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On the Felsenthal Power Index

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Abstract
The paper that introduces the Felsenthal index is titled: 'A well-behaved index of a priori *P*-Power for simple n-person games.' In 2016, Felsenthal introduced his index for simple games. His definition does not base on the axiomatic approach. Then, Felsenthal regarded some properties and proved that his index satisfies a list of six reasonable and compelling postulates. Three of the properties that he regarded refer to the weighted games, but this fact does not reduce the definition of his index to weighted games. He proves that none of seven well-known efficient power indices proposed to date satisfies the list of postulates, indicating for each of them which of the six postulates violate. In this paper we extend some of his postulates, originally defined for weighted games, to simple games. The main objective of the article is to answer three open questions motivated in his article. In particular, we prove that his index may not be the unique one fulfilling the six proposed postulates, provide an axiomatic characterization for his index and, propose an impossibility result, which is obtained by adding a new postulate to a sublist of the postulates he considered. We also remark the existence of some alternative compelling postulates which are not satisfied for the index.

Keywords Distribution of an asset · Efficient power indices · Decision and negotiation · Fair distributions among agents

1 Introduction

Felsenthal and Machover (1998, ch. 6), introduce the notion of *P*-power index, to estimate the expected share in the fixed prize of the members in an *n*-person simple game. By its nature the notion of an a priori *P*-power index implies the

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efficiency property, that stipulates that the collective wealth generated by cooperating is divided among its members without nothing wasted. These study situations are ideal for deciding a fair distribution of a divisible asset with a process of negotiation among the players. Felsenthal and Machover (1995 and 1998, ch. 7), as well as Felsenthal et al., (1998), listed six postulates that considered compelling that a reasonable P-power index should satisfy.

Many examples of P-power indices exist in the literature, just to mention some few of them we refer to the Shapley-Shubik and the relative Banzhaf P-power indices (see Shapley and Shubik 1954 and Banzhaf 1965 respectively). Felsenthal (2016) argues that: '*none of the various P-power indices proposed to date for estimating the expected share in the fixed prize of the members in an n-person simple game satisfies the list of six reasonable and compelling postulates*'. In particular, he proves in Felsenthal (2016) that none of seven known P-power indices satisfy the six postulates, indicating for each of them which of the six postulates violate. These seven indices are: the Shapley-Shubik index (Shapley and Shubik 1954, see also Felsenthal and Machover 1996 and Bernardi and Freixas 2018) which is the restriction to simple games of the Shapley value (Shapley 1953) for cooperative games, the relative Banzhaf index (Banzhaf 1965), the Deegan-Packel index (Deegan and Packel 1978 and Deegan and Packel 1982), the Johnston index (Johnston 1978), the Public Good index (Holler 1978 and Holler 1982), the Shift index (Alonso-Meijide and Freixas 2010 and Alonso-Meijide et al. 2012) and the Minimum sum-representation index (Freixas and Kaniovsky 2014). Any efficient power index can be seen as a P-power index according to the definition by Felsenthal and Machover of P-power, but some efficient power indices can also be seen as I-power indices, which is the case of the Shapley-Shubik (Einy and Haimanko 2011) or the Public Good index (Holler 1978).

Moreover, Felsenthal leaves in Felsenthal (2016) several open questions, which are the main motivation of this work:

- Is the Felsenthal P-power index the unique index satisfying the six reasonable and compelling postulates?
- The previous question indirectly motivates the study of an axiomatic characterization of his index by means of some postulates.
- He also faced the possibility of being reduced to an *impossibility theorem* for P-power indices, since an additional postulate may be suggested which the Felsenthal index does not satisfy. In which case we would obtain an impossibility theorem for P-power indices.

In the sequel, $N = \{1, 2, \dots, n\}$ will denote a fixed but otherwise arbitrary finite set of *players*, called the *grand coalition* or the *assembly* and any subset of N is a *coalition*. The pair (N, v) is a *cooperative game* in characteristic form if $v : 2^N \to \mathbb{R}$ is a function that assigns to every coalition $S \subseteq N$ an attainable payoff $v(S)$ such that $v(\emptyset) = 0$.

For every player set N we denote by $\mathcal{G}^N$ the class of all cooperative games on N. The space of cooperative games is a multi-dimensional Euclidean vector space of dimension $2^n - 1$. For every $S \subseteq N$ the *unanimity game* u_S is defined as

$$u_S(T) = \begin{cases} 1 & \text{if } S \subseteq T \subseteq N \\ 0 & \text{otherwise} \end{cases}$$

The set of unanimity games forms a basis of the game space $\mathcal{G}^N$, i.e.,

$$v = \sum_{S \neq \emptyset} \Delta_v(S) u_S$$

where each scalar is the *Harsanyi dividend* (Harsanyi 1963) of coalition S in game v:

$$\Delta_v(S) = \sum_{T \subseteq S} (-1)^{s-t} v(T) \tag{1}$$

in which $s = |S|$ and $t = |T|$.

A cooperative game v (in N, omitted hereafter) is a *simple game* if: (a) $v(S) = 0$ or 1 for all S, (b) is monotonic, i.e., $v(S) \leq v(T)$ whenever $S \subset T$, and (c) $v(N) = 1$. Let $\mathcal{S}^N$ be the class of all simple games on N.

By monotonicity either the family of *winning* coalitions $\mathcal{W} = \mathcal{W}(v) = \{S \subseteq N : v(S) = 1\}$ or the subfamily of *minimal* winning coalitions $\mathcal{W}^m = \mathcal{W}^m(v) = \{S \in \mathcal{W} : T \subset S \Rightarrow T \notin \mathcal{W}\}$ determines the game. In this paper we also deal with the set of (minimal) *winning coalitions of the least size*, $\mathcal{W}^{ls}(v) = \{S \in \mathcal{W} : |T| < |S| \Rightarrow T \notin \mathcal{W}\}$. Analogously, if $i \in N$ and $v \in \mathcal{S}^N$, we consider: $\mathcal{W}_i(v) = \{S \in \mathcal{W}(v) : i \in S\}$, $\mathcal{W}_i^m(v) = \{S \in \mathcal{W}^m(v) : i \in S\}$ and $\mathcal{W}_i^{ls}(v) = \{S \in \mathcal{W}^{ls}(v) : i \in S\}$.

We consider two operations of simple games. The *join* or *disjunction* of $v \in \mathcal{S}^N$ and $w \in \mathcal{S}^N$, is the simple game $v \vee w$ given by

$$(v \vee w)(S) = \begin{cases} 1 & \text{if either } v(S) = 1 \text{ or } w(S) = 1 \\ 0 & \text{if } v(S) = w(S) = 0 \end{cases}$$

The *meet* or *conjunction* of $v \in \mathcal{S}^N$ and $w \in \mathcal{S}^N$, is the simple game $v \wedge w$ given by

$$(v \wedge w)(S) = \begin{cases} 1 & \text{if } v(S) = w(S) = 1 \\ 0 & \text{if either } v(S) = 0 \text{ or } w(S) = 0 \end{cases}$$

Some special type of players in a simple game are the following. A player who does not belong to any minimal winning coalition is a *null* player. A *dictator* is a player who constitute the sole minimal winning coalition, so that the remaining players are nulls. A player who belongs to every winning coalition is a *vetoer* or a *blocker*, observe in this case that no coalition can win without her. Thus, a dictator is the most radical form of being a vetoer. We denote $V(w)$ the set of vetoers of the game w, observe that $V(w) = \bigcap_{S \in \mathcal{W}(w)} S$.

We define some parameters for every simple game: let d be the number of null players, v be the number of vetoers, $p = |\mathcal{W}^{ls}(v)|$ be the number of winning coalitions of the least size, and $k = |S|$ if $S \in \mathcal{W}^{ls}(v)$, be the number of players of the winning coalitions of the least size. Note that these parameters satisfy: $d \geq 0$, $v \geq 0$, $p \geq 1$, and $k \geq 1$.

Loosely speaking, a power index for $\mathcal{S}^N$ is a function g which assigns to a simple game $v \in \mathcal{S}^N$ a vector $g(v) \in \mathbb{R}^n$; where each component $g_i(v)$ is a measure for the i*th* player in the simple game v according to g. As we deal with a priori *P-power* indices, we regard power indices as measures for estimating the expected payoff that a player can expect in an n-person simple game before playing the game, rather than the probable final result when the game is actually played. The notion of P-power additionally assumes the rational condition of *efficiency*. Formally,

Efficiency: A power index $g : \mathcal{S}^N \to \mathbb{R}^n$ is efficient if for every $v \in \mathcal{S}^N$:

$$\sum_{i=1}^{n} g_i(v) = v(N) \tag{2}$$

Since $v(N)$ represents the collective wealth the players can obtain by themselves, the index g should satisfy the condition in (2), which formulates the requirement that players cannot be divided more than assembly N is able to generate and none of the total amount obtained is wasted.

A simple game is a *weighted game* if there exist natural integers $w_1, \dots, w_n$ such that every coalition S, $S \in \mathcal{W}$ *if and only if* the sum of the w_i's, $i \in S$, is at least equal to some preset quota q. The number w_i is interpreted as the number of votes that the player i owns, and q is the least total number of votes necessary to pass a decision. Such representation for v is represented by $[q; w_1, \dots, w_n]$ and $w(S)$ stands for $\sum_{i \in S} w_i$. For $n \geq 4$ there are simple games which are not weighted, see (Muroga et al. 1962) and (Carreras and Freixas 1996).

In Sect. 2 we list six postulates which were described by Felsenthal (2016) as compelling for a reasonable P-power index. Moreover, we adapt some of his axioms defined for weighted games to simple games in Sect. 2.2. In Sect. 3 the Felsenthal P-power index for simple games is introduced. In Sect. 4 we show that it is not the unique index satisfying the stated postulates in Sect. 2. In Sect. 5, we propose an axiomatic characterization of the Felsenthal index. Section 6 presents an impossibility result for power indices in simple games and shows some weaknesses of the Felsenthal index. Section 7 points out some possible lines for future research.

2 Postulates for a Reasonable A Priori *P*-Power Index According to Felsenthal and Machover

We start this section by describing the original postulates by Felsenthal and Machover (1998) and refer to the Felsenthal paper (Felsenthal 2016) for arguments on the compellingness of them. We assume, de facto, the property of efficiency for any power index.

2.1 Original Postulates by Felsenthal and Machover

The most of these postulates concern the class of all simple games, but the second and the third refer only to weighted simple games.

2.1.1 Null, Ordinary Voter, Vetoer and Dictator

A reasonable P-power index should award no power (0) to a *null*, and it should award the entire power (1) to a *dictator*. The P-power of a vetoer ought to be equal to or larger than that of an *ordinary player*, –i.e., a player, who belongs to some, but not all minimal winning coalitions.

2.1.2 Monotonicity

The postulate of *monotonicity* requires that in a representation of a weighted game, $[q;w_1, \dots, w_n]$ the powers of any two players must not be in reverse order to their weights: $w_b > w_a \Rightarrow g_b(v) \geq g_a(v)$. Note that by monotonicity and the previous property the positivity property for weighted games is deduced, i.e., $g_a(v) \geq 0$ for all $a \in N$.

2.1.3 Donation

Consider two representations of weighted games, u and v, with the same players, the same quota, the same sum of weights, and with the same weights for all players except for players a and b. Thus, the weights of the two representations differ only in one respect: the weight of player a in v is greater by some amount $\epsilon > 0$ than in u, whereas the weight of voter b in v is smaller by the same amount ϵ than b's weight in u. The *donation* postulate stipulates that the P-power index g should satisfy $g_a(v) \geq g_a(u)$ (or, equivalently, $g_b(v) \leq g_b(u)$).

Felsenthal includes, in his list of axioms, a weaker postulate than the donation postulate, which he calls the 'annexation postulate'. The dependency between the two postulates makes irrelevant the inclusion of the weaker one, so we do not include its definition.

2.1.4 Blocker's Share

An index of relative voting power satisfies the *blocker's share* postulate if whenever a is a vetoer (or a blocker) in a simple game v and the least size of a winning coalition of v is k, then $g_a(v) \geq \frac{1}{k}$ whenever a is a vetoer.

2.1.5 Added Blocker

Let u be a simple game with player set N. Let a be a new voter, not a member of N. We say that the simple game v is obtained from u by *adding a as a vetoer* if the assembly of v is $N \cup \{a\}$; *and the winning coalitions of v are obtained from those of u by adding a to each of them*. Thus a winning coalition of v is of the form $S \cup \{a\}$, where S is a winning coalition of u. Clearly, a is a vetoer in v. The added blocker postulate stipulates that $g_b(v) \cdot g_a(u) = g_a(v) \cdot g_b(u)$.

2.2 Extension of Monotonicity and Donation to Simple Games

The postulates of monotonicity and donation (as well as annexation) were originally stated for the subclass of weighted games. Here, we propose two postulates defined for all simple games, which become equivalent to the postulates of monotonicity and donation for weighted games.

2.2.1 Desirability

Let v be a simple game with assembly N and a and b be two players such that $v(S \cup \{a\}) \geq v(S \cup \{b\})$ for all $S \subseteq N \backslash \{a, b\}$. Then the postulate stipulates that a must have at least as power as b, i.e., $g_a(v) \geq g_b(v)$. The intuition under this postulate is that the remaining players in N prefer a to b as a coalition partner, because the over-all gain they obtain by joining player a is greater than or equal to the gain they obtain by joining player b, so it is reasonable to expect that the power of a should be at least the same as the power of b. Note that by desirability and the first property in section 2.1.1, the positivity property for simple games is deduced, i.e., $g_a(v) \geq 0$ for all $a \in N$.

2.2.2 External Monotonicity

Let a and b in N and u and v two simple games on N such that for all $S \subseteq N \setminus \{a, b\}$ it holds: $v(S) = u(S)$, $v(S \cup \{a\}) \geq u(S \cup \{a\})$, and $v(S \cup \{b\}) \leq u(S \cup \{b\})$. Then player a must not have less voting power in v than in u, i.e., $g_a(v) \geq g_a(u)$ (or, equivalently, $g_b(v) \leq g_b(u)$). In words, the relative situation of a with respect to all the other players in v is better than it is in u. Thus, it is reasonable to expect that the power index of a in v should be at least the same as in u (and conversely for b).

The next two lemmas justify the generality of these two postulates for simple games with respect to their counterparts for weighted games.

Lemma 2.1 *Desirability implies Monotonicity.*

Proof Let v be a weighted game with a weighted representation $[q; w_1, \dots, w_n]$ such that $w_a > w_b$ then $w(S) + w_a > w(S) + w_b$ for all $S \subseteq N \setminus \{a, b\}$, which means $v(S \cup \{a\}) \geq v(S \cup \{b\})$ for all $S \subseteq N \setminus \{a, b\}$, which implies that a has at least as much power as b. □

Lemma 2.2 *External monotonicity implies Donation.*

Proof If the weighted game v is obtained from the weighted game u represented by $[q;w_1,\ldots,w_n]$ in which player b donates some (let's say $\epsilon > 0$) of her weight to player a and the remaining weights, as well as the quota, are kept the same. Then, clearly $w(S)+w_a > w(S)+w_a-\epsilon$, and $w(S)+w_b < w(S)+w_b+\epsilon$. The expressions on the left hand side of the two inequalities correspond to weights in game v and the expressions on the right hand side correspond to the weights in game u. Then for all $S \subseteq N \setminus \{a,b\}$, $v(S\cup\{b\}) \le u(S\cup\{b\})$, and $v(S)=u(S)$ and then a's voting power in v should not be smaller than in u. □

Note that if lemma 2.2 does not hold, do we have the paradox of redistribution.

3 The Felsenthal Power Index for Simple Games

In this section we introduce the Felsenthal power index for simple games, it constitutes a slight modification of the probabilistic model that defines the Deegan–Packel index (Deegan and Packel 1978). They differ in the fact that the Felsenthal index uses 'winning coalitions of the least size' instead of 'minimal winning coalitions'. See (Deegan and Packel 1978) and (Felsenthal 2016) for more details.

The Felsenthal power index is obtained as follows. Suppose that the simple game v has $p=|\mathcal{W}^{ls}(v)|$ coalitions whose equal size is k. Then,

$$F_i(v)=\frac{1}{p}\cdot\frac{|\mathcal{W}_i^{ls}(v)|}{k} \tag{3}$$

Hence, the Felsenthal power index distributes power only among players that belong to some winning coalition of the least minimal size.

Example 3.1 (a) Let v be the simple game of four players defined by $\mathcal{W}^m(v)=\{\{1,2\},\{1,3\},\{2,3,4\}\}$, then $\mathcal{W}^{ls}(v)=\{\{1,2\},\{1,3\}\}$, so that $p=k=2$ and $v=d=0$. According to (3) the Felsenthal index is $F(v)=(\frac{1}{2},\frac{1}{4},\frac{1}{4},0)$. Note that the index assigns 0 to the fourth player, although she is not null.

(b) Let v be the simple game of four players defined by $\mathcal{W}^m(v)=\{\{1,2\},\{1,3,4\}\}$, then $\mathcal{W}^{ls}(v)=\{\{1,2\}\}$, so that $k=2$, $p=v=1$ and $d=0$. According to (3) the Felsenthal index is $F(v)=(\frac{1}{2},\frac{1}{2},0,0)$. Note that the index assigns the same payoff to players 1 and 2 although 1 is a vetoer and 2 is not, and, it assigns 0 to players 3 and 4, although they are not nulls.

It is worth noting that a close, but slightly different, concept to the Felsenthal index was proposed by Riker (see, 1962 p. 32 and 1982) for the more restrictive set of weighted simple games, Riker's idea follows the well-known 'Size Principle' introduced by himself.

4 On the Uniqueness of the Felsenthal Power Index

Felsenthal (2016) proves that his index satisfies the five (six if annexation is included) postulates in Sect. 2.1. Moreover he posed the question on whether his index is the unique P-power index that satisfies these axioms. The following result proves that his index is not the unique one that satisfies the list of postulates in section 2.1. The index we define below is as egalitarian as possible, since all players not being dictators, blockers or nulls receive the same payoff.

Proposition 4.1 *For every weighted game u, let k be the minimum size of a winning coalition, v be the number of vetoers in u and d be the number of nulls in u. The following P-power index*

$$\phi_i(u) = \begin{cases} 1 & \text{if } i \text{ is a dictator} \\ 0 & \text{if } i \text{ is a null player} \\ \frac{1}{k} & \text{if } i \text{ is a vetoer} \\ \frac{k-v}{k(n-v-d)} & \text{otherwise} \end{cases} \tag{4}$$

satisfies all the postulates in section 2.1.

Proof We first claim that this index is *efficient*. Indeed, if the game has a dictator the remaining players are nulls and the index is clearly efficient, if the game has v vetoers and d nulls, then

$$\sum_{i=1}^{n} \phi_i(v) = \frac{v}{k} + \frac{(k-v)(n-v-d)}{k(n-v-d)} = 1,$$

if the game has not vetoers, then

$$\sum_{i=1}^{n} \phi_i(v) = \frac{k(n-d)}{k(n-d)} = 1.$$

Null, ordinary voter, vetoer and dictator. If u has a dictator then, the other players are nulls and the property is satisfied. If u has vetoers and the rest of players are nulls, then $k = v$ and the property is satisfied. If u has vetoers, nulls and players non-being vetoers or nulls, then $k > v > 0$ and the property follows from the next equivalences:

$$1 \geq \frac{k-v}{(n-v-d)k} \iff n-v-d \geq k-v \iff n-d \geq k$$

The last inequality is true because all players in a winning coalition of size k are not null.

Monotonicity. Assume that u is a weighted game with a weighted representation such that $w_i > w_j$. If j is null then $\phi_j(u) = 0$, so that $\phi_i(u) \geq \phi_j(u)$. If j is neither a null player nor a vetoer, then i is not null, hence $\phi_i(u) \geq \phi_j(u)$. If j is a vetoer, then i is a vetoer as well, so that $\phi_i(u) = \phi_j(u)$.

Donation. Let u be a weighted game. Assume that i is the only player whose weight increases when converting the game u into v and j is the only player who loses the part of the weight gained by i. It is clear that: if i is null in u then $0 = \phi_i(u) \leq \phi_i(v)$; if i is neither a vetoer nor a null player in u then i is neither a vetoer nor a null player in v, thus $\phi_i(u) \leq \phi_i(v)$; if i is a vetoer in u then i is a vetoer or a dictator in v, thus $\frac{1}{k} = \phi_i(u) \leq \phi_i(v)$; and if i is a dictator in u then i is also a dictator in v, thus $1 = \phi_i(u) = \phi_i(v)$.

Blocker's share. $\phi_i(u) = \frac{1}{k}$ if i is a vetoer in u.

Added blocker. Let $(N \cup \{a\}, u)$ be the simple game obtained after the addition of the vetoer in (N, v). Then expression (4) becomes:

$$\phi_i(u) = \begin{cases} 1 & \text{if } i \text{ is a dictator} \\ 0 & \text{if } i \text{ is a null player} \\ \frac{1}{k+1} & \text{if } i \text{ is a vetoer} \\ \frac{k-v}{(k+1)\cdot(n-v-d)} & \text{otherwise} \end{cases}$$

Let b and c be two players in N. If both are nulls in v, then both receive zero for the index in game u. If both are vetoers in v, then both receive $1/(k+1)$ in u and the proportions remain unalterable. If b is a vetoer and c is neither a vetoer nor a null player, then the proportion of power between the two players in u remain unalterable with respect to the proportion of power in v. □

Observe that the P-power index ϕ applied to Example 3.1-(a) is $\phi(v) = (1/4, 1/4, 1/4, 1/4)$ since the game has neither vetoers nor nulls. The P-power index ϕ for the game defined in Example 3.1-(b) is $\phi(v) = (1/2, 1/6, 1/6, 1/6)$ since the first player is a vetoer and the game has not null players, so players 2, 3 and 4 receive the same payoff according to ϕ.

For those weighted games for which $F \neq \phi$ the Felsenthal index F is different from the P-power index g_α, as defined in Corollary 4.2, which also satisfies the axioms in section 2.1.

Corollary 4.2 *The power index* $g_\alpha = \alpha F + (1-\alpha)\phi$ *for* $\alpha \in [0, 1]$ *satisfies all the postulates in section* 2.1.

We conclude this section by remarking that proposition 4.1 and corollary 4.2 may be easily extended to simple games (Felsenthal just considered weighted games) since desirability and external monotonicity are satisfied for both F and ϕ.

5 An Axiomatic Characterization of the Felsenthal Index

In this section we propose an axiomatization of the Felsenthal index for simple games, which is very close to the first axiomatization of the Shapley value (Shapley 1953) that guarantees uniqueness in the subdomain of simple games, see Dubey (1975). The only difference concerns the transfer postulate so that the power is transferred from one game to another in a different way. In order to introduce the new transfer axiom we use the following notations for every simple game v:

$$k_v = |S| \quad \text{if} \quad S \in \mathcal{W}^{ls}(v),$$
$$p_v = |\mathcal{W}^{ls}(v)|.$$

Transfer axiom for coalitions of minimum size: is defined, for every pair of simple games v and w such that $\mathcal{W}^{ls}(v) \cap \mathcal{W}^{ls}(w) = \emptyset$, i.e., $\min\{k_v, k_w\} < k_{v\wedge w}$.

$$\psi(v \vee w) = \begin{cases} \psi(v) & \text{if } \; k_v < k_w \\ \psi(w) & \text{if } \; k_w < k_v \\ \frac{p_v}{p_v+p_w}\psi(v) + \frac{p_w}{p_v+p_w}\psi(w) & \text{if } \; k_v = k_w \end{cases}$$

In words, the transfer of power is distributed proportionally to membership in winning coalitions of the least size.

The other postulate we use that have not been mentioned before is the property of *symmetry* already used by Shapley (1953) in the axiomatization of his value for cooperative games.

Symmetry: A P-power index $g : \mathcal{S}^N \to \mathbb{R}^n$ is symmetric if for every permutation $\rho : N \to N$:

$$g_{\rho(i)}(\rho v) = g_i(v)$$

where $\rho v \in \mathcal{S}^N$ is such that $\rho v(\rho S) = v(S)$.

We remark that symmetry can be replaced by the weaker property of *equal treatment* (i.e., symmetry implies equal treatment, but the converse is not true). Two players $a, b \in N$ are *equi-desirable* as coalition partners in $v \in \mathcal{S}^N$ if for every $S \subseteq N \setminus \{a, b\}$: $v(S \cup \{a\}) = v(S \cup \{b\})$.

Equal treatment: A P-power index $g : \mathcal{S}^N \to \mathbb{R}^n$ satisfies equal treatment whenever

$$g_a(v) = g_b(v)$$

for every pair of equi-desirable players $a, b \in N$.

This weaker postulate is enough in the proof of the next result, since symmetry is only applied to unanimity games, and for these games the symmetric players are either vetoers or nulls, so that equal treatment is enough.

Theorem 5.1 *The Felsenthal P-power index is the unique index on $\mathcal{S}^N$ that satisfies efficiency, the null-player property, equal treatment and the 'transfer for coalitions of the least size'.*

Proof *Uniqueness:* In all what follows we assume w.l.o.g. that $k_v \leq k_w$. Recall that every simple game has a finite number of minimal winning coalitions, $S_1, S_2, \ldots, S_m$ and can be expressed as $u_{S_1} \vee u_{S_2} \vee \cdots \vee u_{S_m}$.

The proof on the uniqueness of ψ will be by induction on the parameters k_v and p_v.

For $k_v = n$, $v = u_N$ in which case $\psi(v)$ is unique by efficiency and equal treatment.

Suppose $\psi(v)$ has been shown to be unique for all v such that $k_v = k+1, k+2, \ldots, n$.

We now show that $\psi(v)$ is unique when $k_v = k$ and $p_v = 1$. Let S be the unique minimal winning coalition with $|S| = k$. If S is the only minimal winning coalition of v, then $v = u_S$ and by efficiency, equal treatment and null-player $\psi(v)$ is unique. Otherwise let $S_1, \ldots, S_m$ denote all the minimal winning coalitions of v apart from S. These coalitions satisfy $|S_i| > k$ for $1 \leq i \leq m$ since $k_v = 1$. It is

$$(u_{S_1} \vee u_{S_1} \vee \cdots \vee u_{S_m}) \vee u_S = v$$

say, $v' \vee u_S = v$.

It follows that $k_{v'} > k$, $\psi(v')$ is unique by the inductive assumption and $\psi(v)$ is also unique since $\psi(v) = \psi(u_S)$ by the transfer postulate of coalitions of the least size and the vector $\psi(u_S)$ is unique by efficiency, equal treatment and null-player, so is $\psi(v)$.

Suppose now that $\psi(v)$ has been shown to be unique for all v such that either:

$$k_v = k+1, \ldots, n$$

or

$$k_v = k \quad \text{and} \quad p_v = 1, \ldots, j \tag{5}$$

We now show that $\psi(v)$ is unique when $k_v = k$ and $p_v = j+1$.

Indeed, consider the minimal winning coalitions $S_1, \ldots, S_{j+1}$ be the minimal winning coalitions of v with k players each. And let $T_1, \ldots, T_m$ be the remaining minimal winning coalitions of v. By the conditions on k_v and p_v it holds $|T_i| > k$ for $1 \leq i \leq m$. Consider

$$(u_{T_1} \vee \cdots \vee u_{T_m} \vee u_{S_1} \vee \cdots \vee u_{S_j}) \vee u_{S_{j+1}} = v$$

say, $v'' \vee u_{S_{j+1}} = v$.

Game v'' satisfies (5). Therefore $\psi(v'')$ is unique by the inductive assumption. As $k_{v''} = k$ and $k_{u_{S_{j+1}}} = k$, by the transfer property on winning coalitions of the least size, we have

$$\psi(v) = \psi(v'' \vee u_{S_{j+1}}) = \frac{1}{p_{v''} + p_{u_{S_{j+1}}}} \Big(p_{v''}\psi(v'') + p_{u_{S_{j+1}}} \psi(u_{S_{j+1}}) \Big) \tag{6}$$

which proves that $\psi(v)$ is unique.

From the two results we get that $\psi(v)$ is unique for any feasible pair k_v and p_v, i.e., for all simple game in $\mathcal{S}^N$.

Existence: Clearly the Felsenthal index ψ on the class of simple games satisfies the four axioms.

From the uniqueness and the existence it follows that the *Felsenthal index* is the unique index on the class of simple games satisfying the four axioms.

From the proof of uniqueness a recursive construction of F is deduced. Indeed, from (6) we have $p_{u_{S_{j+1}}} = 1$ and $p_{v''} = j$. Thus, Eq. (6) is equivalent to

$$\psi(v) = \frac{1}{j+1} \sum_{i=1}^{j+1} \psi(u_{S_i})$$

So that $\psi = F$. □

6 An Impossibility Result on the Existence of Power Indices

We state in this section a direct impossibility result on P-power indices, its difficulty lies solely in showing that all the axioms used are essential to achieve the aforementioned incompatibility.

Theorem 6.1 *There is not a power index that satisfies efficiency, the null-player property, symmetry, transfer and the added blocker.*

The proof of theorem 6.1 is obvious from Dubey's axiomatization (Dubey 1975) of the Shapley value, since this index is the unique one which satisfies the four first axioms, but the index fails to satisfy the added blocker axiom, see (Felsenthal et al. 1998).

More interestingly is to investigate the independence of these five axioms. The following examples show that the five axioms in theorem 6.1 are essential to achieve the impossibility result.

Proposition 6.2 *The axioms used in theorem 6.1 are independent.*

Proof *Efficiency:* Consider the power index ψ^1 defined by

$$\psi_i^1(v) = \begin{cases} 1 & \text{if } i \text{ is a veto player} \\ 0 & \text{otherwise} \end{cases}$$

This power index is not efficient for every simple game without vetoers or with two or more vetoers. The transfer property is satisfied for ψ^1 because for any pair of simple games v and w with respective set of veto players $V(v)$ and $V(w)$, it holds that: $V(v \wedge w) = V(v) \cup V(w)$ and $V(v \vee w) = V(v) \cap V(w)$. The property of the added blocker is satisfied since $\psi_j^1(v) = \psi_j^1(w)$ for all $j \in N$ where N is the player set where v is defined, and, w is obtained from v with the addition of a blocker. The properties of: null-player and symmetry are trivially satisfied for ψ^1.

The null-player property: Consider the *egalitarian power index* ψ^2 given by

$$\psi_i^2(v) = \frac{v(N)}{n}$$

It satisfies efficiency, symmetry, transfer and the added blocker, but it does not satisfy the null-player property.

Symmetry: Consider a selector $\alpha : 2^N \to N$ with $\alpha(S) \in S$ for all $S \neq \emptyset$. The value ψ^3 is the selector allocation corresponding to α defined by

$$\psi_i^3(v) = \sum_{S \subseteq N \,:\, i=\alpha(S)} \Delta_v(S) \tag{7}$$

where $\Delta_v(S)$ is defined in (1). In particular, we choose the selector $\alpha(S) = \min\{i \in N \;:\; i \in S\}$ so that Eq. (7) becomes

$$\psi_i^3(v) = v(S(i) \cup \{i\}) - v(S(i)), \tag{8}$$

where $S(i) = \{j \in N \;:\; j > i\}$.

The power index $\psi^3(v)$ does not satisfy the symmetry property because of the effect of the selector α. Furthermore, the index is efficient because from (8) it follows:

$$\sum_{i \in N} \psi_i^3(v) = \sum_{i \in N} v(S(i) \cup \{i\}) - v(S(i)) = v(N) = 1.$$

The null-player property is satisfied because, if i is a null-player in v, it holds $v(S(i) \cup \{i\}) - v(S(i)) = 0$, i.e., i's marginal contributions are zero. As $(v \wedge w)(S) = \min\{v(S), w(S)\}$ and $(v \vee w)(S) = \max\{v(S), w(S)\}$, the transfer property easily follows. Finally, the added blocker property is also satisfied because $w(S \cup \{0\}) = v(S)$ for all $S \subseteq N$, where v is defined on N and w is defined on $N \cup \{0\}$, so that 0 is the added blocker in v. Thus, from $w(S \cup \{0\}) = v(S)$ for all $S \subseteq N$, it follows $\psi_i^3(v) = \psi_i^3(w)$ for all $i \in N$.

Transfer: The Felsenthal power index satisfies efficiency, the null-player property, symmetry, as well as the added blocker. But it does not satisfy transfer.

Added blocker: The Shapley-Shubik index is the unique power index on the class of simple games that satisfies efficiency, the null-player property, symmetry and transfer. Nevertheless, it does not satisfy the added blocker, see Felsenthal and Machover (1998, Example 7.9.16, p. 272), who qualified the violation of the postulate as 'flagrant'. □

As is well known the property of symmetry in the characterization of the Shapley value by Dubey (1975) can be replaced by the weaker property of equal-treatment, so this replacement is also valid in theorem 6.1.

Equal treatment. Let v be a simple game with assembly N and a and b be two players such that $v(S \cup \{a\}) = v(S \cup \{b\})$ for all $S \subseteq N \setminus \{a, b\}$. Then the equal treatment postulate stipulates that $g_a(v) = g_b(v)$.

Note that: efficiency, null-player property, equal treatment and added blocker are properties demanded by Felsenthal in his list of postulates. But he demands much more to the index. Thus, the addition of the transfer postulate to his list of postulates reduces to an impossibility result, which answers one of his questions. Of course, it would be compelling to replace transfer in theorem 6.1 for a weaker postulate which kept the impossibility result stated.

With this section we have concluded the three aims of this article. Let's, however, show in what follows two weaknesses of the Felsenthal power index. Felsenthal and Machover propose the natural postulate *vanishing only for nulls* (Postulate 3, page 222 in Felsenthal and Machover 1998).

Vanishing only for nulls. Let v be a simple game with assembly N. The vanishing only for nulls postulate stipulates that $g_a(v) = 0$ *if and only if* a is null in v.

As a matter of example, we remark that this postulates is satisfied by all the seven alternative power indices he studied in Felsenthal (2016). However, as shown in Example 3.1-(a) player 4 is not null in v but $F_4(v) = 0$ or in Example 3.1-(b) were players 3 and 4 are not null in v but $F_3(v) = F_4(v) = 0$. Hence the F-index does not vanish only for nulls. Another postulate, which is verified for the Shapley-Shubik, Banzhaf and Johnston indices is the strict desirability.

Strict desirability. Let v be a simple game with assembly N and a and b be two players such that $v(S \cup \{a\}) \geq v(S \cup \{b\})$ for all $S \subseteq N \setminus \{a, b\}$ and there exists a coalition $T \subseteq N \setminus \{a, b\}$ such that $v(T \cup \{a\}) > v(T \cup \{b\})$. Then the strict desirability postulate stipulates that $g_a(v) > g_b(v)$.

However, this postulates is not verified by the Felsenthal index (3), as shown Example 3.1-(b) for which player 1 is strictly more desirable than player 2, but $F_a(v) > F_b(v)$.

We also want to remark that the Felsenthal index does not satisfy the common internal properties that a P-power index should satisfy, see (Freixas and Gambarelli 1997), which are: efficiency, symmetry, null player property and desirability in its strict version. These properties are verified for several power indices as the Shapley-Shubik, relative Banzhaf, or Johnston among others. The Felsenthal index does not satisfy desirability in its strict version. Moreover, desirability in its strict version together with the null player property implies vanishing only for nulls, which is also not verified for the Felsenthal index. Bertini et. al. (2013) contains some more postulates likely to be studied for the Felsenthal index.

7 Conclusion

This paper was about the Felsenthal index for simple games. We studied some of the questions that were posed in the Felsenthal paper. After extending some properties stated by Felsenthal for weighted games to simple games, we have proved that his index is not the only one that satisfies a list of compelling postulates that Felsenthal and Machover (1998) proposed. We have shown an axiomatic characterization of the index, possibly the first of its kind. We have added the transfer postulate to some other standard postulates and have proven that the index is uniquely characterized. Finally, we have illustrated some other compelling postulates that the Felsenthal index fails to satisfy.

It would be nice to extend the index to a value for cooperative games since there are many contexts in which only the coalitions with the least size among those who have a value distinct of zero count to be measured. It would also be interesting to weaken the transfer postulate in theorem 6.1 in order to find a non-obvious incompatibility result. Another challenging problem would be to study whether there is a P-power index that satisfies: efficiency (postulate de facto), desirability in its strict version, symmetry, blocker's share and added blocker. As far as we know, there is not any recognized power index that satisfies all of them. If someone is able to prove that a power index satisfying all of them does not exist, then it would be interesting to study the independence of the axioms.

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Declarations

Conflict of interest The two authors declare that do not have conflict of interest.

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