in nonWestern contexts. Therefore, studies conducted in other cultural contexts that consider the limitations described above are urgently needed for a fuller understanding of this field.

Conflict of interest statement

None.

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