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RESEARCH

On the time decay for an elastic problem with three porous structures

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Abstract
In this paper, we study the three-dimensional porous elastic problem in the case that three dissipative mechanisms act on the three porosity structures (one in each component). It is important to remark that we consider the case when the material is not centrosymmetric, and therefore, some coupling, not previously considered in the literature concerning the time decay of solutions in porous elasticity, can appear in the system of field equations. The new couplings provided in this situation show a strong relationship between the elastic and the porous components of the material. In this situation, we obtain an existence and uniqueness result for the solutions to the problem using the Lumer-Phillips corollary to the Hille-Yosida theorem. Later, assuming a certain condition determining a "very strong" coupling between the material components, we can use the well-known arguments for dissipative semigroups to prove the exponential stability of the solutions to the problem. It is worth emphasizing that the proposed condition allows bringing the decay of the dissipative porous structure of the problem to the macroscopic elastic structure.

Keywords Elasticity · Porosity · Dissipative mechanisms · Existence · Energy decay

1 Introduction

Much has been written in the last fifty years on the time decay of solutions to different problems in thermomechanics. These types of studies are relevant both from the mathematical and mechanical points of view. Perhaps the starting point was the contributions of Dafermos (1976) regarding thermoelasticity as well as other mechanical situations.

In general, it is not easy to find coupling mechanisms damping the elastic vibrations in dimension greater than one exponentially, but we can recall the recent contribution by Magaña and Quintanilla (2018).

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In this article, we will focus on the study of the temporal decay of the solutions to an elastic problem with voids. We can remember that this theory was proposed by Cowin (1985), Cowin and Nunziato (1983), Nunziato and Cowin (1979). Some basic results can be found in De Cicco and Nappa (2000), Ieşan (1986), Svanadze and De Cicco (2005). It is appropriate to remember that the study of the temporal decay of solutions for this type of materials was initiated about twenty years ago in the contributions (Casas and Quintanilla 2005a,b; Magaña and Quintanilla 2006a,b). Since then, we have been able to see how many contributions aim to study different situations related to elasticity with voids. We can say that the coupling between elasticity and porosity is weak if we restrict ourselves to isotropic materials. Therefore, we always need some special conditions to be able to obtain an exponential decay (Apalara 2017a,b; Bazarra et al. 2022; Feng and Apalara 2019; Fernández et al. 2019; Leseduarte et al. 2010; Pamplona et al. 2011, 2012) since, generally, we will need the presence of two dissipation mechanisms (always well chosen) or the equality of the speed of propagation of the different waves. Even more, despite a large number of contributions, we have to say that the great immensity of them refers to the one-dimensional problem, although we can remember the contributions (Bazarra et al. 2022; Nicaise and Valein 2012) for a dimension greater than one.

In the recent contributions referring to thermoelasticity with various dissipation mechanisms, we have been able to observe that if we work with anisotropic or chiral materials, quite strong coupling mechanisms can appear and that they can allow us to obtain the exponential decay of solutions in a dimension greater than one (Fernández and Quintanilla 2022, 2023a,b,c). Inspired by this fact, we have raised the question of how we can achieve the exponential decay of solutions for elastic materials with voids in dimension greater than one, even though we have to restrict ourselves to chiral materials. Certainly, in this case, we find ourselves with new couplings not contemplated in the chiral case that will allow us to obtain quite interesting decay rates. At the same time, it is pertinent to remember that the study of elasticity with two porous mechanisms (Ieşan and Quintanilla 2014, 2019) was the starting point of many other contributions where the problem of elasticity with several porous mechanisms is considered (Bazarra et al. 2019, 2020, 2022; Mosconi 2005; Scarpetta and Svanadze 2015).

In this article, we study the three-dimensional elasticity problem with three porous mechanisms. We will assume that each of the porous mechanisms is affected by dissipative mechanisms and see that, under adequate hypotheses on the coupling mechanisms, we can obtain the exponential decay of the solutions. It is important to highlight that the conditions that we impose require that the material (its couplings) is not centrosymmetric. At the same time, it is worth noting that we only need a number of couplings equal to the dimension of the domain. This is relevant in comparison with the results provided in Fernández and Quintanilla (2022, 2023a,b). These couplings are more efficient than the ones produced by the heat type.

To compare our approach with other recent works in the literature, we should say that (as far as we know) there are no contributions similar to this one for the porous elasticity since the majority of the contributions on this topic refer to the centrosymmetric case, where the coupling between the elastic and porous components of the material is "weak". Here, we use in a relevant way the coupling provided when the material is not centrosymmetric. In fact, in this situation, the different components of the material could be provided with a stronger coupling than the chiral case. At the same time, our approach also needs a number greater than one of the porous components because we are working in a multidimensional setting. In this sense, our approach recalls the contributions recently obtained for thermoelasticity with several dissipation mechanisms (see Fernández and Quintanilla 2022, 2023a,b,c).

The plan of this paper is the following. The basic equations of the model and the assumptions to prove the results are described in the next section. Then, this problem is written as a Cauchy one in Sect. 3, including also the description of the functional framework and the construction of the main operator which defines the problem. An existence result is proved in Sect. 4 by using Lax-Milgram lemma and the Lumer-Phillips corollary. Finally, in Sect. 5, we obtain the exponential energy decay of these solutions, imposing additional assumptions on the coupling tensors and applying the theory of linear semigroups.

2 Basic equations

In this paper, we will consider a chiral elastic solid, with three porous structures, which occupies a three-dimensional domain $B \subset \mathbb{R}^3$. We will assume that, in the reference configuration, this domain has a boundary ∂B smooth enough to apply the divergence theorem.

The evolution equations are

$$\begin{aligned} \rho \ddot{u}_i &= t_{ij,j}, \\ J_{lj}\ddot{\varphi}_j &= \Pi^l_{i,i} - \sigma^l \quad \text{for} \quad l = 1, 2, 3. \end{aligned} \tag{1}$$

Here, ρ is the material density, u_i represents the displacement field, t_{ij} is the stress tensor, J_{lj} is the matrix of equilibrated inertia, φ_j is the volume fraction of each porous structure, Π^l_i is the vector of equilibrated stresses, and σ^l is the equilibrated body forces.

The constitutive equations are

$$\begin{aligned} t_{ij} &= A_{ijrs}u_{r,s} + D^k_{ijr}\varphi_{k,r} + a^k_{ij}\varphi_k, \\ \Pi^l_j &= c^{kl}_{ij}\varphi_{k,i} + C^{kl}_{ij}\dot{\varphi}_{k,i} + d^{lk}_j\varphi_k + D^l_{ipj}u_{i,p}, \\ \sigma^l &= a^l_{ij}u_{i,j} + d^{kl}_j\varphi_{k,j} + \xi^{kl}\varphi_k. \end{aligned} \tag{2}$$

It is worth noting that, in the centrosymmetric case, the tensors D^k_{ijr} and d^{kl}_j vanish; however, in this article, we will assume that both tensors are nonzero, and, even, we will impose several conditions allowing us to obtain the exponential stability results.

The above tensors used in equations (2) have the following symmetries:

$$A_{ijrs} = A_{rsij}, \quad c^{kl}_{ij} = c^{lk}_{ji}, \quad C^{kl}_{ij} = C^{lk}_{ji}, \quad \xi^{kl} = \xi^{lk}, \quad J_{ij} = J_{ji}. \tag{3}$$

If we introduce the constitutive equations (2) into the evolution equations (1), we obtain the following system of equations:

$$\begin{aligned} \rho \ddot{u}_i &= \left(A_{ijrs}u_{r,s} + D^k_{ijr}\varphi_{k,r} + a^k_{ij}\varphi_k\right)_{,j}, \\ J_{lj}\ddot{\varphi}_j &= \left(c^{kl}_{ij}\varphi_{k,i} + C^{kl}_{ij}\dot{\varphi}_{k,i} + D^l_{pij}u_{i,p} + d^{lk}_j\varphi_k\right)_{,j} \\ &\quad - a^l_{ij}u_{i,j} - d^{kl}_j\varphi_{k,j} - \xi^{kl}\varphi_k. \end{aligned} \tag{4}$$

We will study this system, but to complete the description of the problem, we will impose the initial conditions:

$$\begin{aligned} \boldsymbol{u}(\boldsymbol{x},0) &= \boldsymbol{u}^0(\boldsymbol{x}), \quad \dot{\boldsymbol{u}}(\boldsymbol{x},0) = \boldsymbol{v}^0(\boldsymbol{x}) \quad \text{for a.e. } \boldsymbol{x} \in B, \\ \varphi(\boldsymbol{x},0) &= \varphi^0(\boldsymbol{x}), \quad \dot{\varphi}(\boldsymbol{x},0) = \psi^0(\boldsymbol{x}) \quad \text{for a.e. } \boldsymbol{x} \in B, \end{aligned} \tag{5}$$

and the boundary conditions:

$$\boldsymbol{u}(\boldsymbol{x},t)=\varphi(\boldsymbol{x},t)=0 \quad \text{for a.e. } t>0, \quad \boldsymbol{x}\in\partial B. \tag{6}$$

In the next sections, we will prove some qualitative results for the solutions to problem (4)-(6). So, we will assume that

(i) $\rho(\boldsymbol{x})\geq\rho_0>0$.
(ii) The matrix J_{lj} is positive definite.
(iii) There exists a positive constant C such that

$$A_{ijrs}\xi_{ij}\xi_{rs}+2D_{ijr}^{k}\xi_{ij}\zeta_{kr}+c_{ij}^{kl}\zeta_{ki}\zeta_{lj}+2a_{ij}^{l}\xi_{ij}\eta_l+2d_j^{kl}\zeta_{kj}\eta_l$$
$$+\xi^{kl}\eta_k\eta_l\geq C(\xi_{ij}\xi_{ij}+\zeta_{ij}\zeta_{ij}+\eta_l\eta_l),$$

for every tensors ξ_{ij} and ζ_{ij}, and every vector η_l.
(iv) There exists a positive constant D such that

$$C_{ij}^{kl}\zeta_{ki}\zeta_{lj}\geq D\zeta_{ki}\zeta_{ki},$$

for every tensor ζ_{ki}.

It is relevant to point out that the meaning of assumptions (i) and (ii) is obvious. Assumption (iii) implies that the mechanical energy is positive. This condition is rather natural and related to the studies of elastic stability. Assumption (iv) guarantees that the dissipation is positive definite.

3 Functional statement

In this section, we will transform problem (4)-(6) into a Cauchy problem written in an adequate Hilbert space.

Indeed, we will consider the following space:

$$\mathcal{H}=\boldsymbol{W}_0^{1,2}(B)\times\boldsymbol{L}^2(B)\times\boldsymbol{W}_0^{1,2}(B)\times\boldsymbol{L}^2(B), \tag{7}$$

where $L^2(B)$ and $W_0^{1,2}(B)$ are the usual Sobolev spaces, and $\boldsymbol{W}_0^{1,2}(B)=[W_0^{1,2}(B)]^3$ and $\boldsymbol{L}^2(B)=[L^2(B)]^3$.

In this space, we will use the inner product associated to the norm:

$$\|(\boldsymbol{u},\boldsymbol{v},\boldsymbol{\varphi},\boldsymbol{\psi})\|^2=\frac{1}{2}\int_B\Big(\rho v_i\overline{v_i}+J_{lj}\psi_l\overline{\psi_j}+A_{ijrs}u_{i,j}\overline{u_{r,s}}+D_{ijr}^{k}[u_{i,j}\overline{\varphi_{k,r}}+\overline{u_{i,j}}\varphi_{k,r}]$$
$$+c_{ij}^{kl}\varphi_{k,i}\overline{\varphi_{l,j}}+a_{ij}^{l}[u_{i,j}\overline{\varphi_l}+\overline{u_{i,j}}\varphi_l]+d_j^{kl}[\varphi_{k,j}\overline{\varphi_l}+\overline{\varphi_{k,j}}\varphi_l]+\xi^{kl}\varphi_k\overline{\varphi_l}\Big)\,dv.$$

Here, the bar denotes the conjugated complex.

We note that condition (ii) allows us to see that the norm defined on the variables ψ_j is equivalent to the usual one in the L^2-norm for each component. At the same time, in view of condition (iii), we see that the components associated with the tensors A_{ijrs}, D_{ijr}^{k}, c_{ij}^{kl}, a_{ij}^{l}, d_j^{kl}, and ξ^{kl} define a norm equivalent to the usual norm in the elasticity and to the three porous components. Then, we can conclude that this norm is equivalent to the usual one in the Hilbert space $\mathcal{H}$ defined previously.

We can write our problem as a Cauchy problem as follows,

$$\frac{d}{dt}U(t) = \mathcal{A}U(t), \quad U(0) = U^0 = (\boldsymbol{u}^0, \boldsymbol{v}^0, \boldsymbol{\varphi}^0, \boldsymbol{\psi}^0), \tag{8}$$

where $U = (\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\varphi}, \boldsymbol{\psi})$ and

$$\mathcal{A}\begin{pmatrix} u_i \\ v_i \\ \varphi_i \\ \psi_i \end{pmatrix} = \begin{pmatrix} v_i \\ m_i \\ \psi_i \\ n_i \end{pmatrix},$$

where

$$m_i = \rho^{-1}\left[\left(A_{ijrs}u_{r,s} + D^k_{ijr}\varphi_{k,r} + a^k_{ij}\varphi_k\right)_{,j}\right],$$

$$n_i = H_{il}\left[\left(c^{kl}_{pj}\varphi_{k,p} + C^{kl}_{pj}\psi_{k,p} + D^l_{ipj}u_{i,p} + d^{lk}_j\varphi_k\right)_{,j} - a^l_{pj}u_{p,j} - d^{kl}_j\varphi_{k,j} - \xi^{kl}\varphi_k\right],$$

with H_{il} being the inverse of J_{lj}.

We can see that the domain of the operator $\mathcal{A}$ is made of the elements of the Hilbert space $\mathcal{H}$ such that

$$\boldsymbol{v} \in \boldsymbol{W}^{1,2}_0(B), \quad \boldsymbol{\psi} \in \boldsymbol{W}^{1,2}_0(B),$$

$$\left(A_{ijrs}u_{r,s} + D^k_{ijr}\varphi_{k,r}\right)_{,j} \in \boldsymbol{L}^2(B),$$

$$\left(c^{kl}_{pj}\varphi_{k,p} + C^{kl}_{pj}\psi_{k,p} + D^l_{ipj}u_{i,p}\right)_{,j} \in \boldsymbol{L}^2(B).$$

It is straightforward to show that this subspace is a dense subset of $\mathcal{H}$.

4 Existence of solutions

In this section, we will show the existence of solutions to problem (8) whenever assumptions (i)-(iv) are held. To this end, we will show that the operator $\mathcal{A}$ is dissipative and that zero belongs to the resolvent of the operator. Keeping in mind that the domain of the operator is dense in the Hilbert space $\mathcal{H}$, we can conclude the existence of solutions in virtue of the Lumer-Phillips corollary applied to the Hille-Yosida theorem.

First, considering the field equations (4) and the boundary conditions (6), it follows that, for all $U \in \mathcal{D}(\mathcal{A})$,

$$Re\langle \mathcal{A}U, U\rangle = -\frac{1}{2}\int_B C^{kl}_{ij}\psi_{k,i}\psi_{l,j}\,dv \leq 0.$$

Therefore, we only need to prove that, if $F = (\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3, \boldsymbol{f}_4) \in \mathcal{H}$, then there exists $U = (\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\varphi}, \boldsymbol{\psi}) \in \mathcal{D}(\mathcal{A})$ such that

$$\mathcal{A}U = F. \tag{9}$$

This equation can be written as

$$\begin{aligned}
&\boldsymbol{v}=\boldsymbol{f}_1, \quad \boldsymbol{\psi}=\boldsymbol{f}_3,\\
&\left(A_{ijrs}u_{r,s}+D^k_{ijr}\varphi_{k,r}+a^k_{ij}\varphi_k\right)_{,j}=\rho f_{2i},\\
&\left(c^{kl}_{pj}\varphi_{k,p}+C^{kl}_{pj}\psi_{k,p}+D^l_{ipj}u_{i,p}+d^{lk}_j\varphi_k\right)_{,j}-a^l_{pj}u_{p,j}-d^{kl}_j\varphi_{k,j}-\xi^{kl}\varphi_k=J_{lq}f_{4q}.
\end{aligned} \tag{10}$$

Of course, we have the solution to $\boldsymbol{v}$ and $\boldsymbol{\psi}$. If we substitute the solution to $\boldsymbol{\psi}$, we obtain the system

$$\begin{aligned}
&\left(A_{ijrs}u_{r,s}+D^k_{ijr}\varphi_{k,r}+a^k_{ij}\varphi_k\right)_{,j}=\rho f_{2i},\\
&\left(c^{kl}_{pj}\varphi_{k,p}+d^{lk}_j\varphi_k+D^l_{ipj}u_{i,p}\right)_{,j}-a^l_{pj}u_{p,j}-d^{kl}_j\varphi_{k,j}-\xi^{kl}\varphi_k=J_{lq}f_{4q}-\left(C^{kl}_{pj}f_{3k,p}\right)_{,j}.
\end{aligned}$$

We note that

$$\left(\rho f_{2i}, J_{lq}f_{4q}-\left(C^{kl}_{pj}f_{3k,p}\right)_{,j}\right)\in \boldsymbol{W}^{-1,2}(B)\times\boldsymbol{W}^{-1,2}(B),$$

where $\boldsymbol{W}^{-1,2}(B)$ is the dual space to $\boldsymbol{W}^{1,2}_0(B)$.

On the other hand, we consider the bilinear form:

$$\mathcal{B}[(\boldsymbol{u},\boldsymbol{\varphi}),(\boldsymbol{u}^*,\boldsymbol{\varphi}^*)]=\int_B[\boldsymbol{l}\boldsymbol{u}^*+\boldsymbol{n}\boldsymbol{\varphi}^*]\,dv,$$

where

$$\begin{aligned}
l_i&=\left(A_{ijrs}u_{r,s}+D^k_{ijr}\varphi_{k,r}+a^k_{ij}\varphi_k\right)_{,j},\\
n_l&=\left(c^{kl}_{pj}\varphi_{k,p}+d^{lk}_j\varphi_k+D^l_{ipj}u_{i,p}\right)_{,j}-a^l_{pj}u_{p,j}-d^{kl}_j\varphi_{k,j}-\xi^{kl}\varphi_k.
\end{aligned}$$

In view of condition (iii), it is clear that the operator $\mathcal{B}$ is coercive and bounded in $\boldsymbol{W}^{1,2}_0(B)\times\boldsymbol{W}^{1,2}_0(B)$. The Lax-Milgram lemma implies the existence of a solution to problem (10). Even, we can obtain that $\|U\|\le K\|F\|$ for a suitable constant $K>0$.

So, we can state the following existence result.

Theorem 1 *The operator $\mathcal{A}$ generates a C^0-semigroup of contractions in the space $\mathcal{H}$. Thus, for any initial data $U^0\in\mathcal{D}(\mathcal{A})$, there exists at least one solution to Cauchy problem (8) satisfying*

$$U\in C^1([0,\infty);\mathcal{H})\cap C([0,\infty);\mathcal{D}(\mathcal{A})).$$

5 Exponential decay

In this section, we will show that the solutions obtained in the previous section decay exponentially. To prove this result, we will impose some additional assumptions.

Let us define the operator

$$P_l(\boldsymbol{u})=\left(D^l_{phj}u_{h,p}\right)_{,j}+a^l_{pj}u_{p,j}.$$

We will assume that either assumption

(v) there exists a positive constant L such that

$$\langle \boldsymbol{P}(\boldsymbol{u}), \boldsymbol{u} \rangle \geq L \|\boldsymbol{u}\|^2_{\boldsymbol{W}^{1,2}(B)},$$

for every vector field $\boldsymbol{u}$ vanishing at boundary ∂B, where $\boldsymbol{P}(\boldsymbol{u}) = (P_l(\boldsymbol{u}))$,

or either assumption

(v′) there exists a positive constant L such that

$$\langle \boldsymbol{P}(\boldsymbol{u}), \boldsymbol{u} \rangle \leq -L \|\boldsymbol{u}\|^2_{\boldsymbol{W}^{1,2}(B)}$$

for every vector field $\boldsymbol{u}$ vanishing at boundary ∂B,

hold.

Remark 1 To understand assumption (v), we must consider that $\boldsymbol{u}$ vanishes at the boundary. If we suppose that

$$\int_B D^l_{phj} u_{h,p} u_{l,j} \, dv \geq L_1 \int_B u_{i,j} u_{i,j} \, dv, \tag{11}$$

$$\left| \int_B a^l_{pj} u_{p,j} u_l \, dv \right| \leq L_2 \left(\int_B u_{i,j} u_{i,j} \, dv \right)^{1/2} \left(\int_B u_i u_i \, dv \right)^{1/2}, \tag{12}$$

where L_1 and L_2 are two positive constants depending on the coefficients D^l_{phj} and a^l_{pj}, we can apply Poincaré's inequality to obtain

$$\left| \int_B a^l_{pj} u_{p,j} u_l \, dv \right| \leq L^*_2 \int_B u_{i,j} u_{i,j} \, dv,$$

where L^*_2 depends on the constant L_2 and the topology of the domain B.

Therefore, if $L_1 - L^*_2 > 0$, then assumption (v) is satisfied.

We can observe that a similar comment can be provided regarding assumption (v′).

On the other hand, we note that

$$\int_B a^l_{pj} u_{p,j} u_l \, dv = - \int_B a^l_{pj,j} u_p u_l \, dv - \int_B a^l_{pj} u_p u_{l,j} \, dv.$$

Therefore, when $a^l_{pj,j} = a^l_{jp}$, we find that

$$\int_B a^l_{pj} u_{p,j} u_l \, dv = -\frac{1}{2} \int_B a^l_{pj,j} u_p u_l \, dv.$$

Then, if we assume that $a^l_{pj,j}$ is semi-definite negative, and condition (11) holds, we see that condition (v) is satisfied. A similar comment could be done if we assume that $a^l_{pj,j}$ is semi-definite positive and

$$\int_B D^l_{phj} u_{p,j} u_{l,j} \, dv \leq -L \int_B u_{i,j} u_{i,j} \, dv.$$

The energy decay result is stated as follows.

Theorem 2 *Let us assume that condition* (v) (*or* (v′)) *holds*, *then there exist two positive constants* M, ω *such that*

$$\|U(t)\| \le Me^{-\omega t}\|U(0)\|$$

for every $U(0) \in \mathcal{D}(\mathcal{A})$.

Proof To prove this theorem, we can use the fact that the exponential decay is found whenever the imaginary axis is contained at the resolvent of the operator $\mathcal{A}$, and the asymptotic condition

$$\overline{\lim_{|\lambda|\to\infty}} \|(i\lambda I - \mathcal{A})^{-1}\| < \infty \tag{13}$$

holds (see Liu and Zheng 1999).

To show that the imaginary axis is contained at the resolvent of the operator $\mathcal{A}$, we can follow a usual procedure. Since zero is at the resolvent, if we suppose that there exists $i\tau$ (with $\tau \in \mathbb{R}$) in the spectrum, then there will exist a sequence of real numbers λ_n converging to τ, and a sequence of unit norm vectors $U_n \in \mathcal{D}(\mathcal{A})$, such that

$$\|(i\lambda_n \mathcal{I} - \mathcal{A})U_n\| \to 0. \tag{14}$$

This convergence implies the following ones by components:

$$i\lambda u_l - v_l \to 0 \quad \text{in} \quad W^{1,2}(B), \tag{15}$$

$$v_i - \left(A_{ijrs}u_{r,s} + D^k_{ijr}\varphi_{k,r} + a^k_{ij}\varphi_k\right)_{,j} \to 0 \quad \text{in} \quad L^2(B), \tag{16}$$

$$i\lambda\varphi_l - \psi_l \to 0 \quad \text{in} \quad W^{1,2}(B), \tag{17}$$

$$\begin{aligned} i\lambda J_{lq}\psi_q - &\left(c^{kl}_{pj}\varphi_{k,p} + C^{kl}_{pj}\psi_{k,p} + D^l_{ipj}u_{i,p}\right)_{,j} - a^l_{pj}u_{p,j} \\ &-d^{kl}_j\varphi_{k,j} - \xi^{kl}\varphi_k \to 0 \quad \text{in} \quad L^2(B). \end{aligned} \tag{18}$$

Here, we have omitted the index "n" to simplify the notation.

In view of the dissipation, it follows that $\boldsymbol{\psi} \to \mathbf{0}$ in $\boldsymbol{W}^{1,2}(B)$. Then, $\lambda\boldsymbol{\varphi}$ also tends to zero in $\boldsymbol{W}^{1,2}(B)$. Now, we multiply convergence (18) by u_l to have

$$\langle i\lambda J_{lq}\psi_q, u_l\rangle = \langle iJ_{lq}\psi_q, \lambda u_l\rangle \to 0.$$

We also find that

$$\begin{aligned} \langle\left(c^{kl}_{pj}\varphi_{k,p} + C^{kl}_{pj}\psi_{k,p} + D^l_{ipj}u_{i,p}\right)_{,j}, u_l\rangle &= -\langle c^{kl}_{pj}\varphi_{k,p}, u_{l,j}\rangle - \langle C^{kl}_{pj}\psi_{k,p}, u_{l,j}\rangle \\ &\quad - \langle D^l_{kpj}u_{k,p}, u_{l,j}\rangle. \end{aligned}$$

We note that the first two terms of the right-hand side of this equality tend to zero, and so, we conclude that $\langle \boldsymbol{P}(\boldsymbol{u}), \boldsymbol{u}\rangle \to 0$. In view of the assumption (v) or (v′), it follows that $\boldsymbol{u} \to \mathbf{0}$ in $\boldsymbol{W}^{1,2}_0(B)$.

If we multiply now convergence (16) by $\boldsymbol{u}$, we obtain that $\boldsymbol{v} \to \mathbf{0}$ in $\boldsymbol{L}^2(B)$. Thus, we have arrived to a contradiction because we had assumed that $i\tau$ ($\tau \in \mathbb{R}$) was in the spectrum. Therefore, we obtain that the imaginary axis is contained at the resolvent of the operator $\mathcal{A}$.

To prove that condition (13) holds, we can follow a similar argument. If we assume that (13) is not fulfilled, then there will exist a sequence of real numbers $\lambda_n \to \infty$ and a sequence of unit norm vectors $U_n \in \mathcal{D}(\mathcal{A})$ such that convergences (15)-(18) hold. Then, the previous argument brings us again to a contradiction, and we conclude the proof of the theorem since the key point is that λ_n does not tend to zero. □

6 Conclusions

In recent years, we have seen how coupling the elasticity system with the different "anisotropic or chiral" dissipative mechanisms can bring the system to exponential stability. In this paper, we have shown how selecting three porous mechanisms in the three-dimensional case is sufficient when the material is not centrosymmetric. It leads to a strong coupling between the elastic and the porous components of the material. In this sense, they are "more efficient" than the case corresponding to the heat where, as far as we know, we need a larger number of couplings.

We have obtained an existence and uniqueness result using the Lumer-Phillips corollary to the Hille-Yosida theorem. Under the commented condition on the constitutive anisotropic tensors, we have proved that the dissipative mechanisms lead to the exponential stability of the solutions to the problem, bringing the decay of the dissipative porous structure to the macroscopic elastic structure.

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Declarations

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