NOETHERIAN RINGS OF LOW GLOBAL DIMENSION AND SYZYGETIC PRIME IDEALS

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Abstract. Let R be a Noetherian ring. We prove that R has global dimension at most two if, and only if, every prime ideal of R is of linear type. Similarly, we show that R has global dimension at most three if, and only if, every prime ideal of R is syzygetic. As a consequence, we derive a characterization of these rings using the André-Quillen homology.

Let R be a commutative ring. An ideal I of R is said to be of linear type if the graded surjective morphism $\alpha : \mathbf{S}(I) \to \mathbf{R}(I)$, from the symmetric algebra of I onto the Rees algebra of I, is an isomorphism; I is said to be syzygetic if the second component $\alpha_2 : \mathbf{S}_2(I) \to I^2$ is an isomorphism. It is known that R has weak dimension at most one if, and only if, every ideal of R is of linear type, and equivalently if, and only if, every ideal of R is syzygetic. This leads to a characterization of rings of weak dimension at most one in terms of the André-Quillen homology (see [12]).

Recall that the weak dimension of a ring R, denoted by $\mathrm{w.dim}\,(R)$, is the supremum of the flat dimensions of all R-modules; likewise, the global dimension of R, denoted by $\mathrm{gl\,dim}\,(R)$, is the supremum of the projective dimensions of all R-modules. Clearly $\mathrm{w.dim}\,(R) \leq \mathrm{gl\,dim}\,(R)$, and when R is Noetherian, they agree (see, e.g., [8, Chapter 5]). Since $\mathrm{gl\,dim}\,(R) = \sup\{\mathrm{gl\,dim}\,(R_{\mathfrak{m}}) \mid \mathfrak{m} \in \mathrm{Max}(R)\}$, then for a Noetherian ring R, $\mathrm{gl\,dim}\,(R) \leq N$ is equivalent to $R_{\mathfrak{m}}$ being regular local with $\mathrm{Krull\,dim}\,(R_{\mathfrak{m}}) \leq N$, for every maximal ideal $\mathfrak{m}$ of R (see, e.g., [8, Theorems 5.92, 5.84]).

The purpose of this note is to extend these characterizations of rings of $\mathrm{w.dim}\,(R) \leq 1$ to rings of global dimension at most two and three, but now in the Noetherian context. This is done in quite similar terms. Concretely, we prove the result below. Item (A), shown in general in [12], is included here for the sake of completeness. Unless stated otherwise, R will always be a Noetherian ring.

Theorem. *Let R be a Noetherian ring.*

(A): $\mathrm{gl\,dim}\,(R) \leq 1 \Leftrightarrow$ *every ideal of R is of linear type* $\Leftrightarrow$ *every ideal of R is syzygetic.*
(B): $\mathrm{gl\,dim}\,(R) \leq 2 \Leftrightarrow$ *every prime ideal of R is of linear type.*
(C): $\mathrm{gl\,dim}\,(R) \leq 3 \Leftrightarrow$ *every prime ideal of R is syzygetic.*

Since the linear type and syzygetic conditions are clearly local, to prove this statement one can suppose that $(R, \mathfrak{m}, k)$ is a Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k. Moreover, one can substitute the condition $\mathrm{gl\,dim}\,(R) \leq N$ by the condition "R is regular local of $\mathrm{Krull\,dim}\,(R) \leq N$". Suppose first that R is regular local. If $\mathrm{Krull\,dim}\,(R) \leq 1$, then every nonzero proper ideal of R is generated by a nonzero divisor, hence of linear type and syzygetic (see, e.g., [6, Corollary 3.7]). If $\mathrm{Krull\,dim}\,(R) \leq 2$ or 3, since R is regular local, then $\mathfrak{m}$ is generated by an R-regular sequence, so $\mathfrak{m}$ is of linear type (see, e.g., [6, Corollary 3.8]); moreover R is a UFD, thus every height one prime ideal is principal (generated by a nonzero divisor), and so again of linear type; furthermore, every prime ideal in a regular local ring is generically a complete intersection; it is also perfect when it is of height two, and hence syzygetic (see, e.g., [6, Remark page 91]). This shows the "only if" implications in Theorem (A), (B) and (C).

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Observe that the proof of [6, Corollary 3.8] shows that a Noetherian local ring with syzygetic maximal ideal is regular. Therefore, in order to prove the "if" implication in Theorem (A), it is enough to display, in a two dimensional regular local ring R, a non syzygetic ideal. For, if R were a regular local ring with all its ideals being syzygetic and its Krull dimension were at least two, then, localizing at a height two prime ideal of R, one would get a two dimensional regular local ring with all its ideals being syzygetic, a contradiction. Similarly, to prove the "if" implication in Theorem (B), it suffices to exhibit, in a three dimensional regular local ring, a height two prime ideal which is not of linear type. Finally, to prove the "if" implication in Theorem (C), we exhibit, in a four dimensional regular local ring, a height three prime ideal which is not syzygetic.

In this direction, we show the next result, a kind of rephrasing of [12, Lemma 3], but under the regular local hypothesis, and hence easier to prove it. We give an alternative proof using [13, Corollary 4.11].

Lemma 1. *Let $(R,\mathfrak{m},k)$ be a Noetherian regular local ring of Krull dimension 2. Then $\mathfrak{m}^2$ is not a syzygetic ideal.*

Proof. Let x,y be a regular system of parameters. Let $I=\mathfrak{m}^2=(x^2,xy,y^2)$ and $J=(x^2,y^2)$. Using that x,y is an R-regular sequence, it is easy to check that $xy\notin J$ (see, e.g., [3, Theorem 9.2.2], to relate it to the simplest case of the Monomial Conjecture). Since $(xy)^2\in JI$, then $J:xy\subsetneq JI:(xy)^2$. By [13, Corollary 4.11], we conclude that I is not syzygetic. □

Next lemma displays, in a three dimensional regular local ring, a height two prime ideal which is not of linear type, thus generalizing [5, Corollary 2.7]. There, in the context of the Shimoda Conjecture, one exhibits, in a three dimensional regular local ring, a non-complete intersection height two prime ideal. (Recall that complete intersection implies linear type.)

Lemma 2. *Let $(R,\mathfrak{m},k)$ be a Noetherian regular local ring of Krull dimension 3. Let x,y,z be a regular system of parameters. Let I be the ideal of R generated by*

$$f_1=y^3-x^4\ ,\ f_2=xyz-z^3+x^4-xy^3,$$
$$f_3=x^2y+y^2z-xz^2-x^3y \text{ and } f_4=xy^2-yz^2-x^2y^2+x^3z.$$

Then I is a height two prime ideal minimally generated by four elements. In particular, I is not of linear type.

Finally, the third lemma exhibits a non syzygetic height three prime ideal in a four dimensional regular local ring.

Lemma 3. *Let $(R,\mathfrak{m},k)$ be a Noetherian regular local ring of dimension 4. Let x,y,z,t be a regular system of parameters. Let I be the ideal of R generated by*

$$f_1=yz-xt\ ,\ f_2=z^3-x^5\ ,\ f_3=z^2t-x^4y\ ,$$
$$f_4=zt^2-x^3y^2\ ,\ f_5=t^3-x^2y^3\ ,\ f_6=y^4-x^5\ ,$$
$$f_7=y^3t-x^4z\ ,\ f_8=y^2t^2-x^3z^2.$$

Then I is a height three prime ideal which is not a syzygetic ideal.

The general skeleton of the proofs of Lemmas 2 and 3 are similar to that of the proof of [5, Proposition 2.6]. Namely, once the candidate I is chosen, we show that I is perfect with the desired height, in particular, height unmixed. Then we pick an associated prime $\mathfrak{p}$ to I, which will be of the same height, and, by means of multiplicity theory, we show that $xR+I$ and $xR+\mathfrak{p}$ have the same colength, concluding, by Nakayama's Lemma, that I and $\mathfrak{p}$ are equal.

The ideal I displayed in Lemma 2 is a small variation of [7, Example 3.7]. Concretely, Huneke considers the height two prime ideal defined by the kernel of the homomorphism from the power series ring $\mathbb{C}[\![X,Y,Z]\!]$ to $\mathbb{C}[\![t]\!]$, sending X,Y,Z to t^6,t^7+t^{10},t^8, respectively. He shows that this ideal is generated by the 3×3 minors of a specified 4×3 matrix L, whose entries are either 0, or else one among the monomials X, Y, Z, X^2, XY with a $\pm 1, \pm 2$ integer coefficient. Our example consists in taking these 3×3 minors, but substituting in the matrix L the variables X, Y, Z for the regular parameters x, y, z, and replacing ± 2 by ± 1, in order to avoid characteristic two problems (surprisingly enough, it works).

As for the ideal I considered in Lemma 3, we recover a particular case of a family of prime ideals with unbounded number of generators provided by Bresinsky in [2]. Concretely, we consider the kernel of the homomorphism from $\mathbb{K}[X,Y,Z,T]$, $\mathbb{K}$ any field, to $\mathbb{K}[t]$, sending X, Y, Z, T to $t^{12}, t^{15}, t^{20}, t^{23}$ and then, as before, just substitute the variables for the regular parameters.

Before proceeding to prove Lemmas 2 and 3, we highlight the good behaviour of the syzygetic and linear type conditions through faifhfully flat morphisms of rings. Indeed, this follows from [11, Corollaire 2.3] (see also [13, Theorem 2.4 and Example 2.3]), where one shows that these conditions are characterized in terms of the exactness of a complex of R-modules and noting that, if $R \to S$ is a (faithfully) flat morphism of rings, then $I \otimes_R S \cong IS$.

Proof of Lemma 2. Since $(R, \mathfrak{m})$ is a three dimensional regular local ring with maximal ideal $\mathfrak{m}$ generated by x, y, z, then its completion $(\widehat{R}, \widehat{\mathfrak{m}})$ is a three dimensional regular local ring with maximal ideal $\widehat{\mathfrak{m}} = \mathfrak{m}\widehat{R} = (x,y,z)\widehat{R}$ generated by the regular system of paremeters x, y, z. Let $I = (f_1, f_2, f_3, f_4)$ and $\widehat{I} = I\widehat{R} = (f_1, f_2, f_3, f_4)\widehat{R} \cong I \otimes_R \widehat{R}$. If we prove that $\widehat{I}$ is prime and not of linear type, then $I = I\widehat{R} \cap R$ is prime and not of linear type, because the completion morphism is faithfully flat (see, e.g., [10, § 8]). Therefore we can suppose that R is complete.

First observe that f_1, f_2, f_3, f_4 are, up to a change of sign, the 3×3 minors of the matrix

$$\varphi_2 = \begin{pmatrix} x & xy & z \\ x & y & 0 \\ -z & -x^2 & -y \\ -y & -z & x \end{pmatrix}.$$

In other words, $I = I_3(\varphi_2)$. Since $(f_1, f_2, x) = (x, y^3, z^3)$, then $\operatorname{grade}(f_1, f_2, x) = 3$. By [3, Corollary 1.6.19], f_1, f_2 is an R-regular sequence in $I_3(\varphi_2)$ and so $\operatorname{grade}(I_3(\varphi_2)) \geq 2$. Let φ_1 be the 1×4 matrix defined as (f_1, f_2, f_3, f_4). By the Hilbert-Burch Theorem (e.g., [3, Theorem 1.4.16]),

$$0 \to F_2 = R^3 \xrightarrow{\varphi_2} F_1 = R^4 \xrightarrow{\varphi_1} F_0 = R \to R/I \to 0$$

is a free resolution of R/I. (It is minimal since $\varphi_2(R^3) \subset \mathfrak{m}R^4$ and $\varphi_1(R^4) = I \subset \mathfrak{m}$.) Therefore

$$2 \leq \operatorname{grade}(I) = \min\{i \geq 0 \mid \operatorname{Ext}^i_R(R/I, R) \neq 0\} \leq \operatorname{proj\,dim}_R(R/I) \leq 2,$$

and I is a perfect ideal of grade 2 (see, e.g., [3, Theorem 1.2.5 and page 25]). In particular, I is grade (and height) unmixed (see, e.g., [3, Proposition 1.4.15]) and so $\mathfrak{m}$ is not an associated prime to I.

Let $\mathfrak{p}$ be any associated prime to I and set $D = R/\mathfrak{p}$. Thus D is a one dimensional complete Noetherian local domain (see, e.g., [10, page 63]). Let V be the integral closure of D in its quotient field K. Then V is a finitely generated D-module and a one dimensional integrally closed Noetherian local domain, hence a discrete valuation ring, DVR for short (see, e.g., [15, Theorem 4.3.4]).

Let ν be the valuation on K corresponding to V. Let x, y, z denote also the images of the regular system of parameters of R in V. Set $\nu_x = \nu(x)$, $\nu_y = \nu(y)$ and $\nu_z = \nu(z)$. In V, $f_1 = 0$. Applying ν to the equality $x^4 - y^3 = 0$, one gets $4\nu_x = 3\nu_y$. Thus $\nu_x = 3q$, for some integer $q \geq 1$. In fact, $q > 1$. Indeed, suppose that $q = 1$, $\nu_x = 3$ and $\nu_y = 4$. Since $f_2 = 0$ in V, then $z^3 = x(yz + x^3 - y^3)$. Applying ν to this equality, $3\nu_z \geq \min(12, 7 + \nu_z)$, which implies $\nu_z \geq 4$. Since $f_3 = 0$ in V, then $x^2y = -y^2z + xz^2 + x^3y$. Applying ν to this equality, one gets $10 \geq \min(12, 11, 13)$, a contradiction. Therefore $\nu_x \geq 6$.

Observe that $xR + I = (x, y^3, y^2z, yz^2, z^3)$. Set $S = R/xR$ and consider (by abuse of notation) y, z a regular system of parameters of the regular local ring $(S, \mathfrak{n})$, where $\mathfrak{n} = (y, z)$. Then $R/(xR + I) \cong S/\mathfrak{n}^3$. Since $xR = \operatorname{Ann}_R(S)$, then $\operatorname{length}_R(R/(xR + I)) = \operatorname{length}_S(S/\mathfrak{n}^3)$. Since y, z is a S-regular sequence, there exists a graded isomorphism $k[Y,Z] \cong G(\mathfrak{n})$ of k-algebras, between the polynomial ring in two variables Y, Z over the field $k = S/\mathfrak{n}$ and the associated graded ring of the ideal $\mathfrak{n}$. Using the two exact sequences $0 \to \mathfrak{n}^i/\mathfrak{n}^{i+1} \to S/\mathfrak{n}^{i+1} \to S/\mathfrak{n}^i \to 0$, for $i = 1, 2$, one deduces that $\mathrm{length}_S(S/\mathfrak{n}^3) = 6$. Therefore $\mathrm{length}_R(R/(xR + I)) = 6$.

On the other hand, since $xR + I \subseteq xR + \mathfrak{p}$ and $R/(xR + \mathfrak{p}) \cong (R/\mathfrak{p})/(x \cdot R/\mathfrak{p}) = D/xD$, then

$$6 = \mathrm{length}_R(R/(xR + I)) \geq \mathrm{length}_R(R/(xR + \mathfrak{p})) = \mathrm{length}_D(D/xD).$$

Since $f_1 = 0$ and $f_2 = 0$ in D, then $y^3, z^3 \in xD$, and so xD is an ideal generated by a system of parameters of the one dimensional Cohen-Macaulay local domain $(D, \mathfrak{m}/\mathfrak{p}, k)$. Since V is a finitely generated Cohen-Macaulay D-module of $\mathrm{rank}_D(V) = 1$, then $\mathrm{length}_D(D/xD) = \mathrm{length}_D(V/xV)$ (see [3, Corollary 4.6.11, (c)]). Note that $\mathrm{length}_D(V/xV) = [k_V : k] \cdot \mathrm{length}_V(V/xV)$, where $[k_V : k]$ is the degree of the extension of the residue fields of V and of D and, since V is a DVR, then $\mathrm{length}_V(V/xV) = \nu(x) = \nu_x$. Therefore, $\mathrm{length}_D(D/xD) = [k_V : k] \cdot \nu_x$. Summing up all together,

$$6 = \mathrm{length}_R(R/(xR + I)) \geq \mathrm{length}_R(R/(xR + \mathfrak{p})) = [k_V : k] \cdot \nu_x \geq 6.$$

Hence $\mathrm{length}_R(R/(xR+I)) = \mathrm{length}_R(R/(xR+\mathfrak{p}))$ and, by the additivity of the length with respect to short exact sequences, $xR + I = xR + \mathfrak{p}$.

Note that $x \notin \mathfrak{p}$, otherwise $\mathfrak{p} \supset xR+I \supset (x, y^3, z^3)$ and $\mathfrak{p} = \mathfrak{m}$, a contradiction. Then $\mathfrak{p} \cap xR = x\mathfrak{p}$. In particular, on tensoring $0 \to \mathfrak{p}/I \to R/I \to R/\mathfrak{p} \to 0$ by R/xR, one obtains the exact sequence $0 \to L/xL \to R/(xR + I) \to R/(xR + \mathfrak{p}) \to 0$, where $L = \mathfrak{p}/I$. Since $xR + I = xR + \mathfrak{p}$, then $L = xL$. By Nakayama's Lemma, $L = 0$ and $I = \mathfrak{p}$.

We conclude that I is a prime ideal of R. Since the aforementioned resolution of R/I is minimal, I is minimally generated by 4 elements, which in particular implies that I is not of linear type, because the minimal number of generators of an ideal of linear type is bounded above by the dimension of the ring (see [6, Proposition 2.4]). □

Proof of Lemma 3. Since the proof of the present result is quite analogous to that of Lemma 2, we skip some details and direct the reader to there. For instance, as before, we can suppose that R is complete. Let φ_1 be the 1×8 matrix defined as $(f_1, \ldots, f_8)$. Let φ_2 and φ_3 be the matrices defined as:

$$\varphi_2 = \begin{pmatrix} y^2t & y^3 & t^2 & zt & z^2 & x^3z & x^4 & -yt^2 & x^2y^2 & x^3y & x^4 & 0 \\ 0 & 0 & 0 & 0 & -y & 0 & 0 & x^3 & 0 & 0 & -t & 0 \\ 0 & 0 & 0 & -y & x & 0 & 0 & 0 & 0 & -t & z & x^3 \\ 0 & 0 & -y & x & 0 & 0 & 0 & 0 & -t & z & 0 & 0 \\ 0 & 0 & x & 0 & 0 & 0 & 0 & -xy & z & 0 & 0 & -y^2 \\ 0 & -z & 0 & 0 & 0 & 0 & -t & -x^3 & 0 & 0 & 0 & -x^2y \\ -z & x & 0 & 0 & 0 & -t & y & 0 & 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 & 0 & y & 0 & z & 0 & 0 & 0 & t \end{pmatrix} \text{ and}$$

$$\varphi_3 = \begin{pmatrix} -t & 0 & 0 & -y & 0 \\ 0 & 0 & 0 & t & -x^2y \\ 0 & 0 & -z & 0 & -yt \\ 0 & -z & t & 0 & 0 \\ -x^3 & t & 0 & 0 & 0 \\ z & 0 & 0 & x & 0 \\ 0 & 0 & 0 & -z & x^3 \\ -y & 0 & 0 & 0 & -t \\ 0 & 0 & x & 0 & y^2 \\ 0 & x & -y & 0 & 0 \\ 0 & -y & 0 & 0 & -x^3 \\ x & 0 & 0 & 0 & z \end{pmatrix}.$$

Since $\varphi_3 \cdot \varphi_2 = 0$ and $\varphi_2 \cdot \varphi_1 = 0$, then

$$0 \to F_3 = R^5 \xrightarrow{\varphi_3} F_2 = R^{12} \xrightarrow{\varphi_2} F_1 = R^8 \xrightarrow{\varphi_1} F_0 = R \to R/I \to 0$$

is a complex of R-modules. To prove its exactness we will use the acyclicity criterion of Buchsbaum and Eisenbud (see, e.g., [3, Theorem 1.4.12]). Set $r_i = \sum_{j=i}^{3}(-1)^{j-i}\text{rank}F_j$, so that $r_1 = 1$, $r_2 = 7$ and $r_3 = 5$.

Note that, since $(f_2, f_5, f_6, x) = (x, y^4, z^3, t^3)$, then $\text{grade}(f_2, f_5, f_6, x) = 4$. By [3, Corollary 1.6.19], f_2, f_5, f_6 is an R-regular sequence in $I = I_1(\varphi_1)$. In particular, $\text{grade}(I) \geq 3$ (and $\text{grade}(I_1(\varphi_1)) \geq r_1 = 1$).

In order to prove $\text{grade}(I_7(\varphi_2)) \geq 2$, we look for minors of φ_2 with pure terms in one of the parameters. For instance, up to sign, the minor $g_1 := y^{10} - 2x^5y^6 + x^{10}y^2 \in I_7(\varphi_2)$, with pure term in y, is obtained from the 7×7 submatrix given by the rows $1,2,3,4,5,7,8$ and the columns $2,3,4,5,6,7,12$. Similarly, we get $g_2 := z^8 - 2x^5z^5 + x^{10}z^2 \in I_7(\varphi_2)$ from the 7×7 submatrix given by the rows $1,3,4,5,6,7,8$ and the columns $1,2,5,8,9,10,11$. Since $(g_1, g_2, x) = (x, y^{10}, z^8)$, then $\text{grade}(g_1, g_2, x) = 3$ and g_1, g_2 is an R-regular sequence in $I_7(\varphi_2)$ and $\text{grade}(I_7(\varphi_2)) \geq 2$.

As before, let us seek for minors of φ_3 with pure terms in one of the parameters. Thus $h_1 = y^6 - x^5y^2 \in I_5(\varphi_3)$ is obtained from the 5×5 submatrix given by the rows $1,8,9,10,11$; $h_2 = z^5 - x^5z^2 \in I_5(\varphi_3)$ is obtained from the rows $3,4,6,7,12$ and, finally, $h_3 = t^5 - x^2y^3t^2 \in I_5(\varphi_3)$ is obtained from the rows $1,2,4,5,8$. Since $(h_1, h_2, h_3, x) = (x, y^6, z^5, t^5)$, then h_1, h_2, h_3 is an R-regular sequence in $I_5(\varphi_3)$ and $\text{grade}(I_5(\varphi_3)) \geq 3$.

We conclude that the complex above is a minimal free resolution of R/I. Therefore I is a perfect ideal of grade 3. In particular, I is height unmixed and so $\mathfrak{m}$ is not an associated prime to I.

Let $\mathfrak{p}$ be any associated prime to I and set $D = R/\mathfrak{p}$. Thus D is a one dimensional complete Noetherian local domain. As before, let V be the integral closure of D in its quotient field K. Then V is a finitely generated D-module and a DVR (see [15, Theorem 4.3.4]). Let ν be the valuation on K corresponding to V. Set $\nu_x = \nu(x)$, $\nu_y = \nu(y)$, $\nu_z = \nu(z)$ and $\nu(t) = \nu_t$. In V, $f_2 = z^3 - x^5 = 0$, $f_5 = t^3 - x^2y^3 = 0$ and $f_6 = y^4 - x^5 = 0$. Applying ν to these equalities, one gets $3\nu_z = 5\nu_x$, $3\nu_t = 2\nu_x + 3\nu_y$ and $4\nu_y = 5\nu_x$. The positive vector $(\nu_x, \nu_y, \nu_z, \nu_t) \in \mathbb{Z}^4$, with smallest $\nu_x \geq 1$, satistying these three conditions is $(12, 15, 29, 23)$. (Clealy, this vector also satisfies all the other conditions arising from $f_i = 0$.) In particular, $\nu_x \geq 12$.

Let $(S, \mathfrak{n}, k)$ be the regular local ring with $S = R/xR$ and $\mathfrak{n} = \mathfrak{m}/xR = (y, z, t)$, by abuse of notation. One has $xR + I = (x, yz, z^3, z^2t, zt^2, t^3, y^4, y^3t, y^2t^2)$ and $R/(xR + I) \cong S/J$, where J is the ideal of S defined as $J = (yz, z^3, z^2t, zt^2, t^3, y^4, y^3t, y^2t^2)$. Since $xR = \text{Ann}_R(S)$, then $\text{length}_R(R/(xR + I)) = \text{length}_S(S/J)$. Since y, z, t is a S-regular sequence, there exists a graded isomorphism $k[Y, Z, T] \cong G(\mathfrak{n})$ of k-algebras, where $G = G(\mathfrak{n})$ stands for the associated graded ring of $\mathfrak{n}$. Let J^* denote the homogeneous ideal of G generated by all the initial forms of elements of J. Proceeding as in the proof of [5, Lemma 2.9], one sees that $\text{length}_S(S/J) = \text{length}_S(G/J^*)$. Let L be the ideal of G generated by the initial forms of $yz, z^3, z^2t, zt^2, t^3, y^4, y^3t, y^2t^2$ in G. By [5, Theorem 2.11] (see also [5, Remark 2.10]), $L = J^*$. Hence, $\text{length}_S(G/J^*) = \text{length}_S(G/L)$. Through the isomorsphim $k[Y, Z, T] \cong G$, one deduces that G/L is isomorphic to the k-vector space spanned by $1, Y, YT, YT^2, Y^2, Y^2T, Y^3, Z, ZT, Z^2, T, T^2$. Therefore $\text{length}_R(R/(xR + I)) = \text{length}_S(S/J) = \text{length}_S(G/J^*) = \text{length}_S(G/L) = 12$.

As in Lemma 2, since $xR + I \subseteq xR + \mathfrak{p}$ and $R/(xR + \mathfrak{p}) \cong (R/\mathfrak{p})/(x \cdot R/\mathfrak{p}) = D/xD$, then

$$12 = \text{length}_R(R/(xR + I)) \geq \text{length}_R(R/(xR + \mathfrak{p})) = \text{length}_D(D/xD).$$

Since $f_6 = 0$, $f_2 = 0$ and $f_5 = 0$ in D, then $y^4, z^3, t^3 \in xD$, and so xD is parameter ideal of the one dimensional Cohen-Macaulay local domain $(D, \mathfrak{m}/\mathfrak{p}, k)$. Since V is a finitely generated Cohen-Macaulay D-module of $\text{rank}_D(V) = 1$, then $\text{length}_D(D/xD) = \text{length}_D(V/xV)$ (see [3, Corollary 4.6.11, (c)]). Moreover $\text{length}_D(V/xV) = [k_V : k] \cdot \text{length}_V(V/xV) = [k_V : k] \cdot \nu(x)$ Therefore, $\text{length}_D(D/xD) = [k_V : k] \cdot \nu_x$. Recapitulating,

$$12 = \text{length}_R(R/(xR + I)) \geq \text{length}_R(R/(xR + \mathfrak{p})) = [k_V : k] \cdot \nu_x \geq 12.$$

Hence $\text{length}_R(R/(xR+I)) = \text{length}_R(R/(xR+\mathfrak{p}))$ and $xR+I = xR+\mathfrak{p}$.

Again, note that $x \notin \mathfrak{p}$, otherwise $\mathfrak{p} \supset xR+I \supset (x, y^4, z^3, t^3)$ and $\mathfrak{p} = \mathfrak{m}$, a contradiction. So $\mathfrak{p} \cap xR = x\mathfrak{p}$. Proceeding as in the end of proof of Lemma 2, we conclude that $I = \mathfrak{p}$ is a prime ideal of R. Set $H := (f_1, \ldots, f_7) \subset I$. Since the aforementioned resolution of R/I is minimal, $f_8 \notin H$ and $H : f_8 \subsetneq R$. However, one can check that $f_8^2 = x^2yztf_1^2 - x^4f_1f_5 - x^2f_2f_7 + tf_5f_6 + x^2f_6f_7$. Thus $f_8^2 \in HI$ and $HI : f_8^2 = R$. Therefore, $H : f_8 \subsetneq HI : f_8^2$ and I is not syzygetic (see [13, Lemma 4.2]). □

In terms of the André-Quillen homology (see [1] and [14]; see also [9], for a new and recent treatment), and as a corollary of the Theorem, we state the following characterization of Noetherian rings of low global dimension. Again, just for the sake of completeness, we include item (A), shown in general in [12].

Corollary. *Let R be a Noetherian ring.*

(A): $\operatorname{gl\,dim}(R) \leq 1 \Leftrightarrow H_2(R, S, \cdot) = 0$, *or* $H_2(R, S, S) = 0$, *for every quotient ring* $S = R/I$.
(B): $\operatorname{gl\,dim}(R) \leq 2 \Leftrightarrow H_2(R, S, \cdot) = 0$ *for every quotient domain* $S = R/I$.
(C): $\operatorname{gl\,dim}(R) \leq 3 \Leftrightarrow H_2(R, S, S) = 0$ *for every quotient domain* $S = R/I$.

Note that, unlike Theorem (B), Corollary (B) could be deduced directly from [5, Corollary 2.7], since the vanishing of the second André-Quillen homology $H_2(R, R/I, \cdot)$, in the Noetherian local case, is equivalent to I being generated by an R-regular sequence. We give here a slightly different approach.

Proof of the Corollary. The equivalence between the leftmost and rightmost conditions in Corollary (A), follows immediately from the isomorphism $H_2(R, R/I, R/I) \cong \ker(\alpha_2)$ and the corresponding equivalence between the leftmost and rightmost conditions in Theorem (A) (see, e.g., [11, Corollaire 3.2]). Similary, Corollary (C) follows immediately from Theorem (C).

It remains to prove the first equivalence of Corollary (A) and the equivalence of Corollary (B). To this end, recall that the vanishing of $H_2(R, R/I, \cdot)$ is also equivalent to I being of linear type and I/I^2 being a flat R/I-module (see [11, Théorème 4.2]). Clearly, this characterization together the corresponding "if" implications in Theorem (A) and (B), show the "if" implications of Corollary (A) and (B), respectively. Finally, as said before, if $\operatorname{gl\,dim}(R) \leq 1$, then every nonzero ideal I of R is locally principal, hence its conormal module I/I^2 is R/I-flat. Similarly, if $\operatorname{gl\,dim}(R) \leq 2$, any nonzero prime ideal I of R is either locally principal, or else maximal, hence in either case, its conormal module I/I^2 is again R/I-flat. □

Closing Remark. If we omit the Noetherian assumption on the ring R, we know that the statement of Theorem (A) is true once we substitute $\operatorname{gl\,dim}(R)$ for $\operatorname{w.dim}(R)$ (cf. [12]). Note that $\operatorname{w.dim}(R)$ can be strictly smaller than $\operatorname{gl\,dim}(R)$, for instance, if R is the ring of all algebraic integers (see, e.g., [16, 1.3 Examples]). Therefore the "if" implication of Theorem (A), without the Noetherian hypothesis, is false. This suggests that one should also replace $\operatorname{gl\,dim}(R)$ with $\operatorname{w.dim}(R)$ in the "if" implications of Theorem (B) and (C).

Just to have a flavour of the ins and outs of the non Noetherian setting, and to start with, we show the following simpler statement. Let R be non-necessarily Noetherian.

(1) *If* $\operatorname{gl\,dim}(R) \leq 2$, *then every prime ideal of* R *is of linear type.*

Indeed, since the linear type condition is local, we can suppose again that $(R, \mathfrak{m})$ is local. Then R is either a Noetherian regular local ring (of $\operatorname{Krull\,dim}(R) \leq 2$), a valuation domain, or a so-called umbrella ring (see [16, 2.2 Theorem], for the definitions and a proof). Note that the first case is precisely solved with the "only if" implication of Theorem (B). If R is a valuation domain, then R is a Bézout domain, hence Prüfer and $\operatorname{w.dim}(R) \leq 1$, so every ideal of R is flat and of linear type (see, e.g., [11, Remarque 2.6]). Whenever $\mathfrak{m}$ is principal or non finitely generated, it is shown that R is a valuation domain. If $\mathfrak{m}$ is finitely generated, but not principal, then $\mathfrak{m}$ is generated by two elements,

a, b, say. In such a case, R is a GCD domain with every prime ideal different from $\mathfrak{m}$ being flat, hence of linear type. As for the maximal ideal, there exists an exact sequence $0 \to R \xrightarrow{\varphi} R^2 \xrightarrow{\psi} \mathfrak{m} \to 0$, with $\varphi(1) = (\alpha, \beta)$, say, $\alpha, \beta \in R$, with $\gcd(\alpha, \beta) = 1$, and $\psi(u, v) = ua + vb$. Since $(b, -a) \in \ker(\psi)$, then there exists $\delta \in R$, such that $a = -\delta\beta$ and $b = \delta\alpha$. Note that $\alpha, \beta \in \mathfrak{m}$, otherwise, if for instance α is invertible, then $\delta = \alpha^{-1}b$ and $a = -\delta\beta = (-\alpha^{-1}\beta)b$ and $\mathfrak{m}$ would be principal. Hence $\mathfrak{m} = (a, b)R = \delta(\alpha, \beta)R \subseteq (\alpha, \beta)R \subseteq \mathfrak{m}$, and $\mathfrak{m} = (\alpha, \beta)$ is generated by the R-regular sequence α, β, in particular, $\mathfrak{m}$ is of linear type.

Note that the argument above proves that the conormal module I/I^2 of every prime ideal I is a flat R/I-module. Hence the following statement is also true. Let R be non-necessarily Noetherian.

(2) *If* $\operatorname{gl\,dim}(R) \leq 2$*, then* $H_2(R, S, \cdot) = 0$*, for every quotient domain* $S = R/I$*.*

We do not know whether one can substitute $\operatorname{gl\,dim}(R) \leq 2$ by $\operatorname{w.dim}(R) \leq 2$ in (1) or (2); neither we know if the converse of (1) or (2) are true, even if we replace $\operatorname{gl\,dim}(R) \leq 2$ by $\operatorname{w.dim}(R) \leq 2$. This could be a line of enquiry in future work.

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References

[1] M. André, Homologie des algèbres commutatives. Die Grundlehren der mathematischen Wissenschaften, Band 206. Springer-Verlag, Berlin-New York, 1974.
[2] H. Bresinsky, On prime ideals with generic zero $x_i = t^{n_i}$. Proc. Amer. Math. Soc. 47 (1975), 329-332.
[3] W. Bruns, J. Herzog, Cohen-Macaulay rings. Cambridge Studies in Advanced Mathematics, 39. Cambridge University Press, Cambridge, 1993.
[4] W. Decker, G.M. Greuel, G. Pfister, H. Schönemann, Singular 4-1-2 — A computer algebra system for polynomial computations. http://www.singular.uni-kl.de (2019).
[5] S. Goto, L. O'Carroll, F. Planas-Vilanova, Sally's question and a conjecture of Shimoda. Nagoya Math. J. 211 (2013), 137-161.
[6] J. Herzog, A. Simis, W.V. Vasconcelos, Koszul homology and blowing-up rings. Commutative algebra (Trento, 1981), pp. 79-169, Lecture Notes in Pure and Appl. Math., 84, Dekker, New York, 1983.
[7] C. Huneke, Hilbert functions and symbolic powers. Michigan Math. J. 34 (1987), no. 2, 293-318.
[8] T.Y. Lam, Lectures on modules and rings. Graduate Texts in Mathematics, 189. Springer-Verlag, New York, 1999.
[9] J. Majadas, A.G. Rodicio, Smoothness, regularity and complete intersection. London Mathematical Society Lecture Note Series, 373. Cambridge University Press, Cambridge, 2010.
[10] H. Matsumura, Commutative ring theory. Translated from the Japanese by M. Reid. Cambridge Studies in Advanced Mathematics, 8. Cambridge University Press, Cambridge, 1986.
[11] F. Planas-Vilanova, Sur l'annulation du deuxième foncteur de (co)homologie d'André-Quillen. Manuscripta Math. 87 (1995), no. 3, 349-357.
[12] F. Planas-Vilanova, Rings of weak dimension one and syzygetic ideals. Proc. Amer. Math. Soc. 124 (1996), no. 10, 3015-3017.
[13] F. Planas-Vilanova, On the module of effective relations of a standard algebra. Math. Proc. Cambridge Philos. Soc. 124 (1998), no. 2, 215-229.
[14] D. Quillen, On the (co-) homology of commutative rings. 1970 Applications of Categorical Algebra (Proc. Sympos. Pure Math., Vol. XVII, New York, 1968) pp. 65-87. Amer. Math. Soc., Providence, R.I.
[15] I. Swanson, C. Huneke, Integral closure of ideals, rings, and modules. London Mathematical Society Lecture Note Series, 336. Cambridge University Press, Cambridge, 2006.
[16] W.V. Vasconcelos, The rings of dimension two. Lecture Notes in Pure and Applied Mathematics, Vol. 22. Marcel Dekker, Inc., New York-Basel, 1976.

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