MASTER'S THESIS

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**Title:** Extensions of transversal matroids


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# Extensions of transversal matroids

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I would very much like to thank my supervisor Anna for her dedication and commitment, for always being willing to discuss any issue or idea and for her bright thinking. I would also like to thank my friends and family, who supported me throughout this time, and especially my partner Zaira who was there at all times to help me, encourage me to move forward and, occasionally, even delve with me into this exciting world that is matroid theory.

## Abstract

In this work we introduce the basic notions of matroid theory, transversal matroids and single-element extensions. Our first objective is to study which extensions of a given transversal matroid are also transversal. In particular, we focus our study on the family of uniform matroids. The set of single-element extensions of a matroid is a lattice under the so-called weak order. We try to answer the question of whether the set of transversal extensions of a uniform matroid is also a lattice or not.

We also design an algorithm that catalogs and counts transversal matroids up to a fixed size of the ground set. To do so, we build them from scratch by iteratively extending minimal presentations using the tools that we have previously seen. In view that the strategy does not actually count all transversal matroids, we also study which matroids are not reachable using this strategy and what properties do they satisfy.

### Keywords

matroid theory, transversal matroids, presentations, uniform matroids, matroid extensions, matroid enumeration

# Contents

# 1. Matroids

## 1.1 Introduction

Matroid theory studies the abstract properties of dependency as we usually understand it. It is no surprise, then, that the common way to introduce it is through linear algebra, as it generalizes the notion of linear dependency of vectors in a vector space, and in fact, follows the same terminology. We will talk about dependent or independent sets, bases, and many other well-known terms; however, we will give them an abstract definition instead of the vector space one. According to Oxley's *Matroid Theory* (our reference book for general matroid theory), the founding paper in this field is Whitney's *"On the abstract properties of linear dependence"* with date of 1935.

This work is conducted by the following lines. We will start introducing general matroid theory in Section 1. Starting with basic results and properties, we introduce some important families of matroids and proceed to study two concepts that will be essential in the following sections: geometrical representations of matroids and their lattices of flats. To finish the section we introduce the notion of single-element matroid extensions, we define an order among them (called the *weak order*) and see that it is a lattice under this order.

In Section 2 we focus our study on transversal matroid theory. We will see how transversal matroids are constructed from set systems (presentations), define an order among these presentations, and study what conditions they have to satisfy to be maximal or minimal. Finally, we will approach single-element extension from the transversal matroid point of view, by extending their presentations, and setting the ground for the forthcoming sections to work with this kind of extension.

We proceed to focus even more our study in Section 3; this time we study single-element extensions of a particular family of transversal matroids: the uniform matroids. In this context, we try to answer some open questions, in particular, whether the poset of *transversal* extensions of a uniform matroid is a lattice or not. In order to do so, we study the structure of flats of uniform matroids and apply a well-known result from Mason and Ingleton to determine which extensions are transversal are which are not. With this, we solve the lattice question for low-rank uniform matroids of arbitrary size and give some thoughts on the general case.

We conclude in Section 4 using several of the previously shown results to define an algorithm that attempts to count and catalog transversal matroids up to a certain size of the ground set. The code where we implement the algorithm defined in this section is publicly available, and so is the dataset with the results. We will see that, in fact, our strategy will not find *all* transversal matroids, so we proceed to study the characteristics of the matroids that are excluded from our catalog. Finally, we present the results from our algorithm and discuss their implications.

As reference for fundamental results and definitions about matroid theory, we use the already mentioned Oxley's book [15].

## 1.2 Definitions and families of matroids

Matroids can be motivated in many different ways. Since we introduced the notion as an abstraction of linear dependency, let us define a matroid by its independent sets.

**Definition 1.1.** A *matroid* is a pair $M = (E, \mathcal{I})$ where $\mathcal{I}$ is a non-empty collection of subsets of $E$ and the following are satisfied:

1. $\mathcal{I}$ is a downset w.r.t. inclusion.
2. If $I, J \in \mathcal{I}$ and $|I| < |J|$ then there exists $x \in J \backslash I$ such that $I \cup \{x\} \in \mathcal{I}$.

The set $E$ is called the *ground set* and $\mathcal{I}$ are the *independent sets* of $M$.

*Observation* 1.2. Note that $\emptyset$ is always an independent set, given that $\mathcal{I}$ is a non-empty downset.

**Notation**: Following common matroid theory notation, we write $A \cup x$ instead of $A \cup \{x\}$ for a set $A$ and element $x$. Similarly, we write $A \backslash x$ to denote $A \backslash \{x\}$ and, in general, considering $x$ as the element itself or the set containing it, depending on the context.

**Example 1.3** (Representable matroid)**.** Let $A$ be the following matrix over the real numbers

$$A = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 1 \end{pmatrix}.$$

With the ground set indexing the columns $E = \{1, 2, 3, 4\}$ and the dependencies given by linear dependencies of columns, we have a matroid $M = ([4], \mathcal{I})$ where the independent sets are

$$\mathcal{I} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}.$$

Matroids that arise from matrices as in Example 1.3 are a particular and interesting family of matroids that we may mention at some point in this work, so let us define them properly.

**Definition 1.4.** Let $\mathbb{K}$ be a field. A matroid $M$ is called $\mathbb{K}$-*representable* if there exists a matrix $M$ whose columns are in one-to-one correspondence with $E(M)$ such that a set $I \subseteq E(M)$ is independent if and only if the corresponding columns are linearly independent over $\mathbb{K}$.

The definition of matroid by its independent sets is good for understanding its nature, but we will most of the time work with matroids by describing their bases, which we now define.

**Definition 1.5.** A *basis* of $M$ is a maximal independent set. We denote the set of bases of $M$ by $\mathcal{B} = \mathcal{B}(M)$. A *circuit* is a minimal dependent set. We denote the set of circuits by $\mathcal{C} = \mathcal{C}(M)$.

*Observation* 1.6. Since independent sets are downsets, the set of bases of a matroid characterizes the matroid $M$. Indeed, given $\mathcal{B}(M)$, the independent sets are $\mathcal{I}(M) = \{I \subseteq B \mid B \in \mathcal{B}(M)\}$.

The following lemma shows a couple of simple properties that help us understand the role of circuits in a matroid.

**Lemma 1.7.** *Let $M$ be a matroid and $I \subseteq E(M)$ a subset. Then, the following are equivalent*

*(1) $I$ is independent.*

*(2) $I$ does not contain any circuit.*

*(3) $I$ is contained in some basis.*

*Proof.* Clearly, if $I$ contains a circuit, then it is not independent, as circuits are dependent sets by definition so $I$ would contain a dependent set. Conversely, if $I$ is dependent, then it contains a non-empty minimal dependent set because $\emptyset$ is independent, and that set is, by definition, a circuit. This proves the equivalence between (1) and (2).

Since $\mathcal{I}(M)$ is a downset w.r.t inclusion, a set $I \in \mathcal{I}(M)$ either is maximal or is contained in a maximal set. Thus, if $I$ is independent it must be contained in a basis (which may be itself). Conversely, if $I \subseteq B$ for some basis $B$, then $I$ is also independent, again by the downset property of $\mathcal{I}(M)$. This proves the equivalence between (1) and (3). $\square$

As mentioned, a matroid can also be defined by its bases or circuits. As we have for independent sets, there are conditions that bases and circuits need to satisfy; for the former, we have the following.

**Proposition 1.8.** *Let $\mathcal{B}$ be the set of bases of a matroid $M$ and $B_1, B_2 \in \mathcal{B}$. Then, for any $x \in B_1 \backslash B_2$ there exists $y \in B_2 \backslash B_1$ such that $(B_1 \backslash x) \cup y$ is also a basis.*

This is sometimes referred to as the *basis exchange property* and makes working with bases particularly useful in some situations. In particular, note that this implies that all bases of a matroid have the same size[1]. This motivates the definition of a crucial concept:

**Definition 1.9.** Let $M$ be a matroid. The *rank* of $M$, denoted by $r(M)$, is the size of a basis of $M$.

Elements that appear in all bases or in none of them play an important role throughout this work, so let us define them properly.

**Definition 1.10.** Let $M$ be a matroid and $x \in E(M)$. We say that $x$ is a *loop* if $x$ is not contained in any basis of $M$ (or, equivalently, if $x$ is not contained in any independent set). We say that $x$ is a *coloop* if $x$ is contained in all bases of $M$.

We need to define some operations that we can do over matroids to work with them.

**Definition 1.11.** Let $M$ be a matroid over $E$. For any subset $X \subset E$, we define

[1] This could also be deduced from the second property in Definition 1.1.

- The *deletion of* $X$, denoted by $M\backslash X$, is the matroid with ground set $E\backslash X$ and independent sets

$$\mathcal{I}(M\backslash X) = \{I\backslash X \mid I \in \mathcal{I}(M)\} = \{I \in \mathcal{I}(M) \mid I \subseteq E\backslash X\}.$$

- The *restriction of* $M$ *to* $X$, denoted by $M|X$, is $M\backslash(E\backslash X)$. Its independent sets are

$$\mathcal{I}(M|X) = \{I \in \mathcal{I}(M) \mid I \subseteq X\}.$$

This way we can now define the rank for any subset of $E$.

**Definition 1.12.** Given a subset $X \subseteq E$, the *rank of* $X$, or $r(X)$, is the rank of the matroid $M|X$. Equivalently, $r(X)$ is the size of the largest independent set contained in $X$.

The notion of rank gives rise to a couple of important definitions. Soon enough we will visualize matroids in a geometrical way, and these definitions will acquire a lot of meaning.

**Definition 1.13.** Let $M$ be a matroid over $E$, and $X \subseteq E$. The *closure* of $X$ is

$$cl_M(X) = \{e \in E \mid r(X) = r(X \cup e)\}.$$

We will usually write $cl(X)$ instead of $cl_M(X)$ unless the context requires more specificity.

*Observation* 1.14. A set $X$ is always contained in its closure, namely $X \subseteq cl(X)$. Also, it is not hard to check that if $X \subseteq Y$ we have $cl(X) \subseteq cl(Y)$.

**Definition 1.15.** A *flat* is a set $F \subseteq E(M)$ such that $cl_M(X) = X$, and we denote the set of flats of a matroid $M$ by $\mathcal{F}(M)$. Additionally,

- If a flat $F$ has rank $r(M) - 1$, we say that $F$ is a *hyperplane*.
- If a flat $F$ is also an independent set (or, equivalently, $r(F) = |F|$) we say that $F$ is a *trivial* flat.

**Definition 1.16.** A flat is *cyclic* if it is a union of circuits. We denote the set of cyclic flats of a matroid $M$ by $\mathcal{Z}(M)$.

Given a matroid $M$, one can define the *dual* of $M$, denoted by $M^*$, as the matroid with ground set $E(M)$ and bases $\mathcal{B}(M^*) = \{E(M)\backslash B \mid B \in \mathcal{B}(M)\}$. We will not be using the dual matroid in this work, but we do need it to define the following notion, which will be needed when we study transversal matroids.

**Definition 1.17.** A *cocircuit* of a matroid $M$ is a circuit of $M^*$.

We now show a characterization of cocircuits without explicitly using the dual matroid.

**Lemma 1.18.** *A set $D$ is a cocircuit of $M$ if and only if $D = E(M)\backslash H$ for some hyperplane $H$ of $M$.*

*Proof.* The set $D$ is a cocircuit if and only if it is a minimal dependent set of $M^*$. In other words, $D$ is minimal such that $D \nsubseteq B^*$ for any $B^*$ basis of $M^*$. Since $B^*$ are of the form $E(M)\backslash B$ for bases $B$ of $M$, the condition that $D \nsubseteq B^*$ is equivalent to $E(M)\backslash D \nsupseteq B$ for bases $B$ of $M$. Equivalently, the set $H = E(M)\backslash D$ is maximal such that $B \nsubseteq H$ for any $B$ basis of $M$. In other words, $H$ is maximal satisfying $r(H) < r(M)$, which is equivalent to being a hyperplane. □

We stated that matroid $M$ can be defined by its independent sets or bases, but it can also be defined by its circuits, its flats, or its rank function. For details on these equivalences, we refer the reader to Chapter 39 of the book [16], where this is explained in detail.

We now define a family of matroids which maximizes the number of independent sets (for a fixed rank) and will be the object of our study in Section 3.

**Definition 1.19.** Let $r, n$ be integers satisfying $r \geq 0$, $n \geq 1$, $r \leq n$. The *uniform matroid* $U_{r,n}$ is the matroid over the ground set $[n] = \{1, \dots, n\}$ whose independent sets are

$$\mathcal{I}(U_{r,n}) = \{X \subseteq E \mid |X| \leq r\}.$$

*Observation* 1.20. From the definition of the independent sets, we can deduce the following facts about uniform matroids.

- **Bases:** $\mathcal{B}(U_{r,n}) = \{X \subseteq E \mid |X| = r\}$.
- **Circuits:** $\mathcal{C}(U_{r,n}) = \{C \subseteq E \mid |C| = r + 1\}$.
- **Rank function:** Let $X \subseteq E$. If $|X| < r$, then $r(C) = |C|$. Otherwise, $r(C) = r$.

Uniform matroids may seem simple and not particularly useful at first sight. However, not only do they help us illustrate important notions but they also contain interesting information, and there are several questions about matroids that are already hard to solve even in this family, as we will see later on.

Finally, we introduce another family of matroids that is interesting by nature; we will not dig deep into their properties, but they will be useful later as illustrative examples of some important notions.

**Proposition 1.21.** *Let $G = (V, E)$ be an undirected graph with a set of forests $\mathcal{F}$. Then, $\mathcal{F}$ are the independent sets of a matroid with ground set $E$.*

*Proof.* It is clear that a subset of a forest is also a forest. Also, if $F, F'$ are two forests with $|F| < |F'|$, then $G[F] := (V, F)$ has fewer connected components than $G[F'] := (V, F')$, therefore $F$ contains a path connecting two components of $G[F']$. Adding any edge $e$ of that path to $F'$ cannot create a cycle, thus $F' \cup e$ is also a forest. □

**Definition 1.22.** Matroids of the form $M = (E(G), \mathcal{F}(G))$ where $G$ is a graph with forest set $\mathcal{F}(G)$ are called *graphic* matroids. We denote by $M = M[G]$ the graphic matroid obtained from the graph $G$.

In graphic matroids, the notions of circuit and loop agree with those of the graph, and bases are maximal spanning forests. Loops are allowed, and so are double edges, which are dependent sets of 2 points, and similarly $k$-tuple edges.

**Example 1.23** (Matroid of $K_4$)**.** Consider the complete graph on 4 vertices $K_4$ with edges labeled as in Figure 1. It is clear that any spanning tree has size 3, so that is the rank of the matroid as well. Since there are no double edges, any set of size 2 is independent, and for sets of size 3 it depends on whether they are incident to three or four vertices. For example, the set $\{1, 3, 4\}$ is a basis while the set $\{2, 5, 6\}$ is dependent.

## 1.3 Geometrical representation

As we stated before, we will visualize low-rank matroids in a geometrical way, by representing their dependencies as affine dependencies. To represent a matroid $M$ of rank $r$ we will use an affine space of dimension $r-1$. Each element of $E(M)$ is represented by a point, in such a way that affinely dependent (respectively, independent) sets of points represent dependent (respectively, independent) sets of the matroid. In general, an independent set $I \in \mathcal{I}(M)$ of size $k$ will be represented by $k$ points in general position in a subspace of dimension $k-1$. For example, an independent $I$ of size 2 needs to generate a line, while an independent set of size 3 needs to generate a plane, and so on.

We allow $k$-tuple points; if a set $X$ of size $k$ has $r(X) = 1$ then we allow the $k$ points of this set to be in the same position of the representation. We hardly ever work with triple points or more, but we commonly see double points appear in our examples as they are illustrative for some notions. In the geometrical representation, the $k$ points of a case like this are represented as points that touch one another, so we can label them separately. We also allow curved lines and similar distortions. If a set of three or more points are colinear, a line will be drawn to show that explicitly.

In the case of a loop, an element $e$ that does not belong to any independent set, we will draw it isolated inside a small square, to represent that it does not generate a line with any of the points outside of it.

**Example 1.24.** Recall the graphic matroid $M[K_4]$ from Example 1.23. Since there are no double edges, we will not see any double point; and $K_4$ having no loops tells us that no loops will appear in the geometrical representation either. Figure 1 shows $K_4$ along with a geometrical representation of it, where one can check all these properties. For instance, the circuit $\{2, 5, 6\}$ in $K_4$ appears as a 3-point line in the representation, and the spanning tree $\{1, 3, 4\}$ is represented by 3 non-colinear points.

Note that, to distinguish the graph and the geometrical representation of the matroid, the latter is embedded in the space where it can be represented (in this case, in the plane, because $M[K_4]$ has rank 3).

Geometrical representations are not always as simple as the one presented in Example 1.24, as shown in Example 1.25 below.

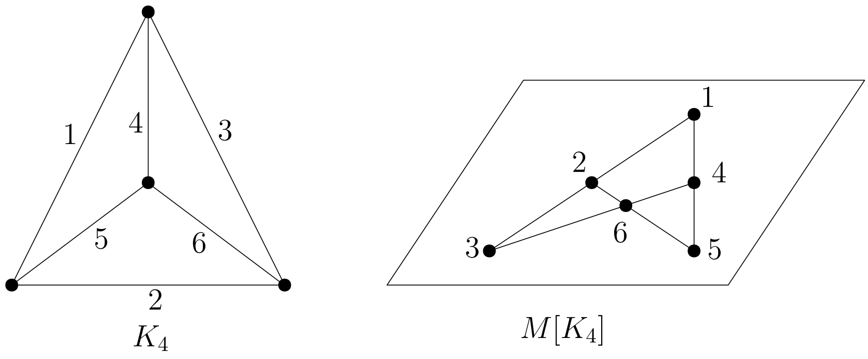


Figure 1: The graph $K_4$ and a geometrical representation of $M[K_4]$

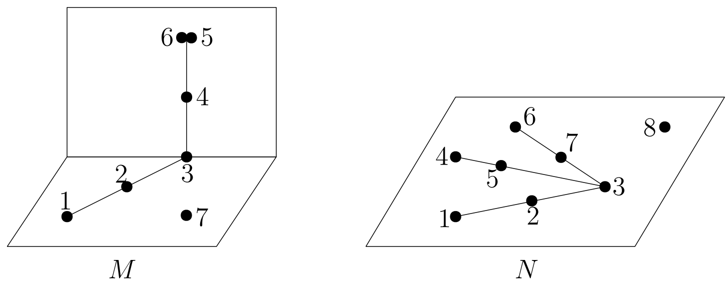


Figure 2: Geometrical representations of two matroids $M$ and $N$

**Example 1.25.** Let $M$ and $N$ be the matroids represented in Figure 2. Since $M$ is represented in a space of dimension 3 we deduce that $r(M) = 4$, and analogously $r(N) = 3$. Take for example the set $\{3, 4, 5, 6\}$ in $M$. Since they are colinear, this is a dependent set. If, instead we consider $\{1, 2, 5, 7\}$ they now generate the whole space, thus telling us that they are an independent set; in fact, a basis, of $M$. Note that not any 4-set is a basis: the set $\{1, 2, 3, 7\}$ is all contained in the lower plane, therefore it is dependent. Moreover, note that any set generating the whole space needs to contain 7, which tells us that it is a coloop.

In $N$, we can observe that any set $I$ with $|I| = 3$ that contains the element 8 will be a basis, as 8 is not colinear with any couple of points. However, not all bases contain 8, for example $\{1, 4, 6\}$ is another basis, hence 8 is not a coloop.

## 1.4 The lattice of flats

### 1.4.1 Properties of flats and rank function

We now introduce some basic results on flats that will be necessary later on and, hopefully, will give the reader some intuition on how they behave.

While the union of flats does not need to be a flat, the intersection does, as we show in Lemma 1.26.

**Lemma 1.26.** *Let $F, G \in \mathcal{F}(M)$ be two flats. Then $F \cap G$ is also a flat.*

*Proof.* We need to prove that $cl(F \cap G) = F \cap G$. We always have $F \cap G \subseteq cl(F \cap G)$ by definition, so let us prove the opposite inclusion. Note that $F \cap G \subseteq F$ and, therefore, $cl(F \cap G) \subseteq cl(F)$. Analogously we have that $cl(F \cap G) \subseteq cl(G)$ and thus $cl(F \cap G) \subseteq cl(F) \cap cl(G)$. Since $F$ and $G$ are flats, we have $cl(F) \cap cl(G) = F \cap G$ and hence $cl(F \cap G) \subseteq F \cap G$. □

Intuition tells us that flats are closely related to independent sets. We now show a result that characterizes independent sets using flats. In particular, this tells us that a matroid is characterized by its set of flats, which we previously stated without proof. Note that this lemma can also be seen as an extension to Lemma 1.7.

**Lemma 1.27.** *Let $M$ be a matroid with ground set $E$, and let $I \subseteq E$. The following are equivalent*

1. *$I$ is independent*
2. *for any $x \in I$ there exists a flat $F \in \mathcal{F}(M)$ such that $x \notin F$ and $I \backslash x \subseteq F$.*

*Proof.* Assume $I$ is independent, let $x \in I$ and consider $F = cl(I \backslash x)$, which satisfies $I \backslash x \subseteq F$. Recall from Definition 1.12 that the rank of $I$ is the size of the biggest independent contained in $I$, therefore $r(I) = |I|$ and $r(I \backslash x) = |I| - 1$. Also, since the closure operator does not alter the rank, we have $r(F) = r(I \backslash x) = |I| - 1$. Now, if we were to have $x \in F$ then we would have $I \subseteq F$ so $r(F) = |I|$. Therefore $x$ cannot be in $F$.

Now assuming (2), suppose $I$ is not independent. Then, by Lemma 1.7 it contains a circuit $C \subseteq I$. Let $x \in C$, and suppose a flat $F$ contains $I \backslash x$. Then, $F$ also contains $C \backslash x$. But since $r(C) = r(C \backslash x)$ then necessarily $x \in cl(C \backslash x) \subseteq F$. This contradicts (2), thus proving that $I$ is independent. □

The following lemma shows that the rank function has the so-called *sub-modular* property. This will help us understand the notion of modular cuts that will show up in the next section.

**Lemma 1.28.** *Let $F, G \in \mathcal{F}(M)$ be two flats of a matroid $M$. Then,*

$$r(F \cup G) + r(F \cap G) \leq r(F) + r(G). \tag{1}$$

*Proof.* Let $I$ be a basis of $F \cap G$. Note that we can extend $I$ using the independent sets property to obtain a basis $J$ of $F \cup G$ such that $I \subseteq J$. We have $r(F \cap G) = |I|$ and $r(F \cup G) = |J|$. Now we partition $I$ into $J = J_1 \cup J_2 \cup J_3$ where

$$\begin{aligned} J_1 &:= J \cap (F \backslash G) \\ J_2 &:= J \cap (G \backslash F) \\ J_3 &:= J \cap (F \cap G). \end{aligned}$$

Since $J_1, J_2, J_3$ are disjoint, it is clear that $|J| = |J_1| + |J_2| + |J_3|$. Note that we constructed $J$ by adding elements to $I$, so $I \subseteq J$ and in particular $I \subseteq J_3$ because $I \subseteq F \cap G$. Since $J_3$ is independent, we must have $J_3 = I$ by maximality of $I$. Putting all these together we have that

$$r(F \cup G) + r(F \cap G) = |J_1| + |J_2| + |J_3| + |I| = |J_1| + |J_2| + 2|I|.$$

On the other hand, note that $I \cup J_1 \subseteq F$ is an independent set, therefore $r(F) \geq |I| + |J_1|$. Analogously, $r(G) \geq |I| + |J_2|$. Finally,

$$r(F \cup G) + r(F \cap G) = |J_1| + |J_2| + 2|I| = (|I| + |J_1|) + (|I| + |J_2|) \leq r(F) + r(G).$$

□

### 1.4.2 Structure

The set of flats of a matroid $M$ will play an important role in studying the extensions of the matroid $M$. This set has a lattice structure, as we show in Proposition 1.29.

**Proposition 1.29.** *The set $\mathcal{F}(M)$ is a lattice ordered by inclusion. If $F, G \in \mathcal{F}(M)$, their join is $F \vee G = cl(F \cup G)$ and their meet is $F \wedge G = F \cap G$.*

*Proof.* Let $F, G \in \mathcal{F}(M)$. The fact that $cl(F \cup G) \in \mathcal{F}(M)$ is a consequence of the closure satisfying the usual closure condition $cl(cl(X)) = cl(X)$ for any set $X \subseteq E(M)$. Also, it is the minimal flat containing both $F$ and $G$ by the definition of closure. For the meet, we know that the intersection is a flat due to Lemma 1.26, and it is clearly the maximal flat contained in both (it is the maximal set contained in both). □

The lattice-theoretic approach to matroid theory is especially interesting, as flat lattices characterize in fact a well-known family of lattices; the *geometrical* lattices (see [15], Theorem 1.7.5).

We will not dig deeper in this part; instead, we now show an example of a very particular lattice of flats that will play an important role later on.

**Example 1.30** (Lattice of flats of $U_{r,n}$)**.** Recalling the definition of $U_{r,n}$, it is clear that all independent sets are flats, namely $\mathcal{I}(U_{r,n}) \subseteq \mathcal{F}(U_{r,n}) = \{X \subseteq [n] \mid |X| < r\}$. Moreover, the only flat that is not independent is $\{[n]\} \in \mathcal{F}(U_{r,n})$. In particular, for ranks lower than $r$, the poset is identical to that part of the boolean lattice.

Figure 3 shows the lattice of flats $\mathcal{F}(U_{3,4})$. Inside the dashed rectangle, the lattice is the same as the boolean lattice $\mathcal{B}_4$, and then it collapses all into the full flat $\{1, 2, 3, 4\}$. This diagram illustrates how the lattice $\mathcal{F}(U_{r,n})$ is in general: a boolean lattice in ranks $0, 1, \dots, r-1$ and then the full flat over it.

We will dig deeper into this lattice in Section 3.

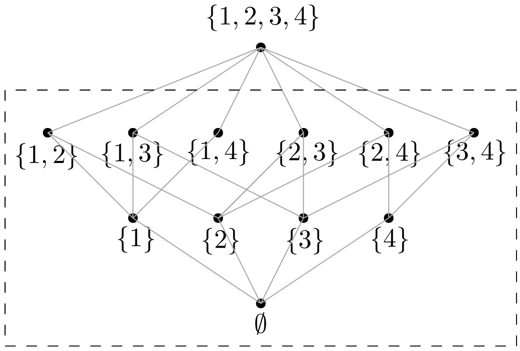


Figure 3: Lattice of flats of $U_{3,4}$

## 1.5 Extensions of a matroid

We proceed to review the fundamental results on matroid extensions and their relation with the lattice of flats. For more details, we refer the reader to [5] and Oxley's book, Section 7.2 [15].

**Definition 1.31.** A *single-element extension of a matroid* $M$ is a matroid $N$ with $E(N) = E(M) \cup x$ such that $N \backslash x = M$. If $r(M) = r(N)$ we say that the extension is *rank-preserving*.

*Observation* 1.32 (on non-rank-preserving extensions). Note that the only way to construct an extension that is not rank-preserving is by adding a coloop: adding the new element to all bases. This extension needs to be taken into account in some contexts, but it is not of much more interest, as it contains essentially the same information as the original matroid. This is why we will mostly work with single-element rank-preserving extensions. From now on, in any context where we talk about extensions of a matroid, we will refer to single-element rank-preserving extensions, unless stated otherwise.

### 1.5.1 Modular cuts

Essentially, we will study extensions of a matroid $M$ by studying how flats change when we adjoin the new element $x$. If $N$ is an extension of $M$ with $M = N \backslash x$, we can partition the flats of $M$ in terms of $N$ as $\mathcal{F}(M) = \mathcal{M}(N) \cup \mathcal{C}(N) \cup \mathcal{I}(N)$ where

$$\mathcal{M}(N) := \{F \in \mathcal{F}(M) \mid F \notin \mathcal{F}(N) \text{ and } F \cup x \in \mathcal{F}(N)\}$$
$$\mathcal{C}(N) := \{F \in \mathcal{F}(M) \mid F \in \mathcal{F}(N) \text{ and } F \cup x \notin \mathcal{F}(N)\}$$
$$\mathcal{D}(N) := \{F \in \mathcal{F}(M) \mid F \in \mathcal{F}(N) \text{ and } F \cup x \in \mathcal{F}(N)\}$$

One can see that, in fact, $\mathcal{C}(N)$ and $\mathcal{D}(N)$ can be obtained from $\mathcal{M}(N)$, thus our main focus of study will be the set $\mathcal{M} = \mathcal{M}(N)$.

We saw in Lemma 1.28 that the rank function was sub-modular; it would be natural to consider for which pairs of flats $F, G$ would equality hold in the sub-modular property.

**Definition 1.33.** Let $F, G \in \mathcal{F}(M)$ be two flats of a matroid $M$. We say that $F, G$ form a *modular pair* if the following is satisfied:

$$r(F \cup G) + r(F \cap G) = r(F) + r(G).$$

**Definition 1.34.** Let $M$ be a matroid and $\mathcal{F}(M)$ its lattice of flats. A subset $\mathcal{M} \subseteq \mathcal{F}(M)$ is a *modular cut* if it is an upset such that, if $F, G \in \mathcal{M}$ form a modular pair, then $F \cap G \in \mathcal{M}$.

*Observation* 1.35. The intersection of modular cuts is a modular cut.

Modular cuts play an essential role in understanding single-element extensions of matroids, as the following result by Crapo shows. For a proof of this result, we refer the reader to [15, Thm 7.2.3].

**Theorem 1.36.** *[[9]] Single-element extensions of a matroid $M$ are in one-to-one correspondence with modular cuts in $\mathcal{F}(M)$.*

Let $M$ be a matroid and $N$ an extension, namely $M = N \backslash x$. We associate the extension $N$ with the modular cut $\mathcal{M} = \mathcal{M}(N) \subseteq \mathcal{F}(M)$ defined at the beginning of this section.

Long story short, the modular cut $\mathcal{M}$ determines in which flats the element $x$ was added, thus defining the extension. The modular pair condition determines in which situations we have to add the new point to the intersection of two flats if we want to add it to both.

**Notation:** Let $M$ be a matroid and $\mathcal{M} \subseteq \mathcal{F}(M)$ a modular cut. We will denote the extension of $M$ given by Theorem 1.36 by $M +_{\mathcal{M}} x$. We will call $M +_{\mathcal{M}} x$ the extension of $M$ corresponding to $\mathcal{M}$.

Since modular cuts are upsets, they are characterized by their minimal elements and that is an important trait that we will make use of. To do so, we define the upset of a collection of sets.

**Definition 1.37.** Let $M$ be a matroid and $\{F_1, \dots, F_k\} \subseteq \mathcal{F}(M)$ a family of flats. We define

$$\{F_1, \dots, F_k\}^+ := \{F' \in \mathcal{F}(M) \mid F_i \subseteq F' \quad \text{for some } i \in [k]\}.$$

We distinguish in the following definition the case $k = 1$ (and write $F^+$ instead of $\{F\}^+$).

**Definition 1.38.** Let $M$ be a matroid and $F \in \mathcal{F}(M)$ a flat. Then, the extension given by the modular cut $\mathcal{M} = F^+$ is called the *principal extension* of $M$ at $F$. We denote it by $M +_F x$ instead of $M +_{\mathcal{M}} x$.

*Observation* 1.39. In Definition 1.37 we did not require $F_1, \dots, F_r$ to be an antichain. In particular, if $F_i \subseteq X$ for some element $X$, we would have $\{F_1, \dots, F_r\}^+ = \{F_1, \dots, F_r, X\}^+$. However, unless stated otherwise, when denoting a modular cut by $\mathcal{M} = \{F_1, \dots, F_r\}^+$ we will assume $F_1, \dots, F_r$ to be the minimal elements of $\mathcal{M}$ (thus, an antichain).

**Example 1.40** (Free extension)**.** Let $M$ be a matroid and $F \in \mathcal{F}(M)$ a flat, and consider the principal extension $M +_F x$. If we take $F = E(M)$, then we obtain the so-called *free extension* of $M$, which is usually denoted by $M + x$. This extension consists of adding $x$ as freely as possible without increasing the rank.

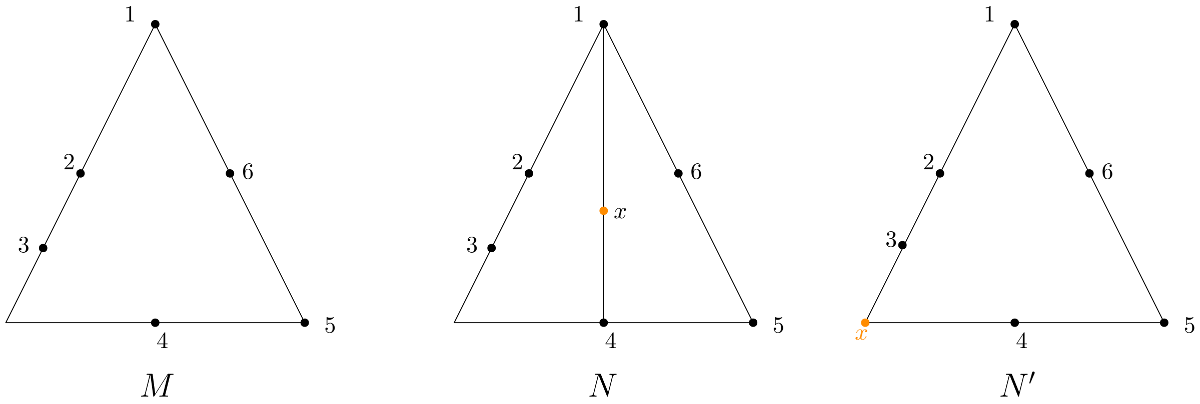

Figure 4: A matroid $M$ and two extensions $N$ and $N'$

Namely, $x$ does not belong to any non-necessary flat, only to the total. Equivalently, if $B$ is a basis of $M$, then $B \cup x$ is a circuit in $M + x$.

In the opposite case, if we take $F = cl(\emptyset)$ and consider $M +_F x$, the extension we obtain has the same independent sets. Equivalently, the new element $x$ is a loop because $\{x\}$ is not an independent set.

Let $e \in E(M)$ be a non-loop element and consider the flat $F = \{e\}$. Then, the corresponding principal extension $M +_F x$ consists of adding $x$ parallel to $e$. Equivalently, if $x \in I$ for some independent set $I$, then $(I \backslash x) \cup e$ is also independent.

In the previously mentioned Oxleys's book Theorem 7.2.3 (from [15]) one can also find the proof of the following result, which identifies the rank function of an extension using its corresponding modular cut.

**Theorem 1.41.** *Let $M$ be a matroid, $\mathcal{M} \subseteq \mathcal{F}(M)$ a modular cut and $N$ the corresponding extension, namely $M = N \backslash x$. Then, the rank function of $N$ is given by*

$$r_N(X) = \begin{cases} r_M(X) + 1 & \text{if } x \in X \text{ and } cl_M(X) \notin \mathcal{M}, \\ r_M(X) & \text{otherwise.} \end{cases}$$

So, essentially, the rank of all subsets $X \subseteq E(M)$ is unaltered, while sets including the new element may increase their rank depending on their closure.

**Example 1.42.** Let $M$ be the matroid given by the geometrical representation[2] in Figure 4 and consider the extensions $N$ and $N'$.

In $N$, the point $x$ was added in the line $\{1, 4\}$, and also in all flats containing that line, which in this case is only the full flat $[6] = \{1, 2, 3, 4, 5, 6\}$. Therefore, the modular cut of this extension is $\mathcal{M} = \{\{1, 4\}, [6]\}$.

In $N'$, the point $x$ was added in the lines $\{1, 2, 3\}$ and $\{4, 5\}$, and again in the full flat $[6]$, so the modular cut of this extension is $\mathcal{M}' = \{\{1, 2, 3\}, \{4, 5\}, [6]\}$.

[2] Note that we omit the ambient space (the plane) because all three matroids are of rank 3.

### 1.5.2 Modular cuts and the weak order

To study the poset structure of the set of extensions of a matroid, let us first define an order among matroids with the same ground set, called the weak order.

**Definition 1.43.** (**Weak order**) Let $M, N$ be two matroids over $E$. We say that $M \leq_w N$ if all independent sets of $M$ are also independent in $N$, namely

$$M \leq_w N \iff \mathcal{I}(M) \subseteq \mathcal{I}(N).$$

This relation $\leq_w$ is called the *weak order*.

Intuitively, $M \leq_w N$ means that $N$ is freer than $M$, as there are less dependencies between the points.

The set of all single-element rank-preserving extensions of a matroid $M$, denoted by $\mathcal{E}(M)$, has a lattice structure under the weak order. We do not discuss in detail this lattice and its properties, as they are not pertinent to the objectives of this work.

We have introduced modular cuts, which are in bijection with extensions, and know that they have a natural order (the inclusion of sets). It is natural to ask what relation is there between the order of two extensions and the order of their modular cuts, and that is what we explain now:

**Proposition 1.44.** *Let $M_1, M_2$ be extensions of a matroid $M$, and $\mathcal{M}_1, \mathcal{M}_2$ the associated modular cuts. Then, $M_1 \leq_w M_2$ if and only if $\mathcal{M}_1 \supseteq \mathcal{M}_2$.*

*Proof.* First, assume $\mathcal{M}_1 \not\supseteq \mathcal{M}_2$ and let us see that $M_1 \not\leq_w M_2$. Let $F \in \mathcal{M}_2 \backslash \mathcal{M}_1$ and $B$ be a basis of $F$. Since $F \in \mathcal{F}(M_1)$, we have $r_{M_1}(B \cup x) = r(F) + 1 = |B \cup x|$ and therefore $B \cup x$ is an independent set in $M_1$. At the same time, Theorem 1.41 shows that $r_{M_2}(F \cup x) = r(F)$ because $F \in \mathcal{M}_2$. This implies that $B \cup x$ is not an independent set in $M_2$, and therefore $M_1 \not\leq_w M_2$.

Now assume $M_1 \not\leq_w M_2$ and let us see that $\mathcal{M}_1 \not\supseteq \mathcal{M}_2$. This means that there is an independent set in $M_1$ that is not independent in $M_2$. Note that this set necessarily includes the newly added element $x$, as otherwise, the rank is unaltered. Thus, let $I \cup x$ be an independent set in $M_1$ that is not independent in $M_2$, and let $F = cl(I)$. Then, we claim that $F \in \mathcal{M}_2 \backslash \mathcal{M}_1$. Clearly $F \notin \mathcal{M}_1$ because $r_{M_1}(F \cup x) = r(F) + 1$, and also $F \in \mathcal{M}_2$ because $r_{M_2}(F \cup x) = r(F)$. □

With this equivalence, we can now explicitly compute the join of two extensions $M_1$ and $M_2$ of a matroid $M$ using their modular cuts.

**Proposition 1.45.** *Let $M_1, M_2$ be extensions of a matroid $M$, and $\mathcal{M}_1, \mathcal{M}_2$ the associated modular cuts. Then, the modular cut of the join $M_1 \vee M_2$ is $\mathcal{M}_1 \vee \mathcal{M}_1 = \mathcal{M}_1 \cap \mathcal{M}_2$.*

*Proof.* By definition, the join $M_1 \vee M_2$ is the smallest extension (with respect to the weak order) such that $M_1 \leq_w M_1 \vee M_2$ and $M_2 \leq_w M_1 \vee M_2$. Using Proposition 1.44, we can reformulate into modular cuts: the modular cut $\mathcal{M}_1 \vee \mathcal{M}_2$ of $M_1 \vee M_2$ is the maximal modular cut satisfying $\mathcal{M}_1 \vee \mathcal{M}_2 \subseteq \mathcal{M}_1$ and $\mathcal{M}_1 \vee \mathcal{M}_2 \subseteq \mathcal{M}_2$. Equivalently $\mathcal{M}_1 \vee \mathcal{M}_2$ is the maximal modular cut satisfying $\mathcal{M}_1 \vee \mathcal{M}_2 \subseteq \mathcal{M}_1 \cap \mathcal{M}_2$. As we observed (and is simple to check) that the intersection of modular cuts is a modular cut, it must be $\mathcal{M}_1 \vee \mathcal{M}_2 = \mathcal{M}_1 \cap \mathcal{M}_2$. □

*Observation* 1.46. Using the same arguments as in the last proof, we deduce that the modular cut $\mathcal{M}_1 \wedge \mathcal{M}_2$ of the meet $M_1 \wedge M_2$ must be the largest modular cut containing both $\mathcal{M}_1 \cup \mathcal{M}_2$. This modular cut does not have such a simple expression as the join; we now try to give some intuition on how it looks like.

We have that $\mathcal{M}_1 \wedge \mathcal{M}_2$ does not need to be a modular cut. Indeed, pairs of elements $A \in \mathcal{M}_1$, $B \in \mathcal{M}_2$ may form modular pairs whose intersection $A \cap B$ does not lie in either $\mathcal{M}_1$ or $\mathcal{M}_2$. To construct the modular cut $\mathcal{M}_1 \wedge \mathcal{M}_2$, one would need to add all such intersections. Moreover, these newly added intersections may form new modular pairs with elements of $\mathcal{M}_1 \cup \mathcal{M}_2$, or even with other intersections of this kind. To compute $\mathcal{M}_1 \wedge \mathcal{M}_2$ one would need to keep repeating this process (finding modular pairs and adding the intersection) until all intersections of modular pairs were present.

# 2. Transversal matroids. Presentations, flats and extensions

In this section, we introduce our main object of study, transversal matroids, and go over the main results that we will need in Sections 3 and 4.

For definitions and fundamental results on transversal matroids, we suggest J. Bonin's introduction [4]. As for main results on single-element rank-preserving transversal extensions, we follow Bonin & de Mier results [5].

## 2.1 Transversal matroids

We already know that matroids can arise from several combinatorial objects. Representable matroids arise from linear dependency, graphical matroids arise from graphs, and so on. The family that we are interested in arises from set systems, which we now define.

**Definition 2.1.** A *set system* $\mathcal{A} = \{A_1, \dots, A_r\}$ over a ground set $E$ is a multiset of subsets of $E$.

Since set systems are multisets, from now on $\{A_1, \dots, A_r\}$ will denote a multiset, thus allowing repetitions among the $A_i$'s.

Let us build a matroid $M = M[\mathcal{A}]$ from a set system $\mathcal{A}$ over a ground set $E$. Let $G$ be the bipartite graph with $V(G) = E \cup \mathcal{A}$ and edges $E(G) = \{(x, A_i) \mid x \in A_i\}$ given by the containment relation. We define $M = M[\mathcal{A}]$ as the matroid with ground set $E$ where a set $X \subseteq E$ is independent if there exists a matching in $G$ that covers $X$. We will prove in this section that this is, indeed, a matroid, but let us look at an example first.

**Example 2.2.** Let the ground set be $E = [6]$ and consider the set system

$$A_1 = \{1, 3, 4\}; A_2 = \{3, 5\}; A_3 = \{2, 5, 6\}.$$

Looking at Figure 5, we can see that

- The set $\{4, 6\}$ is independent because it can be covered by a matching.
- The set $\{1, 3, 6\}$ is a basis because its matching uses all sets of the presentation.
- The set $\{1, 4\}$ is dependent because there is no possible matching that covers both (because, for both, the only neighbor is $A_1$).

This notion can also be stated in terms of partial transversals, which we now define.

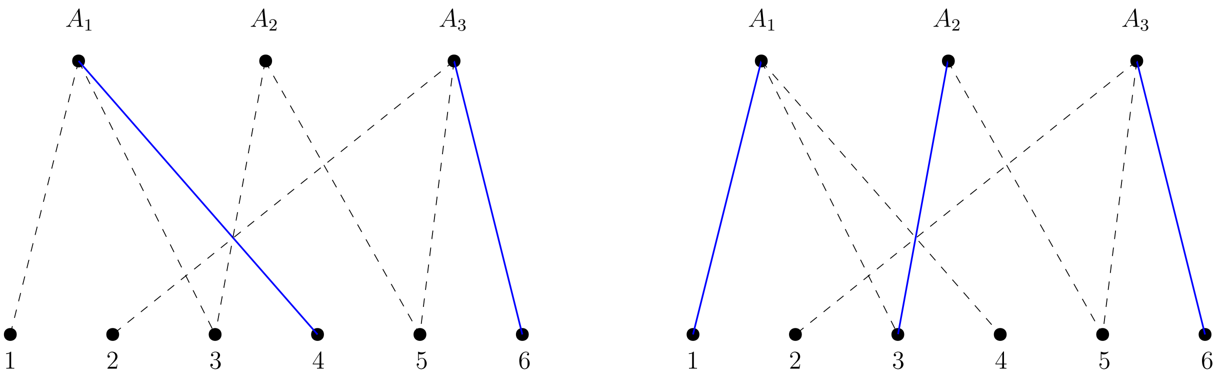

Figure 5: Matchings of $E = [6]$ into a set system $A_1, A_2, A_3$.

**Definition 2.3.** Let $\mathcal{A} = \{A_1, \dots, A_r\}$ be a set system over $E$ and $X = \{x_1, \dots, x_k\} \subseteq E$ a subset. We say that $X$ is a *partial transversal* of $\mathcal{A}$ if there exists an injection $\phi : X \to \mathcal{A}$ such that, for any $x \in X$ we have $x \in \phi(x)$.

*Observation* 2.4. Equivalently, we could define the injection $\phi$ to be into the indices of $\mathcal{A}$, as $\phi : X \to [r]$. In this case, the condition for $X$ to be a partial transversal is that $x \in A_{\phi(x)}$ for any $x \in X$. We may use the former or the latter depending on the situation.

The following result, due to Edmonds & Fulkerson ([10]) shows that $M[\mathcal{A}]$ is indeed a matroid.

**Theorem 2.5.** *Let $\mathcal{A} = \{A_1, \dots, A_r\}$ be a set system over $E$. The partial transversals of $\mathcal{A}$ are the independent sets of a matroid.*

*Proof.* Let $I$ be a partial transversal. We need to verify conditions from Definition 1.1. Condition 1 is clearly satisfied, as the same indices that match $I$ into $\mathcal{A}$ can match any subset of $I$ as well. Now let $J$ be a partial transversal with $|I| < |J|$, and recall the bipartite graph intuition of the partial transversals. Let $E_I$ be the edges covering $I$ and $E_J$ the edges covering $J$. Let us color the edges $E_I$ red, and the edges $E_J$ blue, and consider the graph $G$ induced by these edges $E_I \cup E_J$. Evidently, since $|I| < |J|$ we have $|E_I| < |E_J|$. In Figure 6, we see an example with $I = \{1, 2, 4\}$ (colored RED) and $J = \{1, 3, 4, 9\}$ (colored BLUE). It is not hard to see that the vertex that can be added to $I$ is 3; however, we cannot simply join the matchings, as the blue edge that matches 3 is using $A_1$, which is already incident to a red edge.

If there is a blue edge $(x, A_i)$ such that $x \notin I$ with no red edge being incident to $A_i$, then we can simply add $x$ to $I$ using that edge. This yields a matching of $I \cup x$, proving that $I \cup x$ is independent.

Assume now that there is no such edge, and for each $x \in J$ build the alternate color walk[3] $P_x$ by following the edges (there is no ambiguity, as the degree of each vertex is at most 2).

Let $E_x = \{e_1, \dots, e_t\}$ be the ordered edges in walk $W_x$. There are three possible patterns for the walk $W_x$, pictured in Figure 7, and they are the following:

[3] We use walks instead of paths because we may end up in the same vertex, thus yielding a cycle.

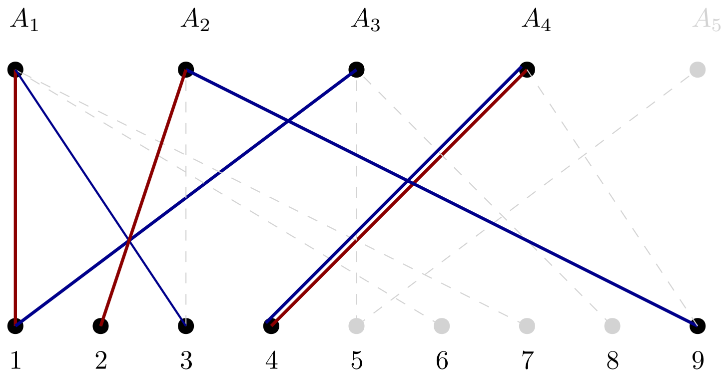


Figure 6: Matchings for two independent sets $I, J$ with $|I| < |J|$

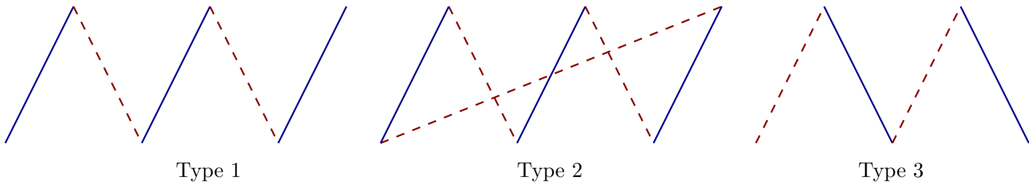


Figure 7: Possible types of alternate walks

1. $W_x$ is a path and the first and last edges have the same color.

2. $W_x$ is a cycle[4], in which case $E_x$ contains as many blue edges as red edges.

3. $W_x$ is a path and the first and last edges have different colors, in which case $E_x$ again contains as many blue edges as red edges.

Since $|E_I| < |E_J|$, there must be at least one walk $W_x$ of type 3 for which the color of the first and last edge is BLUE. In that case, there is a unique vertex $x \in E$ incident in the walk such that $x \in J \backslash I$. All the other vertices, which are in $I$, can use a different edge to form the matching, so we can add $x$ to $I$ using the blue edge that contains it, and the new set $I \cup x$ is therefore independent.

□

**Definition 2.6.** Matroids that can be constructed as $M = M[\mathcal{A}]$ for some set system $\mathcal{A}$ are called *transversal matroids* and the set system $\mathcal{A}$ is said to be a *presentation* of $M$.

The previously mentioned family of uniform matroids, which we will be studying deeper in Section 3, is a simple example of transversal matroids, as shown in the following observation.

*Observation* 2.7. Consider the set system $\mathcal{A} = \{A_1, \ldots, A_r\}$ where $A_i = [n]$ for all $i \in [k]$, and let $X \subseteq [n]$. Note that, if $|X| \leq r$, there will always be a matching of $X$ into $\mathcal{A}$; in fact, the induced bipartite graph is a complete graph. If, instead $|X| > r$, there will never be a matching of $X$ into $\mathcal{A}$. This shows that $\mathcal{A}$ is a

[4] An even cycle, as the graph is bipartite.

presentation of the uniform matroid on $n$ elements and rank $r$, namely $U_{r,n} = M[\mathcal{A}]$. In particular, $U_{r,n}$ is a transversal matroid.

**Definition 2.8.** Let $\mathcal{A} = \{A_1, \dots, A_r\}$ be a presentation of $M$, and $X \subseteq E(M)$ a subset. The *$\mathcal{A}$-support of $X$*, denoted by $s_{\mathcal{A}}(X)$ or simply $s(X)$, is

$$s_{\mathcal{A}}(X) = \{i \mid X \cap A_i \neq \emptyset\}.$$

The following example shows that a matroid can (and most often will) have several presentations.

**Example 2.9.** Consider the uniform matroid $U_{3,6}$. It is clear that one presentation is $A_1 = A_2 = A_3 = [6]$. With this, we are constructing the graph $K_{3,6}$, and any 3-set of $E$ will have a matching (in fact, it will have 6 possible matchings). It is not hard to check that the set system

$$A_1 = \{1, 2, 3, 4\}; \quad A_2 = \{1, 2, 3, 4, 5\}; \quad A_3 = \{1, 2, 3, 4, 6\};$$

is also a presentation of the uniform matroid.

## 2.2 Geometrical representation of transversal matroids

Geometrical representations of transversal matroids are very particular, as the next result from Brylawski ([8]) shows[5].

**Theorem 2.10.** *A matroid is transversal if and only if it can be represented on a simplex in such a way that every cyclic flat $F$ is the set of points in a face of the simplex of dimension $r(F) - 1$.*

In particular, a transversal matroid of rank $r$ can be represented on the $(r - 1)$-simplex. These representations, usually called *simplex-representations*, are the ones that we will use from now on when dealing with transversal matroids.

If the reader is not familiar with the notion, for our purposes, it suffices to consider the $k$-simplex as the convex hull of $k + 1$ affinely independent points in the space $\mathbb{R}^k$. For $k \leq 4$ the simplexes are simply points, line segments, triangles, and tetrahedra.

The convex hull of any non-empty set of a simplex is called a *face*[6]. Intuitively, faces are the simplexes themselves (points, segments, etc.) that make up the whole simplex. Faces of dimension $k - 1$ are called *facets*.

To give some intuition behind this theorem, let us consider some low-rank cases of cyclic flats to see what it means. Suppose that a transversal matroid $M$ has a double point, namely a flat $F$ of size 2 and

[5] Brylawski uses the notion of (transversal) *pregeometries* to denote (transversal) matroids and *free simplicial geometry* to denote a simplex representation with the conditions mentioned Theorem 2.10.

[6] Note that the whole simplex is a face itself.

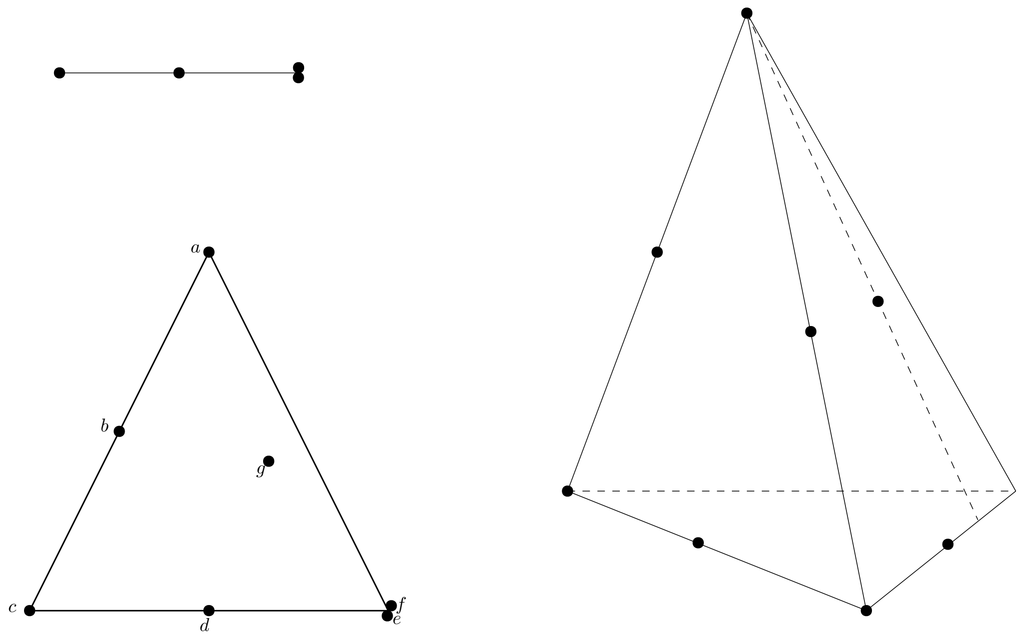


Figure 8: Geometrical representation of transversal matroids of rank 2, 4 and 4, respectively

rank 1. As this flat is cyclic (because it is itself a cycle), the double point needs to be on a vertex in a geometrical representation.

If $M$ also has three or more colinear points, then that cyclic flat will have to be a line of the simplex in the geometrical representation. In general, Brylawski shows that if $M$ is transversal then we will always find a geometrical representation satisfying this for all cyclic flats at the same time. Also, conversely, it shows that if we manage to build such a geometrical representation, then the matroid is transversal.

Example 2.11 below shows examples of simplex-representations for some transversal matroids of different ranks.

**Example 2.11.** In Figure 8 we have geometrical representations of 3 different transversal matroids of rank 2, 3 and 4, represented on the 1-simplex, 2-simplex and 3-simplex, respectively. We use dashed lines to show what points lie in a facet when they are not in the edges, as happens in the rank 4 example.

Let us focus on the labeled one; the matroid of rank 3 drawn in the triangle. For a set to be a basis we need three non-colinear points, such as $\{a, d, e\}$. The only 2-set that is not independent is the double point $\{e, f\}$, as it does not create a segment.

In the geometrical representations, the notions of closure and flats agree with those notions in the affine setting. For example, the closure of the set $\{a, b\}$ is $\{a, b, c\}$ because $c$ lies in the line generated by $\{a, b\}$. The set $\{a, b, c\}$ is, therefore, a flat because it contains all its closure. Cyclic hyperplanes of the matroid will always be contained in a face of the simplex. For example $\{a, b, c\}$ as a flat is one of the three lines in the boundary; but this also tells us that the point $\{a\}$ is itself a flat, as it is a vertex of the triangle. The same happens with the set $\{e, f\}$, as the double point is also a vertex of the triangle.

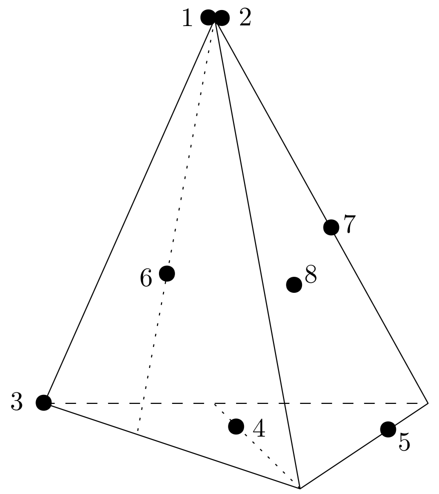


Figure 9: Geometrical representation of a matroid $M$

### 2.2.1 Relation between presentation and geometrical representation

The notions of presentation and geometrical representation of a transversal matroid are, in a way, equivalent. We can build a presentation from a geometrical representation and vice-versa, thus yielding a one-to-one correspondence. Consequently, we will usually identify presentations with their geometrical representations and use the former or the latter at our convenience.

In a presentation, sets correspond to complements of facets of the simplex in a simplex-representation. If a set $A$ belongs to the presentation, then in the corresponding representation there will be a facet of the simplex containing $E(M)\backslash A$.

Conversely, each facet of the simplex in a geometrical representation of $M$ corresponds to a complement of a set in the presentation; thus we can obtain the presentation by computing the complements of each facet of the simplex.

This will probably be much clearer with an example.

**Example 2.12.** Consider the matroid represented in Figure 9, and let us construct the corresponding presentation.

Fix one facet $F_1$, for example, the lower one containing 3, 4, and 5. This corresponds to a set in the presentation as $A_1 = E(M)\backslash F_1 = [9]\backslash\{3,4,5\} = \{1,2,6,7,8\}$. Similarly, from $F_2 = \{1,2,5,7\}$ we get $A_2 = \{3,4,6,8\}$; from $F_3 = \{1,2,3,6\}$ we get $A_3 = \{4,5,7,8\}$ and from $F_4 = \{1,2,3,7\}$ we get $A_4 = \{4,5,6,8\}$.

Note how the double point 1, 2 being in the intersection of three facets means that it only belongs to one set in the presentation. On the other hand, point 8 "floating in the middle" belongs to all sets in the presentation.

To finish this section we state without proof a result that, hopefully, will help to reinforce the relation

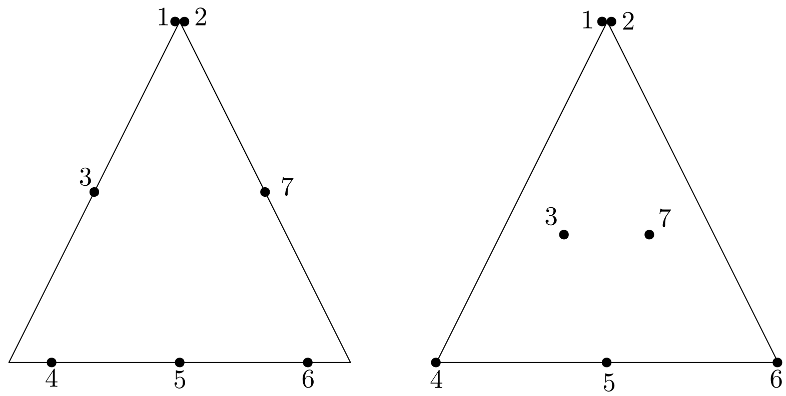


Figure 10: Geometrical representations of minimal presentations $\mathcal{A}_1$ and $\mathcal{A}_2$ of $M$

between cyclic flats and presentations (thus, geometrical representations).

**Proposition 2.13** ([5], Corollary 2.5)**.** *Let $\mathcal{A} = \{A_1, \dots, A_r\}$ be a presentation of a matroid $M$ and $F$ a cyclic flat. Then, $|s_{\mathcal{A}}(F) = r(F)|$. In other words, there exist exactly $r(F)$ integers $i \in [r]$ such that $F \cap A_i \neq \emptyset$.*

## 2.3 Minimal and maximal presentations

We will think of the set of presentations of a matroid as a poset. The order will be given by inclusion of all sets of the presentation.

**Definition 2.14.** Given two presentations $\mathcal{A} = \{A_i \mid i \in [r]\}$ and $\mathcal{B} = \{B_i \mid i \in [r]\}$ of $M$, we write $\mathcal{A} \prec \mathcal{B}$ if, for some permutation $\phi : [r] \to [r]$, we have $A_i \subseteq B_{\phi(i)}$ for all $i \in [r]$.

We are particularly interested in minimal and maximal presentations with respect to this order.

The following example shows that a matroid can have different minimal presentations. In contrast, Proposition 2.16 shows that there is only one maximal presentation.

**Example 2.15.** Consider the following two presentations:

$$\mathcal{A}_1 = \{\{1,2,3,7\}, \{3,4,5,6\}, \{4,5,6,7\}\}$$

$$\mathcal{A}_2 = \{\{1,2,3,7\}, \{3,4,5,7\}, \{3,5,6,7\}\}$$

It is not hard to check that both presentations give the same matroid, namely $M = M[\mathcal{A}_1] = M[\mathcal{A}_2]$. Visually in Figure 10, it is clear that all lines are preserved and no dependencies are added.

At the same time, one can see that the presentation obtained by removing any element from any set in either $\mathcal{A}_1$ or $\mathcal{A}_2$ is not a presentation of $M$[7]. This means that the presentations $\mathcal{A}_1$ and $\mathcal{A}_2$ are minimal

[7] This can be seen with the tools we have provided so far, but it requires a lot of computations. We will soon provide tools to make this check easily and also a geometrical intuition about it.

and that a matroid can have several minimal presentations.

The following result plays a very important role, stating the uniqueness of the maximal presentation, as opposed to that of the minimal presentations. For a proof, see Bondy's paper [2] (a sequel to Bondy & Welsh [3]).

**Proposition 2.16.** *Let $M$ be a matroid of rank $r$. Then, $M$ has a unique maximal presentation with $r$ sets.*

Bondy and Welsh [3] showed how to construct such a unique maximal matroid with the following result, which we implemented in Sage for our algorithm in Section 4.

**Lemma 2.17.** *([3]) Let $\{A_1, \dots, A_r\}$ be a presentation of $M$ and $e \in E(M)\backslash A_i$ for some $i \in [r]$. Then, $\{A_1, \dots, A_i \cup e, \dots, A_r\}$ is a presentation of $M$ if and only if $e$ is a coloop of $M\backslash A_i$.*

As we will see later on, the special case of a transversal matroid having only one presentation is easy to handle in terms of transversal extensions. Moreover, if the matroid has only one minimal presentation we will oftentimes be in a very handleable situation as well.

We now show a simple technical result about presentations that we will need in the future, which shows how to construct a presentation for a deletion of a transversal matroid.

**Lemma 2.18.** *If $\mathcal{A} = \{A_1, \dots, A_r\}$ is a presentation of $M$ and $X \subset E$, then $M\backslash X$ is also transversal and $\mathcal{A}' = \{A_1\backslash X, \dots, A_r\backslash X\}$ is a presentation of $M\backslash X$.*

*Proof.* We will prove that the independent sets of $M\backslash X$ and $M[\mathcal{A}']$ are the same, so $\mathcal{I}(M\backslash X) = \mathcal{I}(M[\mathcal{A}'])$.

Let $I \in \mathcal{I}(M\backslash X) = \{I \in \mathcal{I}(M) : I \subseteq E(M)\backslash X\}$. Then there is a matching $\phi$ of $I$ into $[r]$, namely $e \in A_{\phi(e)}$ for any $e \in I$. Since $I \cap X = \emptyset$, we have $e \in A_{\phi(e)}\backslash X$ for any $e \in I$, which shows that $\phi$ is also a matching of $I$ into $\mathcal{A}'$, proving that $I$ is also independent in $M[\mathcal{A}']$.

Conversely, if $I \in \mathcal{I}(M[\mathcal{A}'])$ then there is a matching $\phi$ of $I$ into $[r]$ using $\mathcal{A}'$, namely $e \in A_{\phi(e)}\backslash X$ for any $e \in I$. In particular $e \in A_{\phi(e)}$, thus proving that $\phi$ is a matching of $I$ into $\mathcal{A}$ and therefore $I \in \mathcal{I}(M)$. Note that $I \cap X = \emptyset$ because, for any $e \in I$ we have $e \in A_{\phi(e)}\backslash X$, so $e \notin X$. Therefore, $I \in \mathcal{I}(M)$ and $I \subseteq E(M)\backslash X$, so $I \in \mathcal{I}(M\backslash X)$. □

### 2.3.1 Minimal and maximal presentations in geometrical representation

To see how minimal and maximal presentations behave in geometrical terms, we show in Figure 11 some different presentations of the matroid represented in Figure 9 and observe the following:

- In the left representation of Figure 11, point 5 was moved from the edge of the simplex inside the lower plane without altering the affine dependencies between the points (thus, representing the same matroid). Equivalently, 5 was removed from the facet $F_2$. Therefore, this representation corresponds to the presentation $\{A_1, A_2 \cup 5, A_3, A_4\}$. In particular, this shows that presentation $\mathcal{A}$ is not maximal.

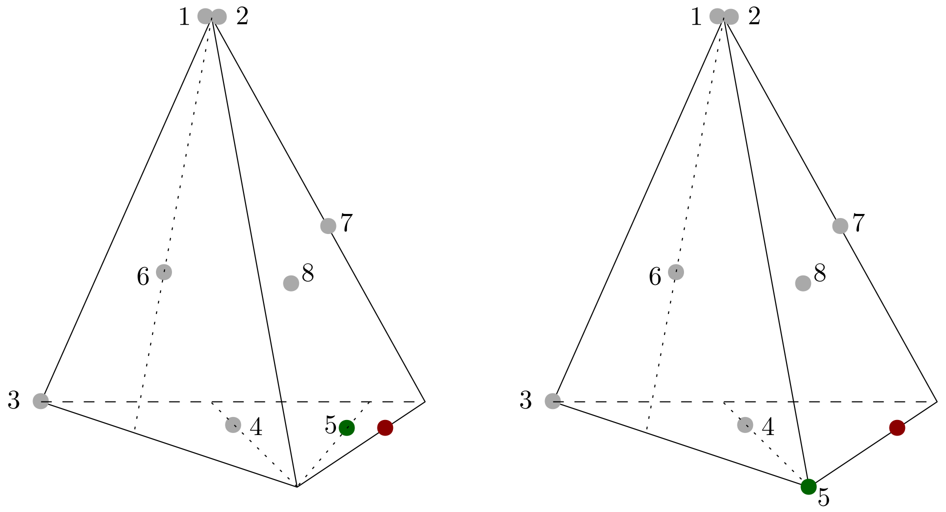


Figure 11: Other geometrical representations of the matroid in Figure 9

- In the right representation of Figure 11, point 5 was moved from the edge of the simplex into the vertex; equivalently, it was added to the facet $F_3$. Therefore, this representation corresponds to the presentation $\{A_1, A_2, A_3 \backslash 5, A_4\}$. In particular, this shows that presentation $\mathcal{A}$ is not minimal.

The following result helps us understand how minimal presentations look in the simplex-representation, in particular on its facets. We refer the reader to [5] for details on this result.

**Proposition 2.19.** *A presentation $\mathcal{A}$ of a transversal matroid $M$ is minimal if and only if each set $A \in \mathcal{A}$ is a cocircuit of $M$.*

This, together with the characterization of cocircuits that we saw in Lemma 1.18, gives us an idea of how to determine if a simplex-representation corresponds to a minimal presentation: each facet of the simplex needs to contain a hyperplane.

## 2.4 Transversal extensions

After discussing general matroid extension theory in Section 1.5, we now focus on our case of interest, transversal matroids.

**Definition 2.20.** Let $\mathcal{A} = \{A_1, \dots, A_r\}$ be a set system, and $I \subseteq [r]$ a subset[8]. We define the extension of $\mathcal{A}$ by $I$, denoted by $\mathcal{A}^I$ as $\{A_1^I, \dots, A_r^I\}$ where

$$A_i^I = \begin{cases} A_i \cup x & \text{if } i \in I, \\ A_i & \text{otherwise.} \end{cases}$$

[8] Note that, in this context, $I$ denotes a subset, not an independent set of a matroid.

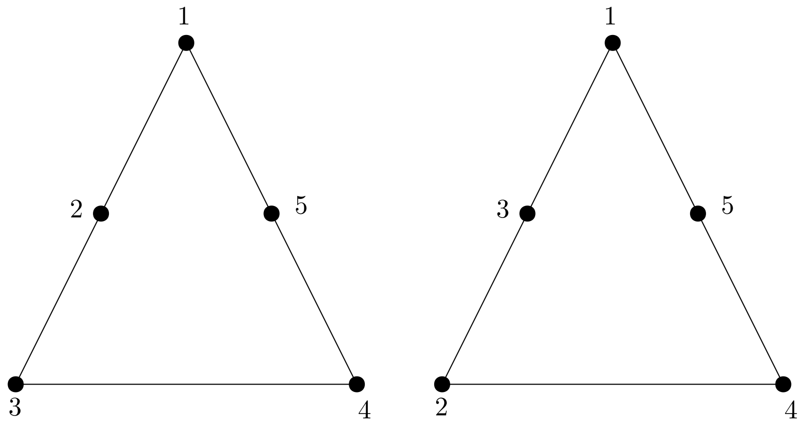


Figure 12: Two representations of $M$.

Let $\mathcal{A}$ be a set system and $M = M[\mathcal{A}]$ the corresponding transversal matroid. Theorem 2.5 tells us that $M[\mathcal{A}^I]$ is a transversal matroid. Moreover, using Proposition 2.13 we can deduce that $M[\mathcal{A}^I]$ is an extension of $M$. If there is no ambiguity between two or more presentations, we may denote by $M^I$ the extension of $M$ obtained by extending the presentation $\mathcal{A}$, namely $M^I = M[\mathcal{A}^I]$.

We stated that, for any matroid $M$, the set of all single-element extensions $\mathcal{E}(M)$ is a lattice. It is natural to ask if, assuming $M$ to be transversal, the set $\mathcal{T}(M) \subseteq \mathcal{E}(M)$ of all transversal extensions of $M$ is also a lattice or not. This is, so far, an open question to which we try to bring some light.

Example 2.21 shows that, taking the meet operation from $\mathcal{E}(M)$, even if $N$ and $N'$ are transversal matroids, $N \wedge N'$ does not need to be transversal. Another interesting example is [5, Example 1].

**Example 2.21.** Consider the set system $\mathcal{A}_1 = \{\{1,2,5\},\{2,3\},\{4,5\}\}$ over $\{1,2,3,4,5\}$ and $M = M[\mathcal{A}_1]$. Let $\mathcal{A}_2 = \{\{1,3,5\},\{2,3\},\{4,5\}\}$ be the set system obtained by swapping elements 2 and 3. It is clear that $\mathcal{A}_1$ and $\mathcal{A}_2$ are presentations of the same matroid, namely $M[\mathcal{A}_1] = M[\mathcal{A}_2]$, as represented in Figure 12.

Now let $I = \{2,3\}$ and consider the transversal extensions of $M$ given by extending presentations $\mathcal{A}_1$ and $\mathcal{A}_2$ by $I$, namely $N_1 = M[\mathcal{A}_1^I]$ and $N_2 = M[\mathcal{A}_2^I]$, represented in Figure 13. In $N_1$, the new element was added to the line $\{3,4\}$ and, in $N_2$, it was added in $\{2,4\}$.

The regular meet of $N_1$ and $N_2$ in the lattice $\mathcal{E}(M)$ is the extension where $x$ is added in both of these lines, as represented in Figure 13. This matroid $N_1 \wedge N_2$ is not transversal. There are several ways to see this, but using the geometrical representations that we have one can see that $N_1 \wedge N_2$ has four three-point lines. If it was transversal, there would be a representation of it such that each of these lines lies in a facet of the 2-simplex, but the 2-simplex only has 3 facets.

*Observation* 2.22. The matroid $N_1 \wedge N_2$ from Example 2.21 is, in fact, a well-known matroid, as it is the graphical matroid of the complete graph $K_4$. Not only it is not transversal, but it is the only matroid on ground set [6] and rank 3 that is not transversal.

We now state without proof some results from [5] that we will require in Section 4, and show the important role of minimal presentations in single-element extensions of transversal matroids.

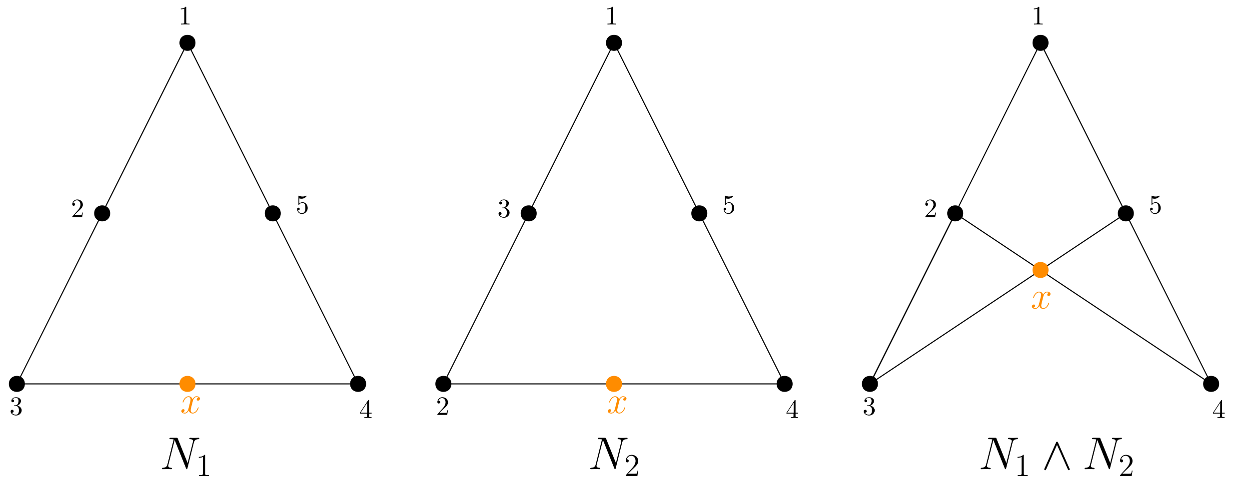


Figure 13: Two transversal extensions of $M$ and their non-transversal meet.

**Proposition 2.23.** *Let $\mathcal{A} = \{A_1, \dots, A_r\}$ be a minimal presentation of a matroid $M = M[\mathcal{A}]$ of rank $r$, and $I \subseteq [r]$. Then, $\mathcal{A}^I$ is a minimal presentation of the extension $N = M^I = M[\mathcal{A}^I]$.*

**Proposition 2.24.** *Let $N$ be a transversal matroid of rank $r$, let $x \in E(N)$, not a coloop, and $M = N \backslash x$. Then, there exists a minimal presentation $\mathcal{A}$ of $M$ and a subset $I \subseteq [r]$ such that $N = M[\mathcal{A}^I]$.*

The following result from [5] identifies the modular cut of the transversal extension that we define by extending a presentation.

**Proposition 2.25.** *Let $\mathcal{A} = \{A_1, \dots, A_r\}$ be a presentation of a transversal matroid $M$, and $I \subseteq [r]$. Then, the modular cut associated to the extension $M[\mathcal{A}^I]$ is*

$$\mathcal{M} = \{F \in \mathcal{F}(M) \mid \textit{for some } X \subseteq F, r_M(X) = |s(X)| \textit{ and } I \subseteq s(X)\}.$$

We will conclude this section with a result that guarantees the transversality of the extension for a particular type of modular cut, which we shall use in the forthcoming sections to reduce the possible cases. To prove it, we first show a lemma that will also be useful in the next section.

**Lemma 2.26.** *Let $M$ be a transversal matroid and $\mathcal{M} = \{F_1, \dots, F_r\}^+$ a modular cut. Consider the extension $N = M +_{\mathcal{M}} x$. Then, if $B$ is a basis of $F_i$ for some $i \in [r]$, the set $B \cup x$ is a circuit in $N$.*

*Proof.* First, let us see that $B \cup x$ is dependent. Note that $cl(B) = F_i$ and thus $cl(B) \in \mathcal{M}$, so by Theorem 1.41 we have that $r_N(B \cup x) = r_M(B) = |B|$. Therefore, $r_N(B \cup x) > |B \cup x|$ so the set is dependent.

Now let us show that for any $e \in B \cup x$, the set $(B \cup x) \backslash e$ is independent. Clearly $(B \cup x) \backslash x$ is independent, as $B$ is a basis, so let $e \in B$ and consider the set $B' = (B \cup x) \backslash e$. Note that $B \backslash e$ is an independent set of size $|B| - 1$, so $cl(B \backslash e)$ is a flat of rank $r(F_i) - 1$ that is contained in $F_i$. Therefore, by minimality of $F_i$, we have that $B\backslash e$ is *not* in the modular cut. In particular, $x \notin cl(B\backslash e)$, so $B'$ is an independent set. □

**Proposition 2.27.** *Let $M$ be a transversal matroid and $F \in \mathcal{F}(M)$ a flat. Then, the following are equivalent:*

*(1) The principal extension $M +_F x$ is transversal.*

*(2) There exists a presentation of $M$ such that $F$ generates a face of the simplex in the corresponding geometrical representation.*

*Proof.* Let $N := M +_F x$. Assuming (1), let $F' = F \cup x$. Since $cl_M(F) = F$ and the modular cut of the extension is $F^+$, Theorem 1.41 tells us that $r_N(F \cup x) = r_N(F) = r_M(F)$. Therefore, $F'$ is a flat of $N$. Moreover, we claim that $F'$ is a cyclic flat. Indeed, if $\mathcal{B}$ is the set of bases of $F'$, then note that $F' = \bigcup_{B \in \mathcal{B}} (B \cup x)$ where $B \cup x$ is a circuit for any $B \in \mathcal{B}$ (shown in Lemma 2.4). Therefore, by Theorem 2.10 there exists a presentation $\mathcal{A}$ of $N$ such that $F'$ generates a face in the simplex-representation. Then, the set system $\{A\backslash x \mid A \in \mathcal{A}\}$ is a presentation of $M$ satisfying (2).

Assuming (2), let $\mathcal{A}$ be a presentation of $M$ such that $F$ generates a face of the simplex-representation. The face must be evidently of dimension $r(F) - 1$. Any such face of the simplex can be expressed as an intersection of $r - r(F)$ facets. As we saw in Section 2.2.1, these facets correspond to complements of the $A_i$'s, namely $E(M)\backslash A_i$ for $i \in [r]$. Thus, we can write $F = \bigcap_{i \in I} E\backslash A_i$ for a set $I \subseteq [r]$ with $|I| = r - r(F)$. Let $J = [r]\backslash I$ and note that $J$ is the $\mathcal{A}$-support of $F$, namely $s(F) = J$. We claim that $\mathcal{A}^J$ is a presentation of $N$.

Let $M^J = M[\mathcal{A}^J]$ and let us show that $N = M^J$. As both are extensions of $M$, it suffices to show that their corresponding modular cuts are the same. We know that the modular cut of $N$ is $F^+$ and, using Proposition 2.25, the modular cut of $M^J$ is

$$\mathcal{M}^J := \{G \in \mathcal{F}(M) \mid \text{for some } Y \subseteq G, r_M(Y) = |s(Y)| \text{ and } J \subseteq s(Y)\}. \tag{2}$$

On one hand, it is clear that $F \in \mathcal{M}^J$ because $r_M(F) = |J| = |s(F)|$ and $s(F) = J$. Since $\mathcal{M}^J$ is a modular cut (thus, an upset), we have $F^+ \subseteq \mathcal{M}^J$. Conversely, let $G \in \mathcal{M}^J$. Then, for some $Y \subseteq G$ we have $J \subseteq s(Y) \subseteq s(G)$. But if $F \nsubseteq G$ then $s(G)$ could not cover $J$, namely $J \nsubseteq s(G)$. So we have $F \subseteq G$, and thus $G \in F^+$.

□

# 3. Transversal extensions of uniform matroids

In this section, we use the tools defined in the previous sections to study single-element rank-preserving transversal extensions of $U_{r,n}$. For the sake of simplicity, we may assume at some points that the ground set of $U_{r,n}$ is $[n]$ (instead of considering a generic $n$-set).

## 3.1 Flats and modular cuts of uniform matroids

In $U_{r,n}$ it is easy to identify the flats and their ranks, as sets are as independent as possible. We have

$$\mathcal{F}(U_{r,n}) = \{F \subseteq [n] \mid |F| < r\} \cup \{[n]\}.$$

The ranks of the proper flats are given by their cardinality:

$$r(F) = \begin{cases} |F| & \text{if } |F| < r, \\ r & \text{if } |F| \geq r. \end{cases}$$

As we saw in Example 1.30, this lattice is very similar to the boolean lattice $\mathcal{B}_n$. The difference is that, when it reaches the $r$-th level, it collapses into the top set $[n]$, omitting all sets of sizes between $r$ and $n-1$, because any such set $X$ has $cl(X) = [n]$.

Since any modular cut $\mathcal{M}$ is an upset, we identify it by its minimal sets $\{F_1, \dots, F_k\}$, which satisfy $\mathcal{M} = \{F_1, \dots, F_k\}^+$. First of all, we deal with the case $k = 1$.

### 3.1.1 Modular cuts with unique minimal element

Modular cuts of the form $\mathcal{M} = F^+$ for some flat $F \in \mathcal{F}(U_{r,n})$ are particularly easy to handle and, as we will see in Proposition 3.2, always yield a transversal extension. This extension $U_{r,n} +_F x$ is the principal extension of $U_{r,n}$ and $F$ (introduced in Definition 1.38). Geometrically, this means adding $x$ as freely as possible inside the flat $F$. For example,

- $\mathcal{M} = \emptyset^+ = \mathcal{F}(U_{r,n})$ yields the extension of adding a loop.
- $\mathcal{M} = \{a\}^+$ yields the extension of doubling the point $a$.
- $\mathcal{M} = \{a, b\}^+$ yields the extension of adding a point in the line $a, b$.

*Observation* 3.1. In the case $\mathcal{M} = \{E(M)\}$, the extension we obtain is the so-called free extension $M + x$ that we saw in Example 1.40. It is not hard to check that, if $\mathcal{A} = \{A_1, \dots, A_r\}$ is a presentation of $M$, then $\mathcal{A} \cup x = \{A_1 \cup x, \dots, A_r \cup x\}$ is a presentation of $M + x$. In particular, the free extension is transversal.

**Proposition 3.2.** *Let $F$ be a flat of $U_{r,n}$. Then, the principal extension $U_{r,n} +_F x$ is transversal.*

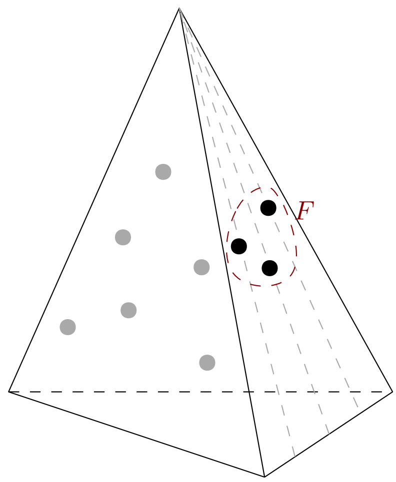


Figure 14: Simplex-representation of $U_{4,n}$ where a flat $F$ lies in a facet.

*Proof.* In the special case that $F = [n]$, the extension $U_{r,n} +_F x$ is the free extension $U_{r,n} + x$ which is transversal as we discussed in Observation 3.1.

Assume now that $F$ is a proper flat ($F \neq [n]$, thus $|F| < r$). Using Proposition 2.27, it suffices to see that there is a presentation of $U_{r,n}$ such that $F$ generates a face in the simplex-representation.

To understand the intuition of the general argument, consider first the case where $r(F) = r - 1$ (equivalently, $F$ is a hyperplane). Then, the set system $\{A_1, \dots, A_r\}$ where $A_1 = A_2 = A_3 = \cdots = A_{r-1} = [n]$ and $A_r = [n] \backslash F$ is a presentation of $U_{r,n}$ that satisfies our condition. In Figure 14 we can see a situation like this: we place the hyperplane (in this case, a plane) in a facet and the rest of the points "floating in the interior in general position".

Now let us apply the same principle to the general case. We know that a face of the simplex of dimension $r(F)$ is the intersection of $r - r(F)$ of its facets. Then, consider the presentation $\mathcal{A} = \{A_1, \dots, A_r\}$ where

$$A_i = \begin{cases} [n] & \text{if } i \leq r(F), \\ [n] \backslash F & \text{if } i > r(F). \end{cases}$$

In the simplex-representation of $\mathcal{A}$, a point $e$ will lie on a facet if and only if $e \in F$. Indeed, if $e \notin F$ then $e \in A_i$ for all $i \in [r]$, therefore no facet $[n] \backslash A_i$ contains it. Conversely, if $e \in F$ then there will be at least one (but in fact $r(F)$) facets that contain it: those of the form $[n] \backslash ([n] \backslash F) = F$.

To finish we need to see that $\mathcal{A}$ is a presentation of $U_{r,n}$. Let $M = M[\mathcal{A}]$. We can compute a presentation of the deletion $M \backslash ([n] \backslash F)$ using Lemma 2.18:

$$A_i \backslash ([n] \backslash F) = \begin{cases} F & \text{if } i \leq r(F), \\ \emptyset & \text{if } i > r(F). \end{cases}$$

Therefore, any element $e$ in $F$ is a coloop of the deletion $M \backslash A_i$ for $i \leq r(F)$. Using Lemma 2.17 we know that replacing $A_i$ by $A_i \cup e$ for any $i \leq r(F)$ is a presentation of the same matroid $M$. This argument can be applied recursively until each set of the presentation is $[n]$, telling us that $M = U_{r,n}$. □

### 3.1.2 Modular cuts with several minimal elements

From now on, we will focus on $k \geq 2$, namely modular cuts of the form $\mathcal{M} = \{F_1, \dots, F_k\}^+$ with $k \geq 2$. These minimal sets $A = \{F_1, \dots, F_k\}$ will form an antichain satisfying $\mathcal{M} = A^+$, and there will not be any modular pair among them (if there was a modular pair $F, F'$ then its intersection would also need to be in $\mathcal{M}$, so $F$ and $F'$ would not be minimal).

This motivates the study of which flats $F, F' \in U_{r,n}$ are modular pairs, which we will do next with a simple yet very useful result.

**Proposition 3.3.** *Let $F, F' \in \mathcal{F}(U_{r,n})$ with $F \not\subseteq F'$, $F' \not\subseteq F$. Then, $F$ and $F'$ are a modular pair if and only if $|F \cup F'| \leq r$.*

*Proof.* Suppose $F$ and $F'$ are a modular pair. First note that both need to have size less than $r$. Otherwise, if for instance $|F| \geq r$, then $F = [n]$ and necessarily $F' \subseteq F$, which we are assuming not to happen.

Now, since $|F|, |F'| < r$, we have $r(F) = |F|$ and $r(F') = |F'|$. In this case, the modular pair relation tells us that $|F| + |F'| = r(F \cup F') + |F \cap F'|$. Combining this with the simple inclusion/exclusion identity $|F| + |F'| = |F \cup F'| + |F \cap F'|$ we obtain that $r(F \cup F') = |F \cup F'|$, which happens exactly when $|F \cup F'| \leq r$.

Now assuming that $|F \cup F'| \leq r$ we know that

$$r(F) = |F|, \quad r(F') = |F'|, \quad r(F \cup F') = |F \cup F'| \quad \text{and} \quad r(F \cap F') = |F \cap F'|.$$

Then the modular pair relation is equivalent to inclusion/exclusion, which is satisfied. □

Using Proposition 3.3 we can find all modular pairs in $U_{r,n}$ for low values of $r$. The tables in Appendix A show all possible cardinalities of $F, F' \in \mathcal{F}(U_{r,n})$ and $F \cap F'$ for them to be a modular pair, assuming $|F| \geq |F'|$ and $F' \not\subseteq F$, for values of $r$ up to 6. Note that Proposition 3.3 implies that the last column $|F \cup F'|$ cannot be greater than $r$. From these tables, one can deduce all possible minimal sets of modular cuts and therefore all possible extensions, as we will do for $r = 3$ in Example 3.4.

**Example 3.4** (Transversal extensions of $U_{3,n}$)**.** From the tables of $r = 3$ in Appendix A one can deduce all possible modular cuts $\mathcal{M}$ of $\mathcal{F}(U_{3,n})$ and, therefore, all extensions of $U_{3,n}$. Let $\mathcal{M}$ be a modular cut of $\mathcal{F}(U_{3,n})$ with minimal sets $\{F_1, \dots, F_k\}$, with $k \geq 2$ (for $k = 1$ see Proposition 3.2). Then, $|F_i| = 2$ for all $i \in [k]$ and also $F_i \cap F_j = \emptyset$ for $i \neq j$. Hence, modular cuts are the up-sets of collections of pairwise disjoint lines.

Moreover, if $k \geq 3$ then the corresponding extension is not transversal, as we cannot have three 3-point lines intersecting in one point in the 3-simplex. Therefore, the only transversal extensions of $U_{3,n}$ are:

- Adding a coloop[9] ($\mathcal{M} = \emptyset$).
- Doubling a point ($\mathcal{M} = \{a\}^+$).
- Adding a point in a line ($\mathcal{M} = \{a, b\}^+$).
- Adding a point in the intersection of two lines ($\mathcal{M} = \{\{a, b\}, \{c, d\}\}^+$ for distinct $a, b, c, d$).
- Free extension ($\mathcal{M} = \{E(U_{3,n})\}$).
- Adding a loop ($\mathcal{M} = \{\emptyset\}^+ = \mathcal{F}(U_{3,n})$).

## 3.2 Cyclic flats in extensions of $U_{r,n}$

Our next objective is to determine which modular cuts yield transversal extensions. To do so, we will use the Mason-Ingleton inequalities, which we explain in Section 3.3. For now, we focus on studying the set of cyclic flats after extending $U_{r,n}$ with the modular cut $\mathcal{M}$, as they are needed in order to apply Mason-Ingleton.

First note that cyclic flats of $U_{r,n}$ are very simple. Indeed, the circuits are

$$\mathcal{C}(U_{r,n}) = \{C \subseteq [n] \mid |C| = r + 1\}$$

and the only flat $F \in \mathcal{F}(U_{r,n})$ with $|F| \geq r$ is $F = [n]$. Since a cyclic flat must be union of circuits, this is the only cyclic flat, namely $\mathcal{Z}(U_{r,n}) = \{[n]\}$.

But what cyclic flats appear after adding $x$ to $U_{r,n}$? To answer this question we first determine which circuits arise in the extension.

### 3.2.1 Circuits

Consider a modular cut $\mathcal{M} = \{F_1, \dots, F_k\}^+$ and the extension $M = U_{r,n} +_{\mathcal{M}} x$. We want to characterize the set $\mathcal{C}(M)$ in order to then find the cyclic flats of that extension. We start by giving some sufficient conditions for a set to be a circuit in $M$.

*Observation* 3.5. If $C$ is a circuit of $U_{r,n}$, then it is also a circuit of $M$ because, obviously, $x \notin C$ and therefore no dependencies are altered in any of the subsets of $C$.

**Lemma 3.6.** *For any $i \in [k]$, the set $F_i \cup x$ is a circuit in $M$.*

*Proof.* Note that $r(F_i) = |F_i|$ means that $F_i$ itself is a basis (the only one) for the flat $F_i$. Then, the result follows from Lemma 2.4. □

**Lemma 3.7.** *Let $C$ be a set with $|C| = r + 1$ and $F_i \nsubseteq C$ for any $i \in [k]$. Then, $C$ is a circuit in $M$.*

[9] Added for completeness, although we usually omit this extension as it is not rank-preserving.

*Proof.* Note that, if $x \notin C$, then $C$ is a circuit of $U_{r,n}$ and, as mentioned in Observation 3.5, it is also a circuit in $M$. Assume, then, that $x \in C$. Clearly, $C \backslash x$ is independent, as it is a subset of $[n]$ with no more than $r$ elements, so it remains to see that $C \backslash e$ is independent for $e \in C \backslash x$. Assume the contrary: for some $e \in C \backslash x$, the set $C \backslash e$ is dependent. Let $F := C \backslash (e \cup x)$ and note that $F$ is a flat in $U_{r,n}$ and also $x \in cl_M(F)$. This means that $F \in \mathcal{M}$, therefore $F_i \subseteq C$ for some $i \in [k]$. But this is false by hypothesis, so we reach a contradiction, thus proving that $C$ is a circuit. □

We have seen some different sufficient conditions on $C$ to be a circuit of $M$. We will see in Proposition 3.10 that it is necessary that one of these conditions hold for $C$ to be a circuit. To do so, we need to show first a couple of technical results.

**Lemma 3.8.** *Let $C$ be a circuit of $M$ with $|C| \leq r$. Then $C$ is a flat with $x \in C$ and $C \backslash x$ is a basis of $C$.*

*Proof.* The fact that $x \in C$ is clear: if $x \notin C$, then $C$ would need to be dependent in $U_{r,n}$ as well because we have not altered its dependencies, but $C$ is independent in $U_{r,n}$ because $|C| \leq r$. It remains to see that $C$ is a flat.

Since $C$ is a circuit, any subset is independent so we have $r(C) = |C| - 1$. Now assume $C$ is not a flat and let us reach a contradiction. If $C$ is not a flat, then there exists $e \in cl(C) \backslash C$. Consider the set $C' = (C \backslash x) \cup e$. Note that $cl(C \backslash x) = cl(C)$ and, moreover, since $e \in cl(C)$, we have $cl(C') = cl(C)$. Thus, $r(C') = r(C) = |C| - 1$. But at the same time, $C'$ is a subset of $[n]$ (it does not contain $x$) with $|C'| \leq r$, so it must be an independent set both in $U_{r,n}$ and $M$. Thus, $r(C') = |C'| = |C|$ which is a contradiction, proving that $C$ is a flat. □

**Lemma 3.9.** *If $C$ is a circuit of $M$ with $|C| \leq r$, then $C = F_i \cup x$ for some $i \in [k]$.*

*Proof.* Let $F = C \backslash x$. We claim that $F \in \mathcal{M}$. First, using Lemma 3.8, $F$ is an independent set (in fact a basis of $C$), so it is a flat in $U_{r,n}$ (where all independent sets are flats). Also, $F \notin \mathcal{F}(M)$ precisely because $x \in cl(F) \backslash F$. Finally, $F \cup x \in \mathcal{F}(M)$, so by definition $F \in \mathcal{M}$.

This implies that $F_i \subseteq F$ for some $i \in [k]$ or, equivalently, $F_i \cup x \subseteq C$. But circuits cannot be contained in other circuits[10], and $F_i \cup x$ is a circuit by Lemma 3.6, so, in fact, it must be that $F_i \cup x = C$. □

Finally putting together all these results, we can characterize the circuits of the extension $M$.

**Proposition 3.10.** *Let $\mathcal{M} = \{F_1, \dots, F_k\}^+$ be a modular cut of $\mathcal{F}(U_{r,n})$ and $M = U_{r,n} +_{\mathcal{M}} x$. Then, $C \in \mathcal{C}(M)$ if and only if one of the following holds:*

*1) $|C| = r + 1$ with $x \notin C$.*

*2) $|C| = r + 1$ with $x \in C$ and $F_i \nsubseteq C$ for all $i \in [k]$.*

*3) $C = F_i \cup x$ for some $i \in [k]$.*

[10] Because recall that circuits are, by definition, minimal dependent sets.

*Proof.* If $C$ satisfies any of the conditions 1, 2, or 3, then Observation 3.5, Lemma 3.6 or Lemma 3.7 (respectively) show that $C$ is a circuit.

Conversely, let $C$ be a circuit of $M$. First note that, if $x \notin C$, then necessarily $C$ is also a circuit in $U_{r,n}$, therefore $C$ is of type 1. Now, assuming $x \in C$, suppose $C$ is not of type 2 and let us see that necessarily it is of type 3. We distinguish two cases:

- If $|C| = r+1$, then it means that $F_i \subseteq C$ for some $i \in [k]$. Moreover, $x \in C$ so we can write $F_i \cup x \subseteq C$. Since $|F_i| < r$ and $|C| = r+1$, there must be an element $e \in C \backslash (F_i \cup x)$, but then $C \backslash e$ is not an independent set, as it contains the dependent set $F_i \cup x$. Thus, in this case, we reach a contradiction, as $C$ is not a circuit.
- If $|C| \neq r+1$, then necessarily[11] $|C| \leq r$. Then Lemma 3.9 tells precisely that $C = F_i \cup x$ for some $i \in [k]$, as we wanted to see.

□

### 3.2.2 Cyclic flats

With Proposition 3.10 we will soon be able to identify all cyclic flats of $M = U_{r,n} +_{\mathcal{M}} x$. Before, a lemma that will help us throughout the proof.

**Lemma 3.11.** *If $F$ is a cyclic flat in $M$, then $x \in F$.*

*Proof.* Let $F$ be a cyclic flat of $M$ and assume $x \notin F$. Since $F$ is cyclic, it must be a union of circuits of $M$, and since $x \notin F$, none of these circuits contains $x$. We have seen in Proposition 3.10 that circuits of $M$ that do not contain $x$ need to have size $r+1$. In particular, $|F| \geq r+1$. The only flat satisfying this is the full flat $[n] \cup x$, which contains $x$, so we reach a contradiction, thus proving that $x \in F$. □

**Proposition 3.12.** *Let $\mathcal{M} = \{F_1, \dots, F_k\}^+$ be a modular cut of $\mathcal{F}(U_{r,n})$ and $M = U_{r,n} +_{\mathcal{M}} x$. Then,*

$$\mathcal{Z}(M) = \{F_i \cup x \mid i \in [k]\} \cup \{[n] \cup x\}.$$

*Proof.* Note that $F_i \cup x$ is a flat by construction, and we know that it is a circuit thanks to Proposition 3.10, so $F_i \cup x \in \mathcal{Z}(M)$. Also, the complete flat $[n] \cup x = E(M)$ is a cyclic flat as long as the matroid $M$ does not contain coloops (which, in our case, is true). It remains to prove that $\mathcal{Z}(M) \subseteq \{F_i \cup x \mid i \in [k]\} \cup \{[n] \cup x\}$.

Lemma 3.11 tells us that any cyclic flat of $M$ will contain $x$, so we only have to study flats of the form $G \cup x \in \mathcal{F}(M)$. In order for $G \cup x$ to be cyclic we need to have that it is the union of some circuits, namely $G \cup x = C_1 \cup \cdots \cup C_t$. Using the notation from Proposition 3.10, if $C_i$ is of type 1 or 2 for some $i \in [t]$, then $|G \cup x| \geq r+1$ and thus $cl(G \cup x) = G \cup x = [n] \cup x$ would be the total flat. Now assuming that

[11] In general, a matroid of rank $r$ cannot have circuits of size strictly bigger that $r+1$, because then all of its subsets of size $r+1$ would need to be independent.

all $C_i$ are of type 3, for $G \cup x$ to be a cyclic flat we need to have $G \cup x = \bigcup_{i \in I} F_i \cup x$ for some $I \subseteq [k]$ or, equivalently $G = \bigcup_{i \in I} F_i$. However, if $|I| \geq 2$ then, again, $|G| > r$ because $|F_i \cup F_j| > r$ for any $i, j \in [k]$ (from Proposition 3.3). Therefore we conclude that

- If $|I| \geq 2$ then $G \cup x = [n] \cup x$ and
- if $|I| = 1$ then $G \cup x = F_i \cup x$

thus proving that these are the only cyclic flats of $M$. □

## 3.3 Applying the Mason-Ingleton inequalities

The following result was first formulated in 1971 by Mason ([13]) and refined[12] a few years later by Ingleton ([11]). In this section, we apply it to the study that we have done in the lattice of flats $\mathcal{F}(U_{r,n})$ to determine which extensions are transversal and which are not.

**Theorem 3.13** (Mason-Ingleton)**.** *A matroid is transversal if and only if, for any non-empty subset (equivalently, antichain) of cyclic flats $\mathcal{F} \subseteq \mathcal{Z}(M)$ the following is satisfied:*

$$r(\cap \mathcal{F}) \leq \sum_{\emptyset \neq \mathcal{F}' \subseteq \mathcal{F}} (-1)^{|\mathcal{F}'|+1} r(\cup \mathcal{F}'). \tag{3}$$

For a proof of this theorem and other characterizations of transversal matroids, see [6].

*Observation* 3.14. Theorem 3.13 states that we only need to check antichains to determine if $M$ is transversal or not. In our case, the full flat $[n] \cup x$ is a cyclic flat but it will never be part of an antichain unless the antichain is exactly $\mathcal{F} = \{[n] \cup x\}$. At the same time, inequality (3) always holds for $\mathcal{F} = \{[n] \cup x\}$. Therefore, we can omit this flat from the formulation and will do so from now on.

Knowing the set $\mathcal{Z}(U_{r,n} +_{\mathcal{M}} x)$, we are now ready to evaluate Equation (3) to determine for what modular cuts $\mathcal{M}$ the extension $M = U_{r,n} +_{\mathcal{M}} x$ is transversal. Let $\mathcal{M}$ be a modular cut in $\mathcal{F}(U_{r,n})$, $M = U_{r,n} +_{\mathcal{M}} x$ the corresponding extension and $\mathcal{F} \subseteq \mathcal{Z}(M)$ a subset of cyclic flats.

**Proposition 3.15.** *Let $\mathcal{M} \subseteq \mathcal{F}(U_{r,n})$ be a modular cut with minimal sets $\{F_1, \dots, F_k\}$, and let $M$ the corresponding extension. Then, $M$ is transversal if and only if, for any $\mathcal{F} \subseteq \{F_1 \cup x, \dots, F_k \cup x\}$, the following holds*

$$r(\cap \mathcal{F}) \leq \sum_{F \in \mathcal{F}} r(F) - r(M)(|\mathcal{F}| - 1). \tag{4}$$

*Proof.* Recall from Proposition 3.12 that $\mathcal{Z}(M) = \{F_1 \cup x, \dots, F_k \cup x, [n] \cup x\}$ and we observed that $[n] \cup x$ does not need to be taken into account. Let $\mathcal{F} \subseteq \mathcal{Z}(M)$ be a collection of cyclic flats. Note that for any

[12] Mason formulated the result using sets of cyclic sets. Ingleton showed 5 years later that the result could be refined using only sets of cyclic flats.

family $\mathcal{F}' \subseteq \mathcal{F}$ of size $|\mathcal{F}'| \geq 2$ we have $r(\cup\mathcal{F}') = r$, because by Proposition 3.3 any two $F, F' \in \mathcal{F}$ have $|F \cup F'| > r$.

With this observation in mind, we separate the sum between subfamilies $\mathcal{F}' \subseteq \mathcal{F}$ of size 1 and the rest. Then, on the right-hand side of inequality (3) we obtain

$$\sum_{\mathcal{F}' \subseteq \mathcal{F}} (-1)^{|\mathcal{F}'|+1} r(\cup\mathcal{F}') = \sum_{F \in \mathcal{F}} r(F) + \sum_{\substack{\mathcal{F}' \subseteq \mathcal{F} \\ |\mathcal{F}'| \geq 2}} (-1)^{|\mathcal{F}'|+1} r = \sum_{F \in \mathcal{F}} r(F) - r \sum_{\substack{\mathcal{F}' \subseteq \mathcal{F} \\ |\mathcal{F}'| \geq 2}} (-1)^{|\mathcal{F}'|} \tag{5}$$

Since the summands of the alternate sum do not depend on $\mathcal{F}'$ anymore except for its parity, we can simply count the subsets $\mathcal{F}' \subseteq \mathcal{F}$ of even and odd cardinality.

$$\sum_{\substack{\mathcal{F}' \subseteq \mathcal{F} \\ |\mathcal{F}'| \geq 2}} (-1)^{|\mathcal{F}'|} = \sum_{i=2}^{|\mathcal{F}|} (-1)^i \binom{|\mathcal{F}|}{i} = -\left( \sum_{i=0}^{1} (-1)^i \binom{|\mathcal{F}|}{i} \right) = |\mathcal{F}| - 1 \tag{6}$$

where we used the well-known fact that the alternate sum of binomial coefficients is zero.

Finally, combining Mason-Ingleton inequality (Equation (3)) with Equations (5) and (6) we obtain the desired expression

$$r(\cap\mathcal{F}) \leq \sum_{F \in \mathcal{F}} r(F) - r(M)(|\mathcal{F}| - 1).$$

□

### 3.3.1 Refining the result

Proposition 3.15 gives us a tool to identify which extensions of $U_{r,n}$ are transversal and which are not. However, we still need to check all subsets of the family $\mathcal{Z}(M) = \{F_1 \cup x, \dots, F_r \cup x, [n] \cup x\}$; our next goal is to relax this condition as much as possible.

Now that we know that we only need to check inequality (4) for $\mathcal{F} \subseteq \{F_i \cup x \mid i \in [k]\}$, we shall prove that this is not necessary for subsets $\mathcal{F}$ of small cardinality.

**Lemma 3.16.** *Let $\mathcal{M}$ be a modular cut of $\mathcal{F}(U_{r,n})$ with minimal elements $\{F_1, \dots, F_k\}$. Let $M$ be the corresponding extension and $\mathcal{F}$ be a subset $\mathcal{F} \subseteq \{F_i \cup x \mid i \in [k]\}$ with at most 2 sets. Then $\mathcal{F}$ satisfies inequality (4).*

*Proof.* Let $\mathcal{F} \subseteq \{F_i \cup x \mid i \in [k]\}$ with $|\mathcal{F}| \leq 2$. The case $|\mathcal{F}| = 1$ is trivial, as inequality (4) becomes $r(F) \leq r(F)$ for the only flat $F \in \mathcal{F}$, so consider the case $\mathcal{F} = \{F \cup x, G \cup x\}$.

Note that we have $F = cl_{U_{r,n}}(F) \in \mathcal{M}$, so using Theorem 1.41 we have that $r_M(F \cup x) = r_{U_{r,n}}(F) = |F|$ (analogously, $r_M(G) = |G|$). Also, $cl_{U_{r,n}}(F \cap G) = F \cap G$ is not in the modular cut $\mathcal{M}$, so Theorem 1.41

tells us this time that $r_M(F \cap G) = r_{U_{r,n}}(F \cap G) + 1 = |F \cap G| + 1$. Then, inequality (4) becomes

$$|F \cap G| + 1 \leq |F| + |G| - r.$$

Equivalently, using the inclusion/exclusion principle on the LHS,

$$|F| + |G| - |F \cup G| + 1 \leq |F| + |G| - r$$

which, simplifying, becomes $|F \cup G| > r$. This is satisfied thanks to Proposition 3.3. $\square$

Gluing up the previous lemmas we can refine Proposition 3.15 as the following corollary.

**Corollary 3.17.** *Let $\mathcal{M} \subseteq \mathcal{F}(U_{r,n})$ be a modular cut with minimal sets $\{F_1, \dots, F_k\}$, and let $M$ the corresponding extension. Then, $M$ is transversal if and only if, for any $\mathcal{F} \subseteq \{F_i \cup x \mid i \in [k]\}$ of size $|\mathcal{F}| \geq 3$, the following holds*

$$r(\cap\mathcal{F}) \leq \sum_{F \in \mathcal{F}} r(F) - r(M)(|\mathcal{F}| - 1). \tag{7}$$

*Observation* 3.18. An equivalent formulation of Corollary 3.17 is that for any $I \subseteq [k]$ with $|I| \geq 3$ the following must be satisfied

$$1 + |\bigcap_{i \in I} F_i| \leq \sum_{i \in I} |F_i| - r(M)(|I| - 1).$$

This expression is obtained from Equation (7) using the fact that any $F \in \mathcal{F}$ is of the form $F_i \cup x$ and $r(F_i \cup x) = |F_i|$ and also

$$r(\mathcal{F}) = r\left(\bigcap_{i \in I} F_i \cup x\right) = r\left(\left(\bigcap_{i \in I} F_i\right) \cup x\right) = \left|\left(\bigcap_{i \in I} F_i\right) \cup x\right| = \left|\bigcap_{i \in I} F_i\right| + 1$$

On the other hand, we can deduce an upper bound on the number of minimal sets that $\mathcal{M}$ can have for $U_{r,n} +_{\mathcal{M}} x$ to be transversal.

**Proposition 3.19.** *Let $\mathcal{M} \subseteq \mathcal{F}(U_{r,n})$ be a modular cut with minimal sets $\{F_1, \dots, F_k\}$, and assume $k \geq r > 1$. Then, $U_{r,n} +_{\mathcal{M}} x$ is not transversal.*

*Proof.* Let $M = U_{r,n} +_{\mathcal{M}} x$ and assume $M$ is transversal. Let $\mathcal{F} = \{F_1 \cup x, \dots, F_k \cup x\}$. By Corollary 3.17, we have that

$$|\cap\mathcal{F}| \leq \sum_{F \in \mathcal{F}} |F| - r(|\mathcal{F}| - 1)$$

where we used the fact that $r_M(F_i \cup x) = r(F_i) = |F_i|$ for all $i \in [k]$.

Considering that $x \in \cap\mathcal{F}$, we have that the following must be satisfied

$$1 \leq |\cap\mathcal{F}| \leq \sum_{i \in [k]} |F_i| - r(k-1) = r + \sum_{i \in [k]} (|F_i| - r).$$

Note that each of the terms $|F_i| - r$ is negative because they are proper flats in $U_{r,n}$. Thus, since $k \geq r$, we have

$$1 \leq r + \sum_{i \in [k]} (|F_i| - r) \leq r + \sum_{i \in [k]} (-1) = r - k < 0$$

which is a contradiction, proving that $M$ is not transversal. □

## 3.4 Structure of the poset of extensions

In this section, we attempt to address the question of whether the poset $\mathcal{T}(U_{r,n})$ of transversal extensions of $U_{r,n}$ is a lattice or not. Since we know that $\mathcal{E}(U_{r,n})$ is indeed a lattice and $\mathcal{T}(U_{r,n}) \subset \mathcal{E}(U_{r,n})$, we will check whether the join/meet of two transversal extensions also lies on $\mathcal{T}(U_{r,n})$ or not.

*Observation* 3.20. Note that what we aim to prove is a sufficient condition for $\mathcal{T}(U_{r,n})$ to be a lattice, but *it is not a necessary* one. Indeed, it could happen that $\mathcal{T}(U_{r,n})$ is not closed under the join/meet operations of $\mathcal{E}(U_{r,n})$ but, instead, one could define another join/meet operations that could satisfy the lattice conditions.

First, we state a simple lemma that will be useful in this section.

**Lemma 3.21.** *Let $F_1, \dots, F_k \in \mathcal{F}(U_{r,n})$ and $\mathcal{M} \subseteq \mathcal{F}(U_{r,n})$ a modular cut. Then, the following properties are satisfied:*

1. $\{F_1, \dots, F_k\}^+ = F_1^+ \cup \cdots \cup F_k^+$.
2. $\{F_1, \dots, F_k\}^+ \subseteq \mathcal{M}$ *if and only if* $F_i \in \mathcal{M}$ *for all* $i \in [k]$.
3. $F_1^+ \cap \cdots \cap F_k^+ = cl(F_1 \cup \cdots \cup F_k)^+$. *In particular, if* $|F_1 \cup \cdots \cup F_k| \geq r$ *then* $F_1^+ \cap \cdots \cap F_k^+ = \{[n]\}$.

*Proof.* :

1. The first property comes from simply applying the definition as

$$\{F_1, \dots, F_k\}^+ = \{A \in \mathcal{F}(U_{r,n}) \mid \exists i \in [k], F_i \subseteq A\} = \bigcup_{i \in [k]} \{A \in \mathcal{F}(U_{r,n}) \mid F_i \subseteq A\} = \bigcup_{i \in [k]} F_i^+$$

2. For the second property, note that one implication is clear; if $F_i \notin \mathcal{M}$ for some $i \in [k]$, the inclusion cannot hold because $F_i \in \{F_1, \dots, F_k\}^+$ but $F_i \notin \mathcal{M}$. Now assume $F_i \in \mathcal{M}$ for all $i \in [k]$ and let $G \in \{F_1, \dots, F_k\}^+$. We have $F_i \subseteq G$ for some $i \in [k]$, and therefore $F_i \subseteq G$. Since $\mathcal{M}$ is an up-set[13], $G \in \mathcal{M}$ concludes this point.

[13] We assumed $\mathcal{M}$ to be a modular cut, which is the case that we will use, but note that it suffices to assume $\mathcal{M}$ to be an up-set.

3. For the third property, let $G \in F_1^+ \cap \cdots \cap F_k^+$. By definition, $G$ needs to satisfy $F_i \subseteq G$ for all $i \in [k]$ or, equivalently, $F_1 \cup \cdots \cup F_k \subseteq G$. Since $G$ is a flat, we also have $cl(F_1 \cup \cdots \cup F_k) \subseteq G$. Similarly, if $G \in cl(F_1 \cup \cdots \cup F_k)^+$, then $cl(F_1 \cup \cdots \cup F_k) \subseteq G$. In particular, $F_1 \cup \cdots \cup F_k \subseteq G$, thus $F_i \subseteq G$ is also satisfied, meaning $G \in F_1^+ \cap \cdots \cap F_k^+$.

   In particular, if $|F_1 \cup \cdots \cup F_k| \geq r$ and $G \in F_1^+ \cap \cdots \cap F_k^+$, then $F_1 \cup \cdots \cup F_k \subseteq G$, so $|G| \geq r$. In $U_{r,n}$ this means that $cl(G) = [n]$..

□

We now proceed to show the form of the modular cut of the join in $\mathcal{E}(U_{r,n})$, which will come in very handy when checking if the extension is transversal or not. In the following lemma and forthcoming sections, $min\{S_1, \dots, S_l\}$ denotes the collection of minimal sets $S_i$ w.r.t inclusion.

**Lemma 3.22.** *Let $\mathcal{M}_1, \mathcal{M}_2$ be two modular cuts of $\mathcal{F}(U_{r,n})$, given by $\mathcal{M}_1 = \{F_1, \dots, F_k\}^+$ and $\mathcal{M}_1 = \{G_1, \dots, G_t\}^+$. Then, the modular cut of $M_1 \vee M_2$ in $\mathcal{E}(U_{r,n})$ is*

$$\mathcal{M}_1 \vee \mathcal{M}_2 = \min\{cl(F_i \cup G_j) \mid i \in [k], j \in [t]\}.$$

*Proof.* We know from Proposition 1.45 that $\mathcal{M}_1 \vee \mathcal{M}_2 = \mathcal{M}_1 \cap \mathcal{M}_2$. We now proceed to compute this intersection using Lemma 3.21. Property 1 tells us that $\mathcal{M}_1 = \{F_1, \dots, F_k\}^+ = F_1^+ \cup \cdots \cup F_k^+$ and, analogously, $\mathcal{M}_2 = G_1^+ \cup \cdots \cup G_t^+$. So, we have

$$\mathcal{M}_1 \cap \mathcal{M}_2 = \left(F_1^+ \cup \cdots \cup F_k^+\right) \cap \left(G_1^+ \cup \cdots \cup G_t^+\right) = \bigcup_{i,j} \left(F_i^+ \cap G_j^+\right) = \bigcup_{i,j} cl(F_i \cup G_j)^+$$

where, in the last step, we used Property 3 of the same lemma. Finally, using Property 1 again,

$$\mathcal{M}_1 \cap \mathcal{M}_2 = \bigcup_{i,j} cl(F_i \cup G_j)^+ = \left\{ \bigcup_{i,j} cl(F_i \cup G_j) \right\}^+ .$$

This already defines the intersection of modular cuts. However, there might be some redundancy in that set of unions, as it is not necessarily an antichain (for example, some unions may generate the whole space which will never be minimal). We need to take the minimal elements of $\left\{\bigcup_{i,j} cl(F_i \cup G_j)\right\}$ to make sure that we are considering only the minimal elements of the modular cut. Therefore,

$$\mathcal{M}_1 \vee \mathcal{M}_2 = \min\{cl(F_i \cup G_j) \mid i \in [k], j \in [t]\}.$$

□

If we use this fact together with Lemma 3.16, we get the following corollary.

**Corollary 3.23.** *Let $M_1$ be a principal extension of $U_{r,n}$ and $M_2$ an extension of $U_{r,n}$ satisfying that its modular cut has at most two minimal elements. Then, $M_1 \vee M_2$ is transversal.*

*Proof.* Let $\mathcal{M}_1 = F^+$ and note that, by Lemma 3.22 the modular cut of $M_1 \vee M_2$ is

$$\mathcal{M}_1 \vee \mathcal{M}_2 = \{\min\{cl(F \cup G) \mid G \in \mathcal{M}_2\}\}$$

which clearly has a size of at most two. Then, it is a straightforward application of Lemma 3.16. □

### 3.4.1 Rank 3 case

In this section, we show that the poset $\mathcal{T}(U_{3,n})$ is indeed a lattice.

**Lemma 3.24.** *Let $\mathcal{M}$ be a modular cut of $U_{3,n}$ with $\mathcal{M} \neq \{[n]\}$. Then, $\mathcal{M}$ is of the form $\mathcal{M} = \{F_1, \dots, F_k\}^+$ where the $F_i$'s are pairwise disjoint lines. Moreover, the extension $U_{3,n} +_{\mathcal{M}} x$ is transversal if and only if $k \leq 2$.*

*Proof.* Let $\{F_1, \dots, F_k\}$ be the minimal sets of $\mathcal{M}$. Since $\mathcal{M}$ is a modular cut, the $F_i$'s must not form modular pairs. Using Proposition 3.3 this means that $|F_i \cup F_j| > 3$ for all $i \neq j$. Additionally, $|F_i| \leq 3$ because otherwise $F_i = [n]$, therefore $|F_i| = 2$ for all $i \in [k]$ (the $F_i$'s must be lines) and $|F_i \cup F_j| = 4$ for all $i \neq j$ (the lines must be disjoint).

Using Lemma 3.16 we know that, if $k \leq 2$, then $\mathcal{M}$ corresponds to a transversal extension. Conversely, assuming $\mathcal{M}$ corresponds to a transversal extension, Proposition 3.19 tells us that $k \leq 2$, which concludes the proof. □

**Proposition 3.25.** *Let $M_1, M_2$ be two transversal extensions of $U_{3,n}$ and $M_1 \vee M_2$ their join in $\mathcal{E}(U_{3,n})$. Then $M_1 \vee M_2$ is transversal.*

*Proof.* Let $\mathcal{M}_1 \vee \mathcal{M}_2$ be the modular cut of $M_1 \vee M_2$. Let $H_1, \dots, H_l$ be the minimal elements of $\mathcal{M}_1 \vee \mathcal{M}_2$. By Lemma 3.16, it suffices to show that $l \leq 2$. Suppose, then, that $l \geq 3$, which by Lemma 3.24 means that $H_1, \dots, H_l$ are pairwise disjoint lines.

Let $\mathcal{M}_1 = \{F_1, \dots, F_k\}^+$ and $\mathcal{M}_2 = \{G_1, \dots, G_t\}^+$ and note that, by Lemma 3.24, $k, t \leq 2$. In particular, if $k = 1$ or $t = 1$, Corollary 3.23 tells us that $M_1 \vee M_2$ is transversal. Suppose, then, that $k = t = 2$, in which case $F_1, F_2$ are pairwise disjoint lines, and so are $G_1, G_2$. We know, by Lemma 3.22 that $H_s = cl(F_i \cup G_j)$ for some $i \in [k]$ and $j \in [t]$. However, $H_s$, $F_i$ and $G_j$ are all lines so the only way for this to hold is that $F_i = G_j = H_s$. Thus, every line $H_s$ needs to be both in $\mathcal{M}_1$ and $\mathcal{M}_2$, therefore $\mathcal{M}_1 \vee \mathcal{M}_2 \leq 2$, which is a contradiction with $l \geq 3$. □

We conclude using the well-known fact that a finite join semi-lattice with a least element is a lattice. This is not hard to see, by defining the meet of two elements as the join of all their common lower bounds (which is a finite set, and it is non-empty due to the existence of the least element). For a formal proof and details on this, see [17, Proposition 3.3.1].

**Corollary 3.26.** *The set $\mathcal{T}(U_{3,n})$ of all transversal extensions of $U_{3,n}$ is a lattice.*

### 3.4.2 Rank 4 case

In this section we show the poset $\mathcal{T}(U_{4,n})$ is indeed a lattice.

**Lemma 3.27.** *Let $\mathcal{M}$ be a modular cut of $U_{4,n}$ with $\mathcal{M} \neq \{[n]\}$ and more than one minimal element. Then, $\mathcal{M}$ has the form of one of the following:*

- $\mathcal{M} = \{F_1, \dots, F_k\}^+$ *where* $|F_i| = 3$ *and* $|F_i \cap F_j| \leq 1$.
- $\mathcal{M} = \{L, F_1, \dots, F_k\}^+$ *where* $|L| = 2$, $|F_i| = 3$, $L \cap F_i = \emptyset$ *and* $|F_i \cap F_j| \leq 1$.

*Moreover, the extension $U_{4,n} +_{\mathcal{M}} x$ is transversal if and only if $\mathcal{M}$ has one of the following forms:*

*a)* $\mathcal{M} = \{L, F\}$ *where* $|L| = 2$, $|F| = 3$ *and* $L \cap F = \emptyset$.

*b)* $\mathcal{M} = \{F_1, F_2\}$ *where* $|F_i| = 3$ *and* $|F_1 \cap F_2| \leq 1$.

*c)* $\mathcal{M} = \{F_1, F_2, F_3\}$ *where* $|F_i| = 3$ *and* $F_1 \cap F_2 \cap F_3 = \emptyset$.

*Proof.* The first part is derived from Proposition 3.3 using the same reasoning as in Lemma 3.24. One can use the modular pair tables in Appendix A for reference.

Now assuming $\mathcal{M}$ is of the form of a) or b), Lemma 3.16 guarantees that the extension will be transversal. If $\mathcal{M} = \{F_1, F_2, F_3\}$, then we need to check Mason-Ingleton for the family $\mathcal{F} = \{F_1 \cup x, F_1 \cup x, F_3 \cup x\}$. We do so using the expression of Observation 3.18:

$$1 + |\bigcap_{i \in I} F_i| = 1 \leq 1 = 9 - 8 = \sum_{i \in I} |F_i| - r(M)(|I| - 1). \qquad (8)$$

Conversely, assume $\mathcal{M}$ yields a transversal extension, and let us show that is of the form of a), b), or c). First, by Proposition 3.19 we know that $\mathcal{M}$ can have at most 3 minimal elements. Since a) and b) cover all possible modular cuts with 2 minimal elements, assume that $\mathcal{M}$ has size 3 and let us see that it is of the form of c). Indeed, note that the inequality (8) is "tight": while the RHS is as big as it can be (because the flats $F_i$ are hyperplanes) if the LHS increases (namely, $|F_1 \cap F_2 \cap F_3| \geq 1$) it will not hold anymore.

Thus, if $\mathcal{M}$ were to contain a line (decreasing the RHS) or the intersection would not be empty (increasing the LHS) the modular cut would not yield a transversal extension, so it has to be of the form of c). □

**Proposition 3.28.** *Let $M_1, M_2$ be two transversal extensions of $U_{4,n}$ and $M_1 \vee M_2$ their join in $\mathcal{E}(U_{4,n})$. Then $M_1 \vee M_2$ is transversal.*

*Proof.* Let $\mathcal{M}_1 \vee \mathcal{M}_2$ be the modular cut of $M_1 \vee M_2$. Using Lemma 3.27, we distinguish the case where $\mathcal{M}_1 \vee \mathcal{M}_2$ contains a line and the one where it does not.

- If $\mathcal{M}_1 \vee \mathcal{M}_2$ **contains a line** $L$, then it must be of the form $\{L, H_2, \dots, H_l\}$. Note that $L$ must be both in $\mathcal{M}_1$ and $\mathcal{M}_2$, as it needs to be $L = cl(F_i \cup G_j)$ for some $i \in [k]$, $j \in [t]$. Then, $\mathcal{M}_1$ is of the form $\mathcal{M}_1 = \{L, F\}$ and $\mathcal{M}_2 = \{L, G\}$. Additionally, $|L \cup F| > 4$ and $|L \cup G| > 4$ as they need to not be modular pairs, therefore $cl(L \cup F) = cl(L \cup G) = [n]$, which tells us that $\mathcal{M}_1 \vee \mathcal{M}_2 = \{L\}$ and therefore the corresponding extension is transversal.

- If $\mathcal{M}_1 \vee \mathcal{M}_2$ **does not contain a line**, then it must be of the form $\{H_1, H_2, \dots, H_l\}$ where all $H_s$ are planes. Note that these planes need to be of the form $F_i \cup G_j$ for some $i \in [k]$, $j \in [t]$. Thus,

  - If $\mathcal{M}_1$ (or analogously, $\mathcal{M}_2$) does not contain any line, then $\mathcal{M}_1$ is of the form $\{F_1, F_2\}$ or $\{F_1, F_2, F_3\}$ where $|F_i| = 3$. Also, for any $s \in [l]$ the plane $H_s$ is of the form $F_i \cup G_j$ for some $i \in [k], j \in [t]$. In particular, $F_i \subseteq H_s$ which, by cardinality, means that $H_s = F_i$. Therefore, either $|\mathcal{M}_1 \vee \mathcal{M}_2| \leq 2$ or $\mathcal{M}_1 \vee \mathcal{M}_2 = \mathcal{M}_1$. In both cases, $M_1 \vee M_2$ is transversal[14].
  - If $\mathcal{M}_1$, $\mathcal{M}_2$ both contain a line, then they are of the form $\mathcal{M}_1 = \{L_1, F\}$, $\mathcal{M}_2 = \{L_2, G\}$. Then, using Lemma 3.22 the modular cut of $M_1 \vee M_2$ is

    $$\mathcal{M}_1 \vee \mathcal{M}_2 = \{\min\{cl(L_1 \cup L_2), cl(L_1 \cup G), cl(L_2 \cup F), cl(F \cup G)\}\}.$$

    We proceed to show that no more than two of these unions can be minimal. If $cl(F \cup G)$ is minimal (thus, $cl(F \cup G) \neq [n]$) then it must be that $F = G$, so $cl(L_1 \cup G) = cl(L_2 \cup F) = [n]$ are not minimal. If, instead, $cl(F \cup G) = [n]$ and $cl(L_1 \cup L_2)$ is minimal (thus, $cl(L_1 \cup L_2) \neq [n]$), then $L_1$ and $L_2$ intersect in a point, but since $L_1 \cap F = L_2 \cap G = \emptyset$, necessarily $|L_1 \cup G| \geq 4$ and $|L_2 \cup F| \geq 4$, meaning that $cl(L_1 \cup G) = cl(L_2 \cup F) = [n]$ are not minimal.

□

**Corollary 3.29.** *The set $\mathcal{T}(U_{4,n})$ of all transversal extensions of $U_{4,n}$ is a lattice.*

### 3.4.3 Thoughts on the general case

In the previous sections, we have proven true the cases $r = 3$ and $r = 4$ of the following conjecture.

**Conjecture 3.30.** *Let $M_1$, $M_2$ be two transversal extensions of $U_{r,n}$ and $M_1 \vee M_2$ their join in $\mathcal{E}(U_{r,n})$. Then $M_1 \vee M_2$ is transversal. In particular, $\mathcal{T}(U_{r,n})$ is a lattice.*

We have not succeeded in proving this general case conjecture; we now review some of the methods we tried to approach the problem.

[14] The former is due to Lemma 3.16, the latter is because $\mathcal{M}_1$ is transversal.

The approach that looks brighter at first glance is the induction method. For this, we try to exploit the following fact.

**Lemma 3.31.** *The collection of transversal extensions of $U_{r,r}$ is isomorphic to the boolean lattice $\mathcal{B}_r$. In particular, $\mathcal{T}(U_{r,r})$ is a lattice.*

*Proof.* The key fact is that $U_{r,r}$ has one unique minimal presentation (up to relabeling). This is $\mathcal{A} = \{A_1, \dots, A_r\}$ where $A_i = \{i\}$ for all $i \in [r]$. This, together with Proposition 2.24 shows that any extension $M$ of $U_{r,r}$ will have a presentation that is an extension of $\mathcal{A}$, namely $M = M[\mathcal{A}^I]$ for some $I \subseteq [r]$.

Therefore, the extension $M$ is uniquely determined by the subset $I \subseteq [r]$ (and each subset gives an extension), yielding a bijection between $\mathcal{T}(U_{r,r})$ and $\mathcal{B}_r$. Moreover, two extensions $M_I = M[\mathcal{A}^I]$ and $M_J = M[\mathcal{A}^J]$ satisfy $M_I \leq_w M_J$ if and only if $I \subseteq J$, so this bijection is in fact an isomorphism. □

With this result as our base case, we could do induction over any of the two $r$ and $n$, because any uniform $U_{r,n}$ can be obtained by repeatedly extending $U_{r,r}$. One approach would be

1. Consider two extensions $M_1, M_2$ of $U_{r,n+1}$ (or $U_{r+1,n}$),
2. associate them with extensions $M_1', M_2'$ of $U_{r,n}$,
3. find the join $M_1' \vee M_2'$ (transversal by hypothesis) and its relation with $M_1 \vee M_2$,
4. and see that $M_1 \vee M_2$ satisfies the Mason-Ingleton inequalities using that $M_1' \vee M_2'$ does.

It is convenient to do the induction on $n$ because the size of the ground set *does not play an explicit role* in the Mason-Ingleton inequalities in general. More specifically, in inequality (4) we see that the size $n$ only affects the maximum size of the sets $F_i$, but it does not play any further role. Instead, the inequality does take into account the rank, and increasing/decreasing it alters one side of the inequality by a factor of $|\mathcal{F}| - 1$. Moreover, altering the rank changes the condition for two sets to form a modular pair, thus altering the modular cuts themselves which may no longer be modular cuts after the change.

We have attempted to build an inductive argument working around this issue but we ended up facing this issue; Mason-Ingleton "punishes" the rank increase and we do not have enough control over the LHS to compensate that.

When trying to do the induction over $n$, we face a different issue: it is very hard to associate extensions of $U_{r,n+1}$ with extensions of $U_{r,n}$ without a huge loss of information. The naive strategy of removing the biggest element $n+1$, for example, makes us lose information about all the minimal elements of the modular cut that contained $n+1$, as they no longer play a special role after the deletion.

Overall, low-dimensional cases seem relatively easy to prove, and probably one could show that cases $r = 5$ and $r = 6$ also work, with some patience, using the modular pair tables from Appendix A, but the general case remains open.

# 4. Catalog of transversal extensions

In this section, we build an algorithm that attempts to count and catalog transversal matroids up to a certain size of the ground set. Following the steps from [1] and [14], we will build them from scratch using single-element extension as our main tool, supported by all the results we have seen in the previous sections.

The code that we built applying the algorithm defined in this section is public and can be found in `https://github.com/esorinas/counting-transversal-matroids`. The repository also contains text files with the counting results and the dataset of all transversal matroids we have found.

Throughout this section, and as we did in Section 3, we assume the ground set of the matroids to be $[n]$.

## 4.1 Preliminaries and strategy

Let $M$ be a matroid on rank $r$ and $\mathcal{A}$ be a minimal presentation of $M$. In Section 2 we introduced how to find transversal extensions of transversal matroids: extending $\mathcal{A}$ by adding $x$ to some of its sets. In particular, Propositions 2.23 and 2.24 showed that:

- Extending a minimal presentation $\mathcal{A}$ with a subset $I \subseteq [r]$ yields a minimal presentation of $M[\mathcal{A}^I]$.
- An extension $N$ always has a presentation that is extended from a minimal presentation of $M = N \backslash x$.

This motivates the idea of thinking that any transversal matroid $M$ can be constructed by repeatedly extending minimal presentations and adding coloops, from the smallest transversal matroid, which is $U_{0,1}$. This is, essentially, the idea of the algorithm.

Starting from the only minimal presentation of $U_{0,1}$, at each step $n$ we will iterate through all minimal presentations $\mathcal{A}$ we have stored in the last iteration. To each of these $\mathcal{A}$, we will add the new element $n+1$ in all possible ways ($\mathcal{A}' = \mathcal{A}^I$ for each $I \subseteq [r]$) storing all these extensions $\mathcal{A}^I$, and we will also add $n+1$ as a coloop ($\mathcal{A}' = \mathcal{A} \cup \{\{n+1\}\}$).

In the end, we will have a list of minimal presentations of matroids with ground set up to $n$. Many of them will be presentations of the same matroid, so in order to *count* matroids we will have to compare them. We will do so by comparing their maximal presentations.

Now we use a notion we defined when introducing transversal matroids (Definition 2.8) to prove a result that will help us evaluate computationally when a set is independent or not.

**Lemma 4.1.** *Let $\mathcal{A} = \{A_1, \dots, A_r\}$ be a presentation of $M$, and $X \subseteq E(M)$ a subset. Then, $X$ is independent if and only if for any $Y \subseteq X$ we have $|Y| \leq |s(Y)|$.*

*Proof.* This is a simple application of the well-known Hall's theorem on bipartite graphs. Recall the bipartite graph approach to the notion of transversal matroid, with vertices $\mathcal{A} \cup E(M)$ and edges given by containment. In this context, a transversal of a set $X \subseteq E(M)$ is basically a matching of $X$ with some $A_i$'s. In particular, $X$ is independent if and only if there is an $X$-perfect matching in this induced subgraph.

Consider the subgraph induced by the edges connecting $X$ to $\{A_i \mid i \in s(X)\}$. Hall's theorem tells us that this subgraph will have an $X$-perfect matching if and only if $|Y| \leq |N(Y)|$ for any subset $Y \subseteq X$, where $N(Y)$ denotes the set of neighbors of $Y$. Since $|N(Y)| = |s(Y)|$, the proof is concluded. □

## 4.2 Completeness of the algorithm

Ideally, after each step, we would like to have found all minimal presentations of all transversal matroids on $[n]$. However, note that Propositions 2.23 and 2.24 are not sufficient for this. It could (and will) happen that, even though we can obtain *at least one* minimal presentation of each matroid, we do not obtain *all* minimal presentations of that matroid. In that case, we may not be finding all extensions of a matroid $M$: Proposition 2.24 guarantees that any extension $N$ is $N = M[\mathcal{A}^I]$ for some $\mathcal{A}$ minimal of $M$, but we cannot be sure that we have that minimal presentation $\mathcal{A}$. Also, the next step may also lack presentations and this will be propagated on.

Let us introduce some notation to work with these concepts. If $M$ is a matroid, we will denote the set of minimal presentations of $M$ by $\mathcal{P}_0(M)$. We want to work on the set of minimal presentations of matroids on ground set $[n]$, namely $\{\mathcal{P}_0(M) \mid E(M) = [n]\}$; however, among this set we do not want to distinguish two presentations if one is just a relabeling of the other (we dig deeper in this problem in Section 4.3). That is why we define $\mathcal{P}_0^n$ as

$$\mathcal{P}_0^n = \{\mathcal{P}_0^n(M) \mid M \text{ is a matroid on ground set } [n]\}/\sim$$

where

$$\mathcal{A}_1 \sim \mathcal{A}_2 \text{ if and only if there exists a permutation } \sigma \text{ such that } \mathcal{A}_2 = \{\sigma(A) \mid A \in \mathcal{A}_1\}.$$

The following operator defines the process of our algorithm at each iteration.

**Definition 4.2.** Let $n \geq 1$ and $\Delta$ be a collection of set systems over $[n]$. We define $\delta_n$ as

$$\delta_n(\Delta) := \{\mathcal{A}^I \mid \mathcal{A} \in \Delta \text{ and } I \subseteq [n]\} \cup \{\mathcal{A} \cup \{n+1\} \mid \mathcal{A} \in \Delta\}.$$

At each iteration, our algorithm uses the output from the previous iteration $\Delta$ and computes the set $\delta_n(\Delta)$. We know that, in the first iteration, $\Delta$ will be $\mathcal{P}_0^1$. Thus, our code would produce all minimal presentations if $\delta_n(\mathcal{P}_0^n) = \mathcal{P}_0^{n+1}$ was satisfied for all $n \geq 1$.

The inclusion $\delta_n(\mathcal{P}_0^n) \subseteq \mathcal{P}_0^{n+1}$ is guaranteed by Proposition 2.23. Unfortunately, the other inclusion *is not satisfied*. The following example shows presentations that lie in $\mathcal{P}_0^{n+1} \setminus \delta_n(\mathcal{P}_0^n)$ for different values of $n$.

**Example 4.3.** Consider the presentation $\mathcal{A} = \{A_1, A_2, A_3\}$ represented in Figure 15, where $A_1 = \{1, 2\}$,

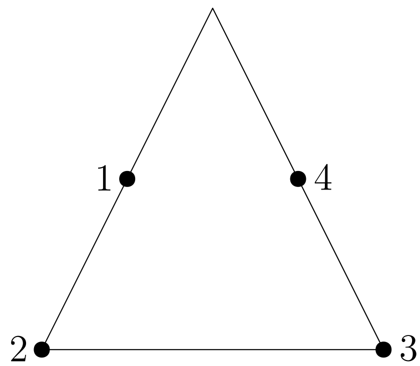


Figure 15: Minimal presentation that is not an extension of any minimal presentation (rank 3).

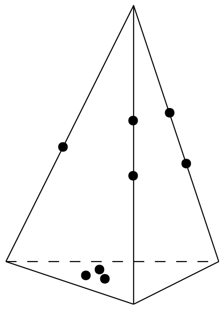

Figure 16: Minimal presentation that is not an extension of any minimal presentation (rank 4).

$A_2 = \{1, 4\}$, $A_3 = \{3, 4\}$. The matroid $M[\mathcal{A}]$ is actually $U_{3,4}$ and any deletion $\mathcal{A}\backslash e$ is a presentation of $U_{3,3}$. However, it is not hard to check that, for any $e \in \{1, 2, 3, 4\}$, the deletion $\mathcal{A}\backslash e$ is a *non-minimal* presentation of $U_{3,3}$, namely $\mathcal{A}\backslash e \notin \mathcal{P}_0(U_{3,3})$, and therefore $\mathcal{A}\backslash e \notin \mathcal{P}_0^3$. Geometrically, after deleting any element in the simplex-representation of Figure 15, either 1 or 4 can move to a vertex without altering the dependencies. In particular, this implies that $\mathcal{A} \in \mathcal{P}_0^4\backslash\delta_n(P_0^3)$.

The same happens with the presentation represented by Figure 16; in this case, it represents a matroid of rank 4 over ground set [8] and one can also see, with a bit more effort, that any deletion yields a non-minimal presentation of a matroid over [7].

In this context, we focus our interest on minimal presentations that cannot be obtained by extending other minimal presentations. Namely, we proceed to study matroids that lie in $\mathcal{P}_0^{n+1}\backslash\delta_n(\mathcal{P}_0^n)$ for some $n$. We call presentations of these kind *non-reachable* presentations.

**Lemma 4.4.** *For $n \geq 1$, let $\mathcal{A} \in \mathcal{P}_0^{n+1}\backslash\delta_n(\mathcal{P}_0^n)$ and $M = M[\mathcal{A}]$. Then, for any $e \in E(M)$, there exists $A \in \mathcal{A}$ such that $e$ is a coloop of $M\backslash A$. Equivalently, any element $e$ is a coloop of at least one facet in the simplex.*

*Proof.* Let $\mathcal{A} = \{A_1, \dots, A_r\}$ be such presentation. Since $\mathcal{A} \notin \delta_n(\mathcal{P}_0^n)$, then for any $e \in E(M)$, the presentation $\mathcal{A}\backslash e = \{A_1\backslash e, \dots, A_r\backslash e\}$ is not minimal of $M\backslash e$.

Using Lemma 2.19, this is equivalent as saying that, for any $e \in E(M)$, there exists $i \in [r]$ such that $A_i \backslash e$ is not a cocircuit of $M \backslash e$. Equivalently, $(E\backslash e)\backslash(A_i\backslash e) = E\backslash(A_i \cup e)$ is not a hyperplane. That means that $r(E\backslash(A_i \cup e)) < r-1$, but at the same time we have that $r(E\backslash A_i) = r-1$ because of the minimality of the presentation $\mathcal{A}$. Therefore, $e$ is a coloop of $M\backslash A_i$. □

**Corollary 4.5.** *Let $n \geq 1$ and $\mathcal{A} \in \mathcal{P}_0^{n+1}\backslash\delta_n(\mathcal{P}_0^n)$. Then, for any $e \in E(M)$ there exists $A \in \mathcal{A}$ such that $e \notin A$.*

*Observation* 4.6. Let $M$ be a matroid with a double point, namely $F = \{e, e'\}$ where $r(F) = 1$. If $e \in I$ for any independent set $I$, then $(I\backslash e) \cup e'$ is also independent. Therefore, neither $e$ nor $e'$ could ever be coloops of a facet in a geometrical representation. Thus, Lemma 4.4 can never be satisfied for a presentation $\mathcal{A}$ of $M$. Therefore, for any $\mathcal{A} \in \mathcal{P}_0(M)$ we have that $\mathcal{A} \in \delta_n(\mathcal{P}_0^n)$. Note that this argument works for multiple points in general, not necessarily of size 2.

The notion we used in the last observation can be extended to the following result. Note that we write $s_{\mathcal{A}}(e)$ instead of $s_{\mathcal{A}}(\{e\})$ when $e$ is an element instead of a set.

**Proposition 4.7.** *Let $\mathcal{A}$ be a minimal presentation of a matroid $M$ on ground set $[n+1]$, with $n \geq 1$. Suppose that there exists a cyclic flat $F \in \mathcal{Z}(M)$ such that, for some $e \in F$, we have $|s_{\mathcal{A}}(e)| = |s_{\mathcal{A}}(F)|$. Then, $\mathcal{A} \in \delta_n(\mathcal{P}_0^n)$.*

*Proof.* Let $F \in \mathcal{Z}(M)$ and $e \in F$ such that $|s_{\mathcal{A}}(\{e\})| = |s\mathcal{A}(F)|$. Note that, since $\{e\} \subseteq F$ we always have $s_{\mathcal{A}}(e) \subseteq s_{\mathcal{A}}(F)$ so, by cardinality, we have $s_{\mathcal{A}}(e) = s_{\mathcal{A}}(F)$. Also, Proposition 2.13 tells us that $s_{\mathcal{A}}(F) = r(F)$; thus we have $s_{\mathcal{A}}(e) = r(F)$.

Let $\mathcal{A} = \{A_1, \dots, A_r\}$ and $E = E(M)$. Using the contrapositive version of Lemma 4.4, it suffices to show that $e$ is not a coloop of $M\backslash A_i$ for any $i \in [r]$. Thus, let $i \in [r]$ and consider the hyperplane $H_i = E\backslash A_i$.

Evidently, if $e \notin H_i$ (equivalently, if $e \in A_i$) then $e$ is not a coloop of $H_i$. Thus, assume $e \notin A_i$ or, equivalently, $i \notin s_{\mathcal{A}}(e)$. Since $s_{\mathcal{A}}(e) = s_{\mathcal{A}}(F)$ that means that $i \notin s_{\mathcal{A}}(F)$, thus $F \cap A_i = \emptyset$. In particular, this means that $F \subseteq E\backslash A_i$, so the whole cyclic flat $F$ lies in $M\backslash A_i$. Observe that for $e$ to be a coloop of $M\backslash A_i$ it would need to be a coloop in the flat $F$ as well. Since $F$ is a cyclic flat, it does not contain coloops, which shows that $e$ cannot be a coloop of $M\backslash A_i$ and the result follows. □

These results show the particularities of non-reachable presentations: they need to satisfy some strict conditions and are, therefore, rare to find. However, they do exist and, therefore, our algorithm will only show a lower bound for the number of transversal matroids. The results we have shown in this are not sufficient in order to control the error in our algorithm in a computationally efficient way; the conditions that we have shown necessary for a presentation to be non-reachable are computationally too hard to compute for each matroid.

### 4.2.1 Manual mitigation

To mitigate the error due to the lack of non-reachable presentations in our algorithm, we manually introduce some non-reachable presentations.

To give some intuition as to how to find these presentations, the contrapositive version of Proposition 4.7 comes in handy: if $\mathcal{A} \in \mathcal{P}_0^{n+1} \backslash \delta_n(\mathcal{P}_0^n)$, then for each cyclic flat $F$ and each $e \in F$ we must have $|s_{\mathcal{A}}(e)| < |s_{\mathcal{A}}(F)|$. Equivalently, cyclic flats need to not have points in their interior in their geometrical representation. In particular, intuition tells us that, the less cyclic flats there are in $M$, the easiest it will be for a presentation $\mathcal{A}$ to satisfy the condition and, thus, to be non-reachable.

This motivates that, in our search for non-reachable presentations, we consider matroids of the form $U_{n,n+1}$. From what we have studied about uniform matroids, it is not hard to see that $U_{n,n+1}$ consists in a single circuit, which is the only cyclic flat of the matroid.

*Observation* 4.8. Matroids the form $U_{n,n+1}$ always have one minimal presentation of the following form: $\mathcal{A} = \{A_1, \dots, A_n\}$ where $A_i = \{i, n+1\}$. Geometrically, this corresponds to one point in each vertex of the simplex and a point "floating in the middle". Essentially, this is the free extension of the unique minimal presentation of $U_{n,n}$. In this section, we refer to this presentation by the *canonical* presentation of $U_{n,n+1}$.

Note that this presentation satisfies the condition of Proposition 4.7, as the only cyclic flat is $F = [n+1]$ and we have $s_{\mathcal{A}}(n+1) = s_{\mathcal{A}}(F) = [n]$. Therefore, $\mathcal{A} \in \sigma_n(\mathcal{P}_0^n)$. Moreover, that this is the only presentation of $U_{n,n+1}$ that satisfies the condition.

With this observation, all non-canonical presentations of $U_{n,n+1}$ are candidates to non-reachable presentations (although they not necessarily are). Therefore, we computed all minimal presentations of $U_{3,4}$ and $U_{4,5}$ (we used a simple brute-force algorithm to do so). It turned out that all the non-canonical ones were in fact non-reachable, as one can easily check.

Figure 17 shows all minimal presentations of $U_{3,4}$ and $U_{4,5}$, with the canonical presentations drawn in gray. All the other presentations, which we found to be non-reachable, were manually added to the code in order to mitigate the problem explained in Section 4.2. They are the following:

- $\mathcal{A}_1 = \{\{1,2\},\{1,4\},\{3,4\}\}$ with $M[\mathcal{A}_1] = U_{3,4}$,
- $\mathcal{A}_2 = \{\{1,5\},\{2,5\},\{1,3\},\{1,4\}\}$ with $M[\mathcal{A}_2] = U_{4,5}$,
- $\mathcal{A}_3 = \{\{1,5\},\{2,5\},\{1,3\},\{2,4\}\}$ with $M[\mathcal{A}_3] = U_{4,5}$.

## 4.3 The isomorphism problem

When we count and catalog transversal matroids we face an intrinsic big issue. We do not want to count *matroids*, but instead, we want to count and catalog *isomorphism classes of matroids*.

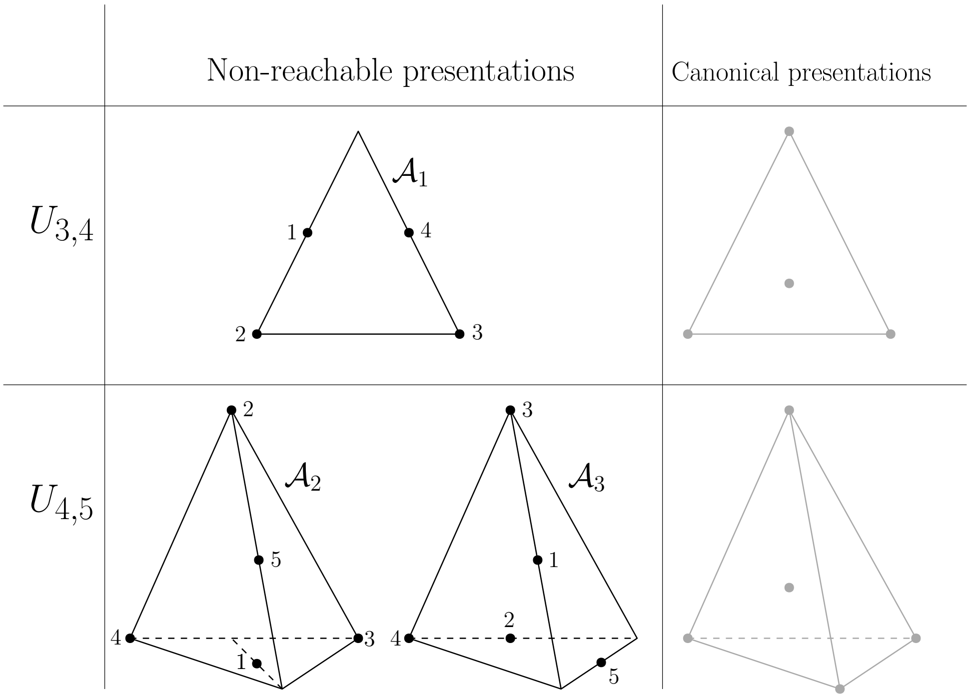


Figure 17: Simplex-representation of all minimal presentations of $U_{3,4}$ and $U_{4,5}$.

**Definition 4.9.** Let $M$ and $N$ be two matroids over the same ground set $E$. We say that $M$ and $N$ are *isomorphic* if there exists a permutation $\phi$ of $E$ such that $X \in \mathcal{I}(M)$ if and only if $\phi(X) \in \mathcal{I}(N)$.

Let us focus on the transversal matroid case. Let $M$, $N$ be transversal matroids over $E$ with maximal presentations $\mathcal{A}_M, \mathcal{A}_N$, respectively. As the maximal presentation is unique, these matroids will be isomorphic if $\mathcal{A}_N$ can be obtained from $\mathcal{A}_M$ via a permutation of $E$.

**Example 4.10.** Let $M = M[\mathcal{A}]$ and $N = M[\mathcal{B}]$ over $E = [4]$ where

$$\mathcal{A} = \{\{1,2\},\{3,4\}\} \quad \mathcal{B} = \{\{1,4\},\{2,3\}\}.$$

It is clear that these matroids are isomorphic, and indeed if we consider the permutation

$$\begin{aligned} \phi : [4] &\longrightarrow [4] \\ i &\longmapsto i+1 \pmod 4 \end{aligned}$$

we have that $\phi(\mathcal{A}) = \mathcal{B}$. Moreover, if we were to look at the (unlabeled) geometrical representations of $M$ and $N$ they would look exactly the same.

This shows the need to count matroids up to isomorphism, as we do not want to count so many multiplicities. This is a challenging part of the code because the problem of determining whether two matroids are isomorphic or not is computationally hard; in fact, it is closely related to the Graph Isomorphism

Problem, which is well-known to be in the NP class.

Recall the bipartite graph that we used to introduce transversal matroids in Section 2.1, with vertex set $E(M) \cup \mathcal{A}$ and edges $\{(a, A_i) \mid e \in A_i\}$. In this context, two transversal matroids are isomorphic if their graphs are. Bipartite Graph Isomorphism is also known to be GI-complete (see [7]).

To address this issue, we need to be careful when storing minimal presentations. Before storing a new minimal presentation $\mathcal{A}$ on ground set $[n]$, we will need to check all relabelings of $\mathcal{A}$ and see if any of them is already stored. We will only store presentations that are not already stored in some relabeled form. For this purpose, our custom class `TransversalMatroid` contains a method `relabel` , which takes as input a permutation and returns a new instance of `TransversalMatroid` . Also, the general method `isDuplicate` takes as input the list of stored presentations and the new candidate, and computes whether it is already stored in any possible relabeling or not.

## 4.4 Auxiliary algorithms

We now translate some of the results we have found into sketched algorithms. We omit some of them, as they are not especially interesting, but recall all the code is publicly available if the reader wants to see all the details.

Using Lemma 4.1 one can build Algorithm 1 to determine when a set $X$ is independent or not in $M$, given a presentation $\{A_1, \dots, A_r\}$ of $M$.

**Algorithm 1** Check if $X$ is independent

**Input:** $\{A_1, \dots, A_r\}$ (presentation of $M$), $X \subseteq [n]$ (candidate to independent set)
**for** $Y \subseteq X$ **do**
    $count \leftarrow 0$
    **for** $i = 1, 2, \dots, r$ **do**
        **if** $|A_i \cap Y| \neq 0$ **then**
            $count \leftarrow count + 1$
        **end if**
    **end for**
    **if** $count < |Y|$ **then**
        return FALSE
    **end if**
**end for**
return TRUE

A computation that we need to do many times is computing the set of bases $\mathcal{B} = \mathcal{B}(M)$ given a presentation $\mathcal{A}$ of $M$. We follow the most naive approach, simply iterating over $\binom{[n]}{r}$ where $r = r(M)$; for each $X \in \binom{[n]}{r}$, we run Algorithm 1 and store it if the output is true. In computational terms, this method is notably slow as it needs to iterate over $\binom{n}{r}$ sets.

The mentioned method to find the bases from a presentation is also used to determine if an element

$e \in M$ is a coloop or not. To do so, we simply iterate over all bases and return false if $e \notin B$ for any $B \in \mathcal{B}(M)$.

An essential calculation that we need to perform is a comparison between matroids. Whenever we obtain a matroid by extending a presentation $M = M[\mathcal{A}']$ we will need to check if we already found this matroid via extending another presentation. To do so, we will compute the maximal presentation $\mathcal{B}$ of $M[\mathcal{A}']$ and compare it with the other previously stored maximal presentations. The following algorithm uses Lemma 2.17 and the previous algorithms to do so.

**Algorithm 2** Compute maximal presentation of $M[\mathcal{A}]$

**for** $A_i \in \mathcal{A}$ **do**
  **for** $e \in E(M) \setminus A_i$ **do**
    **if** $e$ is coloop of $M \setminus A_i$ **then**
      $A_i \leftarrow A_i \cup e$
    **end if**
  **end for**
**end for**
return $\mathcal{A}$

Finally, the last auxiliary algorithm we show is `isDuplicate`. Given the set of stored presentations and a new candidate, Algorithm 3 computes whether the candidate or any of its relabeling is already stored.

**Algorithm 3** `isDuplicate` Check if some relabeling of $\mathcal{A}$ is already stored

**Input:** $\mathcal{P}$ (collection of stored presentations), $\mathcal{A}$ (candidate to new presentation)
**for** $\sigma$ permutation of $[n]$ **do**
  $\mathcal{A}_\sigma = \{\,\}$
  **for** $A \in \mathcal{A}$ **do**
    $\mathcal{A}_\sigma \leftarrow \mathcal{A}_\sigma \cup \sigma(A)$ ▷ where $\sigma(A) = \{\sigma(x) \mid x \in A\}$
  **end for**
  **if** $\mathcal{A}_\sigma \in \mathcal{P}$ **then**
    return TRUE
  **end if**
**end for**
return FALSE

## 4.5 The algorithm

With all the results and ideas we have discussed, we are now ready to define the main algorithm that we use to produce the results that we will see in Section 4.6. The main algorithm is split into two blocks: Algorithm 4 computes a list of minimal presentations, grouped by their ground set size $n$. In particular, if $\mathcal{P}_0$ is the output, then $\mathcal{P}_0[n] \subseteq \mathcal{P}_0^n$. Once this process ends, it just remains to compare pairwise the minimal presentations we have found, to see how many distinct transversal matroids we have found. This is done by Algorithm 5.

**Algorithm 4** Find minimal presentations up to $N$

**Input:** $N$ (iteration limit)
$n \leftarrow 1$
$\mathcal{P}_0 \leftarrow [\overbrace{[\,], \dots, [\,]}^{N}]$ ▷ $\mathcal{P}_0[n]$ will store the list $\mathcal{P}_0^n$
$\mathcal{P}_0[0][0] \leftarrow \{\}$ // presentation of $U_{0,0}$
**while** $n \leq N$ **do**
    **for** $\mathcal{A} \in \mathcal{P}_0[n]$ **do**
        $r \leftarrow r(M[\mathcal{A}])$
        **for** $I \subseteq [r]$ **do**
            $\mathcal{A}' \leftarrow \mathcal{A}^I$
            **if** `isDuplicate(`$\mathcal{P}_0[n]$`, `$\mathcal{A}$`)` == FALSE **then**
                $\mathcal{P}_0[n] \leftarrow \mathcal{P}_0[n] \cup \mathcal{A}'$ ▷ Rank-preserving extension
            **end if**
        **end for**
        $A_{r+1} \leftarrow \{n+1\}$
        $A' \leftarrow \mathcal{A} \cup A_{r+1}$ ▷ Add coloop
        $\mathcal{P}_0[n] \leftarrow \mathcal{P}_0[0] \cup A'$
    **end for**
    $n \leftarrow n+1$
**end while**
return $\mathcal{P}_0$

After the execution of both Algorithms 4 and 5, the number of transversal matroids over $n$ that we have found is simply the length of the list $\mathcal{P}_{max}[n]$, where $\mathcal{P}_{max}$ is the output of the latter algorithm.

### 4.5.1 Technical details

**About performance**: Notably, there are computations that we explicitly calculate several times during the execution as we defined. However, the computations with the most complexity (in particular, comparison of minimal presentations with relabeling and calculation of maximal presentations) are only executed at most once for each pair. We dynamically store a key/value data structure to keep track of computations that have already been done, to avoid recalculating.

Additionally, the algorithm would be slightly more efficient by merging algorithms 4 and 5. Calculating the maximal presentations "on the fly" and doing the comparison would avoid having to iterate over the minimals presentations that we have found. However, this approach does not improve much the performance, as the added complexity is negligible compared to the one of Algorithm 4. For simplicity and readability of the code, we took the decision of keeping them separated.

**About SageMath**: SageMath has some very useful utilities for matroid theory. It allows one to create

**Algorithm 5** Find maximal presentations from a given set of minimals $\mathcal{P}$

**Input:** $\mathcal{P}$ (list of minimal presentations grouped by $n$)
$n \leftarrow 1$
$\mathcal{P}_{max} \leftarrow [\overbrace{[\,], \dots, [\,]}^{N}]$
**while** $n \leq N$ **do**
  **for** $\mathcal{A} \in \mathcal{P}[n]$ **do**
    $\mathcal{A}_{max} \leftarrow$ `getMaxPresentation(`$\mathcal{A}$`)`
    **if** `isDuplicate(`$\mathcal{P}_{max}[n]$`, `$\mathcal{A}_{max}$`)` == FALSE **then**
      $\mathcal{P}_{max}[n] \leftarrow \mathcal{P}_{max}[n] \cup \mathcal{A}_{max}$
    **end if**
  **end for**
  $n \leftarrow n + 1$
**end while**
return $\mathcal{P}_{max}$

matroids (given independents, bases, circuits, and other equivalent sets) and do all kinds of operations with them. However, there is no transversal matroid-specific code. Therefore, we had to define our own utilities for everything related to transversal matroids. We defined a class called `TransversalMatroid` that englobes its core utilities of it (obtaining the maximal presentation, the rank, the bases, etc.).

We also provide some simple unit tests for the internal methods of this class. We believe that, with the proper time to standardize the code, this could be a good contribution to SageMath codebase, but this is out of the scope of this work.

## 4.6 Results

We now show and discuss the results that we have found, how many transversal matroids are there for each ground set size and rank, and what percentage they represent among all matroids on those parameters.

Table 1 contains the counting results we have obtained. For each value of $n$ and $r$, we see two numbers $C^1_{r,n}$ and $C^2_{r,n}$.

- The first value $C^1_{r,n}$ is the exact number of non-isomorphic matroids on ground set $n$ and rank $r$. These numbers are due to Mayhew and Royle [14], who published them along with a dataset of all matroids with at most 9 elements.

- The second value $C^2_{r,n}$ is the number of non-isomorphic transversal matroids that our algorithm has found on ground set $n$ and rank $r$. As we have seen in the previous sections this may not be the exact number, but it is necessarily a lower bound, as all matroids that we find are non-isomorphic and transversal.

Table 1: Number of matroids and transversal matroids

| $r \backslash n$ | 1 | | 2 | | 3 | | 4 | | 5 | | 6 | | 7 | | 8 | | 9 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **0** | 1 | **1** | 1 | **1** | 1 | **1** | 1 | **1** | 1 | **1** | 1 | **1** | 1 | **1** | 1 | **1** | 1 | - |
| **1** | 1 | **1** | 2 | **2** | 3 | **3** | 4 | **4** | 5 | **5** | 6 | **6** | 7 | **7** | 8 | **8** | 9 | - |
| **2** | | | 1 | **1** | 3 | **3** | 7 | **7** | 13 | **13** | 23 | **22** | 37 | **34** | 58 | **50** | 87 | - |
| **3** | | | | | 1 | **1** | 4 | **4** | 13 | **13** | 38 | **37** | 108 | **92** | 325 | **208** | 1275 | - |
| **4** | | | | | | | 1 | **1** | 5 | **5** | 23 | **23** | 108 | **100** | 940 | **421** | 190214 | - |
| **5** | | | | | | | | | 1 | **1** | 6 | **6** | 37 | **37** | 325 | **255** | 190214 | - |
| **6** | | | | | | | | | | | 1 | **1** | 7 | **7** | 58 | **56** | 1275 | - |
| **7** | | | | | | | | | | | | | 1 | **1** | 8 | **8** | 87 | - |
| **8** | | | | | | | | | | | | | | | 1 | **1** | 9 | - |
| **9** | | | | | | | | | | | | | | | | | 1 | - |

Note that necessarily we always have $C^1_{r,n} \geq C^2_{r,n}$. However, we see that for low values of $n$ and/or $r$, we have equality. Indeed, for $n \leq 5$, we have $C^1_{r,n} = C^1_{r,n}$, and for $n = 6$ the numbers may differ in at most 1. This shows that all matroids on ground set $[n]$ for $n \leq 5$ are transversal.

Additionally, note that for $r \geq n-2$, we also have $C^1_{r,n} = C^1_{r,n}$. This is in fact true in general: a matroid on ground set $n$ and rank $r \geq n-2$ is always transversal. This is no coincidence: Ingleton & Piff [12] showed that any matroid with size $n$ and rank $r \geq n-2$ is transversal.

The first entries where equality does not hold are $C^1_{2,4} = C^2_{2,4} + 1$ and $C^1_{3,4} = C^1_{3,4} + 1$. Indeed, for both rank 2 and rank 3, only a singular non-transversal matroid exists on ground set [4]. These distinctive matroids are visually represented in Figure 18. Notably, the rank 3 matroid represented in the figure is the graphic matroid of the graph $K_4$ which we mentioned in Example 1.24. In particular, we know that results are exact up to $n = 6$.

*Observation* 4.11 (on the equality in Table 2 for $n = 7$, $r = 2$). For $n = 7$ and $r = 2$ we still have all the results; there are three non-transversal matroids on those parameters. If $M$ is the only non-transversal matroid on $n = 6$ and $r = 2$, then $M + x$ (free extension), $M +_{\emptyset} x$ (adding a loop) and $M +_F$ where $F = \{a, b\}$ is a double point, all yield a new non-transversal extensions.

In relative terms, Table 2 shows what portion of matroids on ground set $[n]$ and rank $r$ we have found for each given $n$ and $r$.

*Observation* 4.12 (on last columns of Table 1). Our code was not able of computing the results for $n = 9$ in reasonable time. The gap between $n = 8$ and $n = 9$ is huge, as one can observe in the amount of total matroids on those ground sets (for instance, in $r = 5$ there are 325 matroids on ground set [8] and 190214 matroids on ground set [9]). In fact, the numbers $C^1_{r,n}$ for $n \leq 8$ were found in the late 60's by Blackburn, Capo & Higgs (although the resulting paper [1] was not published until years later). However,

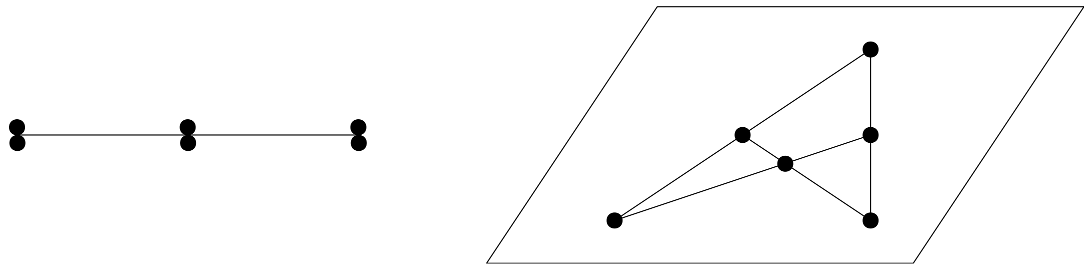

Figure 18: The two only non-transversal matroids on ground set [6].

Table 2: Lower bounds for the % of transversal matroids over total number matroids for each $n$ and $r$.

| $r$ \ $n$ | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** |
|---|---|---|---|---|---|---|---|---|
| **0** | =100% | =100% | =100% | =100% | =100% | =100% | =100% | =100% |
| **1** | =100% | =100% | =100% | =100% | =100% | =95.65% | =100% | =100% |
| **2** | | =100% | =100% | =100% | =100% | =97.37% | =91.89% | ≥86.2% |
| **3** | | | =100% | =100% | =100% | =100% | ≥85.19% | ≥64% |
| **4** | | | | =100% | =100% | =100% | ≥92.59% | ≥44.79% |
| **5** | | | | | =100% | =100% | =100% | ≥78.46% |
| **6** | | | | | | =100% | =100% | ≥96.55% |
| **7** | | | | | | | =100% | =100% |
| **7** | | | | | | | | =100% |

it was not until 2008 that Mayhew and Royle published the catalog with $n = 9$. Even though our objective is not as ambitious as theirs, it seems reasonable that this gap is too much for our code unless it is further optimized.

### 4.6.1 Complexity and benchmarking

Notably, the complexity of the algorithm we propose grows exponentially with $n$. This growth is reflected in the average execution times of Table 3, as $n = 6$ can be executed in reasonable time (less than a minute) but $n = 7$ already takes several hours. The execution of $n = 8$ is excluded from the benchmarking because we used a slightly different strategy. Knowing that it was the last iteration, instead of computing and storing all minimal presentations (which would only be necessary if we attempted to compute $n = 9$) it only checks for maximal ones. This increases the performance by much and allows us to have the results, as otherwise, we would have needed a dedicated machine to execute $n = 8$.

Let $P_n$ be the number of minimal presentations over $[n]$ stored when we enter iteration $n$ of the code. Then, for each of these presentations $\mathcal{A}$, if $M[\mathcal{A}]$ has rank $r$ we may add up to $2^r + 1$ minimal presentations for the next iteration[15]. This rank is bounded by[16] $n$, which means that we potentially add $2^r + 1 \sim 2^n + 1$

[15] For each $I \subseteq [r]$, we may add $\mathcal{A}^I$. Additionally, we may add a coloop.
[16] We have $r \leq n$ and we see a concentration of values around $r = n/2$; most matroids on ground set $[n]$ have rank close to $n/2$.

Table 3: Average execution time of the algorithm for distinct values of $n$. All values are in seconds.

| | n | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| Algorithm 1 | <0.01 | <0.01 | <0.1 | 0.03 | 0.58 | 24.97 | 2871.86 | - |
| Algorithm 2 | <0.01 | <0.01 | 0.03 | 0.14 | 1.03 | 8.65 | 87.58 | - |
| Average Execution Time | <0.01 | <0.01 | 0.03 | 0.17 | 1.61 | 33.62 | 2959.44 | - |

presentations for each $\mathcal{A}$. We cannot control how many of these presentations will be pairwise equal up to relabeling, so we can only state that $P_{n+1} \leq 2^n P_n$. Since we start with $P_0 = 1$, we can say that, roughly, $P_n \leq 2^0 \times 2^1 \times \cdots \times 2^n = 2^{n(n+1)/2}$.

This is a rough approximation, and in fact, the real number $P_n$ is significantly smaller. However, we do note an exponential growth. Note that, additionally, we will compute a comparison up to relabel of each pair of these presentations. This comparison requires comparing two set systems a total of $n!$ times (one for each permutation of the ground set). This gives an idea of why the code is so expensive in terms of time.

## 4.7 Future work

We believe that these algorithms could be optimized and applied to obtain all the minimals for $n = 8$ and maybe even the counting $n = 9$. Algorithms to compare transversal matroids and relabeled presentations could probably be improved with enough time. For the computations shown in this work, we used a personal-use laptop with limited resources.

An approach that we have not investigated and would be of interest is to use the dataset from [14] to count transversal matroids from there instead of building them from scratch as we do. They give each matroid of the catalog using their bases; one could build an algorithm to find the cyclic flats given the bases and then evaluate the Mason-Ingleton inequalities on each of their subsets. This approach is not a simple task but it seems feasible, but we will not dig deeper into it.

It remains open to find a characterization of minimal presentations that do not arise as an extension of a minimal presentation; with that, depending on the characterization, maybe it would be feasible to compute the additional presentations that lack at each step. That would allow us to compute a complete catalog on transversal matroids up to a given $n$.

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# A. Modular pair cadinality tables of low-rank uniform matroids

| Modular pairs for $r = 3$ | | | |
|---|---|---|---|
| $\lvert F \rvert$ | $\lvert F' \rvert$ | $\lvert F \cap F' \rvert$ | $\lvert F \cup F' \rvert$ |
| 2 | 2 | 1 | 3 |
| 2 | 1 | 0 | 3 |
| 1 | 1 | 0 | 2 |

| Non-modular pairs for $r = 3$ | | |
|---|---|---|
| $\lvert F \rvert$ | $\lvert F' \rvert$ | $\lvert F \cap F' \rvert$ |
| 2 | 2 | 0 |

| Modular pairs for $r = 4$ | | | |
|---|---|---|---|
| $\lvert F \rvert$ | $\lvert F' \rvert$ | $\lvert F \cap F' \rvert$ | $\lvert F \cup F' \rvert$ |
| 3 | 3 | 2 | 4 |
| 3 | 2 | 1 | 4 |
| 3 | 1 | 0 | 4 |
| 2 | 2 | 1 | 3 |
| 2 | 2 | 0 | 4 |
| 2 | 1 | 0 | 3 |
| 1 | 1 | 0 | 2 |

| Non-modular pairs for $r = 4$ | | |
|---|---|---|
| $\lvert F \rvert$ | $\lvert F' \rvert$ | $\lvert F \cap F' \rvert$ |
| 3 | 3 | 1 |
| 3 | 3 | 0 |
| 3 | 2 | 0 |

| Modular pairs for $r = 5$ | | | |
|---|---|---|---|
| $\lvert F \rvert$ | $\lvert F' \rvert$ | $\lvert F \cap F' \rvert$ | $\lvert F \cup F' \rvert$ |
| 4 | 4 | 3 | 5 |
| 4 | 3 | 2 | 5 |
| 4 | 2 | 1 | 5 |
| 4 | 1 | 0 | 5 |
| 3 | 3 | 2 | 4 |
| 3 | 3 | 1 | 5 |
| 3 | 2 | 0 | 5 |
| 3 | 2 | 1 | 4 |
| 2 | 2 | 0 | 4 |
| 2 | 2 | 1 | 3 |
| 2 | 1 | 0 | 3 |
| 1 | 1 | 0 | 2 |

| Non-modular pairs for $r = 5$ | | |
|---|---|---|
| $\lvert F \rvert$ | $\lvert F' \rvert$ | $\lvert F \cap F' \rvert$ |
| 4 | 4 | 2 |
| 4 | 4 | 1 |
| 4 | 3 | 1 |
| 4 | 3 | 0 |
| 4 | 2 | 0 |
| 3 | 3 | 0 |

| Modular pairs for $r = 6$ | | | |
|---|---|---|---|
| $\|F\|$ | $\|F'\|$ | $\|F \cap F'\|$ | $\|F \cup F'\|$ |
| 5 | 5 | 4 | 6 |
| 5 | 4 | 3 | 6 |
| 5 | 3 | 2 | 6 |
| 5 | 2 | 1 | 6 |
| 5 | 1 | 0 | 6 |
| 4 | 4 | 2 | 6 |
| 4 | 4 | 3 | 5 |
| 4 | 3 | 1 | 6 |
| 4 | 3 | 2 | 5 |
| 4 | 2 | 0 | 6 |
| 4 | 2 | 1 | 5 |
| 4 | 1 | 0 | 5 |
| 3 | 3 | 0 | 6 |
| 3 | 3 | 1 | 5 |
| 3 | 3 | 2 | 4 |
| 3 | 2 | 0 | 5 |
| 3 | 2 | 1 | 4 |
| 3 | 1 | 0 | 4 |
| 2 | 2 | 0 | 4 |
| 2 | 2 | 1 | 3 |
| 2 | 2 | 0 | 4 |
| 2 | 1 | 0 | 3 |
| 1 | 1 | 0 | 2 |

| Non-modular pairs for $r = 6$ | | |
|---|---|---|
| $\|F\|$ | $\|F'\|$ | $\|F \cap F'\|$ |
| 5 | 5 | 3 |
| 5 | 5 | 2 |
| 5 | 5 | 1 |
| 5 | 5 | 0 |
| 5 | 4 | 2 |
| 5 | 4 | 1 |
| 5 | 4 | 0 |
| 5 | 3 | 1 |
| 5 | 3 | 0 |
| 5 | 2 | 0 |
| 4 | 4 | 1 |
| 4 | 4 | 0 |
| 4 | 3 | 0 |