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Discrete Optimization

An exact approach for the reliable fixed-charge location problem with capacity constraints

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ABSTRACT

Introducing capacities in the reliable fixed charge location problem is a complex task since successive failures might yield in high facility overloads. Ideally, the goal consists in minimizing the total cost while keeping the expected facility overloads under a given threshold. Several heuristic approaches have been proposed in the literature for dealing with this goal. In this paper, we present the first exact approach for this problem, which is based on a cutting planes algorithm. Computational results illustrate its good performance.

1. Introduction

Logistic systems are typically based on costly infrastructures that remain essentially unchanged during long time horizons. However, technical issues or environmental conditions can occasionally disrupt the operation of some of these infrastructures and this can compromise the performance of the entire system (see, for instance Fan, Ma, & Li, 2018; Pariazar & Sir, 2018). This fact has motivated researchers to include reliability issues in their models of the logistic systems as can be seen, for instance, in the review Snyder et al. (2016). This includes the design of reliable distribution networks (Herivin & Mahjoub, 2005) and supply chain networks (Peng, Snyder, Lim, & Liu, 2011; Qi, Max Shen & Snyder, 2010), and the location of facilities able to provide a good service level even upon network failures (Yu & Liu, 2020).

In particular, (un)reliability of the facilities has become an increasingly relevant topic in discrete location since the seminal work Drezner (1987). The studied models include classical ones such as the p-median and the p-center (Albareda-Sambola, Hinojosa, Marín, & Puerto, 2015; Berman, Krass, & Menezes, 2007) or fixed-charge facility location problems (Aboolian, Cui, & Shen, 2013; Alcaraz, Landete, Monge, & Sainz-Pardo, 2016) or path location problems (Puerto, Ricca, & Scozzari, 2014). As far as the nature of the disruptions is concerned, several research lines are active. On the one hand, there is a bunch of literature devoted to preventing or mitigating the effect of intentional disruptions. On this stream we find, for instance Church & Scaparra (2007) and the extensions by the same authors Scaparra & Church (2008), or Afify et al. (2019); Hamidi, Gholamian & Shahanaghi (2017). Another fruitful line of research is concerned about situations where system congestion may provoke disruptions (Mohammadi, Jula, & Tavakkoli-Moghaddamcd, 2019; Zamani, Arkat, Taghi, & Niaki, 2022; Zhang, Batta, & Nagi, 2022, among others). A third line of research involves accidental disruptions. This work belongs to this third class.

The question about how to fix the capacities for facilities in unreliable networks is a hot topic. There is a trend that proposes to update or "harden" the standard capacities, i.e. to increase them by more investment. For instance, in Rohaninejad, Sahraeian, & Tavakkoli-Moghaddam (2018) an exact algorithm is developed for a multi-echelon capacitated problem where external resources may be used for satisfying demands yielding a certain penalty. It is also vital the assumption about the failures correlation. The work Xie, An, & Ouyang (2019) presents a compact mixed-integer mathematical model to optimize the facility location under generally correlated facility disruptions. Related to the facility unreliability is the customers risk attitude. In Berman, Sanajian, & Wang (2017) it is studied how the decision maker's risk attitude can affect the optimal facility locations. In Yu, Haskell, & Liu (2017) it is developed a risk-averse optimization formulation to compute resilient location and customer assignment solutions that control the risks at each individual customer.

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In the problem at hand, it is assumed that we start from a set of customers and a potential set of locations where service plants can be opened. Each plant has a different capacity and installation cost, and each customer has a different demand. In addition, there is a positive probability that installed plants fail and customers have to go to another plant to receive service or are left without service in exchange for compensation. Facility failures are assumed to take place independently of each other. The Reliable Fixed-charge Location Problem with Capacities (RFLCP) is a problem in which several types of decisions are made: it is decided in which potential locations to open the plants, it is decided which plant each of the clients of the network is assigned to when no plant fails and also which plant each client is assigned to in any of the failure scenarios. The solution must meet that the expected overload of the set of installed plants does not exceed a threshold and that the total cost is minimal. In this paper, we present a new exact method to solve the RFLCP, based upon a dynamic algorithm that combines the solutions of two problems, the RFLCP when some capacity constraints are relaxed, and an auxiliary problem used for keeping the expected overload in the demand below the threshold. Previously, there was no model or algorithm in the literature that would allow solving this problem. Obtaining heuristic solutions had been addressed (Albareda-Sambola, Landete, Monge, & Sainz-Pardo, 2016; Gade & Pohl, 2009). Some of the heuristic procedures in the literature are more strict about keeping the overload below the desired threshold, others less so. Thus, obtaining an exact procedure for this problem was a research gap waiting to be addressed. To the best of our knowledge, the only exact method available in the literature for a capacitated location problem with unreliable facilities is Espejo, Marín, & Rodríguez-Chía (2015), where a completely different model is addressed, and only scenarios with one single facility failure are considered.

The RFLCP was introduced in Albareda-Sambola et al. (2016), where the authors devoted special efforts to the design of powerful matheuristic methodologies to solve this reliability facility location problem. However, no work about exact solution procedures has been presented so far for this particular problem. In this work we try to cover this research gap. To do so, we present a dynamic approach that we computationally check.

We can itemize as follows the main contributions of this work:

i. An exact dynamic approach for obtaining exact solutions for the Reliable Fixed-charge Location Problem with Capacities is introduced. It is based in two mixed-integer optimization models: a master problem and an auxiliary assignment problem.
ii. Feasible solutions to the Reliable Fixed-charge Location Problem with Capacities are classified into equivalence classes and properties of these classes are discussed.
iii. Extensive computational experiments are conducted. Instances with up to 50 customers and 75 candidate facilities are solved, and the results are thoroughly reported. They give strong evidence of the efficiency and efficacy of the exact method.

The paper is organized as follows. Section 2 introduces the notation and the essential background on the RFLCP. In Section 3, we introduce the optimization models used in the exact dynamic approach proposed in this work. Next, in Section 4, the algorithm is presented. Section 5 illustrates the new approach by comparing the obtained results to those taken from the literature. Section 6 concludes the work.

2. Notation and preliminaries

Let I be the set of customers, and J the set of potential facility locations. The set J is assumed to include a dummy facility u where non-served customers are allocated. Assignments to this facility, as it is usually done in the literature, are used to represent situations where a particular customer cannot be served by the system. Let d_{ij} be the unitary transportation cost from facility $j \in J$ to customer $i \in I$, h_i the demand of customer i, Q_j the capacity of facility j, and f_j the opening cost of the facility j. In the case of u, the unitary transportation cost d_{iu} is the penalty cost of non-service, $Q_u = \infty$ and $f_u = 0$. The penalty non-service cost, which can represent either an outsourcing cost, or the loss for not servicing a customer, is assumed to be greater than any unitary transportation cost. Let F be the subset of J whose facilities may suffer from service disruption with homogeneous and independent probability q, and NF the subset of J whose facilities never fail, $F \cup NF = J$ and $u \in NF$. Let X_j be a binary variable which takes the value 1 if a facility is opened at location $j \in J$. In order to describe the allocation of each customer for any possible scenario we use assignment levels. Each customer is assigned to different facilities at different levels, and, given a scenario, it will be served form the lowest-level assigned facility that remains available. Let Y_{ijr} be another binary variable which takes the value 1 if customer $i \in I$ is assigned to facility $j \in J$ at level $r \in R$ with $R = \{0, \ldots, |F| - 1\}$. For instance, $Y_{1j_32} = 1$ means that customer 1 would be served by facility j_3 if two specific failures have occurred: if no failure occurs, customer 1 is allocated to the facility j_1 such that $Y_{1j_10} = 1$; if j_1 fails, then customer 1 would be allocated to the facility j_2 such that $Y_{1j_21} = 1$; if both j_1 and j_2 fail, facility j_3 would serve customer 1. The Reliable Fixed-Charge Location Problem with Capacity Constraints (CRFLP) defined by Albareda-Sambola et al. in Albareda-Sambola et al. (2016) was formulated as:

$$\text{(CRFLP)} \min\ \alpha\left(\sum_{j\in J} f_j X_j + \sum_{i\in I} h_i d_{ij} Y_{ij0}\right) + (1-\alpha)\sum_{i\in I} h_i\left(\sum_{j\in NF}\sum_{r\in R} d_{ij} q^r Y_{ijr} + \sum_{j\in F}\sum_{r\in R} d_{ij} q^r (1-q) Y_{ijr}\right)$$

$$\textbf{s.t.}\ X_u = 1 \tag{1}$$

$$\sum_{j\in F} Y_{ijr} + \sum_{j\in NF}\sum_{s=0}^{r} Y_{ijs} = 1 \qquad i \in I, r \in R \tag{2}$$

$$\sum_{r\in R} Y_{ijr} \le X_j \qquad i \in I, j \in J \tag{3}$$

$$\sum_{i\in I} h_i Y_{ij0} \le Q_j X_j \qquad j \in J \tag{4}$$

$$\text{Capacity constraints mostly hold} \tag{5}$$

$$X_j \in \{0, 1\} \qquad j \in J \tag{6}$$

$$Y_{ijr} \in \{0, 1\} \qquad i \in I, j \in J, r \in R \tag{7}$$

where α is a value in (0,1).

The objective function represents the trade-off between the transportation cost from the primary facilities and the transportation costs from the backup facilities. The first component in the objective function computes the cost of serving clients from their primary facilities plus the opening cost. In fact, it is the objective function of the Simple Plant Location problem. The second component in the objective function computes the expected failure cost: each client i is served by their rth backup facility j if the facilities assigned at lower levels have failed (probability q^r) and j itself has not failed (probability $(1-q)$ for $j \in F$ and 1 for $j \in NF$). In other words, it is the transportation cost that will be incurred if

some plants in F fail. Both components are weighted by a scalar $\alpha \in (0,1)$. In the following $v(X,Y)$ indicates the evaluation of the objective function for the solution (X,Y).

Constraint (1) states that the dummy facility is always open. Constraints (2) force that given a pair of customer i and level r, the customer is allocated to a facility at level r or it was allocated to a non-failing facility at a level smaller than r. Constraints (3) indicate that facility j only can give service if it is open. Constraints (4) guarantee not to exceed the capacity of each facility in the first assignment, the scenario in which there are not failures in the system. The group of non-defined constraints (5) are different depending on the emphasis of the model. Ideally, it should keep the expected overload of a solution below a threshold value β.

The expected overload of a solution (X,Y) can be expressed as:

$$E(X,Y) = \sum_{j\in O(X)} \mathbb{E}\left[\left(\xi_j \cdot \underbrace{\sum_{i\in I} h_i\left(\sum_{r\in R} Y_{ijr}\cdot \prod_{s<r}\left(\sum_{j'\in O(X)} Y_{ij's}(1-\xi_{j'})\right)\right)}_{\text{demand at j according to } \xi} - Q_j\right)^+\right], \tag{8}$$

where $(\bullet)^+ = \max\{0,\bullet\}$, $O(X)\subset J$ is the set of locations where facilities have been placed,

$$O(X) = \{j\in J : X_j = 1\}$$

and ξ_j is the Bernoulli random variable that takes value 1 if facility j is operative, and 0 if it has failed. That is, $\xi_j \sim \text{Bernoulli}(1-q)$ for $j\in O_F(X) = O(X)\cap F$, and $\xi_j = 1$ for $j\in O_{NF}(X) = O(X)\cap NF$. The expected overload of the solution is the sum, over all open facilities, of the expected values of the positive difference between the demand allocated to the facility and its capacity. Here, the demand allocated to a facility $j\in O(X)$ is computed as

$$\sum_{i\in I} h_i\left(\sum_{r\in R} Y_{ijr}\cdot \prod_{s<r}\left(\sum_{j'\in O(X)} Y_{ij's}(1-\xi_{j'})\right)\right).$$

Thus, ideally, line (5) in CRFLP should be

$$E(X,Y) \le \beta, \tag{9}$$

but it does not give an affordable formulation. Instead, in Albareda-Sambola et al. (2016) different attempts of controlling the expected overload have been analyzed, namely QRFLP, CRFLP-S, CRFLP-B1 and CRFLP-LR.

- QRFLP is the problem CRFLP when no capacity constraints are considered: line (5) is removed from CRFLP. It is a naive approach which does not pay attention to the overload in levels different from the first.
- CRFLP-S is based on staggered capacities. CRFLP-S is CRFLP when line (5) is replaced by

$$\sum_{s=0}^{r}\sum_{i\in I} h_i Y_{ijs} \le \theta^r Q_j \qquad \forall j\in J, r>1 \tag{10}$$

  where θ is a scale factor. Constraints (10) keep the load of each facility up to each possible assignment level below a value which is proportional to its original capacity.
- CRFLP-B1 imposes a limit on an upper bound for $E(X,Y)$. In Albareda-Sambola et al. (2016) it is proved that if λ_{jr} is the overload at facility j caused by assignments at level r after all assignments at lower levels have been considered, then

$$\sum_{j\in F}\sum_{r>0} q^r(1-q)\lambda_{jr} + \sum_{j\in NF}\sum_{r>0} q^r\lambda_{jr}$$

  is an upper bound for $E(X,Y)$. Then, CRFLP-B1 keeps this bound for expected overload of a solution below the threshold value β. In particular, CRFLP-B1 is CRFLP when line (5) is replaced by

$$\sum_{s=1}^{r}\sum_{i\in I} h_i Y_{ijs} \le Q_j + v_{jr} \qquad \forall j\in J, r\in R \tag{11}$$

$$\lambda_{j1} = v_{j1} \qquad \forall j\in J \tag{12}$$

$$\lambda_{jr} = v_{jr} - v_{j,r-1} \qquad \forall j\in J, r>1 \tag{13}$$

$$\sum_{j\in F}\sum_{r>0} q^r(1-q)\lambda_{jr} + \sum_{j\in NF}\sum_{r>0} q^r\lambda_{jr} \le \beta$$
$$\lambda_{jr}, v_{jr} > 0 \qquad \forall j\in J, r\in R \tag{14}$$

- CRFLP-LR imposes a limit on an estimation of $E(X,Y)$. In Albareda-Sambola et al. (2016) it is empirically proved that a good linear approximation of $E(X,Y)$ ($R^2 = 0.9748$) is

$$\hat{E}(X,Y) = 2.67827q\bar{\lambda}_{\bullet 1} + 1.66348q^2\bar{\lambda}_{\bullet 2} + 1.92325q^3\bar{\lambda}_{\bullet 3} + 4,43350q^4\bar{\lambda}_{\bullet 4}$$

  where $\bar{\lambda}_{\bullet r}$ is the average overload caused by assignments at level r. Thus, CRFLP-LR is CRFLP when line (5) is replaced by

$$(11)-(14)$$
$$\lambda_{\bullet r} = \sum_{j\in J}^{r}\lambda_{jr} \qquad r\in\{1,\dots,4\}$$
$$2.67827q\lambda_{\bullet 1} + 1.66348q^2\lambda_{\bullet 2} + 1.92325q^3\lambda_{\bullet 3} + 4,43350q^4\lambda_{\bullet 4} \le \beta\sum_{j\in J\setminus\{u\}} X_j$$
$$\lambda_{\bullet r} > 0 \qquad \forall r\in R$$

Comparing QRFLP, CRFLP-S, CRFLP-B1 and CRFLP-LR, only CRFLP-B1 ensures solutions to have the expected overload below the requested threshold β. Meanwhile, CRFLP-B1 limits in excess the expected overload providing expensive solutions, in terms of the optimal solution values. Both CRFLP-S and QRFLP do not work in terms of expected overload, while CRFLP-LR just bounds an estimate of the expected overload. CRFLP-LR returns solutions with an expected overload around the requested bound, sometimes below it and sometimes above. In this paper, an exact algorithm which solves CRFLP when line (5) is replaced by (9) is introduced, it has been named CRFLP-EX. Somehow, CRFLP-B1 and CRFLP-LR can be considered as math-heuristic methods for CRFLP-EX.

3. Optimization models

The algorithm proposed for solving CRFLP-EX makes use of two optimization problems: a master problem and an auxiliary assignment problem. The master problem is the CRFLP replacing the expected overload capacity constraints for appropriate sets of constraints. The auxiliary assignment problem indicates which is the set of constraints to be added to the master problem at each iteration. Given a solution to the master problem, the assignment problem gives a feasible solution to the master problem with the same

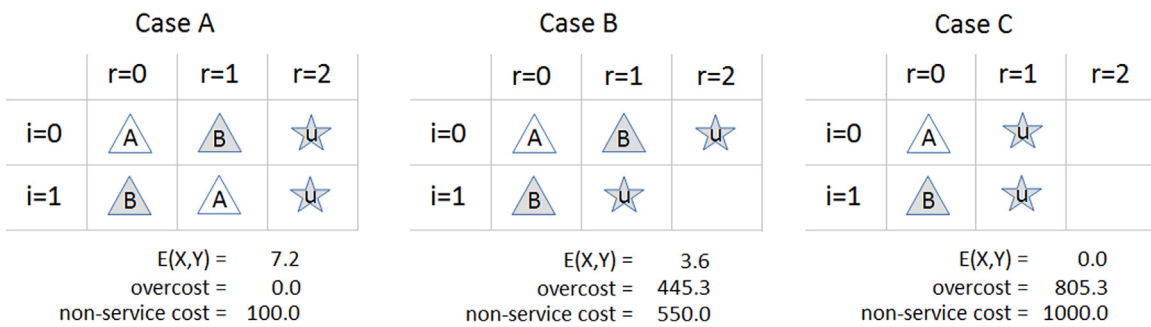

Fig. 1. Example of different assignments.

set of open facilities but with a different allocation which keeps the expected overload at the opened facilities below the threshold β.

The following example illustrates how different allocation patterns for the same set of open facilities can lead to different expected overloads. It shows how the expected overload is reduced by moving the dummy assignments to lower levels.

Example 1. Let $I = \{0, 1\}$ be the set of customers, both with demand $h = 50$. Let $O_F(X) = \{A, B\}$ be the set of open facilities. Let A and B be both facilities in F with capacity 60. In this example, the only facility which does not fail is $O_{NF}(X) = \{u\}$, for the others, $q = 0.1$. Let $d_{0A} = 10$, $d_{0B} = 20$, $d_{1A} = 40$, $d_{1B} = 30$ and let the non-service cost be 100. Fig. 1 illustrates three different assignments. The assignment in Case A is the one provided by MASTER when $W = Z = 0$ and it has an assignment cost of $(0.9 * t10 * 50 + 0.1 * 0.9 * 20 * 50 + 0.1^2 * 100 * 50) + (0.9 * 30 * 50 + 0.1 * 0.9 * 40 * 50 + 0.1^2 * 100 * 50) = 1994.7$ and an expected overload of 7.2 $(0 * 0.9 * 0.9 + (100 - 60)^+ * 0.9 * 0.1 + (100 - 60)^+ * 0.1 * 0.9 + 0 * 0.1 * 0.1)$ corresponding to assign each customer to its cheapest available facility for each level. In Case B, the assignment of customer 1 to the dummy facility is advanced one level. Then, this assignment has a cost of $(0.9 * 10 * 50 + 0.1 * 0.9 * 20 * 50 + 0.1^2 * 100 * 50) + (0.9 * 30 * 50 + 0.1 * 100 * 50) = 2440.0$ and an expected overload of 3.6 $(40 * 0.1 * 0.9 + 0)$. In Case C, which is such that there is no backup facility, the assignment cost is $(0.9 * 10 * 50 + 0.1 * 100 * 50) + (0.9 * 30 * 50 + 0.1 * 100 * 50) = 2800.0$ and the expected overload is 0. Note that overcosts are $2440.0 - 1994.7 = 445.3$ (case B) and $2800 - 1994.7 = 805.3$ (case C). Besides the overcost, the total non-service costs are also remarkable, being: 100, 550 and 1000 units, respectively.

In fact, it can be proved that given a set of open facilities there is always an allocation pattern with expected overload equal to zero or as small as it is want.

Proposition 3.1. *For any set $O(X)$ of open facilities, it is possible to obtain an assignment Y to facilities in $O(X)$ such that the expected overload is bounded by β.*

Proof. Given that in any feasible solution the dummy facility is open, the demand that produces excess of expected overload can be reassigned to it, i.e., the overload may be left unserved. □

Among all the feasible solutions with the same set of open facilities and with expected overload not larger than β, the assignment problem gives the solution with minimum non-service expected cost. Both problems, the master and the auxiliary assignment problem, are iteratively solved. At each iteration, the solution to the assignment problem define new constraints which are dynamically added to the master problem.

3.1. Master problem

The master problem is the QRFLP in which variable W is added together with a set of constraints assigning the proper value to it. Variable W accounts for the additional cost required to modify the MASTER solution assignments in order to render them feasible with respect to the expected overload constraints. In what follows, we will refer to this cost as *overcost*, and we will see that solutions can be classified into equivalence classes where this overcost is constant. The value of W will be established by new constraints dynamically added to the master problem. The formulation for MASTER is as follows:

$$\text{(MASTER)} \min \alpha \left(\sum_{j \in J} f_j X_j + \sum_{i \in I} h_i d_{ij} Y_{ij0} \right) + (1-\alpha) \sum_{i \in I} h_i \left(\sum_{j \in NF} \sum_{r \in R} d_{ij} q^r Y_{ijr} + \sum_{j \in F} \sum_{r \in R} d_{ij} q^r (1-q) Y_{ijr} \right) + W$$

$$(1) - (4), (6), (7)$$

$$W \text{ constraints} \tag{15}$$

$$W \in \mathbb{R}^+ \tag{16}$$

If no W constraints are added, the MASTER problem coincides with QRFLP.

3.2. Auxiliary assignment problem

The facilities can be classified into different equivalence classes depending on their opening costs, failure probabilities and capacities. Two facilities $j, k \in J$ are assumed to belong to the same equivalence class $j \sim k$ if they both have the same opening cost $f_j = f_k$, the same capacity $Q_j = Q_k$ and either they are both in F or both in NF. Let $J/\sim$ be the partition given by the set of equivalence classes and let $[j]$ be the equivalence class of $j \in J$. For each $P \in J/\sim$ and each solution (X, Y) of MASTER, let $O_P(X) = O(X) \cap P$. Two solutions (X_1, Y_1) and (X_2, Y_2) of MASTER are assumed to belong to the same equivalence class if and only if they have the same number of open facilities at each equivalence class, i.e., $|O_P(X_1)| = |O_P(X_2)|$ for all $P \in J/\sim$ and for each $j_1 \in O(X_1)$ there exists $j_2 \in O(X_2)$ such that $j_1 \sim j_2$ and the assignments to j_1 according to Y_1 are the same as the assignments to j_2 according to Y_2.

The auxiliary assignment problem, from now on AUX-CRFLP consists in, given an optimal solution to MASTER, obtaining a solution within its equivalence class that satisfies (9) and has the minimum possible cost. The expected overload of the optimal solution to MASTER might exceed the value β because (9) is not a constraint of MASTER. However, the expected overload of the optimal solution (X, Y) of AUX-CRFLP does not exceed β.

Let (X^M, Y^M) be a solution to MASTER. For all $i \in I$, $j \in O(X^M)$ and $r \in R$, let Y'_{ijr} be binary assignment variables such that (X^M, Y') is a feasible solution to MASTER and satisfies (9). AUX-CRFLP looks for the solution (X, Y) with minimum cost among all the solutions that are equivalent to (X^M, Y'). Notice that the j index for Y' variables moves in $O(X^M)$ while it moves in J for the Y variables.

For all $j \in J$ and $k \in O(X^M)$, we define variable Z_{jk} that takes the value of 1 if facility k in solution (X^M, Y') is replaced by the equivalent facility j in solution (X, Y), and zero otherwise. For example, if $J = \{1, 2, u\}$, J splits into two equivalence classes $\{1, 2\}$ and $\{u\}$, $O(X^M) = \{1, u\}$ and $O(X) = \{2, u\}$, then the non-zero Z variables are $Z_{21} = 1$ and $Z_{uu} = 1$.

Given a solution (X^M, Y^M) to MASTER, there are $2^{|O_F(X^M)|}$ different scenarios that must be taken into account for computing the expected overload. Each of the open facilities might fail or not fail. Let S^M be the set of all possible scenarios, p^s be the probability of scenario s, and ξ_j^s be the constant that indicates the state of facility $j \in O_F(X^M)$ at scenario s. If facility j fails at scenario s, then $\xi_j^s = 0$, otherwise $\xi_j^s = 1$. If $O_F(X^M)$ were the whole F, the set S^M would be extremely large and nonviable to enumerate. However, since usually $|O_F(X^M)| << |F|$, the computational cost of explicitly considering the whole set S^M in AUX-CRFLP is affordable.

To formulate the AUX-CRFLP we will use the additional variables: For each $i \in I$, $j \in O(X^M)$ and each scenario $s \in S^M$, δ_{ij}^s is h_i if, according to assignments Y', i is served from facility j under scenario s, and 0 otherwise. Also, for each facility $j \in O(X^M)$ and each scenario $s \in S^M$, θ_j^s accounts for the overload at facility j, according to assignments in Y'. Note that these demands and overloads would take the same values if computed with respect to assignments Y, although they might be associated with different (equivalent) facilities.

With all the above notation, the AUX-CRFLP can be formulated as follows:

$$(\text{AUX-CRFLP}(X^M)) \min\ \alpha\left(\sum_{i\in I}\sum_{j\in J} h_i d_{ij} Y_{ij0}\right) + (1-\alpha)\sum_{i\in I} h_i\left(\sum_{j\in J}\sum_{r\in R} d_{ij} q^r Y_{ijr} + \sum_{J}\sum_{r\in R} d_{ij} q^r (1-q) Y_{ijr}\right)$$

s.t.

$(1) - (4), (6), (7)$

$$\sum_{j\in P} X_j = |O_P(X^M)| \qquad P \in J/\sim \tag{17}$$

$$\sum_{k\in O_{[j]}(X^M)} Z_{jk} = X_j \qquad j \in J \tag{18}$$

$$\sum_{j\in[k]} Z_{jk} = 1 \qquad k \in O(X^M) \tag{19}$$

$$\sum_{j\in J} jZ_{j,k-1} \le \sum_{j\in J} jZ_{jk} - 1 \qquad k \in O(X^M) \tag{20}$$

$$\sum_{j\in O_P(X^M)} Y'_{ijr} = \sum_{j\in P} Y_{ijr} \qquad i \in I, r \in R, P \in J/\sim \tag{21}$$

$$Y'_{ikr} \ge Y_{ijr} - (1 - Z_{jk}) \qquad i \in I, j \in J, r \in R, k \in O_{[j]}(X^M) \tag{22}$$

$$\sum_{j\in O_F(X^M)} Y'_{ijr} + \sum_{j\in O_{NF}(X^M)}\sum_{s=0}^{r} Y'_{ijs} = 1 \qquad i \in I, r \in R \tag{23}$$

$$\sum_{r\in R} Y'_{ijr} \le 1 \qquad i \in I, j \in O(X^M) \tag{24}$$

$$\sum_{i\in I} h_i Y'_{ij0} \le Q_j X_j \qquad j \in O(X^M) \tag{25}$$

$$h_i\left(\xi_j^s Y'_{ijr} - \sum_{k\in O(X^M):k\neq j}\sum_{t=0}^{r-1} \xi_k^s Y'_{ikt}\right) \le \delta_{ij}^s \qquad i \in I, j \in O(X^M), r \in R, s \in S^M \tag{26}$$

$$\sum_{i\in I} \delta_{ij}^s - Q_j \le \theta_j^s \qquad j \in O(X^M), s \in S^M \tag{27}$$

$$\sum_{j\in O(X^M)}\sum_{s\in S} p^s \theta_j^s \le \beta \tag{28}$$

$$Y'_{ijr} \in \{0, 1\} \qquad i \in I, j \in O(X^M), r \in R \tag{29}$$

$$\delta_{ij}^s, \theta_j^s \in \mathbb{R} \qquad i \in I, j \in O(X^M), s \in S^M \tag{30}$$

The objective function of AUX-CRFLP is the total cost of replacing a solution (X^M, Y^M) by another equivalent solution that has bounded expected overload. Since the opening cost of equivalent solutions is the same, AUX-CRFLP only takes into account the assignment cost. Constraints (1)-(4),(6) and (7) are the corresponding constraints for the assignment variables. Constraints (17) state that the solution (X, Y) to AUX-CRFLP is equivalent to (X^M, Y^M). Constraints (18) and (19) state the bijective relationship between each open facility in $O(X)$ and the associated equivalent solution in $O(X^M)$. Constraints (20) avoid alternative ways of defining this bijective relationship: it is assumed that facility indices are correlative within the same equivalence class. The constraint forces the assignment of elements in J to elements in $O(X)$ to be sorted, i.e. if facility j_1 maps to class k ($Z_{j_1 k} = 1$), then the facility j_2 that maps to $k+1$ is greater than j_1 ($Z_{j_2,k+1} = 0$ for all $j_2 \le j_1$). Constraints (21) and (22) express the relations between Y variables and Y' variables. Besides, the same requirements that affect Y variables must also affect Y' variables. Then, constraints (23)-(25) are necessary. Given a scenario $s \in S^M$, constraints (26) give the values for δ_{ij}^s, the maximum demand supported by facility j from customer i. Constraints (27) provide θ_j^s, the overload for the facility j on the scenario s: the overload is the positive difference between the demand requested and the capacity of a facility. Finally, (28) bounds the expected overload. The left hand side of (28) is the expected overload of the solution (X, Y).

Proposition 3.1 guarantees that the feasible set of AUX-CRFLP is not empty.

Remark 3.1. Let $O(X_1)$ and $O(X_2)$ be two subsets of open facilities. The fact that $O(X_1) \subset O(X_2)$, does not imply that the expected overload or overcost associated with $O(X_1)$ is equal or higher than the overcost associated with $O(X_2)$. The fact is that closing an open facility can reduce the expected overload and even the overcost. For instance, if only the dummy facility is open, the expected overload is zero and thus also the overcost of adjusting it, while if another facility is open, the expected overload will be in general positive, and the overcost might be positive.

Remark 3.2. A *good* initial feasible solution to AUX-CRFLP can be heuristically obtained. At level zero, customers can be sorted by their demand in non increasing order and each of them can be assigned to its nearest open facility with enough available capacity. At the rest of levels, customers are assigned with the same procedure but without exceeding the overload β and with two precautions: (i) if customer i is assigned to facility j at level r, then it is not assigned to the same j at levels $t > r$; (ii) if customer i is assigned to a non-failing facility at level r, then it is not assigned to more facilities at levels $t > r$.

4. Exact dynamic approach

Let R be the equivalence relation on the set $M = \{(X, Y) : (1) - (4), (6), (7)\}$ of MASTER feasible solutions (two solutions (X_1, Y_1) and (X_2, Y_2) of MASTER are assumed to belong to the same equivalence class iff $|O_P(X_1)| = |O_P(X_2)|$ for all $P \in J/\sim$). Let M/R be the partition given by the set of equivalence classes and let $[(X, Y)]$ be the equivalence class of $(X, Y) \in M$.

The exact dynamic approach proposed in this paper iteratively solves both the MASTER problem and the auxiliary assignment problem. The idea is to solve the assignment problem at most once for each equivalence class in M/R (the optimal value is the same for any master solution at the same equivalence class). For each solution of the auxiliary assignment problem, a new set of constraints is added to the master that accounts for the impact of adequately reducing the expected overload of any solution within the given equivalence class. In particular, the constraints are the ones in subsection 4.1 and the dynamic approach is the one in subsection 4.2.

4.1. W constraints for the master problem

Given a solution (X^M, Y^M) to MASTER problem, let v^*(AUX-CRFLP(X^M)) and v^*(MASTER) be the optimal values of AUX-CRFLP(X^M) and of the MASTER problem respectively. It holds that the minimum cost of moving any solution of the equivalence class of (X^M, Y^M) into the feasible set of CRFLP-EX is

$$\Delta = v^*\left(\text{AUX-CRFLP}(X^M)\right) + \alpha \sum_{j \in J} f_j X_j^M - v^*(\text{MASTER}).$$

This observation can be translated into constraints in the following way. Given a feasible solution to MASTER, (X, Y) let d_P^+ and d_P^- be the variables that measure the deviation of $|O_P(X)|$ from $|O_P(X^M)|$ and D_P^+ and D_P^- be the corresponding binary/boolean variables (if d_P^+ is positive, D_P^+ is one, analogously d_P^- and D_P^-). If all D_P^+ and D_P^- are zero, then $(X, Y) \in [(X^M, Y^M)]$ and the cost of rearranging the expected overload is Δ.

$$\sum_{j \in P} X_j + d_P^+ - d_P^- = |O_P(X^M)| \qquad P \in J/\sim \tag{31}$$

$$d_P^+ \leq K \quad D_P^+ \qquad P \in J/\sim \tag{32}$$

$$D_P^+ \leq d_P^+ \qquad P \in J/\sim \tag{33}$$

$$d_P^- \leq K \quad D_P^- \qquad P \in J/\sim \tag{34}$$

$$D_P^- \leq d_P^- \qquad P \in J/\sim \tag{35}$$

$$D_P^+ + D_P^- \leq 1 \qquad P \in J/\sim \tag{36}$$

$$\sum_{P \in J/\sim} (\Delta D_P^+ + \Delta D_P^-) + W \geq \Delta \tag{37}$$

$$D_P^+, D_P^- \in \{0, 1\} \qquad P \in J/\sim \tag{38}$$

$$d_P^+, d_P^- \geq 0 \qquad P \in J/\sim \tag{39}$$

Constraints (31) measure the deviations, positive or negative, to the number of open facilities in the equivalence class. These values are saved in variables d_P^+ and d_P^-, respectively. Constraints (32) to (36) activate D_P^+ and D_P^- variables according to the values of d_P^+ and d_P^-, and ensure that at most one of them is non-null. The large constant K can be set to $|P|$. Constraint (37) states the extra cost. Given that we are minimizing, constraint (37) does nothing if either D_P^+ or D_P^+ are 1 or it forces $W = \Delta$ if they are both equal to 0.

4.2. Main algorithm

In this section, the main algorithm of the exact dynamic approach we propose is described. Parameters and variables in bold are vectors. In particular, the main algorithm is described in Algorithm 1. The function solve(·) gives the optimal value and an optimal solution to the problem (·). If this function has a second input, it is an initial feasible solution. The function heuristic(·) gives a feasible assignment pattern to facilities selected in (·) by using Remark 3.2. The function add_W_constaints(master, Δ, $\mathbf{X}$) adds constraints (31)-(39) to master with parameters Δ and $|O_P(X^M)|$ for all $P \in J/\sim$. Additionally, lines 9–19 of the algorithm can be explained as follows:

Algorithm 1: $(v^*, \mathbf{X}^*, \mathbf{Y}^*)$ = main_algorithm()

```
1  MASTER = model(QRFLP)
2  (c^M, X^M, Y^M) = solve(MASTER)
3  overload = E(X^M, Y^M)
4  if overload > β then
5      v* = +inf
6  else
7      v* = c^M
8      (X*, Y*) = (X^M, Y^M)
9  while v* > c^M & time < t_limit do
10     f = α Σ_{j∈J} f_j X_j^M
11     secondary = model(AUX-CRFLP(X^M))
12     (c^h, X^h, Y^h) = heuristic(X^M)
13     (c, X, Y) = solve(secondary, (c^h, X^h, Y^h))
14     if c + f < v* then
15         v* = c + f
16         Y*=Y
17     Δ = c + f − c^M
18     add_W_constaints(MASTER,Δ, X^M)
19     (c^M, X^M, Y^M) = solve(MASTER)
20 return (v*, X*, Y*)
```

i. Calculate f, the total opening cost of the solutions belonging to the equivalence class of $\mathbf{X}^M$.
ii. Obtain a heuristic assignment $(c^h, \mathbf{X}^h, \mathbf{Y}^h)$ for the equivalence class of $\mathbf{X}^M$, for example according to the procedure described in Remark 3.2.
iii. Solve AUX-CRFLP starting the search of the optimal solution from the feasible solution $(c^h, \mathbf{X}^h, \mathbf{Y}^h)$. Let $(c, \mathbf{X}, \mathbf{Y})$ be the optimal solution obtained.
iv. Calculate the total cost difference Δ between master solution and best solution bounded in equivalence class $\mathbf{X}^M$.
v. Add constraints (31)-(39) to MASTER by *add_W_constraints* procedure in order to increase by Δ the cost of any solution of the equivalence class of $\mathbf{X}^M$, as explained in Section 4.1.

5. Computational experiments

According to the literature, the best performing approximate models for the CRFLP in terms of expected overload are CRFLP-B1 and CRFLP-LR. Therefore, in this section we will use both models to illustrate the usefulness of CRFLP-EX. To this end, the results in this section evaluate the performance of the approximate methods

Table 1
Generated instances.

	#	OR Instances	$\|I\|$	$\|F\|$	$\|NF\|^{(*)}$	$\|J/\sim\|$	q	f_F ($\times 1000$)	f_{NF}/f_F
S20_50_a	180	1 - 10	20	50	0	2	0.05, 0.10, 0.20	1, 2, 3	1, 2
S20_50_b	180	1 - 10	20	35	15	3	0.05, 0.10, 0.20	1, 2, 3	1, 2
S50_50_a	10	1 - 10	50	50	0	2	0.05	2	2
S50_50_b	10	1 - 10	50	35	15	3	0.05	2	2
S20_75_a	10	11 - 20	20	75	0	2	0.05	2	2
S20_75_b	10	11 - 20	20	45	30	3	0.05	2	2

$(*)$ excluding dummy

in terms of optimal value, expected overload, overload distribution and non-serving demand as compared to the ones associated with the optimal solution provided by CRFLP-EX. The results also show that CRFLP-EX is competitive in terms of computational requirements, compared with the previous approximate approaches CRFLP-B1 and CRFLP-LR.

The same experiments described in Albareda-Sambola et al. (2016) have been reproduced, now with CRFLP-EX. The same 400 instances have been used, again divided in different sets depending on the original instances they were generated from, the number of customers, the number of failing and non-failing facilities, the failure probability, and their opening costs. Recall that these instances were generated from the capacitated p-median instances available at the OR-LIBRARY (Beasley, 1990). Their characteristics are summarized in Table 1. The original locations of the instances act both as potential facility locations and as customers. Regarding the weighting factor of the objective function α, this has been fixed in all our experiments to $\alpha = 0.5$; the outsourcing cost to $d_{iu} = 400$ for all $i \in I$, which is much larger than any of the assignment costs; and the capacity level for the non-dummy facilities is given by the OR-LIBRARY.

In Table 1 the columns are: the total number of instances of each group; the data set at the OR-Library; the number of customers; the number of failing and non-failing facilities; the number of equivalence classes; and, the probability of failure q. Moreover, only two different opening costs have been considered for each instance; f_F for each of the facilities in F, and f_{NF} for those in NF (without taking into account the dummy one). Columns with several values denote the use of each one of these values in combination with the values of the other columns. Besides, the limit for the expected overload has been fixed to two values: $\beta = 3$ and $\beta = 6$. In the results tables we use CRFLP-B1(β), CRFLP-LR(β) and CRFLP-EX(β) to differentiate the results obtained with each value.

All the experiments have been conducted on a PC with a 2.33 gigahertz Intel Xeon dual core processor, 8.5 gigabytes of RAM, and operating system LINUX Debian 4.0. The CPLEX v11.0 optimization library has been used and the overall algorithm has been coded in C.

The following tables show the averages over the corresponding sets of instances of several measures of the solution: $v*$ represents the optimal value, $E(X, Y)$ the value of the expected overload in the optimal solution, $P(overload)$ the probability of having overload computed as the sum of the probabilities of all the scenarios with some positive overload, *Dummy* stands for the expected demand at the dummy facility, i.e., the sum of the demands that the dummy facility receives over all the possible scenarios weighted by the probability of the scenario. Finally, #*Open* is the average number of open facilities including the dummy one and *Time* is the time in seconds for solving the problem. The numbers (3) and (6) next to the method name in the first column refer to the value for the expected overload, β.

The results for smaller instances are reported in Tables 2-5, disaggregated by the three different failure probabilities, $q = 0.05$, $q = 0.10$ and $q = 0.20$. For the medium-size instances, they have been summarized in Tables 6 and 7. As shown in Table 1, for these instances one single failure probability has been considered: $q = 0.05$. In all executions we have set a time limit of 3600 seconds. The first groups of instances could be fully solved before this limit was reached. When this is not the case, we have added a last column, labelled *Opt*, that reports the number of instances solved to optimality within the time limit. Note that from Table 1, Tables 2-7 present successive and respectively average values over 120, 60, 60, 120, 20 and 20 instances.

Results regarding the smaller instances, for values of $q = 0.05$ and $q = 0.20$, (Tables 2 and 5) follow a similar pattern independently of the existence or not of non-failing facilities, but it is interesting to differentiate the *S*20_50 instances with $q = 0.1$ by the number of facilities *NF*. Tables 3 and 4 summarize them in order to observe the influence of the number of non-failing facilities in the solutions provided for each method. It should be noted that the expected overload decreases in cases with the possibility of including reliable facilities given that uncertainty also decreases, benefiting the reduction of total costs. A lower uncertainty also allows reducing the number of open facilities in the solutions and the non-served demand; although in our computational experiments the reduction of open facilities directly provokes a reduction of total costs, it is noteworthy that this rule depends on the cost structure and on the tightness of capacity constraints, since it might pay or not to open facilities in *NF*. Similarly, except for the basic model QRFLP, which has no computational difficulty, the computational time is also reduced in instances with reliable facilities, presumably because in these instances the combinatorics are smaller. These and other variations between the instances were thoroughly discussed in Albareda-Sambola et al. (2016) so, in the following, we focus on the differences between the optimal solutions provided by the new exact algorithm and the heuristic solutions provided by the previous approximated approaches.

In Tables 2–5, the solution obtained by the basic model QRFLP is cheaper than the rest of solutions given that QRFLP only takes into account the capacity in the scenario where no failures have happened. Given that CRFLP-B1 is based on restricting an upper bound of the overload not to exceed the requested overload limit, β, this approach provides the most expensive solutions. CRFLP-LR is based on restricting an estimate of the expected overload. It can either overestimate the overload or underestimate it, depending on the situation. Therefore, the true expected overload in the obtained solutions sometimes can be higher than β, and sometimes lower. There could even exist instances for which the CRFLP-LR solution is more expensive than the CRFLP-B1 solution. However, this did not happen in any of our experiments.

Obviously, the value of CRFLP-EX solutions is never larger than the value of CRFLP-B1 solutions, since in the first case we exactly constrain the overload while in the second case we restrict an upper bound. Notice also that, given that CRFLP-LR solutions sometimes exceed the overload requested, one might expect these solutions to be cheaper than those obtained from CRFLP-EX. However, in most of the instances this does not happen, which is an additional advantage of the CRFLP-EX method. In most of the exper-

Table 2
Average values for S20_50 instances with $q = 0.05$.

	v^*	$E(X,Y)$	$\mathbb{P}$(overload)	Dummy	# Open	Time
QRFLP	8997.20	5.19	0.07	0.26	3.44	7.75
CRFLP-B1(3)	9378.19	1.64	0.06	1.92	3.67	30.20
CRFLP-B1(6)	9143.23	3.72*	0.07	1.06*	3.52	32.33
CRFLP-LR(3)	9287.02	2.53	0.07	1.68	3.57	31.71
CRFLP-LR(6)	9051.44	4.65	0.07	0.56	3.47	17.82
CRFLP-EX(3)	9315.85	2.33	0.07	1.52	3.63	27.35
CRFLP-EX(6)	9143.23	3.72	0.07	1.06	3.52	16.05

* Typos detected in Albareda-Sambola et al. (2016)

Table 3
Average values for S20_50_a ($|NF| = 0$) instances with $q = 0.1$.

	v^*	$E(X,Y)$	$\mathbb{P}$(overload)	Dummy	# Open	Time
QRFLP	9355.96	10.26	0.15	1.22	3.60	5.05
CRFLP-B1(3)	10294.36	1.49	0.07	3.02	4.23	112.30
CRFLP-B1(6)	9879.32	3.96	0.12	1.82	4.03	84.77
CRFLP-LR(3)	10172.13	2.20	0.09	2.56	4.21	403.62
CRFLP-LR(6)	9849.81	4.30	0.12	1.75	3.97	72.72
CRFLP-EX(3)	9895.20	2.45	0.07	0.76	4.03	88.31
CRFLP-EX(6)	9836.72	3.79	0.12	1.59	3.93	67.58

Table 4
Average values for S20_50_b ($|NF| = 15$) instances with $q = 0.1$.

	v^*	$E(X,Y)$	$\mathbb{P}$(overload)	Dummy	# Open	Time
QRFLP	9187.60	7.18	0.10	0.61	3.50	8.78
CRFLP-B1(3)	9690.77	0.90	0.05	1.54	3.50	48.52
CRFLP-B1(6)	9473.40	2.47	0.07	0.91	3.72	43.72
CRFLP-LR(3)	9625.14	1.27	0.05	1.28	3.80	84.72
CRFLP-LR(6)	9455.83	3.31	0.08	0.96	3.68	38.28
CRFLP-EX(3)	9458.71	2.98	0.06	0.38	3.72	63.12
CRFLP-EX(6)	9427.90	3.70	0.09	0.79	3.67	35.02

Table 5
Average values for S20_50 instances with $q = 0.20$.

	v^*	$E(X,Y)$	$\mathbb{P}$(overload)	Dummy	# Open	Time	Opt
QRFLP	9995.34	12.73	0.21	2.44	3.91	7.25	120
CRFLP-B1(3)	10975.59	1.27	0.05	2.63	4.58	236.17	120
CRFLP-B1(6)	10512.67	4.60	0.12	1.62	4.34	954.25	120
CRFLP-LR(3)	10636.21	3.48	0.09	1.16	4.36	744.68	120
CRFLP-LR(6)	10503.05	4.49	0.11	1.55	4.23	944.17	120
CRFLP-EX(3)	10292.75	2.70	0.08	2.28	4.12	444.38	118
CRFLP-EX(6)	10323.78	4.68	0.11	1.87	4.04	176.83	120

Table 6
Average values for S50_50 instances with $q = 0.05$.

	v^*	$E(X,Y)$	$\mathbb{P}$(overload)	Dummy	#Open	Time	Opt
QRFLP	17955.67	22.06	0.23	0.00	6.10	719.03	18
CRFLP-B1(3)	21428.69	2.33	0.15	16.37	6.75	187.00	20
CRFLP-B1(6)	20870.85	4.71	0.19	13.37	6.75	215.00	20
CRFLP-LR(3)	21198.76	3.31	0.16	15.12	6.75	1152.60	18
CRFLP-LR(6)	20449.32	6.41	0.21	11.07	6.75	1188.75	16
CRFLP-EX(3)	19848.68	3.00	0.21	10.37	6.10	3525.65	2
CRFLP-EX(6)	19663.85	5.84	0.22	8.69	6.05	3525.65	2

iments the solutions obtained by CRFLP-EX are cheaper than the CRFLP-LR solutions, except for Table 2, i.e, besides providing the best solution to the problem the exact method usually provides cheaper solutions than the approximate method CRFLP-LR.

Concerning the computational requirements of CRFLP-EX, we notice that instances with $\beta = 3$ tend to be harder to solve than those with $\beta = 6$ as expected, since they are more restrictive. Moreover, given that the exact approach proposed here consists of a dynamic method which iteratively solves several models until optimality is proven, a worse performance in terms of computational time is naturally expected, as compared with the rest of methods that only solve a model once. Indeed, this happens in most of the instances, but it is not true in all cases. In Table 2 and Table 3, there are several CRFLP-EX solutions found in less time than the corresponding CRFLP-LR and even than CRFLP-B1 solutions. The worst performance of CRFLP-EX is shown in Table 5. After a limit time of 3600 seconds, the exact method only solves two instances of each group of the 20-customer instances (for $\beta = 3$, and $\beta = 6$). However, it is very remarkable that despite not concluding in most of the instances, the solutions returned are considerably cheaper than the solutions returned by the other two methods, and the expected overload is below the limit overload

Table 7
Average values for S20_75 instances with $q = 0.05$.

	v^*	$E(X, Y)$	$\mathbb{P}$(overload)	Dummy	#Open	Time	Opt
QRFLP	9806.13	5.4	0.12	0.22	3.6	349.25	20
CRFLP-B1(3)	11090.57	1.32	0.10	2.44	4.4	1412.03	19
CRFLP-LR(3)	10885.10	2.57	0.12	1.91	4.3	1337.10	17
CRFLP-EX(3)	10030.98	2.80	0.16	0.93	4.0	772.45	19

allowed. Besides, in these instances the CRFLP-LR model does not work efficiently since it returns solutions exceeding, in average, the requested overloads being their costs higher, too. It can be observed that combinatorics in these instances are extremely large and it is hard to find the optimal solution but, still, CRFLP-EX permits to adjust the assignments so that the expected overaload constraint is tightly satisfied Indeed, in most of the cases, the solution returned by the exact approach is the best known one, even if the method has not run the necessary iterations to prove optimality. As an evidence of the computational difficulty of these instances, the CRFLP-S(1.3) method based on staggered capacities proposed in Albareda-Sambola et al. (2016) could not be solved to optimality for any of these instances after 3600 seconds. On the contrary, instances summarized in Table 6 are also computationally hard. However, in the instances where the exact approach ends, it takes less time than the model based on linear regression which, in turn, takes less computational time than the CRFLP-B1 model. Notice that, in this table, the average expected overload of the solutions obtained by QRFLP is already below 6. Indeed, this is the case for all the instances in this group. For this reason, we did not consider the value $\beta = 6$ in the experiments for these instances.

Finally, the graphics of the left column in Fig. 2 represent the probability distribution of the overload for the optimal solution of the fourth instance in \$50_50_*b* respectively corresponding to models QRFLP, CRFLP-B1(6), CRFLP-LR(6) and CRFLP-EX(6). On the other hand, the right column of this figure represents the probability of demand assigned to the dummy facility when the same instance is solved for each method. In these graphics, we have not drawn the bar corresponding to the 0 value of the overload neither the demand because it has a huge probability as compared with the others. As we can see, for most of the methods, the mass of the overload is concentrated in the lower values, i.e., near to the OY axis, while the most likely non-zero overload in the QRFLP solution has values between $80 - 90$ with a probability of 0.12 existing even overloads higher than 200. Instead, for CRFLP-B1 and CRFLP-LR models, the highest-probability overload range is $0 - 10$ with probability 0.12. Given that CRFLP-B1 method constraints the overload bound, the rest of overload quantities have low probability. Finally, the exact approach distributes the overload a bit more homogeneously, between $0 - 40$ values, but their probability of 0.03 is moderated. CRFLP-EX also has high values of overload around 100 but with a low probability of about 0.005. In summary, the expected overload is 19.62 in QRFLP, 5.80 in CRFLP-B1(6), 7.11 in CRFLP-LR(6) and exactly 6.00 in CRFLP-EX(6) and the values of these solutions are 16764.46, 19611.41, 19202.30 and 17834.11 respectively. The small difference between the overload of CRFLP-B1 and CRFLP-EX solutions shows how the huge advantage of the optimal value of CRFLP-EX approach is due not so much to reduce the overload as to manage it well. In conclusion, the results for this instance evidence the good performance of the exact method not only in terms of overload adjust, but also in terms of distribution of the overload, rebounding these in a better cost.

According to the dummy demand distribution, i.e., the non-served demand distribution, it can be observed that the non-served demand existing in QRFLP solutions is insignificant, that is logical given that QRFLP method only has non-served demand in the scenario where all facilities have failed. Maybe it should be expected to observe the maximum non-served demand in CRFLP-EX given that it is based on introducing restrictions for lower bounding the dummy demand, but in these instances with high combinatorics, the non-served demand is significantly lower in CRFLP-EX approach being its expected values 0, 15.91, 13.71 and 5.27 respectively for QRFLP, CRFLP-B1 CRFLP-LR and CRFLP-EX solutions.

Regarding the instances not concluded, it would be interesting to know the individual performance of the proposed approach for some of these instances. Thus, we have selected two non-concluded instances: namely, instance 1 from group S50_50 and instance 8 from grup S20_75. The detailed results are summarised in Table 7. In all cases, the exact approach gets the cheapest solution with expected overload below the established limit. Curiously, CRFLP-EX returned the same solution for the S50_50 instance for both $\beta = 3$ and $\beta = 6$. For this instance, it can be seen that although the returned solution for QRFLP is still more economical, its expected overload is out of control. Also interestingly, the same solution for instance S20_75 was returned for both QRFLP and CRFLP-EX approaches. This is one of the few instances where fortuitously the overload of QRFLP is already below β.

An interesting aspect to be considered is the impact of the number of equivalence classes in terms of computing time and also in terms of the returned solutions. Given a fixed instance size, each additional class implies one more (17) constraint and $|I| \times |R|$ additional (21) constraints in AUX-CRFLP, while the number of variables does not change. Thus, the feasible region of the auxiliary problem becomes smaller which, in general, means a computationally easier problem. On the other hand, more iterations of the overall method are expected as the number of classes increases, since it would be necessary to get more combinations of the classes of solutions and each additional set of W constraints would affect less sets of open facilities. Then, in the extreme case for which every facility is one class, the auxiliary problem would be trivial but the number of iterations might explode. Let us suppose one class of size $|P|$ is divided into two classes of size $\frac{|P|}{2}$. For computing the expected overload and inserting one constraint referred to $k < \frac{|P|}{2}$ open facilities belonging to the original class we need one master iteration and also to solve one auxiliary problem. But for doing the same with two classes from the original class we would need k master iterations and also to solve k auxiliary problems corresponding to the combinations of the number of open facilities of each class $(k, 0)$, $(k-1, 1), \ldots, (0, k)$ which would seldom be compensated by the decrease in complexity of AUX-CRFLP. Finally, we can conclude that the more classes the harder the problem is, if the rest of features are similar. However, the right number of classes to work with is directly determined by the data. In the extreme case where every facility is one class, we would not get any advantage from the classes and the problem to be solved would be completely combinatorial. This is precisely the main contribution of the proposed method, to save combinatorics by establishing classes from facilities of identical features.

In our computational experiments, we have summarized the computational experience of S20_50 instances solved by CRFLP-EX approach and grouped by the number of classes taken into account in Table 8. Although we only have instances with 2 and 3 classes, the computation time turned out to be significantly lower for the

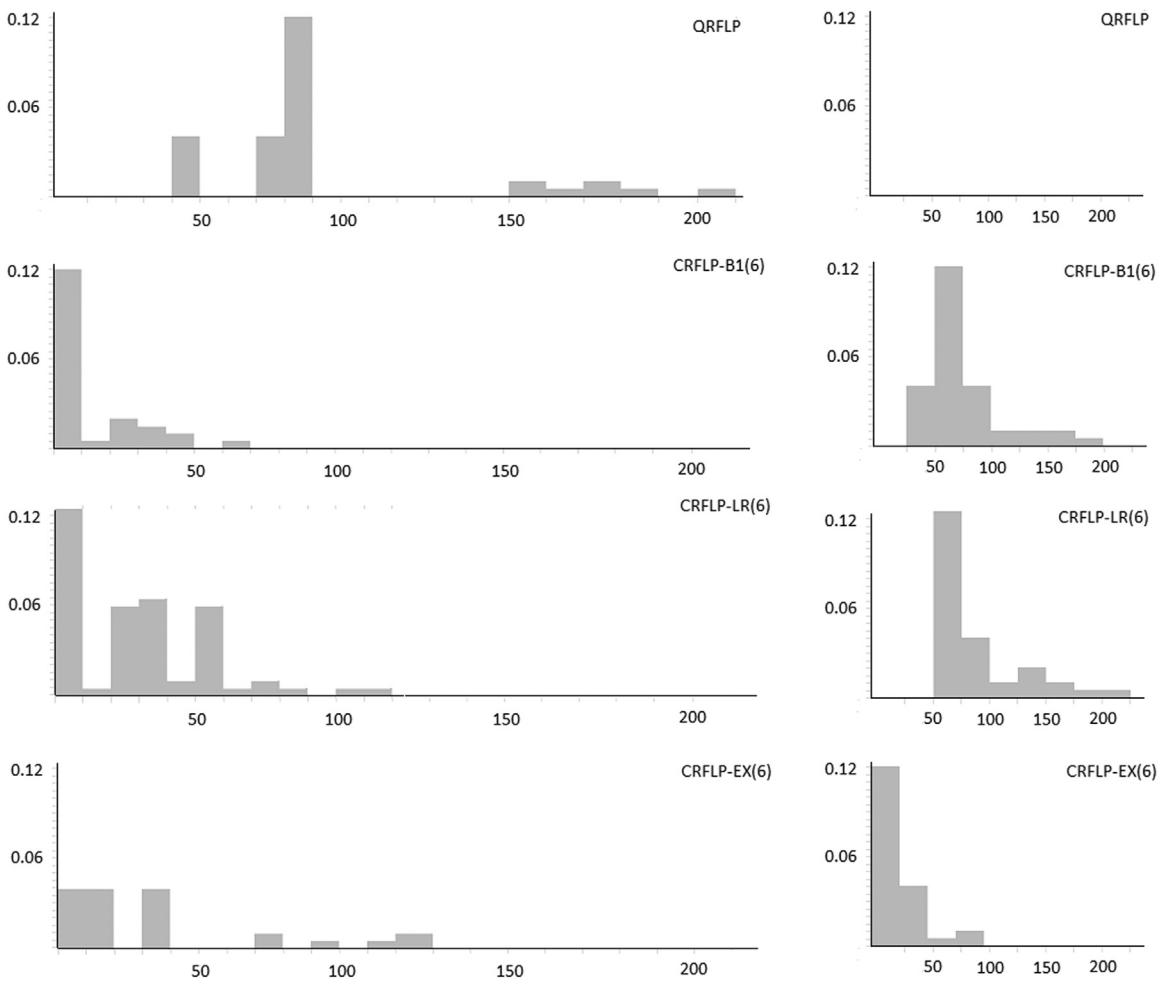

Fig. 2. Probability distribution of overload (left) and non-served demand(right).

Table 8
Some individual CRFLP-EX non-concluded instances.

	v^*	$E(X,Y)$	$\mathbb{P}$(overload)	Dummy	#Open	Time
S50_50 instance 1						
QRFLP	16671.42	14.38	0.23	0.00	6	1947
CRFLP-B1(3)	19252.43	1.98	0.18	10.57	7	379
CRFLP-B1(6)	18708.08	3.82	0.18	7.58	7	576
CRFLP-LR(3)	19012.15	3.20	0.18	9.24	7	3600
CRFLP-LR(6)	18281.62	5.04	0.22	5.18	7	3600
CRFLP-EX(3)	17244.47	3.00	0.23	2.73	6	3600
CRFLP-EX(6)	17244.47	3.00	0.23	2.73	6	3600
S20_75 instance 8						
QRFLP	10839.45	2.99	0.03	0.30	4	1675
CRFLP-B1(3)	11606.37	1.92	0.19	0.02	5	736
CRFLP-LR(3)	11552.60	4.16	0.12	0.02	5	1380
CRFLP-EX(3)	10839.45	2.99	0.03	0.30	4	3600

Table 9
Average values for instances grouped by the number of classes.

# Classes	β	v^*	$E(X,Y)$	$\mathbb{P}$(overload)	Dummy	#Open	#NF Open	Time
2	3	9961.02	2.48	0.08	1.88	4.04	1.00	236.75
3	3	9562.09	2.68	0.07	1.03	3.70	1.71	128.22
2	6	9923.88	4.49	0.11	1.86	3.92	1.00	104.92
3	6	9475.66	3.60	0.08	0.88	3.65	1.70	57.86

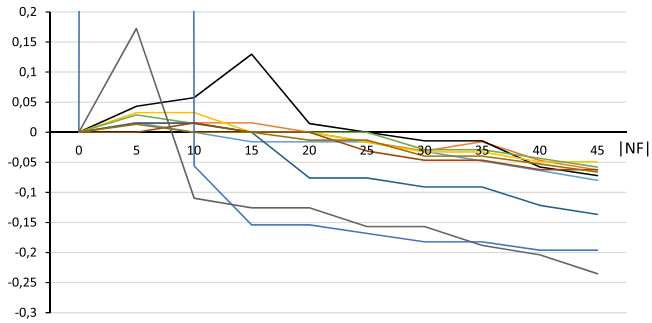

Fig. 3. Percent deviation of time with respect to the instance with $|NF| = 0$.

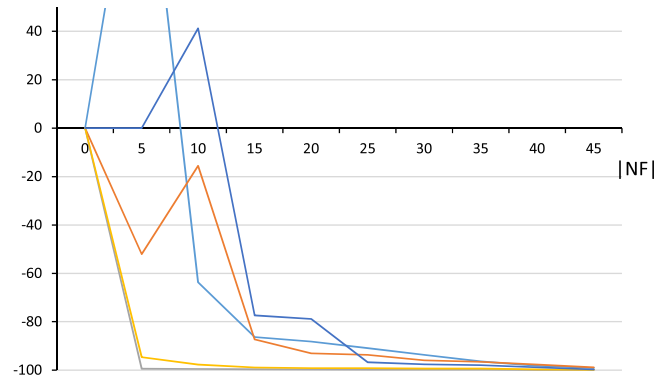

Fig. 4. Percent deviation of time with respect to the instance with $|NF| = 0$.

instances of 3 classes than for the instances of 2, contrary to what we would theoretically expect if we just taken into account the number of classes. This is because it does not just influence the number of classes, but also the characteristics of the classes. Whilst in instances of 2 classes all the facilities can fail except the dummy facility, in instances of 3 classes we have a new class made up of 15 *NF* facilities. This clearly facilitates the resolution of the master and the auxiliary problems because it also happens in general that the larger the number of *NF* facilities, the lower the combinatorics of the problem since less facilities can fail. When $\beta = 3$, we have spent an average of 128.22 seconds to solve instances with 3 classes versus 236.75 for instances with 2 classes. When $\beta = 6$, the differences are of a similar ratio, 57.86 seconds versus 104.92. In terms of goodness of the solution, although we have instances where the cost of opening *NF* facilities is higher than opening *F* facilities, we do open more facilities if all them can fail and also more demand is deviated to the dummy facility, so the solutions with 2 classes are more expensive in average than the solutions with 3 classes.

To further analyze the impact of the number of classes and their characteristics on the computational burden of CRFLP-EX, we have carried out a final series of experiments. To this end, we have considered the subset of the 10 original instances in class S_20_50 with $q = 0.05$, $f_F = 1000$ and $f_{NF} = 1000$, and, from each of them, we considered a sequence of 10 new instances with $|NF|$ ranging from 0 to 45 (excluding NF dummy facility) in steps of 5. We repeated the same experiment starting from the instances in class S_50_50. Note that, like in the case of the previous table, instances with $|NF| = 0$ have two equivalence classes of facilities, while all the others have three.

All instances with 20 customers could be solved within the time limit. In fig. 3, each line corresponds to one of the original instances. For each value of $|NF|$ it gives the percent deviation of the corresponding solution time with respect to the time needed to solve the instance with $|NF| = 0$. One of the lines has been truncated, since, for that particular instance, the percentage for $|NF| = 5$ was much higher than the others. The picture shows how, when the number of classes steps from 2 to 3, the computational effort required increases, but the effect of increasing the proportion of non-failing facilities can compensate this increase. Indeed, from $|NF| = 15$ on, the computational time constantly decreases.

When 50 customers are considered, only for 5 cases we could solve the whole sequence of instances with different $|NF|$ values. Thus, in Fig. 4 we repeat the structure of Fig. 3 but only with the corresponding 5 lines. Fig. 4 shows how the effect of the proportion of non-failing facilities can be higher than that of the number of equivalence classes. Actually, in this group we observe three sequences where even with a low number of non-failing facilities, the time required for the instances with 3 classes is smaller than with 2 classes.

6. Conclusions

It has been proposed an exact dynamic approach in order to solve the problem of strictly bounding the expected overload of the capacitated RFLP. Without the dynamic approach, the expected overload calculation is not viable, even for small instances.

It has been proven that, given a set of open facilities, the corresponding expected overload can be kept arbitrarily small by assigning to the dummy facility the demand that otherwise would exceed the imposed limit on the expected overload. Using this result, we have introduced the problem of rearranging the demand at minimum cost, in order to satisfy this limit. It has also been proven that it is not necessary to perform the rearrangement for all the solutions but only for the representatives of some equivalence classes. The definition of these equivalence classes highly reduces the combinatorial difficulty inherent to the problem. It has been proposed an exact cutting plane dynamic approach that iteratively rearranges the MASTER solutions and produces feasible solutions.

The comprehensive computational analysis shows the good performance of this dynamic approach compared with the approximate methods CRFLP-B1 and CRFLP-LR developed by Albareda-Sambola et al. in Albareda-Sambola et al. (2016). Obviously, the exact method provides the best results in terms of the goal, the cost when the expected overload is bounded. Moreover, in many cases, the exact approach also requires less time than the approximate methods. In all of the instances in which the exact method does not finish within the time limit, its solution is better than the approximate solutions. Besides, the behaviour of the overload and dummy demand of the exact method improves on the behaviour of the approximate methods providing better distributions. It has also been illustrated that, in many cases, it is possible to obtain relevant reductions of the costs only by redistributing the demand, which is another motivation to use this approach.

In the future, it could be considered the development of a heuristic algorithm based on the dynamic method. Additionally, since the idea of controlling the expected overload by exact dynamic approach has worked efficiently in the computational experiments, it could be also considered to extend this idea to other reliability problems.

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