Reliability Analysis in a Wireless ISP

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Abstract

This brief report investigates different quality parameters to assess the reliability in Wireless Internet Service Providers, WISPs. In our analysis we use a Markov chain approach. We investigate the time to failure, failure probability and reliability. We obtain a closed-form reliability formula for the failure of a system subject to the failure of k devices.

1 Introduction

WISPs have emerged to provide access to Internet services in locations where large telecommunications operators have not deployed their infrastructure. WISPs are made up of small companies with small-scale businesses and low profit margins. Thus, a key issue for WISPs is the number of network devices to be deployed, and the reliability benefits they can bring, in order to decide on the most suitable network deployment. In this report we investigate different quality metrics to evaluate the reliability depending on the number of networking devices. In particular, we assume that the WISP infrastructure is divided in clusters. Inside each cluster there is a set of one or more gateways, which we assume connected through optical fiber, and thus, with high reliability. We thus focus on the number of antennas and gateways inside each cluster to assess reliability. The report is an extension of our previous work [2].

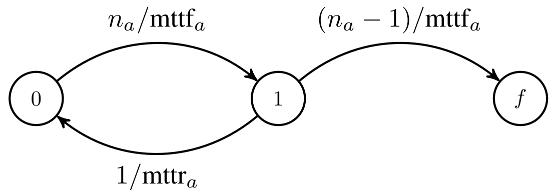
2 Failure time analysis

Assume a cluster with a core of n_a antennas. Let F_k be the RV equal to the time to failure of the core, for a k-edge connected core. We consider that the core fails if k antennas are in failure state for a k-edge connected core. As quality metric we use $\mathrm{E}[F_k]$. Assume that the time to failure of antennas is exponentially distributed with rate $1/\mathrm{mttf}_a$.

The minimum of n_a exponential RVs with rate $1/\mathrm{mttf}_a$ is exponentially distributed with rate n_a/mttf_a. Thus:

$$\mathrm{E}[F_1] = \frac{\mathrm{mttf}_a}{n_a} \tag{1}$$

Assume that the repair time of an antenna is exponentially distributed with rate $1/\mathrm{mttr}_a$. For $k = 2$, $\mathrm{E}[F_2]$ can be computed as the first passage time from state 0 to state f in the continuous time Markov chain:



which is:

$$\mathrm{E}[F_2] = \frac{\mathrm{mttf}_a\,(\mathrm{mttf}_a + \mathrm{mttr}_a\,(n_a - 1))}{\mathrm{mttr}_a\,(n_a - 1)\,n_a} = \frac{\mathrm{mttf}_a + \mathrm{mttr}_a\,(n_a - 1)}{\mathrm{mttr}_a\,(n_a - 1)}\,\mathrm{E}[F_1] \tag{2}$$

Thus, increasing from 1-edge to 2-edge connected core we have a gain:

$$G = \frac{\mathrm{E}[F_2]}{\mathrm{E}[F_1]} = \frac{\mathrm{mttf}_a + \mathrm{mttr}_a\,(n_a - 1)}{\mathrm{mttr}_a\,(n_a - 1)} \tag{3}$$

For instance, for $\mathrm{mttf}_a = 11.4\,\mathrm{y}$, $\mathrm{mttr}_a = 2\,\mathrm{h} \approx 0.00023\,\mathrm{y}$, $n_a = 100$ we have:

$$\mathrm{E}[F_1] = 0.114\,\mathrm{y}$$
$$\mathrm{E}[F_2] \approx 57.2\,\mathrm{y}$$
$$G \approx 500$$

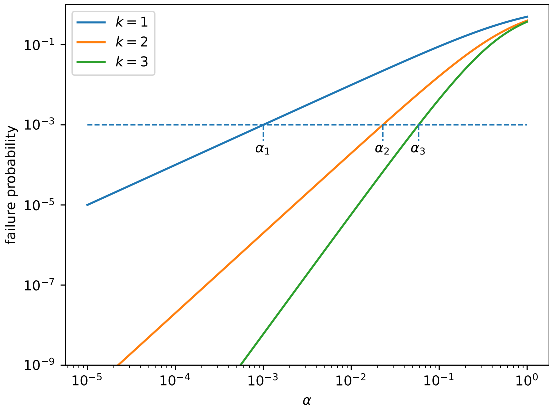
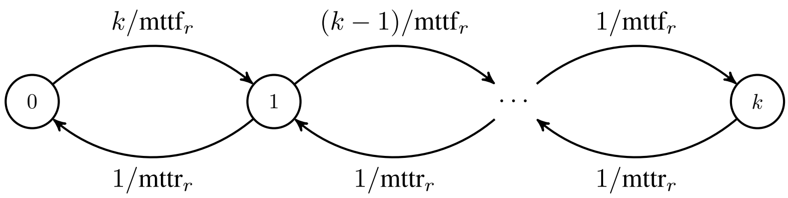


Figure 1: Gateway failure probability

3 Gateway failure probability analysis

Assume now a cluster with k gateways. We assume that the cluster fails when all k gateways fail. Assume that the time to failure of the gateway is exponentially distributed with rate $1/\text{mttf}_r$, and the repair time it is also exponentially distributed with rate $1/\text{mttr}_r$. We have the following continuous Markov chain where the state is the number of failed gateways. We assume also that only one gateway can be simultaneously under repair (there is only one repair team).



As before, the mean time to failure is the first passage time from state 0 to state k. The chain is reversible and the stationary distribution is given by:

$$\pi_i = \frac{\frac{\alpha^i}{(k-i)!}}{\sum_{j=0}^{k} \frac{\alpha^j}{(k-j)!}}, \ i = 0, \cdots, k \tag{4}$$

where $\alpha = \frac{\text{mttr}_r}{\text{mttf}_r}$ and we take $0! = 1$. Thus, the proportion of time the system is in failure state (state k) is given by:

$$\pi_k = \frac{\alpha^k}{\sum_{j=0}^{k} \frac{\alpha^j}{(k-j)!}}. \tag{5}$$

For instance, with $\text{mttf}_r = 22.8$ y and $\text{mttr}_r = 2$ h we have $\alpha = 1/99864$ and:

- For a single gateway ($k = 1$): $\pi_1 = \frac{\alpha}{\sum_{j=0}^{1} \frac{\alpha^j}{(k-j)!}} \approx 10^{-5}$
- For 2 gateways ($k = 2$): $\pi_2 = \frac{\alpha^2}{\sum_{j=0}^{2} \frac{\alpha^j}{(k-j)!}} \approx 2 \times 10^{-10}$

For a failure probability of $\pi_k = 10^{-3}$ we have the values of α_k given in table 1 below (see Fig.3). Thus, increasing from $k = 1$ to $k = 2$, and assuming the same repair time, we reduce the required time to failure in $\alpha_2/\alpha_1 = (\text{mttf}_r)_1/(\text{mttf}_r)_2 \approx 23$ times.

k	α_k
1	1/999
2	0.0229
3	0.0584

Table 1: Values of α_k for a failure probability $\pi_k = 10^{-3}$

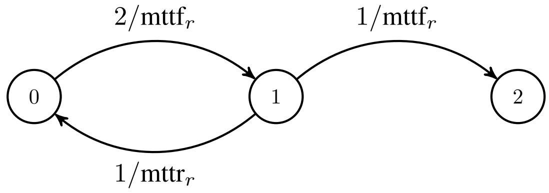
4 Gateway reliability analysis

Let X be a random variable equal to the failure time of a system. The reliability of the system, $R(t)$, is defined as $R(t) = P\{X > t\}$. In other words, it is the probability that the system is working at time t. We shall use the same assumptions as in section 3.

Clearly, for $k=1$ gateways we have:

$$R_1(t) = \mathrm{e}^{-t/\mathrm{mttf}_r},\ t \geq 0 \tag{6}$$

For $k=2$ we have $R_2(t) = 1 - \pi_2(t)$, where $\pi_2(t)$ is the probability to reach state 2 at time t in the chain below.



The infinitesimal generator of the chain is

$$Q = \begin{bmatrix} -2\,f & 2\,f & 0 \\ r & -(r+f) & f \\ 0 & 0 & 0 \end{bmatrix} \tag{7}$$

where we define to simplify the notation $f = 1/\mathrm{mttf}_r$ and $r = 1/\mathrm{mttr}_r$. And we have

$$R_2(t) = \frac{\lambda_1}{\lambda_1 - \lambda_2}\,\mathrm{e}^{\lambda_2\,t} - \frac{\lambda_2}{\lambda_1 - \lambda_2}\,\mathrm{e}^{\lambda_1\,t},\ t \geq 0 \tag{8}$$

where λ_1 and λ_2 are the nonzero eigenvalues of Q:

$$\lambda_1 = \frac{-1}{2}\left(3\,f + r + \sqrt{(3\,f+r)^2 - 8\,f^2}\right)$$
$$\lambda_2 = \frac{-1}{2}\left(3\,f + r - \sqrt{(3\,f+r)^2 - 8\,f^2}\right)$$

Assuming a worst case $f \gg r$, that is, $\mathrm{mttf}_r \ll \mathrm{mttr}_r$ (no taking into account repairs) we can approximate:

$$\lambda_1 \approx -2\,f$$
$$\lambda_2 \approx -f$$

and

$$R_2(t) \approx \tilde{R}_2(t) = 2\,\mathrm{e}^{-f\,t} - \mathrm{e}^{-2f\,t},\ t \geq 0 \tag{9}$$

If we want a 99% reliability in one year, solving for $R_1(1) = 0.99$ and $\tilde{R}_2(1) = 0.99$ in equations (10) and (9), respectively, we get:

$$R_1(1) = 0.99 \Rightarrow \mathrm{mttf}_r \approx 99 \text{ years}$$
$$\tilde{R}_2(1) = 0.99 \Rightarrow \mathrm{mttf}_r \approx 9.5 \text{ years}$$

Thus, increasing the number of gateways from 1 to 2 we reduce the required mttf approximately by a factor of 10. For the sake of comparison, using (12) with $\mathrm{mttf}_r = 10\,\mathrm{mttr}_r$ (i.e. $r = 10\,f$) we get a further reduction of mttf_r to:

$$\mathrm{mttf}_r = 10\,\mathrm{mttr}_r : R_2(1) = 0.99 \Rightarrow \mathrm{mttf}_r \approx 7.85 \text{ years}$$

And if we fix $\mathrm{mttr}_r = 1$ day$= 1/365$ years:

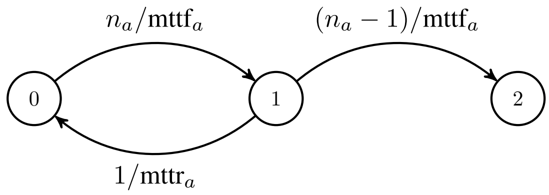
$$\mathrm{mttr}_r = 1 \text{ day} : R_2(1) = 0.99 \Rightarrow \mathrm{mttf}_r \approx 0.73 \text{ years (268 days)}$$

5 Antennas reliability analysis

Assume as in section 2 that we have n_a antennas in a k-edge connected core, such that the core fails when k antennas fail. We can proceed as in section 4 and now for $k=1$ we have:

$$R_1(t) = \mathrm{e}^{-n_a\,t/\mathrm{mttf}_a},\ t \geq 0 \tag{10}$$

For $k=2$ we have the Markov chain:



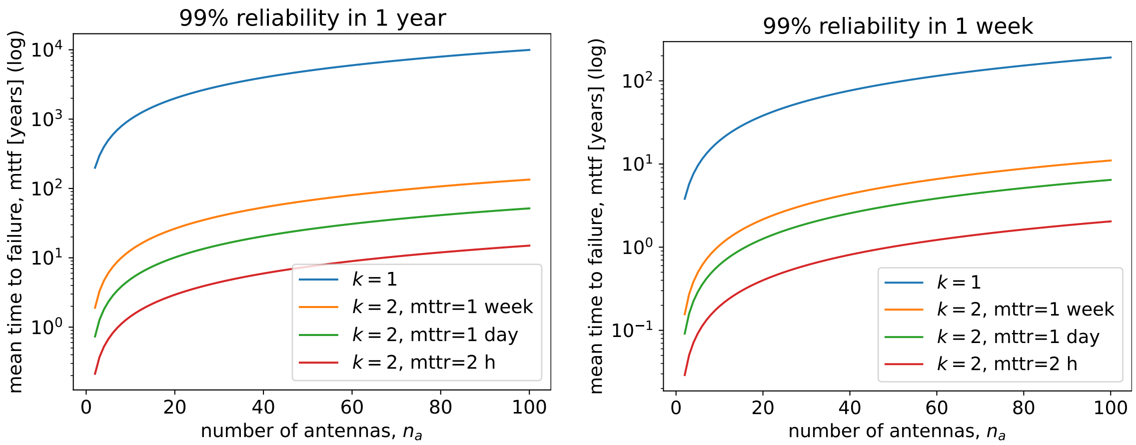


Figure 2: Mean time to failure required to have 99% reliability in 1 year (left) and 1 week (right). Comparison for $k = 1$ and $k = 2$ edge connected cores.

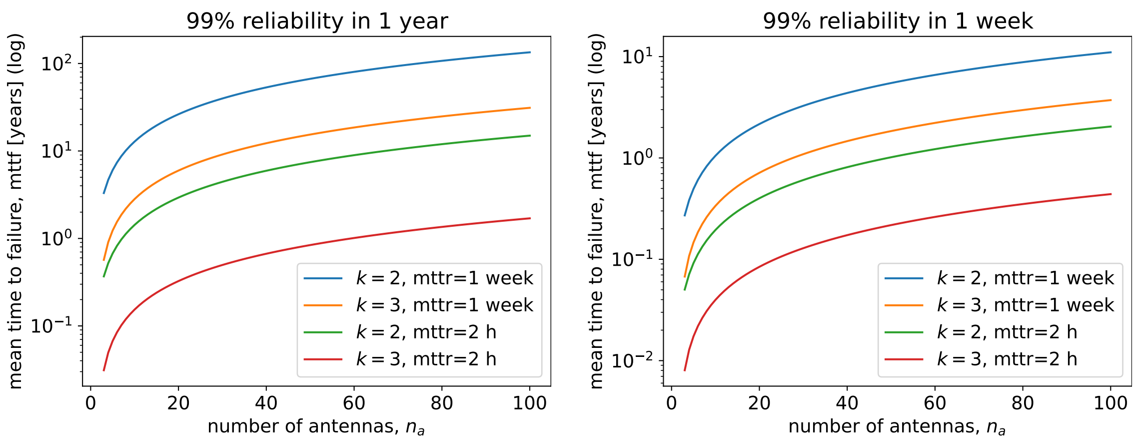


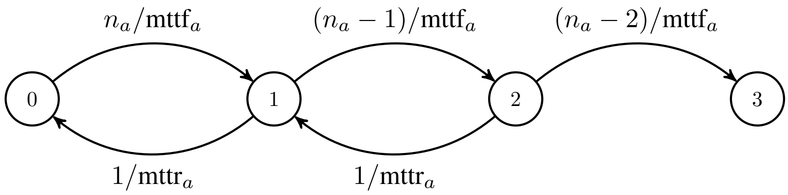
Figure 3: Mean time to failure required to have 99% reliability in 1 year (left) and 1 week (right). Comparison for $k = 2$ and $k = 3$ edge connected cores.

with infinitesimal generator having the nonzero eigenvalues:

$$\lambda_1 = \frac{-1}{2}\left((2\,n_a - 1)\,f + r + \sqrt{((2\,n_a - 1)\,f + r)^2 - 4\,n_a\,(n_a - 1)\,f^2}\right)$$
$$\lambda_2 = \frac{-1}{2}\left((2\,n_a - 1)\,f + r - \sqrt{((2\,n_a - 1)\,f + r)^2 - 4\,n_a\,(n_a - 1)\,f^2}\right)$$

where $f = 1/\text{mttf}_a$, $r = 1/\text{mttr}_a$ and reliability given by (12). Fig. 2 shows the mttf_a that would be required for the antennas varying the number of antennas, for a 99% reliability in 1 year and 1 week.

For $k = 3$ we have the Markov chain:



with infinitesimal generator:

$$Q = \begin{bmatrix} -n_a\,f & n_a\,f & 0 & 0 \\ r & -(r + (n_a - 1)\,f) & (n_a - 1)\,f & 0 \\ 0 & r & -(r + (n_a - 2)\,f) & (n_a - 2)\,f \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{11}$$

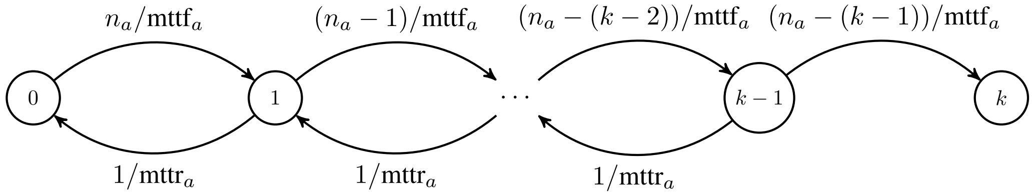
Let $\lambda_1, \lambda_2, \lambda_3$ be the nonzero eigenvalues of Q. We have:

$$R_3(t) = \frac{\lambda_2\,\lambda_3}{(\lambda_1 - \lambda_2)\,(\lambda_1 - \lambda_3)}\,\mathrm{e}^{\lambda_1\,t} + \frac{\lambda_1\,\lambda_3}{(\lambda_2 - \lambda_1)\,(\lambda_2 - \lambda_3)}\,\mathrm{e}^{\lambda_2\,t} + \frac{\lambda_1\,\lambda_2}{(\lambda_3 - \lambda_1)\,(\lambda_3 - \lambda_2)}\,\mathrm{e}^{\lambda_3\,t},\ t \geq 0 \tag{12}$$

Fig. 3 compares the mean time to failure for $k = 2$ and $k = 3$ edge connected cores.

5.1 Generalization

Assume a k-edge connected core (it fails when k devices fail). Failure can be modeled with the absorbing Markov chain:



with infinitesimal generator:

$$Q = \begin{bmatrix} -n_a\, f & n_a\, f & \cdots & 0 & 0 \\ r & -(r + (n_a - 1)\, f) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & \cdots & -(r + (n_a - (k-1))\, f) & (n_a - (k-1))\, f \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \tag{13}$$

where $f = 1/\text{mttf}_a$ and $r = 1/\text{mttr}_a$. Let $\lambda_i,\ i = 1, \cdots k$ be the nonzero eigenvalues of the infinitesimal generator. These are the eigenvalues of the submatrix obtained removing the last row and column of Q. This submatrix is similar to a symmetric tridiagonal matrix with nonzero elements [4]. Thus, its eigenvalues are simple and we can guess the probability of reaching state k at time t by [1]:

$$\pi_k(t) = 1 + \sum_{i=1}^{k} a_i\, \mathrm{e}^{\lambda_i\, t}. \tag{14}$$

Imposing the boundary conditions $\pi_k(0) = 0,\ \left.\frac{d^j \pi_k(t)}{dt^j}\right|_{t=0} = 0,\ j = 1, \cdots k-1$, we have that the unknown coefficients a_i in (14) can be obtained solving:

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ \vdots & & & \\ \lambda_1^{k-1} & \lambda_2^{k-1} & \cdots & \lambda_k^{k-1} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{15}$$

Thus, we have that the coefficients are given by the first column of the inverse of the Vandermonde matrix of (15) with opposite sign [3]. Therefore:

$$R_k(t) = 1 - \pi_k(t) = \sum_{i=1}^{k} -a_i\, \mathrm{e}^{\lambda_i\, t} = (-1)^{k-1} \sum_{i=1}^{k} \frac{\Pi_{j \neq i} \lambda_j}{\Pi_{j \neq i}(\lambda_i - \lambda_j)}\, \mathrm{e}^{\lambda_i\, t},\ t \geq 0,\ k \geq 1. \tag{16}$$

References

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