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RIMS-1973

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June 2023

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June 15, 2023

Abstract

Let p, l be prime numbers. In anabelian geometry for p-adic local fields [i.e., finite extension fields of the field of p-adic numbers], many topics have been discussed. In the present paper, we generalize two of the topics — discovered by S. Mochizuki — to more general complete discrete valuation fields. One is the mono-anabelian reconstruction, under a certain indeterminacy, of the *cyclotomic rigidity isomorphism* between the usual cyclotome $\widehat{\mathbb{Z}}(1)$ associated to a p-adic local field and the cyclotome constructed, in a purely group-theoretic way, from [the underlying topological group structure of] the absolute Galois group of the p-adic local field. The other is the *Neukirch-Uchida-type result*, i.e., the field-theoreticity of an outer isomorphism between the absolute Galois groups of p-adic local fields that preserves the respective ramification filtrations. For our generalizations, we first discuss l-local class field theory for Henselian discrete valuation fields with strongly l-quasi-finite residue fields [i.e., perfect fields such that the maximal pro-l quotients of the absolute Galois groups of their finite extension fields are isomorphic to $\mathbb{Z}_l$] of characteristic p via Artin-Tate's class formation. This theory enables us to reconstruct the l-cyclotomes from the absolute Galois groups of such fields. With regard to cyclotomic rigidity, under a certain assumption, we establish mono-anabelian group/monoid-theoretic reconstruction algorithms for cyclotomic rigidity isomorphisms associated to Henselian discrete valuation fields with quasi-finite residue fields [i.e., perfect residue fields whose absolute Galois groups are isomorphic to $\widehat{\mathbb{Z}}$]. As an application of the reconstructions of cyclotomic rigidity isomorphisms, we determine the structure of the groups of Galois-equivariant automorphisms of various algebraically completed multiplicative groups that arise from complete discrete valuation fields with quasi-finite residues. Moreover, as a byproduct of the argument applied in this determination [especially, in the positive characteristic case], we also determine, in a generalized situation, the structure of a certain indeterminacy "(Ind2)" that appears in S. Mochizuki's *inter-universal Teichmüller theory*. With regard to the Neukirch-Uchida-type result, by combining the reconstruction result of p-cyclotomes above [in the case where $l = p$] with a recent result due to T. Murotani, together with a computation concerning norm maps, we prove an analogous result for mixed characteristic complete discrete valuation fields whose residue fields are [strongly] p-quasi-finite and algebraic over the prime fields.

Contents

2020 Mathematics Subject Classification: Primary 11S20; Secondary 11S31.

Key words and phrases: anabelian geometry; local class field theory; Henselian discrete valuation field; quasi-finite field; cyclotomic rigidity; Neukirch-Uchida theorem.

Introduction

Let p be a prime number. In the present paper, we investigate anabelian geometry for Henselian discrete valuation fields of residue characteristic p [especially, surrounding the case of quasi-finite residue fields — cf. the discussion preceding Theorem B] to generalize various results that have been obtained in anabelian geometry for p-adic local fields [i.e., finite extension fields of the field of p-adic numbers] so far. We discuss mainly two topics:

- Mono-anabelian reconstruction of suitable orbits of the "cyclotomic rigidity isomorphism" [cf. the discussion immediately after Theorem B], and
- Neukirch-Uchida-type result.

First, we explain the background of the first topic. In anabelian geometry [especially, mono-anabelian geometry introduced by S. Mochizuki — cf. [23], Introduction], we often encounter two types of "similar" objects. Let X be a geometric object [such as a scheme or a field]; Π_X a homotopy-theoretic topological group associated to X [such as the étale fundamental group of a scheme or the absolute Galois group of a field]. Then one is an object

$$M(X)$$

constructed directly from X, and the other is an object

$$M(\Pi_X)$$

constructed from [the underlying topological group structure of] Π_X via a functorial group-theoretic algorithm [cf. [23], Remark 1.9.8] that admits a certain natural isomorphism

$$\phi_X : M(X) \xrightarrow{\sim} M(\Pi_X)$$

that is constructed by using the *geometric structure* of X [i.e., a ring/scheme-theoretic structure in the case where X is a field or a scheme]. On the other hand, from the viewpoint of mono-anabelian geometry, one basic question here is:

Question: To what extent is the isomorphism ϕ_X between $M(X)$ and $M(\Pi_X)$ "*rigid*" with respect to a certain input data that involves a structure *strictly weaker* than the full geometric [ring/scheme-theoretic] structure of X?

For instance, let F be a p-adic local field. Then the mono-anabelian reconstruction of suitable orbits of the cyclotomic rigidity isomorphism associated to F may be regarded as an answer to Question in the case where:

- X: F with a choice of an algebraic closure $\overline{F}$ of F.
- Π_X: [the underlying topological group structure] $G_F \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{F}/F)$, i.e., the absolute Galois group of F.
- $M(X)$: $\mu(\overline{F})$, i.e., the group of the roots of unity in $\overline{F}$.
- $M(\Pi_X)$: $\varinjlim_{H \subseteq G_F} H^{\mathrm{ab}}_{\mathrm{tor}}$, where $H \subseteq G_F$ ranges over the open subgroups; $H^{\mathrm{ab}}_{\mathrm{tor}}$ denotes the torsion subgroup of the abelianization of H; the transition maps are induced by the transfers.
- ϕ_X: the isomorphism $\mu(\overline{F}) \xrightarrow{\sim} \varinjlim_{H \subseteq G_F} H^{\mathrm{ab}}_{\mathrm{tor}}$ induced by the reciprocity maps associated to the finite extension fields $\subseteq \overline{F}$ of F.
- Input data: $(G_F \curvearrowright \overline{F}^{\times})$ or $(G_F \curvearrowright \mathcal{O}^{\triangleright}_{\overline{F}})$, where $\mathcal{O}^{\triangleright}_{\overline{F}}$ denotes the multiplicative monoid consisting of nonzero elements of the ring of integers of $\overline{F}$

[cf. [23], Propositions 3.2, 3.3]. Here, we note that the constructions of the reciprocity maps depend on the full ring structures of p-adic local fields, and $\overline{F}^{\times}$, $\mathcal{O}^{\triangleright}_{\overline{F}}$ are purely multiplicative objects. Moreover, we observe that one may not reconstruct the full ring structure of F from G_F [cf., e.g., [34], §2, Theorem]. In particular, the input data is a data *strictly weaker* than the full ring structure of F. Note also that the extent to which the cyclotomic rigidity isomorphism is rigid [i,e., an answer to Question] may depend on the input data [cf. [21], Theorem 2.4; [23], Propositions 3.2, 3.3].

Next, we explain the background of the second topic. The Neukirch-Uchida theorem states that every outer isomorphism between the absolute Galois groups of number fields arises from a unique isomorphism of the number fields [cf. [28], Corollary 12.2.2; [33], Theorem]. Here, we note that Y. Hoshi gave a functorial group-theoretic algorithm for constructing number fields from [the underlying topological group structures of] their absolute Galois groups [cf. [6], Theorem A] and strengthen the Neukirch-Uchida theorem. On the other hand, the analogous statement for p-adic local fields does not hold [cf., e.g., [34], §2, Theorem]. In this situation, S. Mochizuki discovered that, if an outer isomorphism between the absolute Galois groups of p-adic local fields preserves the respective [uppper] ramification filtrations, then this outer isomorphism arises from a unique isomorphism of the p-adic local fields [cf. [19], Theorem].

In the present paper, we generalize various reconstruction results surrounding p-adic local fields, especially, the mono-anabelian group/monoid-theoretic reconstruction algorithms for cyclotomic rigidity isomorphisms and the Neukirch-Uchida-type result for p-adic local fields [obtained by S. Mochizuki] to the case of Henselian discrete valuation fields with certain *infinite* residue fields of characteristic p. In order to state our main results, we begin by preparing some notations.

For each pair of a monoid T and a positive integer n, we shall write $T_{\mathrm{tor}} \subseteq T$ for the torsion subgroup of T; $T[n] \subseteq T$ for the n-torsion subgroup of T; T^{gp} for the groupification of T. For each field F, we shall write $\mu(F)$ for the group of roots of unity $\in F$; F^{sep} for the separable closure [determined up to isomorphisms] of F; $G_F \stackrel{\mathrm{def}}{=} \mathrm{Gal}(F^{\mathrm{sep}}/F)$; $\mathrm{char}(F)$ for the characteristic of F.

Let K be a Henselian discrete valuation field of residue characteristic p. Write k for the residue field of K; $I_K \subseteq G_K$ for the inertia subgroup [i.e., the kernel of the natural surjection $G_K \twoheadrightarrow G_k$]; $P_K \subseteq G_K$ for the wild inertia subgroup [cf. Definition 1.1]. Our first main result is the following [cf. Propositions 1.6, 1.7]:

Theorem A. *The following hold:*

(i) Let $K \subseteq L$ ($\subseteq K^{\mathrm{sep}}$) be a separable field extension such that $G_L \subseteq G_K$ is a nontrivial subnormal closed subgroup. Then the prime number p may be reconstructed, in a purely group-theoretic way, from [the underlying topological group structure of] the absolute Galois group G_L.

(ii) Suppose that every normal closed pro-cyclic subgroup and every pro-p normal closed subgroup of the kernel of the pro-prime-to-p cyclotomic character associated to k are trivial. Then the subgroups $I_K \subseteq G_K$ and $P_K \subseteq G_K$ may be reconstructed, in a purely group-theoretic way, from [the underlying topological group structure of] the absolute Galois group G_K.

Our proof of Theorem A, (i), depends on the *internal indecomposability* [i.e., a property that the centralizer of any nontrivial normal closed subgroup is trivial] of G_L. On the other hand, one may observe that the assumption in Theorem A, (ii), is satisfied in the case where k is algebraic over the prime field or a Hilbertian field [cf. Remark 1.7.1]. It would be interesting to investigate to which extent the assumption in Theorem A, (ii), may be dropped.

Next, for each separable field extension $K \subseteq L$ ($\subseteq K^{\mathrm{sep}}$), we shall write $\mathcal{O}_L$ for the ring of integers of L; $\mathcal{O}_L^{\times}$ ($\subseteq \mathcal{O}_L$) for the group of units of $\mathcal{O}_L$; $\mathcal{O}_L^{\triangleright} \stackrel{\mathrm{def}}{=} \mathcal{O}_L \setminus \{0\}$ [as a multiplicative monoid]. For each finite separable extension $K \subseteq L$ ($\subseteq K^{\mathrm{sep}}$), we shall write

$$\widehat{\mathcal{O}}_L^{\times} \stackrel{\mathrm{def}}{=} \varprojlim_{m \geq 1} \mathcal{O}_L^{\times}/(\mathcal{O}_L^{\times})^m; \quad \widehat{\mathcal{O}}_L^{\triangleright} \stackrel{\mathrm{def}}{=} \varprojlim_{m \geq 1} \mathcal{O}_L^{\triangleright}/(\mathcal{O}_L^{\triangleright})^m; \quad \widehat{L}^{\times} \stackrel{\mathrm{def}}{=} \varprojlim_{m \geq 1} L^{\times}/(L^{\times})^m,$$

where m ranges over the positive integers;

$$\hat{\mathcal{O}}_L^{\triangleright} \subseteq \widehat{\mathcal{O}}_L^{\triangleright} \quad (\text{respectively, } \hat{L}^{\times} \subseteq \widehat{L}^{\times})$$

for the submonoid (respectively, subgroup) generated by $\widehat{\mathcal{O}}_L^{\times}$ and the image of a prime element $\in \mathcal{O}_L^{\triangleright}$ of L via the natural map $\mathcal{O}_L^{\triangleright} \to \widehat{\mathcal{O}}_L^{\triangleright}$ (respectively, $L^{\times} \to \widehat{L}^{\times}$). Finally, we shall write

$$\widehat{\mathcal{O}}_{K^{\mathrm{sep}}}^{\times} \stackrel{\mathrm{def}}{=} \varinjlim_{K \subseteq L} \widehat{\mathcal{O}}_L^{\times}; \quad \hat{\mathcal{O}}_{K^{\mathrm{sep}}}^{\triangleright} \stackrel{\mathrm{def}}{=} \varinjlim_{K \subseteq L} \hat{\mathcal{O}}_L^{\triangleright}; \quad (K^{\hat{\mathrm{sep}}})^{\times} \stackrel{\mathrm{def}}{=} \varinjlim_{K \subseteq L} \hat{L}^{\times},$$

where $K \subseteq L$ ($\subseteq K^{\mathrm{sep}}$) ranges over the finite separable extensions; the transition maps are the natural homomorphisms. It would be worth mentioning that, under a certain mild assumption, these transition maps are injective and behave reasonably well with respect to Galois actions [cf. Lemma 1.12]. The latter property enables us to define the cohomology groups of G_K with coefficients in $\widehat{\mathcal{O}}_{K^{\mathrm{sep}}}^{\times}$, $\hat{\mathcal{O}}_{K^{\mathrm{sep}}}^{\triangleright}$, and $(K^{\hat{\mathrm{sep}}})^{\times}$. On the other hand, one verifies immediately that, if K is a complete discrete valuation field with finite residues, then $\widehat{\mathcal{O}}_K^{\times} = \mathcal{O}_K^{\times}$, $\hat{\mathcal{O}}_K^{\triangleright} = \mathcal{O}_K^{\triangleright}$, and $\hat{K}^{\times} = K^{\times}$. With regard to these objects, one key observation/philosophy in the present paper is the following:

> In order to extend anabelian geometry for p-adic local fields to the case of Henselian discrete valuation fields with infinite residues, it is natural to treat *algebraically [partially] completed multiplicative objects*.

Next, for each field F, we shall say that:

- F is *quasi-finite* if F is perfect, and $G_F \cong \widehat{\mathbb{Z}}$, where $\widehat{\mathbb{Z}}$ denotes the profinite completion of $\mathbb{Z}$.
- F is *μ-finite* if $\mu(F)$ is finite.

In the remainder of Introduction, to simplify various notations and statements, we suppose that

$$\mathrm{char}(K) \;=\; 0.$$

Then our second main result is the following [cf. Theorem 4.7, (ii), (iii)]:

Theorem B. *Suppose that k is quasi-finite and μ-finite. Let G be a topological group isomorphic to G_K. Then the following hold:*

(i) Write

$$\hat{r}_K : \widehat{K}^\times \hookrightarrow G_K^{\mathrm{ab}}$$

for the injective homomorphism [cf. Proposition 4.4] induced by the reciprocity map $r_K : K^\times \to G_K^{\mathrm{ab}}$ [cf. Theorem 4.3]. Then there exists a functorial group-theoretic algorithm

$$G \quad \rightsquigarrow \quad (\mu(G) \subseteq \widehat{\mathcal{O}}^\times(G) \subseteq \hat{\mathcal{O}}^\triangleright(G) \subseteq \hat{K}^\times(G))$$

for constructing — from the topological group G — a data $(\mu(G) \subseteq \widehat{\mathcal{O}}^\times(G) \subseteq \hat{\mathcal{O}}^\triangleright(G) \subseteq \hat{K}^\times(G))$ consisting of submonoids of G^{ab} that satisfies the following condition: if one applies this functorial group-theoretic algorithm to $G = G_K$, then

$$\mu(G) = \hat{r}_K(\mu(K)), \quad \widehat{\mathcal{O}}^\times(G) = \hat{r}_K(\widehat{\mathcal{O}}_K^\times), \quad \hat{\mathcal{O}}^\triangleright(G) = \hat{r}_K(\hat{\mathcal{O}}_K^\triangleright), \quad \hat{K}^\times(G) = \hat{r}_K(\hat{K}^\times)$$

[cf. the discussion at the beginning of the proof of Proposition 3.7].

(ii) By applying the group-theoretic algorithm given in (ii) to every normal open subgroup $H \subseteq G$, one constructs a data $(\mu_s(G) \subseteq \widehat{\mathcal{O}}_s^\times(G) \subseteq \hat{\mathcal{O}}_s^\triangleright(G) \subseteq \hat{K}_s^\times(G))$ consisting of the monoids [equipped with natural actions of G]

$$\mu_s(G) \stackrel{\mathrm{def}}{=} \varinjlim_{H \subseteq G} \mu(H), \quad \widehat{\mathcal{O}}_s^\times(G) \stackrel{\mathrm{def}}{=} \varinjlim_{H \subseteq G} \widehat{\mathcal{O}}^\times(H),$$

$$\hat{\mathcal{O}}_s^\triangleright(G) \stackrel{\mathrm{def}}{=} \varinjlim_{H \subseteq G} \hat{\mathcal{O}}^\triangleright(H), \quad \hat{K}_s^\times(G) \stackrel{\mathrm{def}}{=} \varinjlim_{H \subseteq G} \hat{K}^\times(H)$$

— where the transition maps are induced by the transfers — that satisfies the following condition: if one applies this functorial group-theoretic algorithm to $G = G_K$, then the [completed] reciprocity maps $\{\hat{r}_L\}_{K \subseteq L}$ — where $K \subseteq L$ $(\subseteq K^{\mathrm{sep}})$ ranges over the finite Galois extensions — induce G-equivariant isomorphisms

$$\mu(K^{\mathrm{sep}}) \xrightarrow{\sim} \mu_s(G), \quad \widehat{\mathcal{O}}_{K^{\mathrm{sep}}}^\times \xrightarrow{\sim} \widehat{\mathcal{O}}_s^\times(G), \quad \hat{\mathcal{O}}_{K^{\mathrm{sep}}}^\triangleright \xrightarrow{\sim} \hat{\mathcal{O}}_s^\triangleright(G), \quad (\hat{K^{\mathrm{sep}}})^\times \xrightarrow{\sim} \hat{K}_s^\times(G).$$

In the remainder of Introduction, we shall refer to the G-equivariant isomorphism

$$\varprojlim_{n \geq 1} \mu_n(K^{\mathrm{sep}}) \xrightarrow{\sim} \varprojlim_{n \geq 1} \mu_s(G)[n]$$

induced by the first isomorphism of the final display in Theorem B as the *cyclotomic rigidity isomorphism* associated to K. Note that the proof of Theorem B depends heavily on local class field theory for Henselian discrete valuation fields with quasi-finite residues. Note also that there are many examples of infinite, μ-finite, and quasi-finite fields of characteristic p [cf. Remark 4.2.1].

Now suppose that k is quasi-finite and μ-finite until the end of Corollary D below. Let G be a topological group isomorphic to G_K. Fix a functorial group-theoretic algorithm as in Theorem B, (ii). Write

$$\mu_s(G) \subseteq \widehat{\mathcal{O}}_s^\times(G) \subseteq \hat{\mathcal{O}}_s^\triangleright(G) \subseteq \hat{K}_s^\times(G)$$

for the output data of this algorithm. Let M be a multiplicative monoid such that

- M admits an action of G,
- $(G \curvearrowright M)$ is isomorphic to $(G_K \curvearrowright \widehat{\mathcal{O}}_{K^{\mathrm{sep}}}^\times)$ (respectively, $(G_K \curvearrowright \hat{\mathcal{O}}_{K^{\mathrm{sep}}}^\triangleright)$; $(G_K \curvearrowright (\hat{K^{\mathrm{sep}}})^\times)$).

Write

$$\Gamma \stackrel{\text{def}}{=} \widehat{\mathbb{Z}}^{\times} \text{ (respectively, } \{1\}; \{\pm 1\}); \quad \Lambda(M) \stackrel{\text{def}}{=} \varprojlim_{n \geq 1} (M_{\text{tor}}[n]); \quad \Lambda(G) \stackrel{\text{def}}{=} \varprojlim_{n \geq 1} \mu(G)[n]$$

Then our third main result is the following [cf. Theorem 5.9]:

Theorem C. *There exists a functorial group/monoid-theoretic algorithm*

$$(G \curvearrowright M) \quad \rightsquigarrow \quad \Lambda(M) \stackrel{\sim}{\to} \Lambda(G)$$

for constructing — from $(G \curvearrowright M)$ *— a natural* Γ*-torsor consisting of isomorphisms*

$$\Lambda(M) \stackrel{\sim}{\to} \Lambda(G)$$

that satisfies the following condition: if one applies this group/monoid-theoretic algorithm to one of the following cases where

- $(G \curvearrowright M) = (G_K \curvearrowright \widehat{\mathcal{O}}^{\times}_{K^{\text{sep}}})$;
- $(G \curvearrowright M) = (G_K \curvearrowright \hat{\mathcal{O}}^{\triangleright}_{K^{\text{sep}}})$;
- $(G \curvearrowright M) = (G_K \curvearrowright (\hat{K^{\text{sep}}})^{\times})$,

then the above Γ*-torsor coincides with the natural* Γ*-orbit of the cyclotomic rigidity isomorphism associated to* K*.*

[Note that, in the case where $\Gamma = \widehat{\mathbb{Z}}^{\times}$, a natural Γ-orbit of an isomorphism is nothing but all the isomorphisms. In particular, in this case, there exists no nontrivial mathematical content in Theorem C.] Moreover, as a corollary of Theorem C, we obtain the following [cf. Corollary 6.7]:

Corollary D. *For* $i = 1, 2$*, let* G_i *be a topological group;* M_i *a monoid that admits an action of* G_i*. Write*

$$\text{Isom}(G_1 \curvearrowright M_1,\ G_2 \curvearrowright M_2)$$

for the set of pairs of a continuous isomorphism $G_1 \stackrel{\sim}{\to} G_2$ *and an isomorphism* $M_1 \stackrel{\sim}{\to} M_2$ *compatible with the respective actions;*

$$\text{Isom}(G_1,\ G_2)$$

for the set of continuous isomorphisms $G_1 \stackrel{\sim}{\to} G_2$*. Suppose that* $(G_i \curvearrowright M_i)$ *is isomorphic to* $(G_K \curvearrowright \widehat{\mathcal{O}}^{\times}_{K^{\text{sep}}})$ *(respectively,* $(G_K \curvearrowright \hat{\mathcal{O}}^{\triangleright}_{K^{\text{sep}}})$*;* $(G_K \curvearrowright (\hat{K^{\text{sep}}})^{\times})$*), where* $i = 1, 2$*. Then the natural map*

$$\text{Isom}(G_1 \curvearrowright M_1,\ G_2 \curvearrowright M_2) \longrightarrow \text{Isom}(G_1,\ G_2)$$

is surjective. Moreover, the fibers of this map are Γ*-torsors. [Note that* $\Gamma = \widehat{\mathbb{Z}}^{\times}$ *(respectively,* $\Gamma = \{1\}$*;* $\Gamma = \{\pm 1\}$*).]*

We have discussed the quasi-finite residue case so far. On the other hand, if we restrict our attention to "p-like objects" [such as the group of p-power roots of unity], then one may treat more general case. For each field F, we shall say that:

- F is *p-quasi-finite* if F is perfect, and the maximal pro-p-quotient of G_F is isomorphic to $\mathbb{Z}_p$, where $\mathbb{Z}_p$ denotes the maximal pro-p quotient of $\widehat{\mathbb{Z}}$.
- F is *strongly p-quasi-finite* if every finite extension field of F is p-quasi-finite.

Note that quasi-finite fields are strongly p-quasi-finite. In fact, by applying a similar argument to the argument applied in the proof of local class field theory for Henselian discrete valuation fields with quasi-finite residues [cf. [31]], one may develop p-local class field theory [i.e., local class field theory for pro-p abelian extensions] for Henselian discrete valuation fields with p-quasi-finite residues [cf. §2; [30], §1]. In the case of Henselian discrete valuation fields with strongly p-quasi-finite residues, by applying this p-local class field theory, one may reconstruct various p-like objects. For each separable field extension $K \subseteq L$ ($\subseteq K^{\mathrm{sep}}$), we shall write $\mathfrak{m}_L$ for the maximal ideal of $\mathcal{O}_L$; $\mu_{p^\infty}(L)$ for the group of p-power roots of unity $\in L$;

$$U_{1,L} \stackrel{\mathrm{def}}{=} 1 + \mathfrak{m}_L; \quad U_L^{\mu} \stackrel{\mathrm{def}}{=} U_{1,L}/\mu_{p^\infty}(L).$$

We shall write

$$\mathcal{I}_K \stackrel{\mathrm{def}}{=} \frac{1}{2p} \cdot \log_p(U_{1,K}) \subseteq K,$$

where $\log_p : U_{1,K} \to K$ denotes the p-adic logarithm map [cf. [23], Definition 5.4, (iii)]. Then our forth result is the following [cf. Theorem 7.2, (ii), (iii)]:

Theorem E. *Suppose that k is a strongly p-quasi-finite field. Let G be a topological group isomorphic to G_K. Then the following hold:*

(i) Write

$$\hat{r}_K^p : \varprojlim_{n \geq 1} K^\times/(K^\times)^{p^n} \hookrightarrow (G_K^{\mathrm{ab}})^p$$

for the injective homomorphism [cf. Theorem 3.5, (i)] induced by the reciprocity map $r_K^p : K^\times \to (G_K^{\mathrm{ab}})^p$ [cf. Theorem 2.13]. Then there exists a functorial group-theoretic algorithm

$$G \rightsquigarrow \mu_{p^\infty}(G)$$

for constructing — from the topological group G — a data $\mu_{p^\infty}(G)$ consisting of a subgroup of $(G_K^{\mathrm{ab}})^p$ that satisfies the following condition: if one applies this functorial group-theoretic algorithm to $G = G_K$, then

$$\mu_{p^\infty}(G) = \hat{r}_K^p(\mu_{p^\infty}(K))$$

[cf. the discussion at the beginning of the proof of Proposition 3.7]. Moreover, by applying this group-theoretic algorithm to every normal open subgroup $H \subseteq G$, one constructs the group [equipped with natural action of G]

$$\mu_{p^\infty,s}(G) \stackrel{\mathrm{def}}{=} \varinjlim_{H \subseteq G} \mu_{p^\infty}(H)$$

— where the transition maps are induced by the transfers — that satisfies the following condition: if one applies this functorial group-theoretic algorithm to $G = G_K$, then the [completed] reciprocity maps $\{\hat{r}_L^p\}_{K \subseteq L}$ — where $K \subseteq L$ ($\subseteq K^{\mathrm{sep}}$) ranges over the finite Galois extensions — induce a G-equivariant isomorphism

$$\mu_{p^\infty}(K^{\mathrm{sep}}) \xrightarrow{\sim} \mu_{p^\infty,s}(G).$$

(ii) In the notation of (i), suppose, moreover, that K is complete. Let $I \subseteq G$ be a closed subgroup such that there exists an isomorphism $G \xrightarrow{\sim} G_K$ that induces an isomorphism $I \xrightarrow{\sim} I_K$. Then there exists a functorial group-theoretic algorithm

$$I \subseteq G \rightsquigarrow (U_1(I \subseteq G), \quad \mathcal{I}(I \subseteq G) \subseteq \mathcal{K}(I \subseteq G))$$

for constructing — from the pair of topological groups $I \subseteq G$ — a data $(U_1(I \subseteq G), \mathcal{I}(I \subseteq G) \subseteq \mathcal{K}(I \subseteq G))$ consisting of groups that satisfies the following condition: if one applies this functorial group-theoretic algorithm to $(I \subseteq G) = (I_K \subseteq G_K)$, then

$$U_1(I \subseteq G) = \hat{r}_K^p(U_{1,K}),$$

and $\mathcal{I}(I \subseteq G) \subseteq \mathcal{K}(I \subseteq G)$) is isomorphic to $\mathcal{I}_K \subseteq K$. Moreover, by applying this group-theoretic algorithm to every normal open subgroup $H \subseteq G$ [and the subgroup $I \cap H \subseteq H$], one constructs a data $(U_{1,s}(I \subseteq G), \mathcal{I}_s(I \subseteq G) \subseteq \mathcal{K}_s(G))$ consisting of the groups [equipped with natural actions of G]

$$U_{1,s}(I \subseteq G) \stackrel{\text{def}}{=} \varinjlim_{H \subseteq G} U_1(I \cap H \subseteq H), \quad \mathcal{I}_s(I \subseteq G) \stackrel{\text{def}}{=} \varinjlim_{H \subseteq G} \mathcal{I}(I \cap H \subseteq H),$$

$$\mathcal{K}_s(I \subseteq G) \stackrel{\text{def}}{=} \varinjlim_{H \subseteq G} \mathcal{K}(I \cap H \subseteq H)$$

— where the transition maps are induced by the transfers — that satisfies the following condition: if one applies this functorial group-theoretic algorithm to $(I \subseteq G) = (I_K \subseteq G_K)$, then the [completed] reciprocity maps $\{\hat{r}^p_L\}_{K \subseteq L}$ — where $K \subseteq L$ ($\subseteq K^{\text{sep}}$) ranges over the finite Galois extensions — induce a G-equivariant isomorphism

$$U_{1,K^{\text{sep}}} \xrightarrow{\sim} U_{1,s}(I \subseteq G),$$

and there exists a G-equivariant isomorphism

$$K^{\text{sep}} \xrightarrow{\sim} \mathcal{K}_s(I \subseteq G)$$

that maps $\mathcal{I}_K$ onto $\mathcal{I}(I \subseteq G)$.

Moreover, in light of p-local class field theory for Henselian discrete valuation fields with p-quasi-finite residue fields of characteristic p, together with a similar argument to the argument applied in the positive characteristic analogue of Corollary D, one may determine the structure of a certain indeterminacy "(Ind2)" that appears in S. Mochizuki's *inter-universal Teichmüller theory* [cf. [24]]. In order to state our result, let us recall/introduce the definition of the group of G_K-isometries in a generalized situation. [Note that (Ind2) consists of these isometries.] We shall write

$$\mathrm{Ism}(G_K)$$

for the group of G_K-equivariant automorphisms of $U^{\mu}_{K^{\text{sep}}}$ that, for each finite separable extension $K \subseteq L$ ($\subseteq K^{\text{sep}}$), *preserve* the "lattice" $U^{\mu}_L \subseteq (U^{\mu}_{K^{\text{sep}}})^{G_L}$. We shall refer to each element in $\mathrm{Ism}(G_K)$ as a *G_K-isometry*. Note that U^{μ}_L admits a natural structure of $\mathbb{Z}_p$-module compatible with the natural action of G_K. In particular, there exists a natural injection

$$\mathbb{Z}_p^{\times} \hookrightarrow \mathrm{Ism}(G_K).$$

Then our fifth main result is the following [cf. Theorem 7.4]:

Theorem F. *Suppose that G_k is not a pro-prime-to-p group. Then the natural injection*

$$\mathbb{Z}_p^{\times} \hookrightarrow \mathrm{Ism}(G_K)$$

is bijective.

Theorem F implies that (Ind2) is relatively small.

Finally, we discuss our generalization on the Neukirch-Uchida-type result mentioned above. We shall say that K is a *quasi-p-adic local field* if K is complete, and the residue field k of K is algebraic over the prime field. Then our final main result is the following [cf. Theorem 8.11]:

Theorem G. *Let K_1, K_2 be quasi-p-adic local fields with p-quasi-finite residue fields of characteristic p. Write*

$$\mathrm{Isom}(K_2, K_1)$$

for the set of isomorphisms $K_2 \xrightarrow{\sim} K_1$ of fields;

$$\mathrm{OutIsom}_{\mathrm{Filt}}(G_{K_1}, G_{K_2})$$

for the set of outer isomorphisms $G_{K_1} \xrightarrow{\sim} G_{K_2}$ of profinite groups that preserve the respective upper ramification filtrations. Then the natural map

$$\mathrm{Isom}(K_2, K_1) \longrightarrow \mathrm{OutIsom}_{\mathrm{Filt}}(G_{K_1}, G_{K_2})$$

is bijective.

Theorem G may be regarded as a generalization of the corresponding result for p-adic local fields due to S. Mochizuki [cf. [19], Theorem]. To our best knowledge, Theorem G is the first unconditional result among Neukirch-Uchida-type results for complete discrete valuation fields with *infinite* residues. Our proof of Theorem G may be regarded as an application of T. Murotani's recent work on certain homomorphisms between the absolute Galois groups of mixed characteristic complete discrete valuation fields with perfect residues, together with

- Theorem A, (ii); Theorem E, and
- a certain computation of the intersection of the images of higher unit groups via the norm maps associated to finite field extensions of complete discrete valuation fields with perfect residue fields of characteristic p and its consequence [cf. Propositions 8.4; 8.7].

We recall that there exists a positive characteristic analogue of [19], Theorem, due to V. Abrashkin [cf. [1]]. Therefore, in light of Theorem G, it would be interesting to investigate the extent to which

- the assumption on the residue fields in Theorem G may be weakened, and
- the positive characteristic analogue of Theorem G holds [cf. Remark 8.11.1].

The present paper is organized as follows. In §1, we first prove Theorem A. Next, we introduce certain [multiplicative] monoids that appear as the [algebraic] completions of [multiplicative] monoids associated to Henselian discrete valuation fields, which play important roles throughout the present paper. Moreover, under a certain mild assumption on the residue fields, we observe that these multiplicative monoids behave reasonably well with respect to the finite field extensions and the Galois actions. In §2, by imitating local class field theory for Henselian discrete valuation fields with quasi-finite residues via Artin-Tate's class formation, for each prime number l, we discuss l-local class field theory for Henselian discrete valuation fields with strongly l-quasi-finite residue fields of characteristic p. Here, we note that p-local class field theory for complete discrete valuation fields with p-quasi-finite residue fields of characteristic p was developed by K. Sekiguchi [cf. [30], §1]. On the other hand, if we restrict our attention to the totally wildly ramified abelian extensions, then much more general [and explicit] theory via a generalized version of Neukirch's construction was developed by I. Fesenko [cf. [2]]. In §3, by applying l-local class field theory discussed in §2, we prove that the groups of l-power roots of unity of Henselian discrete valuation fields with strongly l-quasi-finite and stably μ_{l^∞}-finite [cf. Definition 1.9, (ii)] residues are mapped isomorphically onto the torsion subgroups of the maximal pro-l quotients of their absolute Galois groups via the reciprocity maps. In §4, in light of results obtained in §3, we first discuss basic properties of the reciprocity maps in local class field theory for Henselian discrete valuation fields with quasi-finite residues. Then, by applying these properties, we prove Theorem B. In §5, we prove Theorem C [i.e., mono-anabelian group/monoid-theoretic reconstruction algorithms of cyclotomic rigidity isomorphisms] and its positive characteristic analogue. In §6, we apply Theorem C to prove Corollary D and its positive characteristic analogue. Note that, in the positive characteristic case, since Kummer theory does not have enough information, we need a certain additional argument [which leads us to obtain a proof of Theorem F in §7]. In §7, by applying p-local class field theory discussed in §2, we prove Theorems E, F. Finally, in §8, we prove a sophisticated version of Theorem G.

Notations and conventions

Numbers: The notation $\mathbb{Q}$ will be used to denote the field of rational numbers. The notation $\mathbb{Q}_{>0}$ will be used to denote the multiplicative group of positive rational numbers. The notation $\mathbb{Z}$ will be used to denote the ring of integers. The notation $\widehat{\mathbb{Z}}$ will be used to denote the profinite completion of the underlying additive group of $\mathbb{Z}$. If p is a prime number, then the notation $\mathbb{Z}_p$ will be used to denote the maximal pro-p quotient of $\widehat{\mathbb{Z}}$, the notation $\widehat{\mathbb{Z}}^{(p)'}$ will be used to denote the maximal pro-prime-to-p quotient of $\widehat{\mathbb{Z}}$; $\mathbb{F}_p$ will be used to denote the finite field of cardinality p. Note that $\widehat{\mathbb{Z}}$, $\mathbb{Z}_p$, and $\widehat{\mathbb{Z}}^{(p)'}$ admit natural structures of commutative rings.

Monoids: Let M be a [commutative] monoid; n a positive integer. Then we shall write $M^\times \subseteq M$ for the group of invertible elements of M [where, if A is a commutative ring, then we shall use the notation $A^\times$ for the group of invertible elements of the underlying commutative multiplicative monoid of A]; $M_{\mathrm{tor}} \subseteq M$ for the torsion subgroup of M; $M[n] \subseteq M$ for the n-torsion subgroup of M; M^{gp} for the groupification of M. If there exists a group action of a group G on M, then we shall write $M^G \subseteq M$ for the submonoid of M consisting of G-invariant elements.

Torsion abelian groups: Let A be a torsion abelian group; p a prime number. Then we shall write $A[(p^\infty)'] \subseteq A$ for the prime-to-p part of A.

Fields: Let F be a field; p a prime number. Then we shall write F^{sep} for the separable closure [determined up to isomorphisms] of F; $G_F \stackrel{\mathrm{def}}{=} \mathrm{Gal}(F^{\mathrm{sep}}/F)$; $\mathrm{char}(F)$ for the characteristic of F. If $\mathrm{char}(F) \neq p$, then we shall fix a primitive p-th root of unity $\zeta_p \in F^{\mathrm{sep}}$.

Profinite groups: Let G be a profinite group; p a prime number. Then we shall write G^{ab} for the quotient of G by the closure of the commutator subgroup $[G, G] \subseteq G$; G^p for the maximal pro-p quotient of G; $\mathrm{cd}_p(G)$ for the p-cohomological dimension of G; $\mathrm{scd}_p(G)$ for the strict p-cohomological dimension of G; $\mathrm{Aut}(G)$ for the group of continuous automorphisms of G; $\mathrm{Inn}(G) \subseteq \mathrm{Aut}(G)$ for the group of inner automorphisms of G; $\mathrm{Out}(G) \stackrel{\mathrm{def}}{=} \mathrm{Aut}(G)/\mathrm{Inn}(G)$.

Let $H \subseteq G$ be a closed subgroup. Then we shall write $Z_G(H)$ for the centralizer of H in G.

Let M be a discrete/profinite topological G-module; n a nonnegative integer. Then we shall write $H^n(G, M)$ for the n-th continuous group cohomology of G with coefficients in M.

1 Preliminaries on Henselian discrete valuation fields with positive characteristic residue fields

Let p be a prime number; K a Henselian discrete valuation field of residue characteristic p. Write $\mathcal{O}_K$ for the ring of integers of K; $\mathfrak{m}_K \subseteq \mathcal{O}_K$ for the maximal ideal of $\mathcal{O}_K$; $k \stackrel{\mathrm{def}}{=} \mathcal{O}_K/\mathfrak{m}_K$; $I_K \subseteq G_K$ for the inertia subgroup [i.e, the kernel of the natural [outer] surjection $G_K \twoheadrightarrow G_k$].

In the present section, we first investigate some group-theoretic properties of the absolute Galois group G_K that enable us to give group-theoretic characterizations of the prime number p and the subgroups $P_K \subseteq I_K \subseteq G_K$ [cf. Definition 1.1; Propositions 1.6, 1.7]. Next, we introduce certain [multiplicative] monoids that appear as the [algebraic] completions of [multiplicative] monoids associated to K [cf. Definition 1.8], which play important roles throughout the present paper. Moreover, in the case where k contains only finitely many roots of unity, we observe that these completed monoids behave reasonably well with respect to the finite field extensions and the Galois actions [cf. Lemma 1.12]. These properties may be applied to compute certain Galois cohomology groups and to establish mono-anabelian group/monoid-theoretic reconstruction algorithms for cyclotomic rigidity isomorphisms in §5.

Definition 1.1 ([4], §8, Theorem 1, (ii))**.** Let $K \subseteq L$ ($\subseteq K^{\mathrm{sep}}$) be a finite Galois extension. [In particular, L is also a Henselian discrete valuation field.] Write $v_L : L^\times \twoheadrightarrow \mathbb{Z}$ for the discrete valuation of L; $\mathcal{O}_L$ for the ring of integers of L;

$$P_{L/K} \stackrel{\text{def}}{=} \{\sigma \in \mathrm{Gal}(L/K) \mid v_L(\sigma(x) - x) \ \geq \ v_L(x) + 1 \text{ for any } x \in \mathcal{O}_L\};$$

$$P_K \stackrel{\text{def}}{=} \varprojlim_{K \subseteq M} P_{M/K} \subseteq \varprojlim_{K \subseteq M} \mathrm{Gal}(M/K) = G_K,$$

where $K \subseteq M$ ($\subseteq K^{\mathrm{sep}}$) ranges over the finite Galois extensions of K. Note that one may easily verify that $P_K \subseteq I_K$. We shall refer to $P_K \subseteq G_K$ as the *wild inertia subgroup.*

Lemma 1.2. *Suppose that k is perfect. Then it holds that* $\mathrm{cd}_p(P_K) = 1$.

Proof. First, by replacing K by the completion of K, we may assume without loss of generality that K is complete [cf., e.g., [16], Lemma 3.1]. Next, observe that $P_K \subseteq I_K$ is a [necessarily, unique] p-Sylow subgroup [cf. [31], Chapter IV, §2, Corollary 3]. Then it holds that $\mathrm{cd}_p(P_K) = \mathrm{cd}_p(I_K) = 1$ [cf. [29], Corollary 7.3.3, (a); [32], Chapter II, §4, Proposition 12]. This completes the proof of Lemma 1.2. □

Definition 1.3 ([17], Definition 1.1, (v); [17], Proposition 1.2)**.** Let G be a profinite group. Then we shall say that G is *internally indecomposable* if, for each nontrivial normal closed subgroup $H \subseteq G$, it holds that $Z_G(H) = \{1\}$.

Lemma 1.4. *Let*

$$1 \longrightarrow G_1 \longrightarrow G_2 \stackrel{\phi}{\longrightarrow} G_3 \longrightarrow 1$$

be an exact sequence of profinite groups. Write $\rho : G_3 \to \mathrm{Out}(G_1)$ for the natural outer representation associated to the above exact sequence. Then the following hold:

(i) Suppose that

- *G_1 is a torsion-free pro-cyclic group,*
- *G_3 is a torsion-free profinite group, and*
- *every normal closed pro-cyclic subgroup of* $\mathrm{Ker}(\rho)$ *is trivial.*

Then the closed subgroup $G_1 \subseteq G_2$ may be characterized as the maximal normal pro-cyclic subgroup of G_2.

(ii) *Suppose that one of the following conditions holds:*

(a) *G_1 is a pro-prime-to-p group, and every pro-p normal closed subgroup of* $\mathrm{Ker}(\rho)$ *is trivial.*

(b) *G_1 is a nontrivial pro-prime-to-p group, and G_2 is internally indecomposable.*

Then every pro-p normal closed subgroup of G_2 is trivial.

Proof. First, we verify assertion (i). Let $C \subseteq G_2$ be a normal pro-cyclic subgroup. Then it suffices to verify that $C \subseteq G_1$. For each prime numner l, write $C_l \subseteq C$ for the l-part of C. In particular, it suffices to verify that $C_l \subseteq G_1$. Note that since C is pro-cyclic, and G_1, G_3 are torsion-free groups, if C_l is nontrivial, then $C_l \cong \mathbb{Z}_l$. Recall that every nontrivial closed subgroup of $\mathbb{Z}_l$ is open. Then since G_3 is torsion-free, we observe that one of the following holds:

- $C_l \subseteq G_1$.
- C_l is nontrivial, and $C_l \cap G_1 = \{1\}$.

Suppose that C_l is nontrivial, and $C_l \cap G_1 = \{1\}$. Then since G_1 and C_l are normal subgroups of G_2, it holds that $G_1 \subseteq Z_{G_2}(C_l)$. Note that this inclusion implies that $\phi(C_l) \subseteq \mathrm{Ker}(\rho)$. However, since $\phi(C_l) \subseteq \mathrm{Ker}(\rho)$ is a nontrivial normal closed pro-cyclic subgroup, this contradicts our assumption that every normal closed pro-cyclic subgroup of $\mathrm{Ker}(\rho)$ is trivial. Thus, we conclude that $C_l \subseteq G_1$. This completes the proof of assertion (i).

Next, we verify assertion (ii). Let $N \subseteq G_2$ be a pro-p normal closed subgroup. Observe that since N is a pro-p group and G_1 is a pro-prime-to-p group, it holds that $N \cap G_1 = \{1\}$. Thus, it follows immediately from the fact that N and G_1 are normal in G_2 that $N \subseteq Z_{G_2}(G_1)$. In particular, if condition (b) holds, then it holds that $Z_{G_2}(G_1) = \{1\}$, hence that $N = \{1\}$. Therefore, we may assume without loss of generality that condition (a) holds. Note that the inclusion $N \subseteq Z_{G_2}(G_1)$ implies that $\phi(N) \subseteq \mathrm{Ker}(\rho)$. Moreover, since $\phi(N) \subseteq G_3$ is a pro-p normal closed subgroup, $\phi(N) \subseteq \mathrm{Ker}(\rho)$ is a pro-p normal closed subgroup. Thus, since every pro-p normal closed subgroup of $\ker(\rho)$ is trivial, we conclude that $\phi(N) = \{1\}$, hence that $N = N \cap G_1 = \{1\}$. This completes the proof of assertion (ii), hence of Lemma 1.4. □

Proposition 1.5. *The following hold:*

(i) *I_K/P_K is isomorphic to $\widehat{\mathbb{Z}}^{(p)'}(1)$ as a $G_k = G_K/I_K$-module, where "(1)" denotes the Tate twist.*

(ii) *P_K is a torsion-free pro-p group that is not topologically finitely generated. Moreover, if k is perfect, then P_K is a free pro-p group.*

(iii) *G_k and G_K are torsion-free groups.*

(iv) *Let l be a prime number such that $l \neq p$; $N \subseteq G_K$ a pro-l subnormal closed subgroup. Then it holds that $N = \{1\}$.*

Proof. Assertion (i) follows immediately from Kummer theory, together with the Henselian property of tamely ramified extensions of K [cf., e.g, [3], Chapter II, (3.5), Proposition, (1)].

Next, we verify assertion (ii). Note that I_K is not topologically finitely generated [cf. [16], Lemmas 3.1, 3.5; [28], Proposition 6.1.7]. Thus, it follows immediately from assertion (i) that P_K is not topologically finitely generated.

Next, suppose that k is perfect. Then since P_K is a pro-p group, and $\mathrm{cd}_p(P_K) = 1$ [cf. Lemma 1.2], it follows from [29], Theorem 7.7.4, that P_K is a free pro-p group.

Next, we consider the general case. If $\mathrm{char}(K) = p$, then one may construct a complete discrete valuation field M of characteristic p such that the residue field of M is the perfection of k, together with an isomorphism $G_M \xrightarrow{\sim} G_K$ that preserves respective wild inertia subgroups [cf. [16], Lemma 3.1; Cohen's structure theorem]. Therefore, the general case follows immediately from the perfect residue case. Thus, we may assume without loss of generality that $\mathrm{char}(K) = 0$. Let $\{t_i \in k \mid i \in I\}$ be a p-basis of k; $\tilde{t}_i \in \mathcal{O}_K \subseteq K$ a lifting of t_i. For each $i \in I$, we fix a compatible system $\{\tilde{t}_i^{\frac{1}{p^j}} \in K^{\mathrm{sep}}\}_{j \in \mathbb{Z}_{\geq 1}}$ of p-power roots of $\tilde{t}_i$. Write L ($\subseteq K^{\mathrm{sep}}$) for the field obtained by adjoining the elements $\{\tilde{t}_i^{\frac{1}{p^j}} \in K^{\mathrm{sep}}\}_{(i,j) \in I \times \mathbb{Z}_{\geq 1}}$ to K. Then L is a Henselian discrete valuation field with a perfect residue field of characteristic p. Write $K^{\mathrm{tm}} \subseteq L^{\mathrm{tm}}$ ($\subseteq K^{\mathrm{sep}}$) for the maximal tame extensions of K, L, respectively; $L^{\mathrm{tm}}_{p^\infty}$ ($\subseteq K^{\mathrm{sep}}$) for the field obtained by adjoining the p-power roots of unity to L^{tm}. Note that $\zeta_p \in K^{\mathrm{tm}}$. Moreover, in the case where $p = 2$, it follows immediately from the torsion-freeness of G^p_K [cf. [17], Proposition 3.3] that we may assume without loss of generality that $\sqrt{-1} \in K$. Then $\mathrm{Gal}(L^{\mathrm{tm}}_{p^\infty}/K^{\mathrm{tm}})$ is a torsion-free pro-p group. On the other hand, since the wild inertia subgroup P_L of G_L is a free pro-p group [cf. the perfect residue case discussed above], the closed subgroup $G_{L^{\mathrm{tm}}_{p^\infty}} \subseteq P_L$ is also a free pro-p group. Thus, we conclude that $P_K = G_{K^{\mathrm{tm}}}$ is a torsion-free pro-p group. This completes the proof of assertion (ii).

Next, we verify assertion (iii). In light of assertions (i), (ii), it suffices to prove that G_k is torsion-free. However, since k is of positive characteristic, the torsion-freeness follows immediately from Artin-Schreier theorem. This completes the proof of assertion (iii).

Next, we verify assertion (iv). First, observe that it follows immediately from the definition of subnormality that there exists a subnormal closed subgroup $H \subseteq G_K$ such that N is a normal closed subgroup of H. Here, note that every subnormal closed subgroup of G_K is internally indecomposable [cf. [15], Theorem B]. Now suppose that $N \neq \{1\}$. Then since $H \neq \{1\}$, it follows from [15], Proposition 1.10, that $H \cap P_K \subseteq H$ is a nontrivial pro-p normal closed subgroup [cf. assertion (ii)]. However, since $N \subseteq H$ is a nontrivial pro-prime-to-p normal closed subgroup, this contradicts the fact that H is internally indecomposable [cf. Lemma 1.4, (ii)]. Therefore, it holds that $N = \{1\}$. This completes the proof of assertion (iv), hence of Proposition 1.5. □

Proposition 1.6. *Let $K \subseteq L$ ($\subseteq K^{\mathrm{sep}}$) be a separable field extension such that $G_L \subseteq G_K$ is a nontrivial subnormal closed subgroup. Then the prime number p may be reconstructed, in a purely group-theoretic way, from [the underlying topological group structure of] the absolute Galois group G_L.*

Proof. Since $G_L \subseteq G_K$ is a nontrivial subnormal closed subgroup, we observe that $P_K \cap G_L \subseteq G_L$ is a nontrivial pro-p normal closed subgroup [cf. Proposition 1.5, (ii); [15], Theorem B; [15], Proposition 1.10]. Then it follows immediately from Proposition 1.5, (iv), that p is the unique prime number q such that there exists a nontrivial pro-q normal closed subgroup of G_L. This completes the proof of Proposition 1.6. □

Proposition 1.7. *Suppose that every normal closed pro-cyclic subgroup and every pro-p normal closed subgroup of the kernel of the pro-prime-to-p cyclotomic character associated to k are trivial. Then the subgroups $I_K \subseteq G_K$ and $P_K \subseteq G_K$ may be reconstructed, in a purely group-theoretic way, from [the underlying topological group structure of] the absolute Galois group G_K.*

Proof. Recall that p may be reconstructed, in a purely group-theoretic way, from [the underlying topological group structure of] the absolute Galois group G_K [cf. Proposition 1.6]. On the other hand, it follows immediately from our assumption, together with Proposition 1.5, (i), that every normal closed pro-cyclic subgroup and every pro-p normal closed subgroup of the kernel of the natural representation $G_K/I_K \to \mathrm{Aut}(I_K/P_K)$ are trivial. Then it follows immediately from Proposition 1.5, (i), (ii), together with Lemma 1.4, (ii), that the subgroup $P_K \subseteq G_K$ may be characterized as a unique maximal normal closed pro-p subgroup. Moreover, it follows immediately from Proposition 1.5, (i), (iii), together with Lemma 1.4, (i), that the subgroup $I_K \subseteq G_K$ may be characterized as the pull-back of a unique maximal normal pro-cyclic subgroup of G_K/P_K via the natural quotient $G_K \twoheadrightarrow G_K/P_K$. This completes the proof of Proposition 1.7. $\square$

Remark 1.7.1. Suppose that one of the following conditions holds:

(i) k is an algebraic extension field of the prime field.

(ii) k is a Hilbertian field. [In particular,

- finitely generated transcendental extension fields of an arbitrary field, and
- the field of fractions of an arbitrary Noetherian integral domain of dimension ≥ 2 [of characteristic p]

satisfy this condition — cf., e.g., [17], Remark 2.12.1].

Then k satisfies the assumption of Proposition 1.7. Indeed, suppose first that condition (i) holds. Then it follows immediately from [28], Lemma 7.5.4, (ii), that the kernel of the pro-prime-to-p cyclotomic character associated to k is trivial. In particular, the assumption of Proposition 1.7 holds. Next, suppose that condition (ii) holds. Write k^{cyc} ($\subseteq k^{\mathrm{sep}}$) for the maximal cyclotomic extension field of k. Then since k is Hilbertian, and $k \subseteq k^{\mathrm{cyc}}$ is an abelian extension, it follows from [5], Theorem 16.11.3, that k^{cyc} is also Hilbertian. On the other hand, every topologically finitely generated normal closed subgroup of the absolute Galois group of a Hilbertian field is trivial [cf. [14], Theorem 2.1]. Thus, it suffices to verify that every pro-p normal closed subgroup of the absolute Galois group of a Hilbertian field is trivial. However, since every pro-p group is pro-solvable, this follows immediately from [5], Proposition 16.11.6.

Remark 1.7.2. Let k_0 be a field of characteristic p. Suppose that k is isomorphic to the one-parameter power series field $k_0((t))$ over k_0. Then P_K does not coincide with the maximal normal closed pro-p subgroup of G_K. Indeed, observe that the pro-prime-to-p cyclotomic character associated to k factors through the pro-prime-to-p cyclotomic character associated to k_0. Write I_k for the kernel of the natural homomorphism $\widehat{\mathbb{Z}}^{(p)'}(1) \rtimes G_k \to \widehat{\mathbb{Z}}^{(p)'}(1) \rtimes G_{k_0}$, where the respective semi-direct products are determined by the respective pro-prime-to-p cyclotomic characters. Note that I_k is isomorphic to the inertia subgroup of G_k. Write $P_k \subseteq I_k$ for the normal closed pro-p subgroup corresponding to the wild inertia subgroup of G_k. Then $P_k \subseteq \widehat{\mathbb{Z}}^{(p)'}(1) \rtimes G_k$ is a normal closed pro-p subgroup [cf., e.g., Proposition 1.7]. Write Q for the pull-back of P_k via the natural surjection $G_K \twoheadrightarrow \widehat{\mathbb{Z}}^{(p)'}(1) \rtimes G_k$. [cf. Proposition 1.5, (i)]. Then since P_K is a pro-p group [cf. Proposition 1.5, (ii)], it holds that $Q \subseteq G_K$ is a normal closed pro-p subgroup. On the other hand, since $P_k \neq \{1\}$, we observe that $P_K \subsetneq Q$. Thus, we conclude that P_K does not coincide with the maximal normal closed pro-p subgroup of G_K.

Definition 1.8. For each separable extension $K \subseteq L$ ($\subseteq K^{\mathrm{sep}}$), we shall write $\mathcal{O}_L$ for the integral closure of $\mathcal{O}_K$ in L; $\mathcal{O}_L^{\times}$ ($\subseteq \mathcal{O}_L$) for the group of units of $\mathcal{O}_L$; $\mathcal{O}_L^{\triangleright} \stackrel{\mathrm{def}}{=} \mathcal{O}_L \setminus \{0\}$ [as a multiplicative monoid]; $L^{\times} \stackrel{\mathrm{def}}{=} L \setminus \{0\}$ [as a multiplicative group]; $\mathfrak{m}_L$ for the maximal ideal of $\mathcal{O}_L$; $k_L \stackrel{\mathrm{def}}{=} \mathcal{O}_L/\mathfrak{m}_L$;

$$\mathcal{O}_L^{\times,\mathrm{div}} \stackrel{\mathrm{def}}{=} \mathcal{O}_L^{\times} / \bigcap_{m \geq 1} (\mathcal{O}_L^{\times})^m; \quad \widehat{\mathcal{O}}_L^{\times} \stackrel{\mathrm{def}}{=} \varprojlim_{m \geq 1} \mathcal{O}_L^{\times}/(\mathcal{O}_L^{\times})^m;$$

$$\mathcal{O}_L^{\triangleright,\mathrm{div}} \stackrel{\mathrm{def}}{=} \mathcal{O}_L^{\triangleright} / \bigcap_{m \geq 1} (\mathcal{O}_L^{\times})^m; \quad \widehat{\mathcal{O}}_L^{\triangleright} \stackrel{\mathrm{def}}{=} \varprojlim_{m \geq 1} \mathcal{O}_L^{\triangleright}/(\mathcal{O}_L^{\triangleright})^m;$$

$$L^{\times,\mathrm{div}} \stackrel{\mathrm{def}}{=} L^\times / \bigcap_{m\geq 1} (L^\times)^m; \quad \widehat{L}^\times \stackrel{\mathrm{def}}{=} \varprojlim_{m\geq 1} L^\times/(L^\times)^m,$$

where m ranges over the positive integers. Moreover, for each finite separable extension $K \subseteq L$ $(\subseteq K^{\mathrm{sep}})$, we shall write $\hat{\mathcal{O}}^\triangleright_L \subseteq \widehat{\mathcal{O}}^\triangleright_L$ (respectively, $\hat{L}^\times \subseteq \widehat{L}^\times$) for the submonoid (respectively, subgroup) generated by $\widehat{\mathcal{O}}^\times_L$ and the image of a prime element $\in \mathcal{O}^\triangleright_L$ of L via the natural map $\mathcal{O}^\triangleright_L \to \widehat{\mathcal{O}}^\triangleright_L$ (respectively, $L^\times \to \widehat{L}^\times$). Finally, we shall write

$$\mathcal{O}^{\times,\mathrm{div}}_{K^{\mathrm{sep}}} \stackrel{\mathrm{def}}{=} \varinjlim_{K\subseteq L} \mathcal{O}^{\times,\mathrm{div}}_L; \quad \widehat{\mathcal{O}}^\times_{K^{\mathrm{sep}}} \stackrel{\mathrm{def}}{=} \varinjlim_{K\subseteq L} \widehat{\mathcal{O}}^\times_L;$$

$$\mathcal{O}^{\triangleright,\mathrm{div}}_{K^{\mathrm{sep}}} \stackrel{\mathrm{def}}{=} \varinjlim_{K\subseteq L} \mathcal{O}^{\triangleright,\mathrm{div}}_L; \quad \widehat{\mathcal{O}}^\triangleright_{K^{\mathrm{sep}}} \stackrel{\mathrm{def}}{=} \varinjlim_{K\subseteq L} \widehat{\mathcal{O}}^\triangleright_L;$$

$$(K^{\mathrm{sep}})^{\times,\mathrm{div}} \stackrel{\mathrm{def}}{=} \varinjlim_{K\subseteq L} L^{\times,\mathrm{div}}; \quad (\widehat{K^{\mathrm{sep}}})^\times \stackrel{\mathrm{def}}{=} \varinjlim_{K\subseteq L} \widehat{L}^\times;$$

$$\hat{\mathcal{O}}^\triangleright_{K^{\mathrm{sep}}} \stackrel{\mathrm{def}}{=} \varinjlim_{K\subseteq L} \hat{\mathcal{O}}^\triangleright_L; \quad (K^{\hat{\mathrm{sep}}})^\times \stackrel{\mathrm{def}}{=} \varinjlim_{K\subseteq L} \hat{L}^\times,$$

where $K \subseteq L$ $(\subseteq K^{\mathrm{sep}})$ ranges over the finite separable extensions.

Remark 1.8.1. In the notation of Definition 1.8, let $K \subseteq L$ $(\subseteq K^{\mathrm{sep}})$ be a finite separable extension. Suppose that k is a finite field. Then it follows immediately that

$$\bigcap_{m\geq 1} (\mathcal{O}^\times_L)^m = \bigcap_{m\geq 1} (L^\times)^m = \{1\},$$

where m ranges over the positive integers. In particular,

$$\mathcal{O}^{\times,\mathrm{div}}_L = \mathcal{O}^\times_L, \quad \mathcal{O}^{\triangleright,\mathrm{div}}_L = \mathcal{O}^\triangleright_L, \quad L^{\times,\mathrm{div}} = L^\times.$$

Remark 1.8.2. Suppose that K is complete. Then it follows immediately from [16], Lemma 2.6, that the transition maps that appear in the direct limits concerning "div" in Definition 1.8 are injective.

Remark 1.8.3. In the notation of Definition 1.8, suppose that K is complete. Let $K \subseteq L$ $(\subseteq K^{\mathrm{sep}})$ be a finite separable extension. [In particular, L is also a complete discrete valuation field of residue characteristic p.] Then it holds that the natural homomorphism

$$\phi : 1 + \mathfrak{m}_L \longrightarrow \varprojlim_{n\geq 1} (1+\mathfrak{m}_L)/(1+\mathfrak{m}_L)^n$$

is an isomorphism. Indeed, suppose, first, that $\mathrm{char}(K) = 0$. Under this assumption, recall that the p-adic logarithm map $\mathcal{O}^\times_L \to L$ determines an isomorphism $1 + p^2\mathcal{O}_L \xrightarrow{\sim} p^2\mathcal{O}_L$. Then since $\mathcal{O}_L$ is p-adically complete, we observe that the natural homomorphism

$$1 + p^2\mathcal{O}_L \longrightarrow \widehat{1 + p^2\mathcal{O}_L} \stackrel{\mathrm{def}}{=} \varprojlim_{n\geq 1} (1+p^2\mathcal{O}_L)/(1+p^2\mathcal{O}_L)^n$$

is an isomorphism. On the other hand, since L is a mixed characteristic discrete valuation field of residue characteristic p, there exists a positive integer m such that $(1+\mathfrak{m}_L)^{p^m} \subseteq 1 + p^2\mathcal{O}_L$. In particular, we observe that the natural homomorphism

$$\widehat{1 + \mathfrak{m}_L} \stackrel{\mathrm{def}}{=} \varprojlim_{n\geq 1} (1+\mathfrak{m}_L)/(1+p^2\mathcal{O}_L)^n \longrightarrow \varprojlim_{n\geq 1} (1+\mathfrak{m}_L)/(1+\mathfrak{m}_L)^n$$

is an isomorphism. Now we have the following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & 1+p^2\mathcal{O}_L & \longrightarrow & 1+\mathfrak{m}_L & \longrightarrow & (1+\mathfrak{m}_L)/(1+p^2\mathcal{O}_L) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \widehat{1+p^2\mathcal{O}_L} & \longrightarrow & \widehat{1+\mathfrak{m}_L} & \longrightarrow & (1+\mathfrak{m}_L)/(1+p^2\mathcal{O}_L) & \longrightarrow & 1, \end{array}$$

where the vertical arrows denote the natural homomorphisms. Thus, in light of the above observations, together with Snake lemma, we conclude that ϕ is an isomorphism in the case where $\mathrm{char}(K) = 0$. Next, suppose that $\mathrm{char}(K) = p$. Let i be a positive integer. Note that since $\mathrm{char}(K) = p$, it holds that $(1+\mathfrak{m}_L)^{p^i} = \{1 + a^{p^i} \mid a \in \mathfrak{m}_L\}$. Then one may observe that

$$\bigcap_{n\geq 1} (1+\mathfrak{m}_L^n)(1+\mathfrak{m}_L)^{p^i} \;=\; (1+\mathfrak{m}_L)^{p^i}.$$

[Indeed, the inclusion $(1+\mathfrak{m}_L)^{p^i} \subseteq \cap_{n\geq 1} (1+\mathfrak{m}_L^n)(1+\mathfrak{m}_L)^{p^i}$ is immediate. Let $x \in \mathfrak{m}_L$ be such that $1 + x \in \cap_{n\geq 1} (1+\mathfrak{m}_L^n)(1+\mathfrak{m}_L)^{p^i}$. To verify the reverse inclusion, since $(1+\mathfrak{m}_L)^{p^i} = \{1 + a^{p^i} \mid a \in \mathfrak{m}_L\}$, it suffices to verify that there exists an element $a \in \mathfrak{m}_L$ such that $x = a^{p^i}$. Next, observe that since $(1+\mathfrak{m}_L)^{p^i} = \{1 + a^{p^i} \mid a \in \mathfrak{m}_L\}$, and $1 + x \in \cap_{n\geq 1} (1+\mathfrak{m}_L^n)(1+\mathfrak{m}_L)^{p^i}$, it holds that, for each positive integer n, there exist elements $\pi_n \in \mathfrak{m}_L^n$ and $a_n \in \mathfrak{m}_L$ such that $x = a_n^{p^i} + \pi_n$. Here, we note that $(a_n - a_{n+1})^{p^i} = a_n^{p^i} - a_{n+1}^{p^i} = \pi_{n+1} - \pi_n \in \mathfrak{m}_L^n$. In particular, since L is complete, the sequence $\{a_n\}_{n\geq 1}$ converges to an element $a \in \mathfrak{m}_L$. Moreover, we conclude from the equality $x = a_n^{p^i} + \pi_n$ that $x = a^{p^i}$.] Therefore, by varying the positive integer i, we observe that the natural homomorphism

$$\varprojlim_{i\geq 1} (1+\mathfrak{m}_L)/(1+\mathfrak{m}_L)^{p^i} \;\longrightarrow\; \varprojlim_{i\geq 1} (1+\mathfrak{m}_L)/(1+\mathfrak{m}_L^{p^i})$$

is injective. On the other hand, one observes easily that, if m is a positive integer coprime to p, then the m-th power map on $1+\mathfrak{m}_L$ is an automorphism. Thus, since L is complete [so $1+\mathfrak{m}_L \xrightarrow{\sim} \varprojlim_{n\geq 1} (1+\mathfrak{m}_L)/(1+\mathfrak{m}_L^n)$], we also conclude that ϕ is an isomorphism in the case where $\mathrm{char}(K) = p$.

Remark 1.8.4. In the notation of Definition 1.8, let m be a positive integer. Suppose that m is coprime to p if $\mathrm{char}(K) = p$. Then it holds that $\widehat{\mathcal{O}}^{\times}_{K^{\mathrm{sep}}}$ and $(\hat{K^{\mathrm{sep}}})^{\times}$ are m-divisible. Indeed, for each finite separable extension $K \subseteq L$ ($\subseteq K^{\mathrm{sep}}$), we have a natural exact sequence

$$\varprojlim_{n\geq 1} \mathcal{O}_L^{\times}/(\mathcal{O}_L^{\times})^n \;\longrightarrow\; \varprojlim_{n\geq 1} \mathcal{O}_L^{\times}/(\mathcal{O}_L^{\times})^n \longrightarrow \mathcal{O}_L^{\times}/(\mathcal{O}_L^{\times})^m \longrightarrow 1,$$

where the first arrow denotes the m-th power map. On the other hand, it follows immediately from our assumption on m that

$$\varinjlim_{K\subseteq L} \mathcal{O}_L^{\times}/(\mathcal{O}_L^{\times})^m \;=\; \{1\},$$

where $K \subseteq L$ ($\subseteq K^{\mathrm{sep}}$) ranges over the finite separable extensions. Thus, we conclude that $\widehat{\mathcal{O}}^{\times}_{K^{\mathrm{sep}}}$ is m-divisible, hence also that $(\hat{K^{\mathrm{sep}}})^{\times}$ is m-divisible.

Definition 1.9. Let L be a field; l a prime number; m a positive integer. Write

$$\mu_m(L) \overset{\mathrm{def}}{=} \{x \in L \mid x^m = 1\}; \quad \mu_{l^\infty}(L) \overset{\mathrm{def}}{=} \bigcup_{n\geq 1} \mu_{l^n}(L); \quad \mu(L) \overset{\mathrm{def}}{=} \bigcup_{n\geq 1} \mu_n(L),$$

where n ranges over the positive integers. Then we shall say that:

(i) L is *μ-finite* if $\mu(L)$ is finite.

(ii) L is *stably μ_{l^∞}-finite* if, for every finite field extension $L \subseteq M$, it holds that $\mu_{l^\infty}(M)$ is finite.

(iii) L is *stably μ-finite* if, for every finite field extension $L \subseteq M$, it holds that $\mu(M)$ is finite.

Lemma 1.10. *Let L be a field. Write L_0 ($\subseteq L$) for the algebraic closure of the prime field L_{prm} ($\subseteq L$) in L. Then the following hold:*

(i) The natural [outer] homomorphism $G_L \to G_{L_0}$ is surjective.

(ii) Suppose that L is of positive characteristic. Then it holds that $L_0 = L_{\text{prm}}(\mu(L))$.

(iii) Suppose that L is of positive characteristic. Then L is μ-finite if and only if the image of the natural [outer] homomorphism $G_L \to G_{L_{\text{prm}}}$ is open.

(iv) Suppose that L is μ-finite and of positve characteristic. Then L is stably μ-finite.

Proof. Assertion (i) follows immediately from elementary Galois theory. Assertion (ii) follows immediately from the fact that every nonzero element $\in L_0$ is a root of unity. Assertion (iii) follows immediately from assertions (i), (ii). Assertion (iv) follows immediately from assertion (iii). This completes the proof of Lemma 1.10. □

Lemma 1.11. *In the notation of Definition 1.8, let l be a prime number. Then K is stably μ_{l^∞}-finite (respectively, stably μ-finite) if and only if k is stably μ_{l^∞}-finite (respectively, μ-finite).*

Proof. In light of Lemma 1.10, (iv), Lemma 1.11 follows immediately from the Henselian property of [every finite extension field of] K. □

Lemma 1.12. *In the notation of Definition 1.8, suppose that, for each prime number l, it holds that k is stably μ_{l^∞}-finite. Then the transition maps that appear in the direct limits in Definition 1.8 are injective. Moreover, for each finite separable extension $K \subseteq F$ ($\subseteq K^{\text{sep}}$), it holds that $((\widehat{K^{\text{sep}}})^\times)^{G_F} = \widehat{F}^\times$. In particular, for each separable extension $K \subseteq M$ ($\subseteq K^{\text{sep}}$), it holds that*

$$((\widehat{K^{\text{sep}}})^\times)^{G_M} = \bigcup_{K \subseteq F\ (\subseteq M)} \widehat{F}^\times,$$

where $K \subseteq F$ ($\subseteq M$) ranges over the finite field extensions $\subseteq M$.

Proof. Note that every finite extension field of K is also a Henselian discrete valuation field. Thus, to verify Lemma 1.12, it suffices to prove the following assertions:

(a) The natural homomorphism $\widehat{K}^\times \to \widehat{L}^\times$ is injective.

(b) $((\widehat{K^{\text{sep}}})^\times)^{G_K} = \widehat{K}^\times$.

First, we verify assertion (a). By replacing L by a finite separable extension field of L, we may assume without loss of generality that $K \subseteq L$ is a finite Galois extension. Then since the kernel of the natural homomorphism $K^\times/(K^\times)^m \to L^\times/(L^\times)^m$ is

$$(K^\times \bigcap (L^\times)^m)/(K^\times)^m \;=\; ((L^\times)^m)^{\mathrm{Gal}(L/K)}/(K^\times)^m,$$

it suffices to prove that

$$\varprojlim_{m\geq 1} \; ((L^\times)^m)^{\mathrm{Gal}(L/K)}/(K^\times)^m \;=\; \{1\}.$$

Next, since k is stably μ_{l^∞}-finite for each prime numer l, it follows immediately from Lemma 1.11 that $\mu_{l^\infty}(L)$ is finite for each prime number l. In particular, for each positive integer m, the natural ascending chain

$$\mu_m(L) \;\subseteq\; \cdots \;\subseteq\; \mu_{m^j}(L) \;\subseteq\; \mu_{m^{j+1}}(L) \;\subseteq\; \cdots$$

stabilizes. Then it follows immediately from the various definitions involved that, for each positive integer i, we have

$$\varprojlim_{m\geq 1} \; H^i(\mathrm{Gal}(L/K), \mu_m(L)) \;=\; \{1\}.$$

On the other hand, by considering the long exact sequence associated to the natural exact sequence of $\mathrm{Gal}(L/K)$-modules

$$1 \longrightarrow \mu_m(L) \longrightarrow L^\times \longrightarrow (L^\times)^m \longrightarrow 1,$$

we have a natural injection

$$((L^\times)^m)^{\mathrm{Gal}(L/K)}/(K^\times)^m \;\hookrightarrow\; H^1(\mathrm{Gal}(L/K), \mu_m(L)).$$

Thus, by taking the inverse limit, we conclude that

$$\varprojlim_{m\geq 1} \; ((L^\times)^m)^{\mathrm{Gal}(L/K)}/(K^\times)^m \;\hookrightarrow\; \varprojlim_{m\geq 1} \; H^1(\mathrm{Gal}(L/K), \mu_m(L)) \;=\; \{1\},$$

hence that the natural homomorphism $\widehat{K}^\times \to \widehat{L}^\times$ is injective. This completes the proof of assertion (a). Note that the proof of assertion (a) also implies that the inverse system

$$\{(K^\times \bigcap (L^\times)^m)/(K^\times)^m\}_{m\geq 1}$$

satisfies the Mittag-Leffler condition.

Next, we verify assertion (b). For each finite Galois extension $K \subseteq L$, the long exact sequence considered above, together with Hilbert's theorem 90, induces an injection

$$H^1(\mathrm{Gal}(L/K), (L^\times)^m) \;\hookrightarrow\; H^2(\mathrm{Gal}(L/K), \mu_m(L)).$$

Then, by taking the inverse limit, we observe that

$$\varprojlim_{m\geq 1} \; H^1(\mathrm{Gal}(L/K), (L^\times)^m) \;\hookrightarrow\; \varprojlim_{m\geq 1} \; H^2(\mathrm{Gal}(L/K), \mu_m(L)) \;=\; \{1\}.$$

Thus, in light of the Mittag-Leffler condition discussed above, by considering the long exact sequence associated to the natural exact sequence of $\mathrm{Gal}(L/K)$-modules

$$1 \longrightarrow (L^\times)^m \longrightarrow L^\times \longrightarrow L^\times/(L^\times)^m \longrightarrow 1,$$

we conclude that the natural homomorphism

$$\widehat{K}^{\times} \;=\; \varprojlim_{m\geq 1} \; K^{\times}/(K^{\times})^{m} \;\longrightarrow\; \varprojlim_{m\geq 1} \; (L^{\times}/(L^{\times})^{m})^{\mathrm{Gal}(L/K)} \;=\; (\widehat{L}^{\times})^{\mathrm{Gal}(L/K)}$$

is surjective, hence bijective [cf. (a)]. Finally, in light of this bijection, it follows immediately from the various definitions involved that $((\widehat{K^{\mathrm{sep}})^{\times}})^{G_K} = \widehat{K}^{\times}$. This completes the proof of assertion (b), hence of Lemma 1.12. □

2 l-local class field theory for Henselian discrete valuation fields with strongly l-quasi-finite residues via Artin-Tate's class formation

Let p, l be prime numbers. In the present section, we discuss l-local class field theory for Henselian discrete valuation fields with strongly l-quasi-finite residue fields [cf. Definition 2.8] of characteristic p via Artin-Tate's class formation for our later use. In light of the richness of extensions, the case where $l = p$ is the most interesting one. On the other hand, for our later use, we also discuss l-local class field theory in the case where $l \neq p$. Historically, p-local class field theory for complete discrete valuation fields with p-quasi-finite residue fields of characteristic p via Artin-Tate's class formation was developed by Sekiguchi [cf. [30], §1]. Moreover, if we restrict our attention to the totally ramified p-extensions, then much more general [and explicit] theory via a generalized version of Neukirch's construction was developed by Fesenko [cf. [2]].

Let K be a Henselian discrete valuation field whose residue field k is a perfect field of characteristic p. For each finite field extension $K \subseteq L$ and each positive integer n, write $\widetilde{L}$ for the completion of L; $U_{n,L} \overset{\mathrm{def}}{=} 1 + \mathfrak{m}_L^n$; $N_{L/K} : L^{\times} \to K^{\times}$ for the norm map. For each [possibly, infinite] Galois extension $K \subseteq L$ and each real number $v \geq -1$, write $\mathrm{Gal}(L/K)_v \subseteq \mathrm{Gal}(L/K)$ (respectively, $\mathrm{Gal}(L/K)^v \subseteq \mathrm{Gal}(L/K)$) for the higher ramification group of index v, relative to the lower (respectively, upper) numbering [cf. [31], Chapter IV, §3]. For each finite Galois extension $K \subseteq L$ and each real number v, write

$$\psi_{L/K}(v) \;\overset{\mathrm{def}}{=}\; \int_0^v \; [\mathrm{Gal}(L/K) : \mathrm{Gal}(L/K)^w] \; dw$$

[cf. [31], Chapter IV, §3].

First, we record an elementary observation:

Lemma 2.1. *Let n be a positive integer. Then it holds that $U_{n,K} = U_{n,\widetilde{K}} \cap K^{\times}$.*

Proof. Lemma 2.1 follows immediately from the various definitions involved. □

Next, we recall the following well-known facts on the behaviors of the norm maps of the finite unramified Galois extensions of K.

Proposition 2.2. *Let $K \subseteq L$ be a finite unramified Galois extension; π_K a uniformizer of K [hence, of L]. For each positive integer n, by using the uniformizer π_K, we identify k (respectively, k_L) and $U_{n,K}/U_{n+1,K}$ (respectively, $U_{n,L}/U_{n+1,L}$) in a natural way. Then the following hold:*

(i) Let n be a positive integer. Then it holds that $N_{L/K}(U_{n,L}) \subseteq U_{n,K}$.

(ii) The norm map $N_{L/K}$ induces the norm map $k_L^{\times} \to k^{\times}$.

(iii) Let n be a positive integer. Then the norm map $N_{L/K}$ induces the trace map

$$Tr_{k_L/k} : k_L \;=\; U_{n,L}/U_{n+1,L} \;\longrightarrow\; U_{n,K}/U_{n+1,K} \;=\; k.$$

[Note that since $k \subseteq k_L$ is a separable extension, $Tr_{k_L/k}$ is surjective.]

Proof. Proposition 2.2 follows immediately from [3], Chapter III, (1.2), Proposition, together with Lemma 2.1. □

Next, we recall the following well-known facts on the behaviors of the norm maps of the totally ramified cyclic extensions of K of prime degree.

Proposition 2.3. *Suppose that $l \neq p$. Let $K \subseteq L$ be a totally ramified cyclic extension of degree l; π_K a uniformizer of K; π_L a uniformizer of L such that $\pi_L^l = \pi_K$. For each positive integer n, by using the uniformizer π_K (respectively, π_L), we identify k and $U_{n,K}/U_{n+1,K}$ (respectively, $U_{n,L}/U_{n+1,L}$) in a natural way. Then the following hold:*

(i) Let n be a positive integer. Then it holds that $N_{L/K}(U_{ln,L}) \subseteq U_{n,K}$.

(ii) The norm map $N_{L/K}$ induces the l-th power map on $k^\times$.

(iii) Let n be a positive integer. Then the norm map $N_{L/K}$ induces an isomorphism [of $\mathbb{F}_p$-vector spaces]

$$k \;=\; U_{ln,L}/U_{ln+1,L} \;\xrightarrow{\sim}\; U_{n,K}/U_{n+1,K} \;=\; k$$

that maps $k \ni x \mapsto lx \in k$.

Proof. Proposition 2.3 follows immediately from [3], Chapter III, (1.3), Proposition, together with Lemma 2.1. □

Proposition 2.4. *Let $K \subseteq L$ be a totally ramified cyclic extension of degree p; π_K a uniformizer of K; π_L a uniformizer of L such that $N_{L/K}(\pi_L) = \pi_K$; $\sigma \in \mathrm{Gal}(L/K)$ a generator. Write s for the maximal positive integer among the positive integers m such that $\mathrm{Gal}(L/K)^m = \mathrm{Gal}(L/K)$; $\eta \in \mathcal{O}_L^\times$ for the unit such that*

$$\frac{\sigma(\pi_L)}{\pi_L} \;=\; 1 + \eta\pi_L^s.$$

For each positive integer n, by using the uniformizer π_K (respectively, π_L), we identify k and $U_{n,K}/U_{n+1,K}$ (respectively, $U_{n,L}/U_{n+1,L}$) in a natural way. Then the following hold:

(i) Let n be a positive integer. Then it holds that $N_{L/K}(U_{\psi_{L/K}(n),L}) \subseteq U_{n,K}$. Moreover, if $n \leq s+1$, then it holds that $N_{L/K}(U_{n,L}) \subseteq U_{n,K}$.

(ii) The norm map $N_{L/K}$ induces the p-th power map on $k^\times$.

(iii) Let n be a positive integer such that $n \leq s$. Then the norm map $N_{L/K}$ induces a homomorphism [of $\mathbb{F}_p$-vector spaces]

$$N_n : k \;=\; U_{n,L}/U_{n+1,L} \;\longrightarrow\; U_{n,K}/U_{n+1,K} \;=\; k.$$

(iv) Let n be a positive integer such that $n < s$. Then, for each $x \in k$, it holds that

$$N_n(x) \;=\; x^p.$$

[Note that since k is perfect, N_n is bijective.]

(v) For each $x \in k$, it holds that

$$N_s(x) \ = \ x^p - \overline{\eta}^{p-1}x,$$

where $\overline{\eta} \in k^\times$ denotes the image of η via the natural surjection $\mathcal{O}_L^\times \twoheadrightarrow k^\times$. [Note that, if k is a p-closed field, i.e., a field that admits no nontrivial cyclic extension of degree p, then N_s is surjective.]

(vi) Let n be a positive integer such that $n > s$. Then the norm map $N_{L/K}$ induces an isomorphism [of $\mathbb{F}_p$-vector spaces]

$$N_n : k \ = \ U_{\psi_{L/K}(n),L}/U_{\psi_{L/K}(n)+1,L} \ \overset{\sim}{\to} \ U_{n,K}/U_{n+1,K} \ = \ k.$$

that maps $k \ni x \mapsto -\overline{\eta}^{p-1}x \in k$.

Proof. Proposition 2.4 follows immediately from [3], Chapter III, (1.5), Proposition, together with Lemma 2.1. □

Next, by applying the above propositions, we observe some important properties concerning the images of norm maps.

Lemma 2.5. *Let m be a positive integer; $K \subseteq L$ a finite Galois extension of degree l-power. Then the following hold:*

(i) Suppose that $l = p$, $\mathrm{char}(K) = 0$, and the finite Galois extension $K \subseteq L$ is of degree p. Write e for the absolute ramification index of K. Then it holds that $U_{(2m+1)e,K} \subseteq (U_{2me,K})^p \subseteq N_{L/K}(U_{m,L})$.

(ii) Suppose that $l = p$, $\mathrm{char}(K) = p$, and the finite Galois extension $K \subseteq L$ is of degree p. Then there exists a positive integer n such that $U_{n,K} \subseteq N_{L/K}(U_{m,L})$.

(iii) There exists a positive integer n such that $U_{n,K} \subseteq N_{L/K}(U_{m,L})$.

Proof. First, we verify assertion (i). Since $K \subseteq L$ is a field extension of degree p, and $2me > m$, the second inclusion is immediate. On the other hand, in the case where K is complete, by applying an elementary property of the p-adic logarithm [cf. Lemma 8.5 below], or alternatively, [3], Chapter I, (5.8), Corollary 2, we observe that the first inclusion is also immediate. The first inclusion in the general Henselian case follows immediately from the complete case, together with Lemma 2.1; the fact that K is algebraically closed in $\widetilde{K}$. This completes the proof of assertion (i).

Next, we verify assertion (ii). Let π_K be a uniformizer of K. In the case where $K \subseteq L$ is a totally ramified extension, we may assume without loss of generality that $\pi_K = N_{L/K}(\pi_L)$, where π_L is a uniformizer of L. Suppose that $K \subseteq L$ is a(n) unramified (respectively, totally ramified) extension. We set $s \in \mathbb{Z}$ to be 0 (respectively, the positive integer "s" appearing in Proposition 2.4). Then, in light of Artin-Schreier theory, there exists a unit $u \in \mathcal{O}_K^\times$ such that $L = K(\lambda)$, where λ is a root of the Artin-Schreier equation

$$y^p - y \ = \ \frac{u}{\pi_K^s}$$

[cf. the proof of [3], Chapter III, (2.1), Lemma (respectively, the proof of [3], Chapter III, (2.4), Proposition)]. Now we claim the following:

Claim 2.5.A: It holds that $\frac{u}{\pi_K^s} + \mathfrak{m}_K^m \subseteq N_{L/K}(\lambda + \mathfrak{m}_K^m \cdot \mathcal{O}_L)$.

Indeed, let $a \in \mathfrak{m}_K^m$. Then it follows from our assumption that K is Henselian that there exists $b \in \mathfrak{m}_K$ such that $a = b^p - b$. Moreover, one verifies immediately [by considering the valuations] that $b \in \mathfrak{m}_K^m$. Thus, since $L = K(\lambda + b)$, and

$$(\lambda + b)^p - (\lambda + b) \ = \ \frac{u}{\pi_K^s} + a,$$

we conclude that $\frac{u}{\pi_K^s} + a = N_{L/K}(\lambda + b) \in N_{L/K}(\lambda + \mathfrak{m}_K^m \cdot \mathcal{O}_L)$. This completes the proof of Claim 2.5.A.

Here, we observe that $N_{L/K}(\lambda) = \frac{u}{\pi_K^s}$. In particular, if we set $u_L \in L^\times$ to be λ (respectively, $\pi_L^s \cdot \lambda$), then we have $u = N_{L/K}(u_L)$. [Note that this equality implies that u_L is, in fact, a unit of $\mathcal{O}_L$.] Thus, in light of Claim 2.5.A, we conclude that $u + \mathfrak{m}_K^{s+m} \subseteq N_{L/K}(u_L + \mathfrak{m}_L^{ps+m})$, hence that $U_{s+m,K} \subseteq N_{L/K}(U_{ps+m,L}) \subseteq N_{L/K}(U_{m,L})$. This completes the proof of assertion (ii).

Next, we verify assertion (iii). In light of the transitivity of the norm maps, since every nontrivial finite l-group has nontrivial center, we may assume without loss of generality that the finite Galois extension $K \subseteq L$ is of degree l. In the case where $l \neq p$, it follows immediately from the Henselian property of K, together with the various definitions involved, that $U_{m,K} = U_{m,K}^l \subseteq N_{L/K}(U_{m,L})$. Therefore, it suffices to consider the case where $l = p$. Then assertion (iii) follows immediately from assertions (i), (ii). This completes the proof of Lemma 2.5. □

Corollary 2.6. *Let $K \subseteq L$ be a finite Galois extension of degree l-power. Then the following hold:*

(i) Suppose that $K \subseteq L$ is an unramified extension. Then, for each positive integer n, it holds that $N_{L/K}(U_{n,L}) = U_{n,K}$.

(ii) Suppose that k is an l-closed field. In the case where $l \neq p$, suppose, moreover, that $\zeta_l \in k$. Then the norm map $N_{L/K} : L^\times \to K^\times$ is surjective.

Proof. First, since every nontrivial finite l-group has nontrivial center, we may assume without loss of generality that the finite Galois extension $K \subseteq L$ is a cyclic extension of degree l. Then, in light of Lemma 2.5, (iii), assertion (i) follows immediately from Proposition 2.2. Next, we verify assertion (ii). Note that it follows immediately from our assumptions on k that $k^\times$ is l-divisible. Thus, in the case where $l \neq p$ (respectively, $l = p$), we conclude from Proposition 2.3 (respectively, Proposition 2.4) that the norm map $N_{L/K} : L^\times \to K^\times$ is surjective. This completes the proof of assertion (ii), hence of Corollary 2.6. □

Next, we introduce the l-Brauer groups associated to fields.

Definition 2.7. Let F be a field. Then we shall write $F \subseteq F^l$ ($\subseteq F^{\mathrm{sep}}$) for the maximal pro-$l$ extension;

$$ {}_lB_F \stackrel{\mathrm{def}}{=} H^2(G_F^l, (F^l)^\times). $$

Definition 2.8. Let F be a perfect field. Then:

(i) We shall say that F is an *l-quasi-finite field* if the maximal pro-l quotient of the abolute Galois group of F is isomorphic to $\mathbb{Z}_l$.

(ii) We shall say that F is a *strongly l-quasi-finite field* if every finite extension field of F is an l-quasi-finite field.

Proposition 2.9. *Suppose that, if $l \neq p$, then k is strongly l-quasi-finite. Then the following hold:*

(i) It holds that $H^2(G_k, \overline{k}^\times)[l] = \{0\}$.

(ii) It holds that $H^2(G_k^l, (k^l)^\times) = \{0\}$.

(iii) Suppose, moreover, that, if $l = p$, then G_k is not a pro-prime-to-p group. Then it holds that $\mathrm{cd}_l(G_k) = 1$.

Proof. First, suppose that

$$l = p.$$

Then assertion (i) follows immediately from the fact that k is perfect. Next, we verify assertion (ii). Let $k \subseteq k^\dagger$ be a finite Galois extension of degree p-power. Then since $\mathrm{Gal}(k^\dagger/k)$ is a finite p-group, $H^2(\mathrm{Gal}(k^\dagger/k), (k^\dagger)^\times)$ is annihilated by a p-power. On the other hand, since $k^\dagger$ is perfect, the endomorphism of $H^2(\mathrm{Gal}(k^\dagger/k), (k^\dagger)^\times)$ given by multiplication by p is an automorphism. Thus, we conclude that $H^2(\mathrm{Gal}(k^\dagger/k), (k^\dagger)^\times) = \{0\}$. In particular, by varying $k \subseteq k^\dagger$, we observe that $H^2(G^p_k, (k^p)^\times) = \{0\}$. This completes the proof of assertion (ii) in the case where $l = p$. Assertion (iii) follows immediately from [32], Chapter I, §3, Corollary 2 [cf. our assumption that G_k is not pro-prime-to-p]; [32], Chapter II, §2, Proposition 3. This completes the proof of Proposition 2.9 in the case where $l = p$. Therefore, in the remainder of the proof, we may assume without loss of generality that

$$l \neq p,$$

hence that k is strongly l-quasi-finite.

Next, we verify assertion (ii) in the case where $\zeta_l \in k$. Since every element of $H^2(G^l_k, (k^l)^\times)$ is annihilated by an l-power [cf. the above discussion], it suffices to verify that the endomorphism of $H^2(G^l_k, (k^l)^\times)$ given by multiplication by l is injective. Let us observe that the natural exact sequence of G^l_k-modules

$$1 \longrightarrow \mathbb{F}_l \longrightarrow (k^l)^\times \xrightarrow{\bullet^l} (k^l)^\times \longrightarrow 1$$

[cf. our assumption that $\zeta_l \in k$] induces an exact sequence

$$H^2(G^l_k, \mathbb{F}_l) \longrightarrow H^2(G^l_k, (k^l)^\times) \xrightarrow{\times l} H^2(G^l_k, (k^l)^\times).$$

Thus, since $H^2(G^l_k, \mathbb{F}_l) = H^2(\mathbb{Z}_l, \mathbb{F}_l) = \{0\}$ [cf. our assumption that k is l-quasi-finite], we conclude the desired injectivity. This completes the proof of assertion (ii) in the case where $\zeta_l \in k$.

Next, we verify assertion (i). First, we claim the following assertion:

Claim 2.9.A: Every element of $H^2(\mathrm{Gal}(k(\zeta_l)^l/k), (k(\zeta_l)^l)^\times)$ is annihilated by $l - 1$.

Indeed, since $H^1(G^l_{k(\zeta_l)}, (k(\zeta_l)^l)^\times) = \{0\}$ [cf. Hilbert's theorem 90], by considering the Hochschild-Serre spectral sequence associated to the natural exact sequence of profinite groups

$$1 \longrightarrow G^l_{k(\zeta_l)} \longrightarrow \mathrm{Gal}(k(\zeta_l)^l/k) \longrightarrow \mathrm{Gal}(k(\zeta_l)/k) \longrightarrow 1,$$

we obtain an exact sequence

$$0 \longrightarrow H^2(\mathrm{Gal}(k(\zeta_l)/k), k(\zeta_l)^\times) \longrightarrow H^2(\mathrm{Gal}(k(\zeta_l)^l/k), (k(\zeta_l)^l)^\times) \longrightarrow H^2(G^l_{k(\zeta_l)}, (k(\zeta_l)^l)^\times).$$

On the other hand, since k is strongly l-quasi-finite, $k(\zeta_l)$ is also l-quasi-finite. Then it follows from the second paragraph of the present proof that $H^2(G^l_{k(\zeta_l)}, (k(\zeta_l)^l)^\times) = \{0\}$. In particular, we have an isomorphism

$$H^2(\mathrm{Gal}(k(\zeta_l)/k), k(\zeta_l)^\times) \xrightarrow{\sim} H^2(\mathrm{Gal}(k(\zeta_l)^l/k), (k(\zeta_l)^l)^\times).$$

Then Claim 2.9.A follows immediately from the fact that every element of $H^2(\mathrm{Gal}(k(\zeta_l)/k), k(\zeta_l)^\times)$ is annihilated by $[k(\zeta_l) : k]$. This completes the proof of Claim 2.9.A.

Next, we observe that since $H^1(G_{k(\zeta_l)^l}, \overline{k}^\times) = \{0\}$ [cf. Hilbert's theorem 90], by considering the Hochschild-Serre spectral sequence associated to the natural exact sequence of profinite groups

$$1 \longrightarrow G_{k(\zeta_l)^l} \longrightarrow G_k \longrightarrow \mathrm{Gal}(k(\zeta_l)^l/k) \longrightarrow 1,$$

we obtain an exact sequence

$$0 \longrightarrow H^2(\mathrm{Gal}(k(\zeta_l)^l/k), (k(\zeta_l)^l)^\times) \longrightarrow H^2(G_k, \overline{k}^\times) \longrightarrow H^2(G_{k(\zeta_l)^l}, \overline{k}^\times).$$

Here, we note that it follows from the Merkurjev-Suslin theorem [cf. [12], Theorem 11.5] that

$$H^2(G_{k(\zeta_l)^l}, \overline{k}^\times)[l] \ = \ \{0\}.$$

Thus, we conclude from Claim 2.9.A that $H^2(G_k, \overline{k}^\times)[l] = \{0\}$. This completes the proof of assertion (i).

Next, we verify assertion (ii) in general. Observe that since $H^1(G_{k^l}, \overline{k}^\times) = \{0\}$ [cf. Hilbert's theorem 90], by considering the Hochschild-Serre spectral sequence associated to the natural exact sequence of profinite groups

$$1 \longrightarrow G_{k^l} \longrightarrow G_k \longrightarrow G_k^l \longrightarrow 1,$$

we obtain an exact sequence

$$0 \longrightarrow H^2(G_k^l, (k^l)^\times) \longrightarrow H^2(G_k, \overline{k}^\times) \longrightarrow H^2(G_{k^l}, \overline{k}^\times).$$

In particular, it follows immediately from assertion (i) that we have

$$H^2(G_k^l, (k^l)^\times)[l] \ \hookrightarrow \ H^2(G_k, \overline{k}^\times)[l] \ = \ \{0\}.$$

Therefore, since every element of $H^2(G_k^l, (k^l)^\times)$ is annihilated by an l-power, we conclude that

$$H^2(G_k^l, (k^l)^\times) \ = \ \{0\}.$$

This completes the proof of assertion (ii).

Finally, we verify assertion (iii). Let us first observe that $\mathrm{cd}_l(G_k) \geq 1$ [cf. [28], Chapter I, §3, Corollary 2; our assumption that G_k is l-quasi-finite, hence that G_k is not pro-prime-to-l]. Then since, for every finite field extension $k \subseteq k'$, $H^1(G_{k'}, \overline{k}^\times) = \{0\}$ [cf. Hilbert's theorem 90] is l-divisible, it follows from assertion (i) that $\mathrm{cd}_l(G_k) = 1$ [cf. [32], Chapter II, §2, Proposition 4]. This completes the proof of assertion (iii), hence of Proposition 2.9. □

Proposition 2.10. *Suppose that $l \neq p$, and k is an l-quasi-finite field. Then $\zeta_l \notin K$ if and only if $G_K^l \cong \mathbb{Z}_l$.*

Proof. Note that since $G_k^l \cong \mathbb{Z}_l$, it follows immediately from the various definitions involved that the condition that $G_K^l \cong \mathbb{Z}_l$ is equivalent to the following condition:

$$K^l \ \subseteq \ K^{\mathrm{ur}}.$$

First, suppose that $\zeta_l \in K$. Let π_K be a uniformizer of K. Then the finite field extension generated by any l-th root of π_K over K is a totally ramified Galois extension of degree l. Thus, we conclude that $K^l \nsubseteq K^{\mathrm{ur}}$. Conversely, suppose that there exists a finite Galois extension $K \subseteq L$ of degree l-power whose ramification index $e_{L/K}$ is > 1. In particular, we have two intermediate fields $K \subseteq K' \subseteq L' \subseteq L$ such that $K' \subseteq L'$ is a totally ramified Galois extension of degree l. Then since [as is well-known] L' is generated by an l-th root of $\pi_{K'}$ over K', it follows from the Galoisness of $K' \subseteq L'$ that we have $\zeta_l \in L'$ $(\subseteq L)$. Therefore, since $[L : K]$ is a power of l, we conclude that $[K(\zeta_l) : K] = 1$, hence that $\zeta_l \in K$. This completes the proof of Proposition 2.10. □

Now we prove a key property of ${}_lB_K$, which leads us to construct Artin-Tate's class formation.

Proposition 2.11. *The following hold:*

(i) Suppose that k is an l-closed field. In the case where $l \neq p$, suppose, moreover, that $\zeta_l \in k$. Then it holds that

$$ {}_lB_K \ = \ \{0\}. $$

(ii) In the case where $l \neq p$, suppose that k is strongly l-quasi-finite. Then there exists a natural isomorphism

$$ {}_lB_K \ \xrightarrow{\sim} \ \mathrm{Hom}(G_k^l, \mathbb{Q}/\mathbb{Z}). $$

Proof. First, we verify assertion (i). Observe that it follows immediately from Corollary 2.6, (ii) (respectively, Hilbert's theorem 90) that, for each finite Galois extension $K \subseteq L$ ($\subseteq K^l$) of degree l-power, the 0-th (respectively, 1-th) Tate cohomology of $\mathrm{Gal}(L/K)$ with coefficients in $L^\times$ vanishes. Then, by applying [31], Chapter IX, §5, Theorem 8, we observe that $H^2(\mathrm{Gal}(L/K), L^\times) = \{0\}$. Thus, by varying the finite Galois extensions $K \subseteq L$ of degree l-power, we conclude that ${}_lB_K = \{0\}$. This completes the proof of assertion (i).

Next, we verify assertion (ii). Write $K \subseteq K^{l\text{-ur}}$ ($\subseteq K^l$) for the maximal unramified pro-l extension. Then, in light of Hilbert's theorem 90, we have the following exact sequence

$$ 0 \longrightarrow H^2(\mathrm{Gal}(K^{l\text{-ur}}/K), (K^{l\text{-ur}})^\times) \longrightarrow {}_lB_K \longrightarrow {}_lB_{K^{l\text{-ur}}}. $$

Note that the residue field of $K^{l\text{-ur}}$ is an l-closed field. In particular, if $l = p$ or $\zeta_l \in k$ in the case where $l \neq p$, then it follows from assertion (i) that ${}_lB_{K^{l\text{-ur}}} = \{0\}$. On the other hand, if $l \neq p$, and $\zeta_l \notin k$, then $K \subseteq K^l$ is an unramified extension [cf. the proof of Proposition 2.10]. Thus, we obtain a natural isomorphism

$$ H^2(\mathrm{Gal}(K^{l\text{-ur}}/K), (K^{l\text{-ur}})^\times) \ \xrightarrow{\sim} \ {}_lB_K. $$

Next, let $K \subseteq L$ ($\subseteq K^{l\text{-ur}}$) be an unramified Galois extension of degree l-power. Then, in light of the existence of a splitting $L^\times \xrightarrow{\sim} \mathcal{O}_L^\times \times \mathbb{Z}$ [as $\mathrm{Gal}(L/K)$-modules], for each nonnegative integer n, we have a split exact sequence

$$ 0 \longrightarrow H^n(\mathrm{Gal}(L/K), \mathcal{O}_L^\times) \longrightarrow H^n(\mathrm{Gal}(L/K), L^\times) \longrightarrow H^n(\mathrm{Gal}(L/K), \mathbb{Z}) \longrightarrow 0. $$

In particular, it follows from Hilbert's theorem 90 that $H^1(\mathrm{Gal}(L/K), \mathcal{O}_L^\times) = \{0\}$. On the other hand, the natural exact sequence of $\mathrm{Gal}(L/K)$-modules

$$ 1 \longrightarrow U_{1,L} \longrightarrow \mathcal{O}_L^\times \longrightarrow k_L^\times \longrightarrow 1 $$

induces an injection

$$ H^1(\mathrm{Gal}(L/K), U_{1,L}) \ \hookrightarrow \ H^1(\mathrm{Gal}(L/K), \mathcal{O}_L^\times). $$

Then it holds that $H^1(\mathrm{Gal}(L/K), U_{1,L}) = \{0\}$. Moreover, it follows from Corollary 2.6, (i), that the 0-th Tate cohomology of $\mathrm{Gal}(L/K)$ with coefficients in $U_{1,L}$ vanishes. Thus, we conclude from [31], Chapter IX, §5, Theorem 8, that $H^n(\mathrm{Gal}(L/K), U_{1,L}) = \{0\}$ for each positive integer n. Therefore, by considering the long exact sequence associated to the natural exact sequence

$$ 1 \longrightarrow U_{1,L} \longrightarrow \mathcal{O}_L^\times \longrightarrow k_L^\times \longrightarrow 1, $$

we obtain an isomorphism

$$ H^2(\mathrm{Gal}(L/K), \mathcal{O}_L^\times) \ \xrightarrow{\sim} \ H^2(\mathrm{Gal}(k_L/k), k_L^\times). $$

Here, we note that $H^2(\mathrm{Gal}(L/K), \mathbb{Z}) \overset{\sim}{\to} H^1(\mathrm{Gal}(L/K), \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(\mathrm{Gal}(L/K), \mathbb{Q}/\mathbb{Z})$. Then the above split exact sequence induces a split exact sequence

$$0 \longrightarrow H^2(\mathrm{Gal}(k_L/k), k_L^\times) \longrightarrow H^2(\mathrm{Gal}(L/K), L^\times) \longrightarrow \mathrm{Hom}(\mathrm{Gal}(L/K), \mathbb{Q}/\mathbb{Z}) \longrightarrow 0.$$

Thus, in light of Proposition 2.9, (ii), by varying the unramified Galois extensions $K \subseteq L$ of degree l-power, we conclude that there exists a natural isomorphism

$$_lB_K \overset{\sim}{\to} \mathrm{Hom}(G_k^l, \mathbb{Q}/\mathbb{Z}),$$

as desired. This completes the proof of assertion (ii), hence of Proposition 2.11. □

Theorem 2.12. *In the case where $l = p$ (respectively, $l \neq p$), suppose that k is a p-quasi-finite field (respectively, a strongly l-quasi-finite field). Write X for the set of finite subextension fields of $K \subseteq K^l$;*

$$\mathrm{inv}_K : {}_lB_K \hookrightarrow \mathbb{Q}/\mathbb{Z}$$

for the injective homomorphism induced by the isomorphism of Proposition 2.11, (ii). Then the data

$$(G_K^l, \ \{G_L^l\}_{L \in X}, \ (K^l)^\times, \ \mathrm{inv}_K)$$

determines a class formation in the sense of Artin-Tate.

Proof. In light of the construction of inv_K, together with Hilbert's theorem 90, Theorem 2.12 follows immediately from the various definitions involved [cf. the discussion in the proof of [31], Chapter XIII, §3, Proposition 7]. □

Theorem 2.13. *In the case where $l = p$ (respectively, $l \neq p$), suppose that k is a p-quasi-finite field (respectively, a strongly l-quasi-finite field). Then there exists a reciprocity map*

$$r_K^l : K^\times \longrightarrow (G_K^l)^{\mathrm{ab}}$$

such that, for each finite Galois extension $K \subseteq L$ ($\subseteq K^l$) of degree l-power, r_K^l induces an isomorphism

$$K^\times/\mathrm{Im}(N_{L/K}) \overset{\sim}{\to} \mathrm{Gal}(L/K)^{\mathrm{ab}}.$$

[In particular, the image of r_K^l is dense.] Moreover, the following hold:

(i) Let $K \subseteq L$ ($\subseteq K^l$) be a field extension. Write $K \subseteq M$ for the maxmimal abelian l-extension contained in L. Then it holds that

$$\mathrm{Im}(N_{M/K}) = \mathrm{Im}(N_{L/K}).$$

In particular, the subgroup $\mathrm{Im}(N_{L/K}) \subseteq K^\times$ is of finite index that divides the degree $[K : L]$ of the field extension $K \subseteq L$, and this index is equal to $[K : L]$ if and only if $K \subseteq L$ is an abelian l-extension.

(ii) The assignment $L \mapsto \mathrm{Im}(N_{L/K})$ determines a bijection between the set of abelian l-extensions of K and the set of norm subgroups of $K^\times$. Furthermore, if $K \subseteq L_1$ and $K \subseteq L_2$ are abelian l-extensions, then it holds that

$$\mathrm{Im}(N_{L_1L_2/K}) = \mathrm{Im}(N_{L_1/K}) \cap \mathrm{Im}(N_{L_2/K}); \quad \mathrm{Im}(N_{L_1 \cap L_2/K}) = \mathrm{Im}(N_{L_1/K}) \cdot \mathrm{Im}(N_{L_2/K}).$$

Proof. In light of general theory of Artin-Tate's class formation [cf. [31], Chpater XI], Theorem 2.13 follows immediately from Theorem 2.12. □

Remark 2.13.1. In the notation of Theorem 2.13, suppose that $l \neq \mathrm{char}(K)$, and $\zeta_l \in K$. Let n be a positive integer. Note that, in light of Kummer theory and Ponrjyagin duality, the cup product

$$H^1(G_K^l, \mathbb{Z}/l^n\mathbb{Z}) \times H^1(G_K^l, \mu_{l^n}(K^{\mathrm{sep}})) \longrightarrow H^2(G_K^l, \mu_{l^n}(K^{\mathrm{sep}})) \overset{\sim}{\to} \mathbb{Z}/l^n\mathbb{Z}$$

[cf. Lemma 3.1, (i) below] determines a homomorphism

$$K^\times/(K^\times)^{l^n} \longrightarrow G_K^{\mathrm{ab}}/l^n G_K^{\mathrm{ab}}.$$

Then this homomorphism coincides with the homomorphism induced by the reciprocity map r_K^l. Indeed, if we recall the construction of the reciprocity map based on Tate-Nakayama's theorem, then this follows from standard homological algebra [cf. [13], Proposition 0.14; the discussion in [13], Lemma 1.7].

Next, in the case where k is a p-quasi-finite field, we discuss an important relation between the higher unit groups and the higher ramification groups via the reciprocity map r_K^p.

Lemma 2.14. *Let*

$$P : k^p \twoheadrightarrow k^p$$

be an additive polynomial [i.e., a polynomial that preserves addition] coefficients in k of degree p-power. Suppose that k is a p-quasi-finite field, and $\mathrm{Ker}(P) \subseteq k$. *Then it holds that* $k/P(k) \cong \mathrm{Ker}(P)$.

Proof. Recall that $H^1(G_k^p, k^p) = \{0\}$. Then since $G_k^p \cong \mathbb{Z}_p$, the long cohomology exact sequence associated to the natural exact sequence

$$0 \longrightarrow \mathrm{Ker}(P) \longrightarrow k^p \longrightarrow k^p \longrightarrow 0$$

implies that $k/P(k) \cong \mathrm{Ker}(P)$. This completes the proof of Lemma 2.14. □

Proposition 2.15. *Let n be a positive integer; $K \subseteq L$ a totally ramified Galois extension of degree p-power. Write* $G \overset{\mathrm{def}}{=} \mathrm{Gal}(L/K)$; $\psi \overset{\mathrm{def}}{=} \psi_{L/K}$. *Then the following hold:*

(i) $N_{L/K}(U_{\psi(n),L}) \subseteq U_{n,K}$, *and* $N_{L/K}(U_{\psi(n)+1,L}) \subseteq U_{n+1,K}$.

(ii) Write

$$N_n : (k =)\ U_{\psi(n),L}/U_{\psi(n)+1,L} \longrightarrow U_{n,K}/U_{n+1,K}\ (= k)$$

for the homomorphism induced by $N_{L/K}$ [cf. (i)]. Then N_n is determined by an additive polynomial P_n whose separable degree coincides with the cardinality of $G_{\psi(n)}/G_{\psi(n)+1}$. Moreover, $\mathrm{Ker}(N_n) \cong G_{\psi(n)}/G_{\psi(n)+1}$.

(iii) Suppose that $G_{\psi(n)} = \{1\}$ (respectively, $G_{\psi(n)+1} = \{1\}$). Then it holds that $N_{L/K}(U_{\psi(n),L}) = U_{n,K}$ (respectively, $N_{L/K}(U_{\psi(n)+1,L}) = U_{n+1,K}$).

(iv) Suppose that k is a p-quasi-finite field. Then it holds that

$$U_{n,K}/(U_{n+1,K} \cdot N_{L/K}(U_{\psi(n),L})) \cong G_{\psi(n)}/G_{\psi(n)+1}.$$

(v) Suppose that k is a p-quasi-finite field, and the totally ramified Galois extension $K \subseteq L$ is an abelian extension. Then the family $\{U_{n,K}/N_{L/K}(U_{\psi(n),L})\}_{n\geq 1}$ forms a decreasing filtration of subgroups of $K^\times/\mathrm{Im}(N_{L/K})$ *that correponds to the family $\{G^n\}_{n\geq 1}$ of subgroups of G via the reciprocity map.*

Proof. Assertion (i) follows immediately from [31], Chapter V, §6, Proposition 8, together with Lemma 2.1. Assertion (ii) follows immediately from [31], Chapter V, §6, Proposition 9. Assertion (iii) follows immediately from [31], Chapter V, §6, Corollary 3, together with Lemma 2.5, (iii). Assertion (iv) follows immediately from assertion (ii), together with Lemma 2.14. In light of assertions (ii), (iii), (iv), assertion (v) follows from a similar argument to the argument applied in the proof of [31], Chapter XV, §2, Corollary 3. This completes the proof of Proposition 2.15. $\square$

Theorem 2.16. *Suppose that k is a p-quasi-finite field. Let n be a positive integer. Then the image of $U_{n,K}$ ($\subseteq K^\times$) via the reciprocity map r_K^p is a dense subgroup of $((G_K^p)^{\mathrm{ab}})^n$.*

Proof. Let $K \subseteq L$ be a finite abelian p-extension. Then it suffices to verify that the isomorphism $K^\times/\mathrm{Im}(N_{L/K}) \xrightarrow{\sim} \mathrm{Gal}(L/K)$ maps the subgroup $U_{n,K} \cdot \mathrm{Im}(N_{L/K})/\mathrm{Im}(N_{L/K})$ onto $\mathrm{Gal}(L/K)^n$. To verify this, it follows immediately from Corollary 2.6, (i), that we may assume without loss of generality that $K \subseteq L$ is a totally ramified p-extension. However, this is nothing but Proposition 2.15, (v). This completes the proof of Theorem 2.16. $\square$

Finally, we record some basic properties of Hilbert and Artin-Schreier pairings.

Definition 2.17. Let $\chi \in \mathrm{Hom}(G_K^l, \mathbb{Q}/\mathbb{Z}) = H^1(G_K^l, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} H^2(G_K^l, \mathbb{Z})$; $b \in K^\times = H^0(G_K^l, (K^l)^\times)$. Then we shall write

$$(\chi, b) \ \in \ {}_lB_K$$

for the cup product of χ and b.

Definition 2.18. In the case where $l = p$ (respectively, $l \neq p$), suppose that k is a p-quasi-finite field (respectively, a strongly l-quasi-finite field).

(i) Let m be a positive integer; $a, b \in K^\times$. Suppose that $l \neq \mathrm{char}(K)$, and $\mu_{l^m}(K^{\mathrm{sep}}) \subseteq K^\times$. Fix a primitive l^m-th root of unity $\zeta_{l^m} \in (K^{\mathrm{sep}})^\times$ and identify $\mu_{l^m}(K^{\mathrm{sep}})$ and $\mathbb{Z}/l^m\mathbb{Z}$ via the assignment $\zeta_{l^m}^n \mapsto n$. Write

$$\chi_a \ \in \ H^1(G_K^l, \mathbb{Z}/l^m\mathbb{Z}) \ = \ \mathrm{Hom}(G_K^l, \mathbb{Z}/l^m\mathbb{Z}) \ \subseteq \ \mathrm{Hom}(G_K^l, \mathbb{Q}/\mathbb{Z})$$

— where we identify $\mathbb{Z}/l^m\mathbb{Z}$ with $\frac{1}{l^m}\mathbb{Z}/\mathbb{Z}$ ($\subseteq \mathbb{Q}/\mathbb{Z}$) in a natural way — for the image of a via the Kummer map; $(a, b) \stackrel{\mathrm{def}}{=} (\chi_a, b) \in {}_lB_K$. Then we shall write

$$(a, b)_v \ \stackrel{\mathrm{def}}{=} \ \zeta_{l^m}^{l^m \cdot \mathrm{inv}_K((a,b))} \ \in \ \mu_{l^m}(K^{\mathrm{sep}}).$$

(ii) Let $a \in K$; $b \in K^\times$. Suppose that $\mathrm{char}(K) = p = l$. Write

$$\psi_a \ \in \ H^1(G_K^p, \mathbb{Z}/p\mathbb{Z}) \ = \ \mathrm{Hom}(G_K^p, \mathbb{Z}/p\mathbb{Z}) \ \subseteq \ \mathrm{Hom}(G_K^p, \mathbb{Q}/\mathbb{Z})$$

— where we identify $\mathbb{Z}/p\mathbb{Z}$ with $\frac{1}{p}\mathbb{Z}/\mathbb{Z}$ ($\subseteq \mathbb{Q}/\mathbb{Z}$) in a natural way — for the image of a via the Artin-Schreier map; $[a, b) \stackrel{\mathrm{def}}{=} (\psi_a, b) \in {}_pB_K$. Then we shall write

$$[a, b)_v \ \stackrel{\mathrm{def}}{=} \ p \cdot \mathrm{inv}_K([a, b)) \ \in \ \mathbb{Z}/p\mathbb{Z}.$$

Proposition 2.19. *In the case where $l = p$ (respectively, $l \neq p$), suppose that k is a p-quasi-finite field (respectively, a strongly l-quasi-finite field). Let m be a positive integer; $a, a', b, b' \in K^\times$. Suppose, moreover, that $l \neq \mathrm{char}(K)$, and $\mu_{l^m}(K^{\mathrm{sep}}) \subseteq K^\times$. Then the following hold:*

(i) $(a, b)_v$ *is indepedent of the choice of* ζ_{l^m}.

(ii) $(aa', b)_v = (a, b)_v \cdot (a', b)_v$, *and* $(a, bb')_v = (a, b)_v \cdot (a, b')_v$.

(iii) Write $K_m \overset{\mathrm{def}}{=} K(a^{\frac{1}{l^m}})$. *Then* $(a, b)_v = 1$ *if and only if* $b \in \mathrm{Im}(N_{K_m/K})$.

(iv) $(a, -a)_v = 1 = (a, 1 - a)_v$, *and* $(a, b)_v \cdot (b, a)_v = 1$.

(v) Suppose that $(a, c)_v = 1$ *for any* $c \in K^\times$. *Then it holds that* $a \in (K^\times)^{l^m}$.

(vi) It holds that

$$(K^\times)^{l^m} = \bigcap_{K \subseteq L} \mathrm{Im}(N_{L/K}),$$

where $K \subseteq L$ *ranges over the cyclic extensions of degree dividing* l^m.

Proof. Proposition 2.19 follows from a similar argument to the argument applied in the proof of [31], Chapter XIV, §2, Propositions 6, 7, together with their Corollaries. □

Proposition 2.20. *Suppose that k is a p-quasi-finite field, and $\mathrm{char}(K) = p$. Let $a, a' \in K$; $b, b' \in K^\times$. Then the following hold:*

(i) $[a + a', b)_v = [a, b)_v + [a', b)_v$, *and* $[a, bb')_v = [a, b)_v + [a, b')_v$.

(ii) Write K_a $(\subseteq K^{\mathrm{sep}})$ *for the minimal splitting field of the Artin-Schreier equation* $y^p - y = a$. *Then* $[a, b)_v = 0$ *if and only if* $b \in \mathrm{Im}(N_{K_a/K})$.

(iii) $[a, a)_v = 0$.

(iv) Suppose that $[a, c)_v = 0$ *for any* $c \in K^\times$. *Then it holds that* $a \in \wp(K)$.

(v) Suppose that K is complete. Then it holds that

$$(K^\times)^p = \bigcap_{K \subseteq L} \mathrm{Im}(N_{L/K}),$$

where $K \subseteq L$ *ranges over the cyclic extensions of degree* p.

Proof. Proposition 2.20 follows from a similar argument to the argument applied in the proof of [31], Chapter XIV, §5, Propositions 14, 16. □

3 Groups of l-power roots of unity associated to Henselian discrete valuation fields with strongly l-quasi-finite residues

Throughout the present section, we maintain the notation of §2 [except the assumption that k is perfect!]. In the present section, in the case where K is a Henselian discrete valuation field with strongly l-quasi-finite and stably μ_{l^∞}-finite residues [cf. Definitions 1.9, (ii); 2.8], by applying l-local class field theory discussed in §2, we prove that the group of l-th power roots of unity $\mu_{l^\infty}(K)$ maps isomorphically onto the torsion subgroup of $(G_K^l)^{\mathrm{ab}}$ via the reciprocity map [cf. Proposition 3.7].

First, we begin by discussing elementary properties of certain Galois cohomology groups.

Lemma 3.1. *Suppose that $l \neq \mathrm{char}(K)$. Then the following hold:*

(i) Suppose, moreover, that the following conditions hold:

- *If $l = p$ (respectively, $l \neq p$), then k is a p-quasi-finite field (respectively, a strongly l-quasi-finite field).*
- *It holds that $\zeta_l \in K$.*

Let n be a positive integer; $K \subseteq L$ $(\subseteq K^l)$ a finite Galois extension of degree l-power. Then there exists a natural isomorphism

$$H^2(G_L^l, \mu_{l^n}(K^l)) \xrightarrow{\sim} \mathbb{Z}/l^n\mathbb{Z}.$$

(ii) Let $K \subseteq M$ be a field extension. Then the natural homomorphism

$$H^2(G_M, \mathbb{F}_l) \longrightarrow H^2(G_{M(\zeta_l)}, \mathbb{F}_l)$$

is injective.

(iii) The natural homomorphism

$$H^2(G_K^l, \mathbb{F}_l) \longrightarrow H^2(G_K, \mathbb{F}_l)$$

is injective. Moreover, if $l = p$, and k is perfect, then this homomorphism is an isomorphism.

(iv) The natural homomorphism

$$H^2(G_K^l, \mathbb{F}_l) \longrightarrow H^2(G_{K(\zeta_l)}^l, \mathbb{F}_l)$$

is injective.

Proof. First, we verify assertion (i). Note that since k is l-quasi-finite, and the finite Galois extension $K \subseteq L$ is of degree l-power, it holds that k_L is also l-quasi-finite. On the other hand, since $\zeta_l \in K$, it follows immediately from Hilbert's theorem 90 that $H^2(G_L^l, \mu_{l^n}(K^l))$ is isomorphic to $H^2(G_L^l, (K^l)^\times)[l^n]$. Thus, we conclude from Proposition 2.11, (ii), that there exists a natural isomorphism $H^2(G_L^l, \mu_{l^n}(K^l)) \xrightarrow{\sim} \mathbb{Z}/l^n\mathbb{Z}$. This completes the proof of assertion (i).

Next, we verify assertion (ii). In light of the Hochschild-Serre spectral sequence associated to the natural exact sequence

$$1 \longrightarrow G_{M(\zeta_l)} \longrightarrow G_M \longrightarrow \mathrm{Gal}(M(\zeta_l)/M) \longrightarrow 1,$$

it suffices to prove that

$$H^1(\mathrm{Gal}(M(\zeta_l)/M), H^1(G_{M(\zeta_l)}, \mathbb{F}_l)) = \{0\}, \quad H^2(\mathrm{Gal}(M(\zeta_l)/M), \mathbb{F}_l) = \{0\}.$$

However, these equalities follow immediately from the fact that $[M(\zeta_l) : M]$ is coprime to l. This completes the proof of assertion (ii).

Next, we verify assertion (iii). Write $N \stackrel{\text{def}}{=} \mathrm{Ker}(G_K \twoheadrightarrow G_K^l)$. Again, in light of the Hochschild-Serre spectral sequence associated to the natural exact sequence

$$1 \longrightarrow N \longrightarrow G_K \longrightarrow G_K^l \longrightarrow 1,$$

to verify the injectivity (respectively, the bijectivity) of the natural homomorphism $H^2(G_K^l, \mathbb{F}_l) \to H^2(G_K, \mathbb{F}_l)$, it suffices to prove that

$$H^1(N, \mathbb{F}_l) \ = \ \{0\} \quad \text{(respectively,} \quad H^1(N, \mathbb{F}_l) \ = \ H^2(N, \mathbb{F}_l) \ = \ \{0\}).$$

However, since the equality $H^1(N, \mathbb{F}_l) = \{0\}$ follows immediately from the fact that $N^l = \{1\}$, it suffices to verify that $H^2(N, \mathbb{F}_l) = \{0\}$ under the assumption that $l = p$, and k is perfect. Here, we observe that the ramification index of the field extension $K \subseteq K^p$ is divisible by p^∞. In particular, if k is perfect, then it follows immediately from [31], Chapter XII, §3, Exercise 2, that

$$H^2(G_{K^p(\zeta_p)}, \mathbb{F}_p) \ \stackrel{\sim}{\to} \ \varinjlim_{K \subseteq L} \ H^2(G_{L(\zeta_p)}, \mathbb{F}_p) \ = \ \{0\},$$

where $K \subseteq L$ ($\subseteq K^p$) ranges over the Galois extensions of degree p-power. Thus, we conclude from assertion (ii) that $H^2(N, \mathbb{F}_p) = H^2(G_{K^p}, \mathbb{F}_p) = \{0\}$. This completes the proof of assertion (iii).

Finally, we verify assertion (iv). Observe that we have the following commutative diagram of natural homomorphisms

$$\begin{array}{ccc} H^2(G_K^l, \mathbb{F}_l) & \longrightarrow & H^2(G_K, \mathbb{F}_l) \\ \downarrow & & \downarrow \\ H^2(G_{K(\zeta_l)}^l, \mathbb{F}_l) & \longrightarrow & H^2(G_{K(\zeta_l)}, \mathbb{F}_l). \end{array}$$

Thus, we conclude from assertions (ii), (iii), that the natural homomorphism $H^2(G_K^l, \mathbb{F}_l) \to H^2(G_{K(\zeta_l)}^l, \mathbb{F}_l)$ is injective. This completes the proof of assertion (iv), hence of Lemma 3.1. □

Proposition 3.2. *Let r be a positive integer; G a pro-l group. Suppose that*

$$\dim_{\mathbb{F}_l} \ H^2(G, \mathbb{F}_l) \ \leq \ r.$$

Then, for each positive integer m, it holds that $G^{\mathrm{ab}}[l^m]$ is generated by r elements.

Proof. Fix a free pro-l group F and an epimorphism $F \twoheadrightarrow G$ that induces an isomorphism $F^{\mathrm{ab}}/lF^{\mathrm{ab}} \stackrel{\sim}{\to} G^{\mathrm{ab}}/lG^{\mathrm{ab}}$. [Note that the existence of such an epimorphism follows immediately from the fact that F is projective — cf. [29], Theorem 7.5.1; [29], Remark 7.7.1, (a), (b); the proof of [29], Lemma 7.7.3.] Write R for the kernel of the epimorphism $F \twoheadrightarrow G$. Then, by considering a five term exact sequence associated to the exact sequence of pro-l groups

$$1 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 1,$$

we obtain an isomorphism $H^1(R, \mathbb{F}_l)^G \stackrel{\sim}{\to} H^2(G, \mathbb{F}_l)$. Then it follows from [29], Proposition 7.8.2, together with our assumption, that the normal closed subgroup $R \subseteq F$ is topologically F-normally generated by r elements $\in R$. Thus, we conclude that the kernel of the natural surjection $F^{\mathrm{ab}} \twoheadrightarrow G^{\mathrm{ab}}$ [induced by the epimorphism $F \twoheadrightarrow G$] is topologically generated by r elements. This observation immediately implies the desired conclusion. This completes the proof of Proposition 3.2. □

Proposition 3.3. *Suppose that k is a strongly l-quasi-finite field. Then, for each positive integer m, it holds that $(G_K^l)^{\mathrm{ab}}[l^m]$ is a cyclic l-group.*

Proof. Let m be a positive integer. In the case where $\mathrm{char}(K) = p = l$, since G_K^p is a free pro-p group, it holds that $(G_K^p)^{\mathrm{ab}}[p^m]$ is trivial. Next, suppose that $l \neq \mathrm{char}(K)$. Note that since k is a strongly l-quasi-finite field, the residue field of $K(\zeta_l)$ is also l-quasi-finite. In particular, it follows immediately from Lemma 3.1, (i), (iv), that there exists a natural injective homomorphism $H^2(G_K^l, \mathbb{F}_l) \hookrightarrow \mathbb{F}_l$. Thus, we conclude from Proposition 3.2 that $(G_K^l)^{\mathrm{ab}}[l^m]$ is a cyclic l-group, as desired. This completes the proof of Proposition 3.3. □

Lemma 3.4. *Let M, N be abelian groups; $\phi : N \to M$ a homomorphism of abelian groups. Then the following hold:*

(i) Suppose that

- *the natural homomorphism $N/lN \to M/lM$ [induced by ϕ] is injective, and*
- *the natural homomorphism $N[l] \to M[l]$ [induced by ϕ] is surjective.*

Then, for each positive integer m, the natural homomorphism $N/l^mN \to M/l^mM$ [induced by ϕ] is injective.

(ii) Suppose that

- *ϕ is an inclusion,*
- *$N[l] = M[l]$, and*
- *the natural homomorphism $N/lN \to M/lM$ is bijective.*

Then the homomorphism $M/N \to M/N$ induced by multiplication by l is bijective.

Proof. First, we verify assertion (i). In light of the natural factorization $N \to \mathrm{Im}(\phi) \subseteq M$ of ϕ, we may assume without loss of generality that ϕ is injective or surjective.

Suppose, first, that ϕ is injective. Then it follows immediately from our assumptions, together with Snake lemma, that $M/\phi(N)[l]$ is trivial. This implies that, for each positive integer m, it holds that $M/\phi(N)[l^m]$ is trivial. Thus, by applying Snake lemma again, we conclude that the natural homomorphism $N/l^mN \to M/l^mM$ is injective.

Next, suppose that ϕ is surjective. Then it follows immediately from our assumptions, together with Snake lemma, that $\mathrm{Ker}(\phi) = l\mathrm{Ker}(\phi)$. This implies that, for each positive integer m, it holds that $\mathrm{Ker}(\phi) = l^m\mathrm{Ker}(\phi)$. Thus, by applying Snake lemma again, we conclude that the natural homomorphism $N/l^mN \to M/l^mM$ is injective. This completes the proof of assertion (i).

Assertion (ii) follows immediately from Snake lemma. This completes the proof of Lemma 3.4. □

Theorem 3.5. *Suppose that the following conditions hold:*

(a) If $l = p$ (respectively, $l \neq p$), then k is a p-quasi-finite field (respectively, a strongly l-quasi-finite field).

(b) If $l = p$, then K is algebraically closed in $\widetilde{K}$ [Note that since K is Henselian, this condition automatically holds in the case where $\mathrm{char}(K) = 0$.*]*

Then the following hold:

(i) For each positive integer m, the natural homomorphism

$$r_m : K^\times/(K^\times)^{l^m} \longrightarrow G_K^{\mathrm{ab}}/l^m G_K^{\mathrm{ab}}$$

induced by the reciprocity map $r_K^l : K^\times \to (G_K^l)^{\mathrm{ab}}$ is injective. In particular, it holds that $\mathrm{Ker}(r_K^l)$ consists of the l-divisible elements of $K^\times$.

(ii) Suppose that $l \neq p$. Then, for each positive integer m, the injection r_m [cf. (i)] is bijective.

(iii) Suppose that $l = p$. Then the composite homomorphism

$$U_{1,K} \subseteq K^\times \xrightarrow{r_K^p} (G_K^p)^{\mathrm{ab}}$$

is injective.

Proof. First, we verify assertion (i). Observe that, for each positive integer m, it follows immediately from Theorem 2.13 that

$$\mathrm{Ker}(r_m) = \Big(\bigcap_{K\subseteq L} \mathrm{Im}(N_{L/K})\Big)/(K^\times)^{l^m},$$

where $K \subseteq L$ ranges over the abelian extensions of degree dividing l^m.

Next, suppose that $\mathrm{char}(K) = p = l$. Then since K is Henselian and algebraically closed in $\widetilde{K}$ [cf. condition (b)], by replacing K by $\widetilde{K}$, we may assume without loss of generality that K is complete. Theorefore, in light of the above equality, the injectivity of r_1 follows from Proposition 2.20, (v). On the other hand, since $\mathrm{char}(K) = p$, we observe that G_K^{ab} has no p-torsion elements. Thus, we conclude from Lemma 3.4, (i), together with the injectivity of r_1, that r_m is injective for each positive integer m.

Next, suppose that $l \neq \mathrm{char}(K)$. Note that, in light of the above equality, it follows immediately from Proposition 2.19, (vi), that r_m is injective in the case where $\mu_{l^m}(K^{\mathrm{sep}}) \subseteq K^\times$. Write $K_{l^m} \stackrel{\mathrm{def}}{=} K(\mu_{l^m}(K^{\mathrm{sep}}))$. Then it follows immediately from the well-known computation of the cohomology groups of finite cyclic groups [cf., e.g., [31], Chapter VIII, §4] that $H^1(\mathrm{Gal}(K_{l^m}/K), \mu_{l^m}(K_{l^m})) = \{0\}$. On the other hand, we have an exact sequence

$$H^1(\mathrm{Gal}(K_{l^m}/K), \mu_{l^m}(K_{l^m})) \longrightarrow H^1(G_K, \mu_{l^m}(K^{\mathrm{sep}})) \longrightarrow H^1(G_{K_{l^m}}, \mu_{l^m}(K^{\mathrm{sep}}))$$

induced by the natural exact sequence

$$1 \longrightarrow G_{K_{l^m}} \longrightarrow G_K \longrightarrow \mathrm{Gal}(K_{l^m}/K) \longrightarrow 1.$$

In particular, the natural homomorphism $K^\times/(K^\times)^{l^m} \to K_{l^m}^\times/(K_{l^m}^\times)^{l^m}$ is injective. Therefore, in light of the well-known functoriality of the reciprocity maps with respect to the field extensions [cf. [31], Chapter XI, §3], it holds that r_m is injective. This completes the proof of assertion (i).

Next, we verify assertion (ii). Since $l \neq p$, one verifies easily that the cardinality of $G_K^{\mathrm{ab}}/l^m G_K^{\mathrm{ab}}$ is $\leq 2l^m$, hence finite. In particular, to verify the surjectivity of r_m, it follows from Nakayama's lemma that we may assume without loss of generality that $m = 1$. On the other hand, since K is a discrete valuation field, the $\mathbb{F}_l$-vector space $K^\times/(K^\times)^l$ is of dimension ≥ 1. Therefore, in light of the injectivity of r_1, together with Proposition 2.10, that we may assume without loss of generality that $\zeta_l \in K$. In this case, K admits a totally ramified cyclic extension of degree l. This implies that the $\mathbb{F}_l$-vector space $K^\times/(K^\times)^l$ is of dimension ≥ 2. Thus, since the cardinality of $G_K^{\mathrm{ab}}/lG_K^{\mathrm{ab}}$ is $\leq 2l$, we conclude from the injectivity of r_1 that r_1 is bijective. This completes the proof of assertion (ii).

Assertion (iii) follows immediately from assertion (i). This completes the proof of Theorem 3.5. $\square$

Remark 3.5.1. In the notation of Theorem 3.5, one may not drop the assumption that K is algebraically closed in $\widetilde{K}$ in the case where $\mathrm{char}(K) = p = l$. Indeed, suppose that $\mathrm{char}(K) = p$. Then, if the homomorphism $K^\times/(K^\times)^p \to G_K^{\mathrm{ab}}/pG_K^{\mathrm{ab}}$ is injective, then the natural homomorphism

$$K^\times/(K^\times)^p \longrightarrow \widetilde{K}^\times/(\widetilde{K}^\times)^p$$

is injective, hence that there exists no nontrivial purely inseparable extension of K in $\widetilde{K}$. Note that since K is Henselian, it holds that K is separably closed in $\widetilde{K}$. Thus, we conclude that K is algebraically closed in $\widetilde{K}$. On the other hand, we recall that there exists a Henselian discrete valuation field of characteristic p that is not algebraically closed in its completion. For instance, by forming the Henselization of the well-known non-Nagata discrete valuation field constructed in [11], Example 2.31, one obtains such a field.

Proposition 3.6. *The following hold:*

(i) Suppose that, if $l = p$ (respectively, $l \neq p$), then k is perfect (respectively, strongly l-quasi-finite). Then it holds that $1 \leq \mathrm{cd}_l(G_K^l) \leq 2$.

(ii) Suppose that $l \neq \mathrm{char}(K)$, and k is a strongly l-quasi-finite field. Then $\zeta_l \notin K$ if and only if G_K^l is a [nontrivial] free pro-l group.

(iii) Suppose that k is a strongly l-quasi-finite and stably μ_{l^∞}-finite field. Then it holds that $\mathrm{scd}_l(G_K^l) = 2$. [Note that k is a stably μ_{p^∞}-finite field.]

Proof. First, we verify assertion (i). In the case where $l \neq p$, and $\zeta_l \notin K$, it follows from Proposition 2.10 that $\mathrm{cd}_l(G_K^l) = 1$. In the case where $l \neq p$, and $\zeta_l \in K$, one may observe that the maximal pro-l extension of K contains the l-power roots of a uniformizer of K. In particular, since k is l-quasi-finite, it holds that G_K^l is an extension of $\mathbb{Z}_l$ [cf. Proposition 1.5, (i)], hence that $\mathrm{cd}_l(G_K^l) = 2$. Next, in the case where $\mathrm{char}(K) = p = l$, since G_K^p is a nontrivial free pro-p group, it holds that $\mathrm{cd}_p(G_K^p) = 1$. Thus, we may assume without loss of generality that $l = p$, and $\mathrm{char}(K) = 0$. On the other hand, we note that every p-adic local field admits a $\mathbb{Z}_p$-extension whose ramification index is divisible by p^∞. In particular, by forming the composite of K with such a $\mathbb{Z}_p$-extension of a p-adic local field $\subseteq K$, we observe that there exists a $\mathbb{Z}_p$-extension $K \subseteq L$ whose ramification index is divisible by p^∞. Then, by applying a similar argument to the argument applied in the proof of Lemma 3.1, (iii), we conclude that $H^2(G_L^p, \mathbb{F}_p) = \{0\}$, hence that $\mathrm{cd}_p(G_L^p) = 1$. Therefore, since $\mathrm{cd}_p(\mathbb{Z}_p) = 1$, it follows immediately from [29], Proposition 7.4.2 that $\mathrm{cd}_p(G_K^p) \leq 2$. This completes the proof of assertion (i).

Next, we verify assertion (ii). In light of Proposition 2.10, together with the discussion in the first paragraph of the present proof, we may assume without loss of generality that

$$l = p, \quad \mathrm{char}(K) = 0.$$

Then sufficiency follows immediately from Theorem 3.5, (iii). Next, we verify necessity. By replacing K by the p-adic completion of K, we may assume without loss of generality that K is complete. Then since k is a p-quasi-finite field, it follows from [32], Chapter II, §4, Proposition 12, together with Proposition 2.9, (iii), that $\mathrm{cd}_p(G_K) = 2$. Next, write D_2 for the p-dualizing module of G_K, i.e.,

$$D_2 \overset{\mathrm{def}}{=} \varinjlim_{m \geq 1} \varinjlim_{K \subseteq L} \mathrm{Hom}(H^2(G_L, \mathbb{Z}/p^m\mathbb{Z}), \mathbb{Q}/\mathbb{Z}),$$

where $K \subseteq L$ ($\subseteq K^{\mathrm{sep}}$) ranges over the finite Galois extensions; m ranges over the positive integers; the transition maps of $\varinjlim_{K \subseteq L}$ are induced by the corestriction maps. Then since $\mathrm{cd}_p(G_K) = 2$, and k is a strongly p-quasi-finite field, it follows immediately from [31], Chapter XII, §3, Theorem 2, together with [28], Theorem 3.4.4 [cf. also [32], Chapter I, §3, Proposition 14], that, for each finite Galois extension $K \subseteq L$ ($\subseteq K^{\mathrm{sep}}$) and positive integer n,

$$\mathbb{Z}/p^n\mathbb{Z} \overset{\sim}{\to} \mathrm{Hom}(H^2(G_L, \mu_{p^n}(K^{\mathrm{sep}})), \mathbb{Q}/\mathbb{Z}) \overset{\sim}{\to} \mathrm{Hom}_{G_L}(\mu_{p^n}(K^{\mathrm{sep}}), D_2).$$

In particular, we obtain a canonical isomorphism of G_K-modules

$$\mathbb{Z}/p^n\mathbb{Z} \overset{\sim}{\to} \mathrm{Hom}(\mu_{p^n}(K^{\mathrm{sep}}), D_2[p^n]).$$

Observe that the homomorphism $f_n : \mu_{p^n}(K^{\mathrm{sep}}) \to D_2[p^n]$ that corresponds to $1 \in \mathbb{Z}/p^n\mathbb{Z}$ via the above isomorphism is bijective. Therefore, by varying n, we obtain an isomorphism

$$D_2 \overset{\sim}{\to} \mu_{p^\infty}(K^{\mathrm{sep}}).$$

On the other hand, by applying [28], Theorem 3.4.4, again, we observe that

$$\mathrm{Hom}(H^2(G_K, \mathbb{F}_p), \mathbb{Q}/\mathbb{Z}) \overset{\sim}{\to} \mathrm{Hom}_{G_K}(\mathbb{F}_p, D_2).$$

Thus, since $\zeta_p \notin K$, we conclude that $\mathrm{Hom}(H^2(G_K, \mathbb{F}_p), \mathbb{Q}/\mathbb{Z}) = \{0\}$, hence that $H^2(G_K, \mathbb{F}_p) = \{0\}$. Then it follows from Lemma 3.1, (iii), that $H^2(G_K^p, \mathbb{F}_p) = \{0\}$, hence from [29], Theorem 7.7.4, that G_K^p is a free pro-p group. This completes the proof of necessity, hence of assertion (ii).

Finally, we verify assertion (iii). Note that it follows from assertion (i) that $1 \leq \mathrm{cd}_l(G_K^l) \leq 2$. In the case where $\mathrm{cd}_l(G_K^l) = 1$, we observe that

$$H^2(G_K^l, \mathbb{Z})[l] \overset{\sim}{\to} H^1(G_K^l, \mathbb{Q}/\mathbb{Z})[l] \overset{\sim}{\to} H^1(G_K^l, \mathbb{F}_l) \neq \{0\}.$$

Then it follows from [28], Corollary 3.3.4, that $\mathrm{scd}_l(G_K^l) = 2$. Therefore, to verify that $\mathrm{scd}_l(G_K^l) = 2$, we may assume without loss of generality that

$$\mathrm{cd}_l(G_K^l) = 2.$$

In particular, it follows immediately from assertion (ii), together with the fact that G_K^p is a free pro-p group in the case where $\mathrm{char}(K) = p$, that

$$l \neq \mathrm{char}(K), \quad \zeta_l \in K.$$

Next, write $D_{2,l}$ for the l-dualizing module of G_K^l, i.e.,

$$D_{2,l} \overset{\mathrm{def}}{=} \varinjlim_{m \geq 1} \varinjlim_{K \subseteq L} \mathrm{Hom}(H^2(G_L^l, \mathbb{Z}/l^m\mathbb{Z}), \mathbb{Q}/\mathbb{Z}),$$

where $K \subseteq L$ ($\subseteq K^l$) ranges over the finite Galois extensions of degree l-power; m ranges over the positive integers; the transition maps of $\varinjlim_{K \subseteq L}$ are induced by the corestriction maps. In this situation, since $\zeta_l \in K$, it follows immediately from Lemma 3.1, (i), together with [28], Theorem 3.4.4, that, for each finite Galois extension $K \subseteq L$ ($\subseteq K^l$) of degree l-power and positive integer n,

$$\mathbb{Z}/l^n\mathbb{Z} \overset{\sim}{\to} \mathrm{Hom}(H^2(G_L^l, \mu_{l^n}(K^l)), \mathbb{Q}/\mathbb{Z}) \overset{\sim}{\to} \mathrm{Hom}_{G_L^l}(\mu_{l^n}(K^l), D_{2,l}).$$

In particular, we obtain a canonical isomorphism of G_K^l-modules

$$\mathbb{Z}/l^n\mathbb{Z} \overset{\sim}{\to} \mathrm{Hom}(\mu_{l^n}(K^l), D_{2,l}[l^n]).$$

Observe that the homomorphism $f_n : \mu_{l^n}(K^l) \to D_{2,l}[l^n]$ that corresponds to $1 \in \mathbb{Z}/l^n\mathbb{Z}$ via the above isomorphism is bijective. Thus, by varying n, we conclude that

$$D_{2,l} \overset{\sim}{\to} \mu_{l^\infty}(K^l),$$

hence from [28], Corollary 3.4.5, together with Lemma 1.11 [cf. our assumption that k is stably μ_{l^∞}-finite], that $\mathrm{scd}_l(G_K^l) = 2$. This completes the proof of assertion (iii), hence of Proposition 3.6. □

Remark 3.6.1. Suppose that $l = p$, and k is perfect. Then, in light of Proposition 3.6, (i), it would be interesting to investigate the extent to which the assumption that k is a strongly p-quasi-finite field in Proposition 3.6, (ii), (iii), may be dropped.

Proposition 3.7. *Suppose that k is a strongly l-quasi-finite and stably μ_{l^∞}-finite field. [Note that k is a stably μ_{p^∞}-finite field.] Then the reciprocity map $r^l_K : K^\times \to (G^l_K)^{\mathrm{ab}}$ induces an isomorphism*

$$\mu_{l^\infty}(K) \overset{\sim}{\to} ((G^l_K)^{\mathrm{ab}})_{\mathrm{tor}}.$$

Proof. Let m be a positive integer. Observe that since k is a stably μ_{l^∞}-finite field, it follows immediately from Theorem 3.5, (i), that the natural homomorphism $\mu_{l^m}(K) \to (G^l_K)^{\mathrm{ab}}[l^m]$ [induced by r^l_K] is injective. Therefore, it suffices to verify that this injection is bijective. In the case where $l \neq \mathrm{char}(K)$, and $\zeta_l \notin K$ (respectively, $\mathrm{char}(K) = p = l$), it follows from Proposition 3.6, (ii) (respectively, the [well-known] fact that G^p_K is a free pro-p group) that $(G^l_K)^{\mathrm{ab}}[l^m] = \{0\}$. In particular, there is nothing to prove. Thus, we may assume without loss of generality that $l \neq \mathrm{char}(K)$, and $\zeta_l \in K$. In this situation, the Galois extension $K \subseteq K(\zeta_{l^m})$ is of degree l-power. On the other hand, it follows from Proposition 3.3 that $(G^l_{K(\zeta_{l^m})})^{\mathrm{ab}}[l^m]$ is a cyclic l-group. In particular, we have an isomorphism $\mu_{l^m}(K(\zeta_{l^m})) \overset{\sim}{\to} (G^l_{K(\zeta_{l^m})})^{\mathrm{ab}}[l^m]$. Here, we recall that $\mathrm{scd}_l(G^l_K) = 2$ [cf. Proposition 3.6, (iii)]. Thus, we conclude from [28], Theorem 3.6.4, that

$$\mu_{l^m}(K) = \mu_{l^m}(K(\zeta_{l^m}))^{\mathrm{Gal}(K(\zeta_{l^m})/K)} \overset{\sim}{\to} ((G^l_{K(\zeta_{l^m})})^{\mathrm{ab}}[l^m])^{\mathrm{Gal}(K(\zeta_{l^m})/K)} = (G^l_K)^{\mathrm{ab}}[l^m].$$

This completes the proof of Proposition 3.7. □

4 Mono-anabelian group-theoretic reconstruction algorithms associated to Henselian discrete valuation fields with quasi-finite residues

Let p be a prime number; K a Henselian discrete valuation field of residue characteristic p. Throughout the present section, we maintain the notation of §1, especially Definition 1.8.

In the present section, from the viewpoint of mono-anabelian geometry, under certain assumptions on the residue field k of K, we establish certain group-theoretic algorithms/procedures to reconstruct, from [the underlying topological group structure of] the absolute Galois group G_K of K, the multiplicative groups introduced in Definition 1.8. Our main tool is local class field theory for Henselian discrete valuation fields with quasi-finite residues [cf. Definition 4.1; Theorem 4.3 below], and the main result of the present section [cf. Theorem 4.7] may be regarded as a generalization of [20], Proposition 1.2.1.

Definition 4.1 ([31], Chapter XIII, §2)**.** Let L be a perfect field. Then we shall say that L is *quasi-finite* if the absolute Galois group of L is isomorphic to $\widehat{\mathbb{Z}}$.

Remark 4.1.1. Note that we do not fix a specific isomorphism between the absolute Galois group of L and $\widehat{\mathbb{Z}}$. In particular, the above definition of quasi-finite fields is slightly different from the definition given in [31], Chapter XIII, §2.

Remark 4.1.2. Let L be a quasi-finite field; l a prime number. Then it follows immediately from the various definitions involved that every finite extension field of a pro-prime-to-l extension field of L is a strongly l-quasi-finite field.

In the remainder of the present section, suppose that

the residue field k is quasi-finite.

Proposition 4.2. *Suppose that k is μ-finite. Write $k_{\rm prm}$ for the prime field $\subseteq k$; $\overline{k}_{\rm prm}$ for the algebraic closure of $k_{\rm prm}$ in the fixed algebraic closure $k^{\rm sep}$ of the perfect field k. Then the following hold:*

(i) There exists a unique topological generator $F \in G_k$ that acts on $\overline{k}_{\rm prm}$ as some positive integer power of the Frobenius automorphism $x \mapsto x^p$.

(ii) G_K/P_K is naturally isomorphic to

$$\widehat{\mathbb{Z}}^{(p)'}(1) \rtimes G_k \quad (\overset{\sim}{\to}\ \widehat{\mathbb{Z}}^{(p)'}(1) \rtimes \widehat{\mathbb{Z}}),$$

where $F \in G_k$ acts on $\widehat{\mathbb{Z}}^{(p)'}(1)$ by multiplication by some positive integer power of p. Moreover, the natural homomorphism $G_k \to \mathrm{Aut}(\widehat{\mathbb{Z}}^{(p)'}(1))$ is injective.

Proof. Write $k_0 \overset{\rm def}{=} \overline{k}_{\rm prm} \cap k$. Note that since k is μ-finite, it follows from Lemma 1.10, (ii), that k_0 is a finite field. Then we have the natural composite homomorphism

$$G_k \longrightarrow G_{k_0} \longrightarrow G_{k_{\rm prm}} \longrightarrow \mathrm{Aut}(\widehat{\mathbb{Z}}^{(p)'}(1)),$$

where the first arrow denotes the natural surjection [cf. Lemma 1.10, (i)]; the second arrow denotes the natural open injection. Observe that since k_0 and k are quasi-finite fields, it holds that $G_{k_0} \cong G_k \cong \widehat{\mathbb{Z}}$. In particular, the surjection $G_k \twoheadrightarrow G_{k_0}$ is, in fact, an isomorphism. Write $f_0 \overset{\rm def}{=} [k_0 : k_{\rm prm}]$. Observe that the inverse image of the Frobenius automorphism $\in G_{k_0}$ that maps $x \mapsto x^{p^{f_0}}$ via the isomorphism $G_k \overset{\sim}{\to} G_{k_0}$ satisfies the condition of F in assertion (i). On the other hand, the uniqueness portion of assertion (i) follows immediately from the easily verified fact that $\widehat{\mathbb{Z}}^\times \cap \mathbb{Q}_{>0} = \{1\}$. This completes the proof of assertion (i). Finally, since the surjection $G_k \twoheadrightarrow G_{k_0}$ is an isomorphism, assertion (ii) follows immediately from assertion (i), together with [28], Proposition 7.5.2; [28], Lemma 7.5.4, (ii). This completes the proof of Proposition 4.2. □

Remark 4.2.1. One may construct many examples of infinite, μ-finite, and quasi-finite fields of characteristic p as follows: Let k_0 be an infinite perfect field of characteristic p such that the natural homomorphism $\phi : G_{k_0} \to G_{\mathbb{F}_p}$ has an open image. Note that $\mathrm{Im}(\phi) \overset{\sim}{\to} \widehat{\mathbb{Z}}$ [so there are many splittings of the surjection $G_{k_0} \twoheadrightarrow \mathrm{Im}(\phi)$]. Let $s : \mathrm{Im}(\phi) \hookrightarrow G_{k_0}$ be a splitting of the surjection $G_{k_0} \twoheadrightarrow \mathrm{Im}(\phi)$. Write $k_1 \subseteq \overline{k}_0$ for the subfield fixed by $\mathrm{Im}(s)$. Then it follows immediately from Lemma 1.10, (iii), together with the various definitions involved, that k_1 is an infinite, μ-finite, and quasi-finite field.

Next, we recall local class field theory for K. The following reciprocity map r is obtained by restricting the reciprocity map in local class field theory for the completion of K [cf. [31], Chapter XIII, §4, Corollaries of Propositions 8, 13; [31], Chapter XIV, §2, Corollary of Proposition 7; [31], Chapter XIV, §5, Proposition 16; [3], Chapter V, §3, Corollary 1, together with Remarks 4.3.1; 4.3.2; 4.3.3 below]:

Theorem 4.3. *There exists a reciprocity map*

$$r_K : K^\times \ \longrightarrow\ G_K^{\rm ab}$$

such that the following conditions hold:

(a) $\mathrm{Im}(r_K) \subseteq G_K^{\mathrm{ab}}$ *is a dense subgroup, and*

$$\mathrm{Ker}(r_K) \;=\; \bigcap_{m\geq 1} (K^{\times})^m.$$

In particular, r_K *induces an injective homomorphism* $K^{\times,\mathrm{div}} \hookrightarrow G_K^{\mathrm{ab}}$ *with a dense image.*

(b) For each finite abelian extension $K \subseteq L$ $(\subseteq K^{\mathrm{sep}})$*,* r_K *induces an isomorphism*

$$K^{\times}/\mathrm{Im}(N_{L/K}) \;\xrightarrow{\sim}\; \mathrm{Gal}(L/K).$$

(c) r_K *induces an injection* $\mathcal{O}_K^{\times,\mathrm{div}} \hookrightarrow \mathrm{Im}(I_K \subseteq G_K \twoheadrightarrow G_K^{\mathrm{ab}})$ *with a dense image. [Note that if* K *is complete, then* k *is finite if and only if this injection is bijective.]*

(d) Let $\pi_K \in \mathcal{O}_K^{\triangleright,\mathrm{div}}$ *be a prime element. Then* $r_K(\pi_K)$ *is a lifting* $\in G_K^{\mathrm{ab}}$ *of a topological generator* $\in G_k$*.*

Remark 4.3.1. Note that, in [3], Chapter V, §3, Corollary 1, the equality in the display of condition (a) [in the complete case] is proved by applying the existence theorem in local class field theory for complete discrete valuation fields with quasi-finite residues. On the other hand, we note that this equality may be obtained by applying Proposition 4.4 [i.e., a corollary of Theorem 3.5] below.

Remark 4.3.2. Write $\mathfrak{Primes}$ for the set of prime numbers. In light of Remark 4.1.2, we have a family

$$\{r_K^l : K^{\times} \;\longrightarrow\; (G_K^l)^{\mathrm{ab}}\}_{l\in\mathfrak{Primes}}$$

of the reciprocity maps [cf. Theorem 2.13]. This family induces a homomorphism

$$K^{\times} \;\longrightarrow\; G_K^{\mathrm{ab}}.$$

Then this homomorphism coincides with the reciprocity map r_K that appears in Theorem 4.3. Indeed, this follows immediately from the construction of the reciprocity map via Artin-Tate's class formation [cf. [31], Chapter XI, §3, diagram (1)].

Remark 4.3.3. One easily verifies that Lemma 2.5, (iii), may be generalized to a similar result for arbitrary finite abelian extensions. Thus, in light of Lemma 2.1, together with this generalization, one may observe that condition (b) follows from condition (b) for the completion of K. On the other hand, conditions (c), (d) follow immediately from conditions (a), (b), together with conditions (c), (d) for the completion of K.

Remark 4.3.4. Let n be a positive integer coprime to $\mathrm{char}(K)$. Note that, in light of Kummer theory and Ponrjyagin duality, the cup product

$$H^1(G_K, \mathbb{Z}/n\mathbb{Z}) \;\times\; H^1(G_K, \mu_n(K^{\mathrm{sep}})) \;\longrightarrow\; H^2(G_K, \mu_n(K^{\mathrm{sep}})) \;\xrightarrow{\sim}\; \mathbb{Z}/n\mathbb{Z}$$

[cf. [31], Chapter XIII, §3, Proposition 6] determines a homomorphism

$$K^{\times}/(K^{\times})^n \;\longrightarrow\; G_K^{\mathrm{ab}}/nG_K^{\mathrm{ab}}.$$

Then this homomorphism coincides with the homomorphism induced by the reciprocity map r_K. Indeed, if we recall the construction of the reciprocity map based on Tate-Nakayama's theorem, then this follows from standard homological algebra [cf. [13], Proposition 0.14; the discussion in [13], Lemma 1.7].

Proposition 4.4. *Let n be a positive integer. Then the reciprocity map $r_K : K^\times \to G_K^{\mathrm{ab}}$ induces an injective homomorphism*

$$K^\times/(K^\times)^n \hookrightarrow G_K^{\mathrm{ab}}/nG_K^{\mathrm{ab}}.$$

Suppose, moreover, that one of the following conditions holds:

- *n is coprime to p.*
- *K is complete, and k is finite.*

Then this injective homomorphism is bijective.

Proof. First, it follows immediately from the various definitions involved that we may assume without loss of generality that n is a power of some prime number l. Then, in light of Remark 4.3.2, we conclude from Theorem 3.5, (i) (respectively, Theorem 3.5, (ii)) that the natural homomorphism $K^\times/(K^\times)^n \to G_K^{\mathrm{ab}}/nG_K^{\mathrm{ab}}$ is injective (respectively, bijective in the case where $l \neq p$). On the other hand, in the case where K is complete, and k is finite, since r_K is continuous [relative to the discrete valuation topology on $K^\times$ and the profinite topology on G_K^{ab}], and $\mathrm{Im}(r_K) \subseteq G_K^{\mathrm{ab}}$ is a dense subgroup [cf. the former part of condition (a) of Theorem 4.3; condition (b) of Theorem 4.3; [31], Chapter XIV, §6, Theorem 1], the surjectivity of the natural homomorphism $K^\times/(K^\times)^n \to G_K^{\mathrm{ab}}/nG_K^{\mathrm{ab}}$ follows immediately from the fact that $K^\times/(K^\times)^n$ is compact, and $G_K^{\mathrm{ab}}/nG_K^{\mathrm{ab}}$ is Hausdorff. This completes the proof of Proposition 4.4. □

Remark 4.4.1. Suppose that k is infinite. Then the injection

$$K^\times/(K^\times)^p \hookrightarrow G_K^{\mathrm{ab}}/pG_K^{\mathrm{ab}}$$

is not surjective. Indeed:

(i) Suppose that $\mathrm{char}(K) = 0$. Since k is infinite, it holds that $K^\times/(K^\times)^p$ is an infinite dimensional $\mathbb{F}_p$-vector space. In particular, the cardinality of $K^\times/(K^\times)^p$ is different from the cardinality of $\mathrm{Hom}(K^\times/(K^\times)^p, \mathbb{F}_p)$. On the other hand, in the case where $\zeta_p \in K^\times$, it follows immediately from Pontryagin duality, together with Kummer theory, that there exists an isomorphism

$$G_K^{\mathrm{ab}}/pG_K^{\mathrm{ab}} \xrightarrow{\sim} \mathrm{Hom}(K^\times/(K^\times)^p, \mathbb{F}_p).$$

In particular, the above injection is not surjective in the case where $\zeta_p \in K^\times$. Write $F \overset{\mathrm{def}}{=} K(\zeta_p)$. Then the general case follows immediately from the observations:

- The cardinality of $K^\times/(K^\times)^p$ coincides with the cardinality of $F^\times/(F^\times)^p$.
- The cardinality of $G_K^{\mathrm{ab}}/pG_K^{\mathrm{ab}}$ coincides with the cardinality of $G_F^{\mathrm{ab}}/pG_F^{\mathrm{ab}}$.

(ii) Suppose that $\mathrm{char}(K) = p$. Write $\{G^i \subseteq G_K^{\mathrm{ab}}\}_{i\geq 1}$ for the upper ramification filtration. Then it follows immediately from [31], Chapter XV, Corollary 3 [of Theorem 1], that the above injection induces an injection

$$(k \xrightarrow{\sim})\ 1 + \mathfrak{m}/(1 + \mathfrak{m}^2) \hookrightarrow G^1/G^2.$$

On the other hand, it follows immediately from Artin-Schreier theory that

$$\mathrm{Hom}(G^1/G^2, \mathbb{F}_p) \xrightarrow{\sim} \mathrm{Ker}(\mathrm{Hom}(G^1, \mathbb{F}_p) \twoheadrightarrow \mathrm{Hom}(G^2, \mathbb{F}_p)) \xrightarrow{\sim} k,$$

hence from Pontryagin duality that

$$G^1/G^2 \xrightarrow{\sim} \mathrm{Hom}(k, \mathbb{F}_p).$$

Thus, we conclude from a similar argument [of cardinality] to the argument applied in (i) that the above injection is not surjective.

Remark 4.4.2. In the notation of Proposition 4.4, suppose that $n = p$, and $\mathrm{char}(K) = 0$. Then the injectivity of the natural homomorphism $K^\times/(K^\times)^n \to G_K^{\mathrm{ab}}/nG_K^{\mathrm{ab}}$ was proved by Litvak [cf. [10], Proposition].

Here, we record the following result that corresponds to Proposition 3.6, (iii). On the other hand, this result may not be applied in the remainder of the present paper.

Proposition 4.5. *Let l be a prime number. Then it holds that $\mathrm{scd}_l(G_K) = 2$ or $\mathrm{scd}_l(G_K) = 3$. Moreover, $\mathrm{scd}_l(G_K) = 2$ if and only if k is stably μ_{l^∞}-finite. In particular, it holds that $\mathrm{scd}_p(G_K) = 2$.*

Proof. First, by replacing K by the completion of K, we may assume without loss of generality that K is complete. Next, in the case where $\mathrm{char}(K) = p$, and $l = p$, it holds that $\mathrm{cd}_p(G_K) = 1$, hence that $\mathrm{scd}_p(G_K) = 2$. Thus, we may assume without loss of generality that $l \neq \mathrm{char}(K)$. Then since k is quasi-finite, it follows immediately from [32], Chapter II, §4, Proposition 12, that $\mathrm{cd}_l(G_K) = 2$, hence that $\mathrm{scd}_l(G_K) = 2$ or $\mathrm{scd}_l(G_K) = 3$.

Next, write D_2 for the dualizing module of G_K, i.e.,

$$D_2 \stackrel{\mathrm{def}}{=} \varinjlim_{m\geq 1} \varinjlim_{K\subseteq L} \mathrm{Hom}(H^2(G_L, \mathbb{Z}/m\mathbb{Z}), \mathbb{Q}/\mathbb{Z}),$$

where $K \subseteq L$ ($\subseteq K^{\mathrm{sep}}$) ranges over the finite separable extensions; m ranges over the positive integers coprime to $\mathrm{char}(K)$; the transition maps of $\varinjlim_{K\subseteq L}$ are induced by the corestriction maps. Then, in light of [31], Chapter XIII, §3, Proposition 6; [28], Theorem 3.4.4, it holds that, for each finite separable extension $K \subseteq L$ ($\subseteq K^{\mathrm{sep}}$) and positive integer n,

$$\mathbb{Z}/n\mathbb{Z} \stackrel{\sim}{\to} \mathrm{Hom}(H^2(G_L, \mu_n(K^{\mathrm{sep}})), \mathbb{Q}/\mathbb{Z}) \stackrel{\sim}{\to} \mathrm{Hom}_{G_L}(\mu_n(K^{\mathrm{sep}}), D_2).$$

In particular, we obtain a canonical isomorphism of G_K-modules

$$\mathbb{Z}/n\mathbb{Z} \stackrel{\sim}{\to} \mathrm{Hom}(\mu_n(K^{\mathrm{sep}}), D_2[n]).$$

Observe that the homomorphism $f_n : \mu_n(K^{\mathrm{sep}}) \to D_2[n]$ that corresponds to $1 \in \mathbb{Z}/n\mathbb{Z}$ via the above isomorphism is bijective. Thus, by varying n, we conclude that $D_2 \stackrel{\sim}{\to} \mu(K^{\mathrm{sep}})$, hence from [28], Corollary 3.4.5, that $\mathrm{scd}_l(G_K) = 2$ if and only if K is stably μ_{l^∞}-finite. Finally, by applying Lemma 1.11, we obtain the desired conclusion. This completes the proof of Proposition 4.5. □

Proposition 4.6. *Suppose that, for each prime number l, it holds that k is stably μ_{l^∞}-finite. Then the reciprocity map $r_K : K^\times \to G_K^{\mathrm{ab}}$ induces an isomorphism*

$$\mu(K) \stackrel{\sim}{\to} (G_K^{\mathrm{ab}})_{\mathrm{tor}}.$$

Proof. In light of Remarks 4.1.2; 4.3.2, Proposition 4.6 follows immediately from Proposition 3.7. □

Theorem 4.7. *In the notation of Definition 1.8, suppose that k is quasi-finite and μ-finite. Then the following hold:*

(i) Let G be a topological group isomorphic to the absolute Galois group of a Henselian discrete valuation field with a quasi-finite and μ-finite residue field of positive characteristic. Then there exists a functorial group-theoretic algorithm

$$G \rightsquigarrow (p(G),\ P(G) \subseteq I(G) \subseteq G,\ F(G) \in G/I(G),\ f_0(G),\ \mathrm{char}(G))$$

for constructing — from the topological group G — a data $(p(G),\ P(G) \subseteq I(G) \subseteq G,\ F(G) \in G/I(G), \mathrm{char}(G))$ consisting of a prime number $p(G)$, normal subgroups $P(G) \subseteq I(G) \subseteq G$ of G, an element $F(G) \in G/I(G)$, a positive integer $f_0(G)$, and a nonnegative integer $\mathrm{char}(G)$ that satisfies the following condition: if one applies this functorial group-theoretic algorithm to $G = G_K$, then

$$p(G) \ = \ p, \quad P(G) \ = \ P_K, \quad I(G) \ = \ I_K, \quad F(G) \ = \ F, \quad f_0(G) \ = \ f_0, \quad \mathrm{char}(G) \ = \ \mathrm{char}(K)$$

— where $f_0 = [k_0 : k_{\mathrm{prm}}]$ [cf. the proof of Proposition 4.2, (i)].

(ii) Let G be a topological group isomorphic to G_K. Write

$$\hat{r}_K : \widehat{K}^{\times} \ \hookrightarrow \ G_K^{\mathrm{ab}}$$

for the injective homomorphism [cf. Proposition 4.4] induced by the reciprocity map $r_K : K^{\times} \to G_K^{\mathrm{ab}}$. Suppose that if $\mathrm{char}(K) = p$, then K is complete, and k is finite. Then there exists a functorial group-theoretic algorithm

$$G \quad \rightsquigarrow \quad (\mu(G) \ \subseteq \ \widehat{\mathcal{O}}^{\times}(G) \ \subseteq \ \hat{\mathcal{O}}^{\rhd}(G) \ \subseteq \ \hat{K}^{\times}(G))$$

for constructing — from the topological group G — a data $(\mu(G) \ \subseteq \ \widehat{\mathcal{O}}^{\times}(G) \ \subseteq \ \hat{\mathcal{O}}^{\rhd}(G) \ \subseteq \ \hat{K}^{\times}(G))$ consisting of submonoids of G^{ab} that satisfies the following condition: if one applies this functorial group-theoretic algorithm to $G = G_K$, then

$$\mu(G) \ = \ \hat{r}_K(\mu(K)), \quad \widehat{\mathcal{O}}^{\times}(G) \ = \ \hat{r}_K(\widehat{\mathcal{O}}_K^{\times}), \quad \hat{\mathcal{O}}^{\rhd}(G) \ = \ \hat{r}_K(\hat{\mathcal{O}}_K^{\rhd}), \quad \hat{K}^{\times}(G) \ = \ \hat{r}_K(\hat{K}^{\times})$$

[cf. the discussion at the beginning of the proof of Proposition 3.7].

(iii) Let G be a topological group isomorphic to G_K. Suppose that if $\mathrm{char}(K) = p$, then K is complete, and k is finite. Then, by applying the group-theoretic algorithm given in (ii) to every normal open subgroup $H \subseteq G$, one constructs a data $(\mu_s(G) \ \subseteq \ \widehat{\mathcal{O}}_s^{\times}(G) \ \subseteq \ \hat{\mathcal{O}}_s^{\rhd}(G) \ \subseteq \ \hat{K}_s^{\times}(G))$ consisting of the monoids [equipped with natural actions of G]

$$\mu_s(G) \ \stackrel{\mathrm{def}}{=} \ \varinjlim_{H \subseteq G} \ \mu(H), \quad \widehat{\mathcal{O}}_s^{\times}(G) \ \stackrel{\mathrm{def}}{=} \ \varinjlim_{H \subseteq G} \ \widehat{\mathcal{O}}^{\times}(H),$$

$$\hat{\mathcal{O}}_s^{\rhd}(G) \ \stackrel{\mathrm{def}}{=} \ \varinjlim_{H \subseteq G} \ \hat{\mathcal{O}}^{\rhd}(H), \quad \hat{K}_s^{\times}(G) \ \stackrel{\mathrm{def}}{=} \ \varinjlim_{H \subseteq G} \ \hat{K}^{\times}(H)$$

— where the transition maps are induced by the transfers — that satisfies the following condition: if one applies this functorial group-theoretic algorithm to $G = G_K$, then the [completed] reciprocity maps $\{\hat{r}_L\}_{K \subseteq L}$ — where $K \subseteq L$ $(\subseteq K^{\mathrm{sep}})$ ranges over the finite Galois extensions — induce G-equivariant isomorphisms

$$\mu(K^{\mathrm{sep}}) \ \stackrel{\sim}{\to} \ \mu_s(G), \quad \widehat{\mathcal{O}}_{K^{\mathrm{sep}}}^{\times} \ \stackrel{\sim}{\to} \ \widehat{\mathcal{O}}_s^{\times}(G), \quad \hat{\mathcal{O}}_{K^{\mathrm{sep}}}^{\rhd} \ \stackrel{\sim}{\to} \ \hat{\mathcal{O}}_s^{\rhd}(G), \quad (K^{\hat{\mathrm{sep}}})^{\times} \ \stackrel{\sim}{\to} \ \hat{K}_s^{\times}(G).$$

In the remainder of the present paper, we shall refer to the G-equivariant isomorphism

$$\varprojlim_{n \geq 1} \ \mu_n(K^{\mathrm{sep}}) \ \stackrel{\sim}{\to} \ \varprojlim_{n \geq 1} \ \mu_s(G)[n]$$

induced by the first isomorphism as the cyclotomic rigidity isomorphism associated to K.

Proof. First, we verify assertion (i). To verify assertion (i), it suffices to give a group-theoretic characterization of

$$p, \ P_K \ \subseteq \ I_K \ \subseteq \ G_K, \ F \ \in \ G_k, \ f_0, \ \mathrm{char}(K)$$

in terms of the underlying topological group structure of G_K. Note that, in light of Propositions 1.6; 1.7, we have already given group-theoretic characterizations of p and $P_K \subseteq I_K \subseteq G_K$. Next, it follows immediately from Proposition 4.2, (i), (ii), that the topological generator $F \in G_K/I_K = G_k$ may be characterized as a unique topological generator that acts on I_K/P_K ($\xrightarrow{\sim} \widehat{\mathbb{Z}}^{(p)'}(1)$) by multiplication by some positive integer power of p. Moreover, the positive integer f_0 may be characterized as the positive integer m such that F acts on I_K/P_K by multiplication by p^m. Next, observe that $\mathrm{char}(K) = 0$ (respectively, $\mathrm{char}(K) = p$) if and only if there exists an (respectively, no) open subgroup $H \subseteq G_K$ such that the p-torsion subgroup $H^{\mathrm{ab}}[p] \subseteq H^{\mathrm{ab}}$ is nontrivial [cf. Proposition 4.6]. This completes the proof of assertion (i).

Next, we verify assertion (ii). To verify assertion (ii), it suffices to give a group-theoretic characterization of monoids

$$\mu(K) \ \subseteq \ \widehat{\mathcal{O}}_K^{\times} \ \subseteq \ \hat{\mathcal{O}}_K^{\triangleright} \ \subseteq \ \hat{K}^{\times}$$

in terms of the underlying topological group structure of G_K that reflects the condition concerning $\hat{r}_K$. Let $K \subseteq L$ ($\subseteq K^{\mathrm{sep}}$) be a finite Galois extension. Observe that since k is quasi-finite and μ-finite, it follows immediately from Proposition 4.6, together with Lemma 1.11, that $\mu(L)$ may be characterized as the torsion subgroup of G_L^{ab} via the reciprocity map. Moreover, by applying the theory of transfer, we obtain a group-theoretic characterization of $\mu(K^{\mathrm{sep}})$.

Note that, if $\mathrm{char}(K) = p$, then since K is complete, and k is finite, the injection $\hat{r}_K$ is an isomorphism. On the other hand, if $\mathrm{char}(K) = 0$, then for each positive integer m, we have the cup-product

$$H^1(G_K, \mu_m(K^{\mathrm{sep}})) \times H^1(G_K, \mathbb{Z}/m\mathbb{Z}) \ \longrightarrow \ H^2(G_K, \mu_m(K^{\mathrm{sep}})).$$

In particular, in light of Kummer theory, Pontryagin duality, and [31], Chapter XIII, §3, Proposition 6, we obtain a group-theoretic characterization of the image of the injection

$$K^{\times}/(K^{\times})^m \ \hookrightarrow \ G_K^{\mathrm{ab}}/mG_K^{\mathrm{ab}}$$

[cf. Remark 4.3.4; Proposition 4.4]. Then, by varying m, we also obtain a group-theoretic characterization of $\mathrm{Im}(\hat{r}_K)$. Thus, it follows immediately from Theorem 4.3 that

- $\widehat{\mathcal{O}}_K^{\times}$ may be characterized as $\mathrm{Im}(I_K \subseteq G_K \twoheadrightarrow G_K^{\mathrm{ab}}) \cap \mathrm{Im}(\hat{r}_K)$ [cf. (i)];
- $\hat{\mathcal{O}}_K^{\triangleright}$ may be characterized as the submonoid $\subseteq G_K^{\mathrm{ab}}$ generated by $\mathrm{Im}(I_K \subseteq G_K \twoheadrightarrow G_K^{\mathrm{ab}}) \cap \mathrm{Im}(\hat{r}_K)$ and F [cf. (i)];
- $\hat{K}^{\times}$ may be characterized as the subgroup $\subseteq G_K^{\mathrm{ab}}$ generated by $\mathrm{Im}(I_K \subseteq G_K \twoheadrightarrow G_K^{\mathrm{ab}}) \cap \mathrm{Im}(\hat{r}_K)$ and F.

This completes the proof of assertion (ii). Assertion (iii) follows immediately from assertion (ii), together with its proof. This completes the proof of Theorem 4.7. □

Remark 4.7.1. At the time of writing of the present paper, the authors do not know

> *whether or not the assumption that K is complete, and k is finite if* $\mathrm{char}(K) = p$ *that appears in Theorem 4.7, (ii) [hence also (iii)], may be dropped.*

5 Mono-anabelian group/monoid-theoretic reconstruction algorithms for cyclotomic rigidity isomorphisms

Let p be a prime number; K a Henselian discrete valuation field of residue characteristic p. Throughout the present section, we maintain the notation of §1 [especially, Definition 1.8], and suppose that

the residue field k of K is quasi-finite and μ-finite

[cf. Definitions 1.9, (i); 4.1].

In the present section, from the viewpoint of mono-anabelian geometry, we establish group/monoid-theoretic algorithms/procedures to construct certain orbits of the cyclotomic rigidity isomorphism associated to K [cf. Theorems 4.7, (iii); 5.9].

Definition 5.1. Let G be a topological group isomorphic to G_K. Suppose that there exist functorial group-theoretic algorithms

$$G \quad \rightsquigarrow \quad (p = p(G),\ P(G) \subseteq I(G) \subseteq G,\ F(G) \in G/I(G),\ \mathrm{char}(K) = \mathrm{char}(G)),$$

$$G \quad \rightsquigarrow \quad (G \curvearrowright \mu_s(G) \subseteq \mathcal{O}_s^{\times}(G) \subseteq \mathcal{O}_s^{\triangleright}(G) \subseteq K_s^{\times}(G))$$

satisfying conditions that appear in Theorem 4.7, (i), (iii). [Note that, if we assume that K is complete, and k is finite in the case where $\mathrm{char}(K) = p$, then the existence of such algorithms follows from Theorem 4.7, (i), (iii) — cf. Remark 4.7.1.] Fix such functorial group-theoretic algorithms. Write

$$\widehat{\mathbb{Z}}(G) \stackrel{\mathrm{def}}{=} \begin{cases} \widehat{\mathbb{Z}} & \text{if } \mathrm{char}(K) = 0, \\ \widehat{\mathbb{Z}}^{(p)'} & \text{if } \mathrm{char}(K) = p. \end{cases}$$

Let M be a multiplicative monoid such that

- M admits an action of G,
- $(G \curvearrowright M)$ is isomorphic to $(G_K \curvearrowright \widehat{\mathcal{O}}^{\times}_{K^{\mathrm{sep}}})$ (respectively, $(G_K \curvearrowright \hat{\mathcal{O}}^{\triangleright}_{K^{\mathrm{sep}}})$; $(G_K \curvearrowright (\hat{K^{\mathrm{sep}}})^{\times})$).

Write

$$\Gamma(G) \stackrel{\mathrm{def}}{=} \widehat{\mathbb{Z}}(G)^{\times} \stackrel{\mathrm{def}}{=} \mathrm{Aut}(\widehat{\mathbb{Z}}(G)) \text{ (respectively, } \{1\}; \{\pm 1\});$$

$$\Lambda(M) \stackrel{\mathrm{def}}{=} \varprojlim_{n \geq 1} M_{\mathrm{tor}}[n]; \quad \Lambda(G) \stackrel{\mathrm{def}}{=} \varprojlim_{n \geq 1} \mu_s(G)[n];$$

$$G^{\mathrm{ur}} \stackrel{\mathrm{def}}{=} G/I(G); \quad M^{\mathrm{ur}} \stackrel{\mathrm{def}}{=} M^{I(G)} \ (\subseteq M);$$

$$(M^{\mathrm{ur}})^{*} \stackrel{\mathrm{def}}{=} M^{\mathrm{ur}} \text{ (respectively, } (M^{\mathrm{ur}})^{\times};\ \mathcal{O}^{\times}(M^{\mathrm{ur}})),$$

where, if $(G \curvearrowright M)$ is isomorphic to $(G_K \curvearrowright (\hat{K^{\mathrm{sep}}})^{\times})$, then $\mathcal{O}^{\times}(M^{\mathrm{ur}})$ denotes the kernel of the [unique up to composition with an automorphism of $\mathbb{Z}$] surjection $M^{\mathrm{ur}} \twoheadrightarrow \mathbb{Z}$ [cf. Lemma 5.2 below].

Lemma 5.2. *In the notation of Definition 5.1, suppose that $(G \curvearrowright M)$ is isomorphic to $(G_K \curvearrowright (\hat{K^{\mathrm{sep}}})^{\times})$. Then there exists a surjection $M^{\mathrm{ur}} \twoheadrightarrow \mathbb{Z}$ unique up to composition with an automorphism of $\mathbb{Z}$.*

Proof. First, it follows immediately from Lemma 1.12, together with the various definitions involved, that

$$M^{\mathrm{ur}} \cong \bigcup_{K \subseteq L} \hat{L}^{\times} \cong \mathbb{Z} \times \bigcup_{K \subseteq L} \widehat{\mathcal{O}}_L^{\times},$$

where $K \subseteq L$ ($\subseteq K^{\mathrm{sep}}$) ranges over the finite unramified extensions. Thus, to verify Lemma 5.2, it suffices to prove that, for each Henselian discrete valuation field L whose residue field is perfect and of characteristic p, every homomorphism $v : \widehat{\mathcal{O}}_L^{\times} \to \mathbb{Z}$ is trivial.

Next, we make the following *observations*:

- $k_L^{\times}$ is uniquely p-divisible [so $\varprojlim_{n\geq 1} k_L^{\times}/(k_L^{\times})^n$ is also uniquely p-divisible].
- Let m be a positive integer coprime to p. Then $1+\mathfrak{m}_L$ is uniquely m-divisible [so $\varprojlim_{n\geq 1} 1+\mathfrak{m}_L/(1+\mathfrak{m}_L)^n$ is also uniquely m-divisible].
- There exists a natural exact sequence

$$1 \longrightarrow \varprojlim_{n\geq 1} 1+\mathfrak{m}_L/(1+\mathfrak{m}_L)^n \longrightarrow \widehat{\mathcal{O}}_L^{\times} \longrightarrow \varprojlim_{n\geq 1} k_L^{\times}/(k_L^{\times})^n \longrightarrow 1.$$

Note that $\mathbb{Z}$ admits no nontrivial l-divisible element for any prime number l. Then it follows immediately from the second observation that

$$v(\varprojlim_{n\geq 1} 1+\mathfrak{m}_L/(1+\mathfrak{m}_L)^n) = \{0\}.$$

Thus, we conclude from the first and third observations that $v(\widehat{\mathcal{O}}_L^{\times})$ is a p-divisible subgroup of $\mathbb{Z}$, hence that $v(\widehat{\mathcal{O}}_L^{\times}) = \{0\}$. This completes the proof of Lemma 5.2. $\square$

In the remainder of the present section, to establish group/monoid-theoretic algorithms mentioned above, we discuss various cohomology groups. These cohomology groups are well-defined due to Lemma 1.12.

Lemma 5.3. *In the notation of Definition 5.1, the natural G-equivariant injection $M_{\mathrm{tor}} \hookrightarrow M^{\mathrm{gp}}$ induces a natural isomorphism*

$$\begin{cases} H^2(G, M_{\mathrm{tor}}) \xrightarrow{\sim} H^2(G, M^{\mathrm{gp}}) & \text{if } \mathrm{char}(K) = 0, \\ H^2(G, M_{\mathrm{tor}}) \xrightarrow{\sim} H^2(G, M^{\mathrm{gp}})[(p^{\infty})'] & \text{if } \mathrm{char}(K) = p. \end{cases}$$

Proof. First, it follows immediately from Proposition 4.6 that M_{tor} is isomorphic to $\mu(K^{\mathrm{sep}})$. On the other hand, we note that M^{gp} is isomorphic to $\widehat{\mathcal{O}}_{K^{\mathrm{sep}}}^{\times}$ or $(\hat{K^{\mathrm{sep}}})^{\times}$. In particular, it follows immediately from Remark 1.8.4 that

- if $\mathrm{char}(K) = 0$, then $M^{\mathrm{gp}}/M_{\mathrm{tor}}$ admits the structure of a $\mathbb{Q}$-vector space;
- if $\mathrm{char}(K) = p$, then $M^{\mathrm{gp}}/M_{\mathrm{tor}}$ admits the structure of a $\mathbb{Z}_{(p)}$-module, where $\mathbb{Z}_{(p)}$ denotes the localization of $\mathbb{Z}$ at the prime ideal (p).

Thus, we conclude that, for each positive integer i,

$$\begin{cases} H^i(G, M^{\mathrm{gp}}/M_{\mathrm{tor}}) = \{0\} & \text{if } \mathrm{char}(K) = 0, \\ H^i(G, M^{\mathrm{gp}}/M_{\mathrm{tor}})[(p^{\infty})'] = \{0\} & \text{if } \mathrm{char}(K) = p. \end{cases}$$

Then, by considering the cohomology long exact sequence associated to the exact sequence of G-modules

$$0 \longrightarrow M_{\mathrm{tor}} \longrightarrow M^{\mathrm{gp}} \longrightarrow M^{\mathrm{gp}}/M_{\mathrm{tor}} \longrightarrow 0,$$

we obtain the desired conclusion. This completes the proof of Lemma 5.3. $\square$

Lemma 5.4. *Let M be an abelian group. Write*

$$\widehat{M} \stackrel{\text{def}}{=} \varprojlim_{m \geq 1} M/mM,$$

where m ranges over the positive integers. Then the following hold:

(i) Let n be a positive integer. Then the natural homomorphism $M/nM \to \widehat{M}/n\widehat{M}$ is an isomorphism.

(ii) Suppose that M is a reduced abelian group, and that, for each prime number l, the natural injection $M \hookrightarrow \widehat{M}$ induces an isomorphism $M[l] \xrightarrow{\sim} \widehat{M}[l]$. Then the cokernel $\widehat{M}/M$ of this injection is uniquely divisible. In particular, for each finite separable extension $K \subseteq L$ ($\subseteq K^{\text{sep}}$), it holds that $\widehat{\mathcal{O}}_L^{\times}/\mathcal{O}_L^{\times,\text{div}}$ is uniquely divisible [cf. Proposition 4.6].

Proof. First, we verify assertion (i). In light of the Mittag-Leffler condition, by taking the inverse limit of the natural exact sequences

$$0 \longrightarrow (M[n] + mM)/nmM \longrightarrow M/nmM \longrightarrow nM/nmM \longrightarrow 0,$$

we observe that $n\widehat{M} = \varprojlim_{m\geq 1} nM/nmM$. This observation immediately implies that the natural homomorphism $M/nM \to \widehat{M}/n\widehat{M}$ is an isomorphism. This completes the proof of assertion (i).

Next, we verify assertion (ii). Let l be a prime number. Then it suffices to verify that $\widehat{M}/M$ is uniquely l-divisible. Note that it follows from assertion (i) that the natural homomorphism $M/lM \to \widehat{M}/l\widehat{M}$ is an isomorphism. Thus, since the natural injection $M \hookrightarrow \widehat{M}$ induces an isomorphism $M[l] \xrightarrow{\sim} \widehat{M}[l]$, we conclude from Lemma 3.4, (ii), that $\widehat{M}/M$ is uniquely l-divisible. This completes the proof of assertion (ii), hence of Lemma 5.4. ☐

Lemma 5.5. *In the notation of Definition 5.1, suppose that*

$$M \cong \hat{\mathcal{O}}^{\triangleright}_{K^{\text{sep}}}, \text{ or } M \cong (K^{\hat{\text{sep}}})^{\times}.$$

Then the Leray-Serre spectral sequence associated to the exact sequence

$$1 \longrightarrow I(G) \longrightarrow G \longrightarrow G^{\text{ur}} \longrightarrow 1$$

induces a natural isomorphism

$$H^2(G^{\text{ur}}, (M^{\text{ur}})^{\text{gp}}) \xrightarrow{\sim} H^2(G, M^{\text{gp}}).$$

Proof. Write

$$D \stackrel{\text{def}}{=} \bigcup_{K\subseteq L} \bigcap_{n\geq 1} (\mathcal{O}_L^{\times})^n \subseteq \mathcal{O}^{\times}_{K^{\text{sep}}},$$

where $K \subseteq L$ ($\subseteq K^{\text{sep}}$) ranges over the finite Galois extensions. Note that since the inverse limit of nonempty finite sets is nonempty, it holds that, for each $m \geq 1$, the m-th power map on $\cap_{n\geq 1}(\mathcal{O}_L^{\times})^n$ is surjective. Moreover, since L contains only finitely many roots of unity, this m-th power map is bijective. Thus, we observe that D admits the structure of a $\mathbb{Q}$-vector space. Then since the natural action of G_K on D is continuous, it follows immediately that, for each $j \geq 1$,

$$H^j(I_K, D) = \{0\}.$$

In particular, for each $i = 1, 2$, the cohomology long exact sequence associated to the exact sequence

$$1 \longrightarrow D \longrightarrow (K^{\mathrm{sep}})^{\times} \longrightarrow (K^{\mathrm{sep}})^{\times,\mathrm{div}} \longrightarrow 1$$

induces a natural isomorphism

$$H^i(I_K, (K^{\mathrm{sep}})^{\times}) \ \stackrel{\sim}{\to}\ H^i(I_K, (K^{\mathrm{sep}})^{\times,\mathrm{div}}).$$

Thus, it follows from Hilbert's theorem 90 and the well-known calculation of the Brauer groups that, for each $i = 1, 2$,

$$H^i(I_K, (K^{\mathrm{sep}})^{\times,\mathrm{div}}) \ = \ \{0\}.$$

On the other hand, it follows immediately from Lemma 1.12 that

$$(\hat{K^{\mathrm{sep}}})^{\times} \ = \ \bigcup_{K \subseteq L} \hat{L}^{\times},$$

where $K \subseteq L$ $(\subseteq K^{\mathrm{sep}})$ ranges over the finite extensions. Then since $\hat{L}^{\times}/L^{\times,\mathrm{div}} = \widehat{\mathcal{O}}_L^{\times}/\mathcal{O}_L^{\times,\mathrm{div}}$, by applying Lemma 5.4, (ii), we observe that $(\hat{K^{\mathrm{sep}}})^{\times}/(K^{\mathrm{sep}})^{\times,\mathrm{div}}$ admits the structure of a $\mathbb{Q}$-vector space. In particular, for each $i = 1, 2$,

$$H^i(I_K, (\hat{K^{\mathrm{sep}}})^{\times}) \ = \ \{0\}.$$

Note that $M^{\mathrm{gp}} \cong (\hat{K^{\mathrm{sep}}})^{\times}$. Then it holds that, for each $i = 1, 2$,

$$H^i(I(G), M^{\mathrm{gp}}) \ = \ \{0\}.$$

Thus, since $(M^{\mathrm{gp}})^{I(G)} = (M^{\mathrm{ur}})^{\mathrm{gp}}$ [cf. Lemma 1.12], it follows immediately from the Leray-Serre spectral sequence associated to the exact sequence

$$1 \longrightarrow I(G) \longrightarrow G \longrightarrow G^{\mathrm{ur}} \longrightarrow 1$$

that we have a natural isomorphism

$$H^2(G^{\mathrm{ur}}, (M^{\mathrm{ur}})^{\mathrm{gp}}) \ \stackrel{\sim}{\to}\ H^2(G, M^{\mathrm{gp}}).$$

This completes the proof of Lemma 5.5. $\square$

Lemma 5.6. *In the notation of Definition 5.1, the natural G^{ur}-equivariant surjection*

$$(M^{\mathrm{ur}})^{\mathrm{gp}} \ \twoheadrightarrow\ (M^{\mathrm{ur}})^{\mathrm{gp}}/(M^{\mathrm{ur}})^{*}$$

induces a natural isomorphism

$$H^2(G^{\mathrm{ur}}, (M^{\mathrm{ur}})^{\mathrm{gp}}) \ \stackrel{\sim}{\to}\ H^2(G^{\mathrm{ur}}, (M^{\mathrm{ur}})^{\mathrm{gp}}/(M^{\mathrm{ur}})^{*}).$$

Proof. We maintain the notation of the proof of Lemma 5.5. Write $K \subseteq K^{\mathrm{ur}}$ $(\subseteq K^{\mathrm{sep}})$ for the maximal unramified extension. Let us note that $D \subseteq K^{\mathrm{ur}}$. Then since D admits the structure of a $\mathbb{Q}$-vector space, it follows immediately that, for each $j \geq 1$,

$$H^j(G_K/I_K, D) \ = \ \{0\}.$$

In particular, for each $i \geq 1$, the cohomology long exact sequence associated to the exact sequence

$$1 \longrightarrow D \longrightarrow \mathcal{O}_{K^{\mathrm{ur}}}^{\times} \longrightarrow (\mathcal{O}_{K^{\mathrm{sep}}}^{\times,\mathrm{div}})^{I_K} \longrightarrow 1$$

induces a natural isomorphism

$$H^i(G_K/I_K, \mathcal{O}^\times_{K^{\mathrm{ur}}}) \ \stackrel{\sim}{\to}\ H^i(G_K/I_K, (\mathcal{O}^{\times,\mathrm{div}}_{K^{\mathrm{sep}}})^{I_K}).$$

Recall that, for each $i \geq 1$,

$$H^i(G_K/I_K, \mathcal{O}^\times_{K^{\mathrm{ur}}}) \ = \ \{0\}.$$

On the other hand, it follows immediately from Lemma 1.12 that

$$(\widehat{\mathcal{O}}^\times_{K^{\mathrm{sep}}})^{I_K} \ = \ \bigcup_{K \subseteq L} \widehat{\mathcal{O}}^\times_L,$$

where $K \subseteq L$ ($\subseteq K^{\mathrm{sep}}$) ranges over the finite unramified extensions. Then, by applying Lemma 5.4, (ii), we observe that $(\widehat{\mathcal{O}}^\times_{K^{\mathrm{sep}}})^{I_K}/\mathcal{O}^\times_{K^{\mathrm{ur}}}$ admits the structure of a $\mathbb{Q}$-vector space. In particular, for each $i \geq 2$,

$$H^i(G_K/I_K, (\widehat{\mathcal{O}}^\times_{K^{\mathrm{sep}}})^{I_K}) \ = \ \{0\}.$$

Note that $(M^{\mathrm{ur}})^* \cong (\widehat{\mathcal{O}}^\times_{K^{\mathrm{sep}}})^{I_K}$. Then it holds that, for each $i \geq 2$,

$$H^i(G^{\mathrm{ur}}, (M^{\mathrm{ur}})^*) \ = \ \{0\}.$$

Thus, by considering the cohomology long exact sequence associated to the exact sequence of G^{ur}-modules

$$1 \longrightarrow (M^{\mathrm{ur}})^* \longrightarrow (M^{\mathrm{ur}})^{\mathrm{gp}} \longrightarrow (M^{\mathrm{ur}})^{\mathrm{gp}}/(M^{\mathrm{ur}})^* \longrightarrow 1,$$

we have a natural isomorphism

$$H^2(G^{\mathrm{ur}}, (M^{\mathrm{ur}})^{\mathrm{gp}}) \ \stackrel{\sim}{\to}\ H^2(G^{\mathrm{ur}}, (M^{\mathrm{ur}})^{\mathrm{gp}}/(M^{\mathrm{ur}})^*).$$

This completes the proof of Lemma 5.6. □

Lemma 5.7. *In the notation of Definition 5.1, suppose that*

$$M \ \cong\ \hat{\mathcal{O}}^{\triangleright}_{K^{\mathrm{sep}}}, \text{ or } M \ \cong\ (\hat{K^{\mathrm{sep}}})^\times.$$

Then there exists a natural $\Gamma(G)$*-torsor consisting of isomorphisms*

$$H^2(G^{\mathrm{ur}}, (M^{\mathrm{ur}})^{\mathrm{gp}}/(M^{\mathrm{ur}})^*) \ \stackrel{\sim}{\to}\ \mathbb{Q}/\mathbb{Z}.$$

Proof. We have a natural isomorphism $G^{\mathrm{ur}} \stackrel{\sim}{\to} \widehat{\mathbb{Z}}$ that maps $F(G)$ to 1. Moreover, we have a natural $\Gamma(G)$-torsor consisting of isomorphisms $(M^{\mathrm{ur}})^{\mathrm{gp}}/(M^{\mathrm{ur}})^* \stackrel{\sim}{\to} \mathbb{Z}$. [Note that, in the case where $M \cong \hat{\mathcal{O}}^{\triangleright}_{K^{\mathrm{sep}}}$, we have a canonical generator $\in M^{\mathrm{ur}}/(M^{\mathrm{ur}})^*$.] On the other hand, by considering the cohomology long exact sequence associated to the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0,$$

we have a natural isomorphism

$$H^2(\widehat{\mathbb{Z}}, \mathbb{Z}) \ \stackrel{\sim}{\to}\ H^1(\widehat{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z}) \ \stackrel{\sim}{\to}\ \mathbb{Q}/\mathbb{Z}.$$

Thus, we obtain a natural $\Gamma(G)$-torsor consisting of isomorphisms

$$H^2(G^{\mathrm{ur}}, (M^{\mathrm{ur}})^{\mathrm{gp}}/(M^{\mathrm{ur}})^*) \ \stackrel{\sim}{\to}\ \mathbb{Q}/\mathbb{Z}.$$

This completes the proof of Lemma 5.7. □

Proposition 5.8. *In the notation of Definition 5.1, there exists a functorial group/monoid-theoretic algorithm*

$$(G \curvearrowright M) \quad \rightsquigarrow \quad H^2(G, \Lambda(M)) \ \overset{\sim}{\to} \ \widehat{\mathbb{Z}}(G)$$

for constructing — from $(G \curvearrowright M)$ *— a natural* $\Gamma(G)$*-torsor consisting of isomorphisms*

$$H^2(G, \Lambda(M)) \ \overset{\sim}{\to} \ \widehat{\mathbb{Z}}(G)$$

that satisfies the following condition: if one applies this group/monoid-theoretic algorithm to one of the following cases where

- $(G \ \curvearrowright \ M) \ = \ (G_K \ \curvearrowright \ \widehat{\mathcal{O}}^{\times}_{K^{\mathrm{sep}}})$;
- $(G \ \curvearrowright \ M) \ = \ (G_K \ \curvearrowright \ \hat{\mathcal{O}}^{\triangleright}_{K^{\mathrm{sep}}})$;
- $(G \ \curvearrowright \ M) \ = \ (G_K \ \curvearrowright \ (K^{\hat{\mathrm{sep}}})^{\times})$*,*

then the above $\Gamma(G)$*-torsor coincides with the natural* $\Gamma(G)$*-orbit of the natural isomorphism*

$$H^2(G_K, \varprojlim_{n \geq 1} \ \mu_n(K^{\mathrm{sep}})) \ \overset{\sim}{\to} \ \widehat{\mathbb{Z}}(K) \ \overset{\mathrm{def}}{=} \ \begin{cases} \widehat{\mathbb{Z}} & \text{if } \mathrm{char}(K) \ = \ 0, \\ \widehat{\mathbb{Z}}^{(p)'} & \text{if } \mathrm{char}(K) \ = \ p. \end{cases}$$

[cf. [28], Corollary 2.7.6; [31], Chapter XIII, §3, Proposition 6].

Proof. If $\Gamma(G) = \widehat{\mathbb{Z}}(G)^{\times}$, then we have nothing to prove. Thus, we may assume without loss of generality that M is isomorphic to $\hat{\mathcal{O}}^{\triangleright}_{K^{\mathrm{sep}}}$ or $(K^{\hat{\mathrm{sep}}})^{\times}$. Then it follows immediately from Lemmas 5.3, 5.5, 5.6, 5.7, that there exists a functorial group/monoid-theoretic algorithm whose input data is

$$(G \ \curvearrowright \ M)$$

and whose output data is a natural $\Gamma(G)$-torsor consisting of isomorphisms

$$\begin{cases} H^2(G, M_{\mathrm{tor}}) \ \overset{\sim}{\to} \ \mathbb{Q}/\mathbb{Z} & \text{if } \mathrm{char}(K) \ = \ 0, \\ H^2(G, M_{\mathrm{tor}}) \ \overset{\sim}{\to} \ \mathbb{Q}/\mathbb{Z}[(p^{\infty})'] & \text{if } \mathrm{char}(K) \ = \ p. \end{cases}$$

Thus, by applying the functor $\mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, -)$ to the above $\Gamma(G)$-torsor, we obtain a natural $\Gamma(G)$-torsor consisting of isomorphisms

$$H^2(G, \Lambda(M)) \ \overset{\sim}{\to} \ \widehat{\mathbb{Z}}(G).$$

Here, we observe that the above discussion to construct the $\Gamma(G)$-torsor $H^2(G, \Lambda(M)) \overset{\sim}{\to} \widehat{\mathbb{Z}}(G)$ is essentially similar to the proof of [31], Chapter XIII, §3, Proposition 6 [except the discussions concerning algebraically completed objects]. Thus, we obtain a suitable functorial group/monoid-theoretic algorithm, as desired. This completes the proof of Proposition 5.8. □

Theorem 5.9. *In the notation of Definition 5.1, there exists a functorial group/monoid-theoretic algorithm*

$$(G \curvearrowright M) \quad \rightsquigarrow \quad \Lambda(M) \ \overset{\sim}{\to} \ \Lambda(G)$$

for constructing — from $(G \curvearrowright M)$ *— a natural* $\Gamma(G)$*-torsor consisting of isomorphisms*

$$\Lambda(M) \ \overset{\sim}{\to} \ \Lambda(G)$$

that satisfies the following condition: if one applies this group/monoid-theoretic algorithm to one of the following cases where

- $(G \curvearrowright M) = (G_K \curvearrowright \widehat{\mathcal{O}}^{\times}_{K^{\mathrm{sep}}})$;
- $(G \curvearrowright M) = (G_K \curvearrowright \hat{\mathcal{O}}^{\triangleright}_{K^{\mathrm{sep}}})$;
- $(G \curvearrowright M) = (G_K \curvearrowright (\hat{K^{\mathrm{sep}}})^{\times})$),

then the above $\Gamma(G)$*-torsor coincides with the natural* $\Gamma(G)$*-orbit of the cyclotomic rigidity isomorphism associated to* K *[cf. Theorem 4.7, (iii)].*

Proof. Fix a functorial group/monoid-theoretic algorithm

$$(G \curvearrowright M) \quad \rightsquigarrow \quad H^2(G, \Lambda(M)) \overset{\sim}{\to} \widehat{\mathbb{Z}}(G)$$

as in Proposition 5.8. Then, by applying this functorial group/monoid-theoretic algorithm to $(G \curvearrowright M)$ and

$$(G \curvearrowright \widehat{\mathcal{O}}^{\times}_{s}(G)) \quad (\text{respectively, } (G \curvearrowright \hat{\mathcal{O}}^{\triangleright}_{s}(G));\ (G \curvearrowright \hat{K}^{\times}_{s}(G))),$$

we obtain $\Gamma(G)$-torsors consisting of isomorphisms

$$H^2(G, \Lambda(M)) \overset{\sim}{\to} \widehat{\mathbb{Z}}(G), \quad H^2(G, \Lambda(G)) \overset{\sim}{\to} \widehat{\mathbb{Z}}(G).$$

Thus, we obtain a unique $\Gamma(G)$-torsor consisting of isomorphisms

$$\Lambda(M) \overset{\sim}{\to} \Lambda(G)$$

that determines a $\Gamma(G)$-torsor consisting of isomorphisms

$$H^2(G, \Lambda(M)) \overset{\sim}{\to} H^2(G, \Lambda(G))$$

compatible with the above $\Gamma(G)$-torsors. Thus, in light of the condition that appears in Theorem 4.7, (iii), we obtain a suitable functorial group/monoid-theoretic algorithm, as desired. This completes the proof of Theorem 5.9. □

6 Galois-equivariant isomorphisms between various multiplicative groups associated to complete discrete valuation fields with quasi-finite residues

In the present section, as an application of the theory of cyclotomic rigidity discussed in the previous section, we determine the structure of the group of Galois-equivariant isomorphisms between various multiplicative groups that arise from complete discrete valuation fields with quasi-finite residues [cf. Corollary 6.7].

Definition 6.1. In the notation of Definition 5.1 [i.e., $(G \curvearrowright M)$ is isomorphic to $(G_K \curvearrowright \widehat{\mathcal{O}}^{\times}_{K^{\mathrm{sep}}})$ (respectively, $(G_K \curvearrowright \hat{\mathcal{O}}^{\triangleright}_{K^{\mathrm{sep}}})$; $(G_K \curvearrowright (\hat{K^{\mathrm{sep}}})^{\times})$)], we shall write

$$\mathrm{Aut}(G \curvearrowright M)$$

for the group of pairs of a continuous automorphism of G and an automorphism of M compatible with the action of G on M;

$$\psi_{\Gamma(G)} : \mathrm{Aut}(G \curvearrowright M) \longrightarrow \Gamma(G)$$

for the composite of the natural homomorphism

$$\mathrm{Aut}(G \curvearrowright M) \longrightarrow \mathrm{Aut}(M)$$

obtained by forgetting G with the natural homomorphism

$$\mathrm{Aut}(M) \longrightarrow \mathrm{Aut}(\Lambda(M)) = \Gamma(G) = \widehat{\mathbb{Z}}(G)^{\times}$$

(respectively, for the trivial homomorphism; for the natural homomorphism

$$\mathrm{Aut}(G \curvearrowright M) \longrightarrow \mathrm{Aut}(M^{\mathrm{ur}}/\mathcal{O}^{\times}(M^{\mathrm{ur}})) = \Gamma(G) = \{\pm 1\});$$

$$\psi_G : \mathrm{Aut}(G \curvearrowright M) \longrightarrow \mathrm{Aut}(G)$$

for the natural homomorphism obtained by forgetting M;

$$\psi \stackrel{\mathrm{def}}{=} \psi_{\Gamma(G)} \times \psi_G : \mathrm{Aut}(G \curvearrowright M) \longrightarrow \Gamma(G) \times \mathrm{Aut}(G).$$

Lemma 6.2. *In the notation of Definition 5.1, the following hold:*

(i) Let $H \subseteq G$ be an open subgroup; n a positive integer. Suppose that n is coprime to $p(G)$ in the case where $\mathrm{char}(G) = p(G)$. Then the multiplication by n determines an exact sequence

$$0 \longrightarrow M^{\mathrm{gp}}[n] \longrightarrow M^{\mathrm{gp}} \longrightarrow M^{\mathrm{gp}} \longrightarrow 0$$

of H-modules.

(ii) Let $H \subseteq G$ be an open subgroup. Then the exact sequence of H-modules of assertion (i) induces a Kummer map

$$\kappa_H : (M^H \hookrightarrow)\ (M^{\mathrm{gp}})^H \longrightarrow H^1(H, \Lambda(M))$$

— where, if $\mathrm{char}(G) = 0$ (respectively, $\mathrm{char}(G) = p(G)$), then n ranges over the set of positive integers (respectively, the set of positive integers coprime to $p(G)$).

(iii) In the situation of assertion (ii), suppose that $\mathrm{char}(G) = 0$. Then κ_H is injective.

(iv) In the situation of assertion (ii), suppose that $\mathrm{char}(G) = p(G)$. Then the kernel of κ_H consists of the prime-to-$p(G)$-divisible elements of $(M^{\mathrm{gp}})^H$.

(v) The family $\{\kappa_H\}_{H \subseteq G}$ of the Kummer maps [where $H \subseteq G$ ranges over the open subgroups] determines a Kummer map

$$\kappa_M : M \longrightarrow {}_{\infty}H^1(G, \Lambda(M)) \stackrel{\mathrm{def}}{=} \varinjlim_{H \subseteq G} H^1(H, \Lambda(M)).$$

Proof. Assertion (i) follows immediately from Remark 1.8.4. Assertions (ii), (iv) follow immediately from the various definitions involved. Assertion (iii) follows immediately from Lemma 5.4, (i). Assertion (v) follows immediately from assertion (ii), together with Lemma 1.12. This completes the proof of Lemma 6.2. □

Theorem 6.3. *In the notation of Definition 6.1, suppose that $\mathrm{char}(K) = 0$ [so $\widehat{\mathbb{Z}}(G) = \widehat{\mathbb{Z}}$]. Then ψ is bijective.*

Proof. Note that one may construct easily a splitting $s : \Gamma(G) \to \mathrm{Aut}(G \curvearrowright M)$ of $\psi_{\Gamma(G)}$ such that $\mathrm{Im}(s)$ induces the identity automorphism of G. Then the surjectivity follows immediately from the various definitions involved [cf. Theorem 4.7, (iii)].

Next, we verify the injectivity. Let $\sigma \in \mathrm{Ker}(\psi)$ be an element. We consider the injective Kummer map

$$\kappa_M : M \hookrightarrow {}_\infty H^1(G, \Lambda(M))$$

[cf. Lemma 6.2, (iii), (v)], and the natural actions of σ on the domain and codomain of κ_M that is compatible with κ_M.

First, suppose that $(G \curvearrowright M) \cong (G_K \curvearrowright \widehat{\mathcal{O}}^{\times}_{K^{\mathrm{sep}}})$. Then since $\sigma \in \mathrm{Ker}(\psi)$, it holds that σ acts trivially on ${}_\infty H^1(G, \Lambda(M))$. Thus, since κ_M is injective, we conclude that σ is trivial.

Next, suppose that $(G \curvearrowright M) \cong (G_K \curvearrowright (\hat{K^{\mathrm{sep}}})^{\times})$. Then since $\sigma \in \mathrm{Ker}(\psi)$, it follows immediately from Lemma 1.12, together with the various definitions involved, that the automorphism of M induced by σ preserves the subgroup of M corresponding to the subgroup $\hat{\mathcal{O}}^{\triangleright}_{K^{\mathrm{sep}}} \subseteq (\hat{K^{\mathrm{sep}}})^{\times}$ [with respect to any isomorphism $(G \curvearrowright M) \xrightarrow{\sim} (G_K \curvearrowright (\hat{K^{\mathrm{sep}}})^{\times})$].

In summary, to verify the injectivity of ψ, we may assume without loss of generality that $(G \curvearrowright M) \cong (G_K \curvearrowright \hat{\mathcal{O}}^{\triangleright}_{K^{\mathrm{sep}}})$. In this case, it follows immediately from Theorem 5.9 that κ_M and the cyclotomic rigidity isomorphism $\Lambda(M) \xrightarrow{\sim} \Lambda(G)$ induce an injective homomorphism

$$\kappa_G : M \hookrightarrow {}_\infty H^1(G, \Lambda(G)) \stackrel{\mathrm{def}}{=} \varinjlim_{H \subseteq G} H^1(H, \Lambda(G))$$

— where $H \subseteq G$ ranges over the open subgroups — compatible with the natural actions of σ on the domain and codomain of κ_G. Thus, since $\sigma \in \mathrm{Ker}(\psi_G)$, we conclude that σ is trivial. This completes the proof of Theorem 6.3. □

Next, we verify a positive characteristic analogue of Theorem 6.3. However, since Kummer theory has less information in the positive characteristic case, we need to apply local class field theory.

Proposition 6.4. *Let l be a prime number; M a torsion-free abelian pro-l group; $\sigma \in \mathrm{Aut}(M)$. Suppose that $\sigma(N) = N$ for each open subgroup $N \subseteq M$. Then there exists a unique element $a \in \mathbb{Z}_l^{\times}$ such that, for each $m \in M$, it holds that $\sigma(m) = am$.*

Proof. First, it follows immediately from our assumptions that $\sigma(H) = H$ for each *closed* subgroup $H \subseteq M$. In particular, since M is torsion-free, it holds that, for each $m \in M \setminus \{0\}$, there exists a unique element $a_m \in \mathbb{Z}_l^{\times}$ such that $\sigma(m) = a_m m$. It suffices to prove that a_m is independent of m. Note that, for any $b \in \mathbb{Z}_l \setminus \{0\}$, it holds that $a_{bm} = a_m$. On the other hand, for each $m_1, m_2 \in M$, it holds that

$$a_{m_1+m_2}(m_1 + m_2) = \sigma(m_1 + m_2) = \sigma(m_1) + \sigma(m_2) = a_{m_1}m_1 + a_{m_2}m_2.$$

Then, if $m_2 \notin \mathbb{Z}_l \cdot m_1$, and $m_1 \notin \mathbb{Z}_l \cdot m_2$, then $a_{m_1} = a_{m_1+m_2} = a_{m_2}$. Thus, we obtain the desired conclusion. This completes the proof of Proposition 6.4. □

Lemma 6.5. *Write $U_K \stackrel{\mathrm{def}}{=} 1 + \mathfrak{m}_K$. [Note that since U_K has no nontrivial divisible element, U_K may be regarded as a subgroup of $\widehat{\mathcal{O}}^{\times}_K$ ($\subseteq \hat{K}^{\times}$).] Suppose that K is complete. Then the following hold:*

(i) The subgroup $U_K \subseteq \widehat{\mathcal{O}}^{\times}_K$ consists of the prime-to-p-divisible elements of $\widehat{\mathcal{O}}^{\times}_K$. Moreover, the subgroup of p-divisible elements of $\widehat{\mathcal{O}}^{\times}_K$ determines a unique splitting of the natural surjection $\widehat{\mathcal{O}}^{\times}_K \twoheadrightarrow \widehat{\mathcal{O}}^{\times}_K/U_K$. In particular, we have a canonical splitting

$$\widehat{\mathcal{O}}^{\times}_K \xrightarrow{\sim} (\widehat{\mathcal{O}}^{\times}_K/U_K) \times U_K$$

that is preserved by any automorphism of $\widehat{\mathcal{O}}^{\times}_K$.

(ii) The subgroup $\widehat{\mathcal{O}}_K^\times \subseteq \hat{K}^\times$ *is generated by* l*-divisible elements and* p*-divisible elements of* $\hat{K}^\times$*, where* l *is a prime number* $\neq p$*. Moreover, any automorphism of* $\hat{K}^\times$ *preserves the subgroup* $U_K \subseteq \hat{K}^\times$*.*

(iii) The kernels of the natural homomorphisms

$$\widehat{\mathcal{O}}_K^\times \twoheadrightarrow \varprojlim_n \mathcal{O}_K^\times/(\mathcal{O}_K^\times)^n; \quad \hat{K}^\times \to \varprojlim_n K^\times/(K^\times)^n$$

— where n *ranges over the positive integers coprime to* p *— coincide with* U_K*.*

Proof. First, since k is perfect, assertion (i) follows immediately from Remark 1.8.3, together with our assumption that K is complete. Next, since $\mathbb{Z}$ has no nontrivial l-divisible element for any prime number l, assertions (ii), (iii) follow immediately from assertion (i). This completes the proof of Lemma 6.5. $\square$

Theorem 6.6. *In the notation of Definition 6.1, suppose that* $\mathrm{char}(K) = p$ *[so* $\widehat{\mathbb{Z}}(G) = \widehat{\mathbb{Z}}^{(p)'}$*], and* K *is complete.*

(i) Suppose, moreover, that $(G \curvearrowright M) \cong (G_K \curvearrowright \widehat{\mathcal{O}}_{K^{\mathrm{sep}}}^\times)$*. Then it holds that* ψ *is surjective, and* $\mathrm{Ker}(\psi) \xrightarrow{\sim} \mathbb{Z}_p^\times$*.*

(ii) Suppose, moreover, that $(G \curvearrowright M) \cong (G_K \curvearrowright (\widehat{K^{\mathrm{sep}}})^\times)$ *or* $(G \curvearrowright M) \cong (G_K \curvearrowright \hat{\mathcal{O}}_{K^{\mathrm{sep}}}^\triangleright)$*. Then it holds that* ψ *is bijective.*

Proof. The surjectivity portions of assertions (i), (ii), follow from a similar argument to the argument applied in the surjectivity portion of the proof of Theorem 6.3.

In the remainder, we compute $\mathrm{Ker}(\psi)$. For each finite Galois extension $K \subseteq L$ ($\subseteq K^{\mathrm{sep}}$), write $U_L \stackrel{\mathrm{def}}{=} 1 + \mathfrak{m}_L$;

$$\widehat{N}_{L/K} : \hat{L}^\times \longrightarrow \hat{K}^\times$$

for the natural homomorphism induced by the norm map $N_{L/K} : L^\times \to K^\times$.

Next, we verify that $\mathrm{Ker}(\psi) \xrightarrow{\sim} \mathbb{Z}_p^\times$ in the case where $(G \curvearrowright M) \xrightarrow{\sim} (G_K \curvearrowright \hat{\mathcal{O}}_{K^{\mathrm{sep}}}^\times)$. Note that it follows immediately from Lemma 6.5, (i), that, for each finite Galois extension $K \subseteq L$ ($\subseteq K^{\mathrm{sep}}$), there exists a canonical splitting

$$\widehat{\mathcal{O}}_L^\times \xrightarrow{\sim} (\widehat{\mathcal{O}}_L^\times/U_L) \times U_L.$$

that is preserved by any automorphism of $\widehat{\mathcal{O}}_L^\times$. Let $a \in \mathbb{Z}_p^\times$ be an element. Write σ_a for the automorphism of $\widehat{\mathcal{O}}_{K^{\mathrm{sep}}}^\times$ such that, for each finite Galois extension $K \subseteq L$ ($\subseteq K^{\mathrm{sep}}$), σ_a maps

$$(\widehat{\mathcal{O}}_L^\times/U_L) \times U_L \ni (x, y) \mapsto (x, ay) \in (\widehat{\mathcal{O}}_L^\times/U_L) \times U_L.$$

Then we observe that the assignment $a \mapsto \sigma_a$ determines an injection $\mathbb{Z}_p^\times \hookrightarrow \mathrm{Ker}(\psi)$. Next, let $\sigma \in \mathrm{Ker}(\psi)$ be an element. Then, in light of Lemmas 6.2, (iv); 6.5, (i), (iii), it follows from a similar argument to the argument applied in the proof of the injectivity portion of Theorem 6.3 that, for each finite Galois extension $K \subseteq L$ ($\subseteq K^{\mathrm{sep}}$), σ acts naturally on $\widehat{\mathcal{O}}_L^\times$ and trivially on $\widehat{\mathcal{O}}_L^\times/U_L$. Thus, it suffices to verify the following assertion:

> Claim 6.6.A: Let $K \subseteq F$ ($\subseteq K^{\mathrm{sep}}$) be a finite Galois extension. Then there exists a unique element $a \in \mathbb{Z}_p^\times$ such that $\sigma(x) = ax$ for each $x \in U_F$.

Indeed, by replacing K by a finite Galois extension of K, we may assume without loss of generality that $K = F$. Note that since $\sigma \in \mathrm{Ker}(\psi)$, it holds that

$$\sigma(N_{L/K}(U_L)) \ = \ N_{L/K}(U_L)$$

for each finite Galois extension $K \subseteq L$ ($\subseteq K^{\mathrm{sep}}$). Next, write

$$P_K^* \ \subseteq \ (G_K^{\mathrm{ab}})^p$$

for the $\mathbb{Z}_p$-submodule that arises as the image of $P_K \subseteq G_K$ via the natural surjection $G_K \twoheadrightarrow (G_K^{\mathrm{ab}})^p$. Then, in light of local class field theory [cf. Theorem 4.3], together with the equalities $\{\sigma(N_{L/K}(U_L)) = N_{L/K}(U_L)\}_{K \subseteq L}$, the following hold:

- The reciprocity map induces an injective homomorphism

  $$U_K \ \hookrightarrow \ \varprojlim_{K \subseteq L} \ U_K/N_{L/K}(U_L) \ \stackrel{\sim}{\to} \ \varprojlim_{K \subseteq L} \ K^\times/N_{L/K}(L^\times) \ \stackrel{\sim}{\to} \ \varprojlim_{K \subseteq L} \ \mathrm{Gal}(L/K) \ = \ P_K^*,$$

  where $K \subseteq L$ ($\subseteq K^{\mathrm{sep}}$) ranges over the totally ramified abelian extensions of degree p-power.

- σ induces an automorphism of P_K^* that preserves arbitrary open subgroups of P_K^* and is compatible with σ via the injection $U_K \hookrightarrow P_K^*$.

Here, we observe that since $\mathrm{char}(K) = p$, it holds that $(G_K^{\mathrm{ab}})^p$, hence also P_K^*, is torsion-free. Thus, we conclude from Proposition 6.4 that there exists a unique element $a \in \mathbb{Z}_p^\times$ such that $\sigma(x) = ax$ for each $x \in U_K$. This completes the proof of Claim 6.6.A, hence of assertion (i).

Next, we verify the injectivity portion of assertion (ii). We consider the case where $(G \curvearrowright M) \stackrel{\sim}{\to} (G_K \curvearrowright (\hat{K^{\mathrm{sep}}})^\times)$ only. [The proof in the case where $(G \curvearrowright M) \stackrel{\sim}{\to} (G_K \curvearrowright \hat{\mathcal{O}}^{\triangleright}_{K^{\mathrm{sep}}})$ is similar to the proof of this case.] Let $\sigma \in \mathrm{Ker}(\psi)$ be an element. Then, in light of Lemmas 6.2, (iv); 6.5, (ii), (iii), it follows from a similar argument to the argument applied in the proof of the injectivity portion of Theorem 6.3 that, for each finite Galois extension $K \subseteq L$ ($\subseteq K^{\mathrm{sep}}$), σ acts naturally on $\hat{L}^\times$ and trivially on $\hat{L}^\times/U_L$ Write

$$\sigma_K : \hat{K}^\times \ \stackrel{\sim}{\to} \ \hat{K}^\times$$

for the automorphism induced by σ. Next, we verify the following assertion:

Claim 6.6.B: Let $\pi \in \hat{K}^\times$ be a prime element. Then $\sigma_K(\pi) = \pi$.

Indeed, write $(\mathbb{Z} \stackrel{\sim}{\to})\ T_\pi \subseteq \hat{K}^\times$ for the subgroup generated by π. Then it follows immediately from local class field theory [cf. Theorem 4.3] that

$$\bigcap_{K \subseteq L} \ \mathrm{Im}(\widehat{N}_{L/K}) \ = \ T_\pi,$$

where $K \subseteq L$ ($\subseteq K^{\mathrm{sep}}$) ranges over the finite Galois extensions such that $\pi \in \mathrm{Im}(\widehat{N}_{L/K})$. On the other hand, since $\sigma \in \mathrm{Ker}(\psi)$, it holds that $\sigma_K(\mathrm{Im}(\widehat{N}_{L/K})) = \mathrm{Im}(\widehat{N}_{L/K})$ for each finite Galois extension $K \subseteq L$ ($\subseteq K^{\mathrm{sep}}$). In particular, it holds that $\sigma_K(\pi) \in T_\pi$. Thus, since σ_K acts trivially on $\hat{K}^\times/U_K$, we conclude that $\sigma_K(\pi) = \pi$. This completes the proof of Claim 6.6.B.

In light of Claim 6.6.B, by varying the prime elements $\in \hat{K}^\times$, we observe that σ_K is trivial. Thus, by applying the above argument to the open subgroups of G_K, we conclude that σ is trivial. This completes the proof of assertion (ii), hence of Theorem 6.6. □

Corollary 6.7. *We retain the notation of Definition 1.8. For $i = 1, 2$, let G_i be a topological group; M_i a monoid that admits an action of G_i. Write*

$$\mathrm{Isom}(G_1 \curvearrowright M_1,\ G_2 \curvearrowright M_2)$$

for the set of pairs of a continuous isomorphism $G_1 \overset{\sim}{\to} G_2$ and an isomorphism $M_1 \overset{\sim}{\to} M_2$ compatible with the respective actions;

$$\mathrm{Isom}(G_1,\ G_2)$$

for the set of continuous isomorphisms $G_1 \overset{\sim}{\to} G_2$. Suppose that

- *k is quasi-finite and μ-finite;*
- *$(G_i \curvearrowright M_i)$ is isomorphic to $(G_K \curvearrowright \widehat{\mathcal{O}}^{\times}_{K^{\mathrm{sep}}})$ (respectively, $(G_K \curvearrowright \hat{\mathcal{O}}^{\triangleright}_{K^{\mathrm{sep}}})$; $(G_K \curvearrowright (\widehat{K^{\mathrm{sep}}})^{\times})$), where $i = 1, 2$.*

Then the following hold:

(i) Suppose, moreover, that $\mathrm{char}(K) = 0$. Then the natural map

$$\mathrm{Isom}(G_1 \curvearrowright M_1,\ G_2 \curvearrowright M_2) \longrightarrow \mathrm{Isom}(G_1,\ G_2)$$

is surjective. Moreover, the fibers of this map are $\Gamma(G_2)$-torsors. [Note that $\Gamma(G_2) \overset{\sim}{\to} \widehat{\mathbb{Z}}^{\times}$ (respectively, $\Gamma(G_2) = \{1\}$; $\Gamma(G_2) = \{\pm 1\}$).]

(ii) Suppose, moreover, that $\mathrm{char}(K) = p$, K is complete, and k is finite. Then the natural map

$$\mathrm{Isom}(G_1 \curvearrowright M_1,\ G_2 \curvearrowright M_2) \longrightarrow \mathrm{Isom}(G_1,\ G_2)$$

is surjective. Moreover, the fibers of this map are $\Gamma(G_2) \times \mathbb{Z}_p^{\times}$ (respectively, $\Gamma(G_2)$; $\Gamma(G_2)$)-torsors. [Note that $\Gamma(G_2) \overset{\sim}{\to} (\widehat{\mathbb{Z}}^{(p)'})^{\times}$ (respectively, $\Gamma(G_2) = \{1\}$; $\Gamma(G_2) = \{\pm 1\}$).]

Proof. In light of Theorem 4.7, (iii), assertion (i) (respectively, assertion (ii)) follows immediately from Theorem 6.3 (respectively, Theorem 6.6). This completes the proof of Corollary 6.7. □

Remark 6.7.1. In the notation of Corollary 6.7, suppose that K is a complete discrete valuation field with finite residues, and

$$(G_i \curvearrowright M_i) \ \cong\ (G_K \curvearrowright \mathcal{O}^{\times}_{K^{\mathrm{sep}}}/\mu(K^{\mathrm{sep}})),$$

where $i = 1, 2$. Then it follows immediately from the surjectivity portion of Corollary 6.7 that the natural map

$$\mathrm{Isom}(G_1 \curvearrowright M_1,\ G_2 \curvearrowright M_2) \longrightarrow \mathrm{Isom}(G_1,\ G_2)$$

is surjective. Thus, in light of Corollary 6.7, it would be interesting to determine the structure of fibers of this surjection. However, at the time of writing of the present paper, the authors fail to determine the structure of fibers due to the lack of the cyclotome. [Note that the injectivity portions of Theorems 6.3, 6.6 are proved by applying cyclotomic rigidity isomorphisms.] The authors hope to be able to address such an issue in a future paper.

7 Mono-anabelian group-theoretic reconstruction algorithms associated to mixed characteristic complete discrete valuation fields with strongly p-quasi-finite residues

Throughout the present section, we maintain the notation of §2 [except the assumption that k is perfect!]. In the present section, in the case where K is a mixed characteristic complete discrete valuation field with strongly p-quasi-finite residues [cf. Definition 2.8], by applying p-local class field theory discussed in §2, we establish mono-anabelian group-theoretic algorithms/procedures to reconstruct various p-like objects associated to K. Moreover, in light of this p-local class field theory, by applying a similar argument to the argument applied in the proof of Theorem 6.6 [i.e., an application of local class field theory], we determine the structure of the group of G_K-isometries [i.e., (Ind2)] that appears in Mochizuki's inter-universal Teichmüller theory [cf. [24], Example 1.8, (iv)] in a generalized situation [cf. Theorem 7.4].

First, we begin by observing that, in the case where K is a mixed characteristic complete discrete valuation field with strongly p-quasi-finite residues, various p-like objects associated to K may be reconstructed in a similar way to the case where K is a mixed characteristic complete discrete valuation field with quasi-finite residues.

Definition 7.1. For each separable extension $K \subseteq L$ ($\subseteq K^{\mathrm{sep}}$), we shall write

$$U_{1,L} \stackrel{\mathrm{def}}{=} 1 + \mathfrak{m}_L; \quad U_L^{\mu} \stackrel{\mathrm{def}}{=} U_{1,L}/\mu_{p^\infty}(L).$$

Suppose that $\mathrm{char}(K) = 0$, and K is complete. Then we shall write

$$\mathcal{I}_K \stackrel{\mathrm{def}}{=} \frac{1}{2p} \cdot \log_p(U_{1,K}) \ \subseteq \ K,$$

where $\log_p : U_{1,K} \to K$ denotes the p-adic logarithm map. [Note that $\mathcal{O}_K \subseteq \mathcal{I}_K$ — cf. Lemma 8.5 below. We shall refer to $\mathcal{I}_K$ as the *log shell* associated to K.]

Theorem 7.2. *Suppose that k is a strongly p-quasi-finite field. Then the following hold:*

(i) Let G be a topological group. Suppose that there exists a prime number l such that G is isomorphic to the absolute Galois group of a Henselian discrete valuation field with a strongly l-quasi-finite residue field of characteristic l. Then there exists a functorial group-theoretic algorithm

$$G \quad \rightsquigarrow \quad (p(G),\ \mathrm{char}(G))$$

for constructing — from the topological group G — a data $(p(G), \mathrm{char}(G))$ consisting of a prime number $p(G)$, and a nonnegative integer $\mathrm{char}(G)$ that satisfies the following condition: if one applies this functorial group-theoretic algorithm to $G = G_K$, then

$$p(G) \ = \ p, \quad \mathrm{char}(G) \ = \ \mathrm{char}(K).$$

(ii) Suppose that $\mathrm{char}(K) = 0$. Let G be a topological group isomorphic to G_K. Write

$$\hat{r}_K^p : \varprojlim_{n \geq 1} \ K^\times/(K^\times)^{p^n} \ \hookrightarrow \ (G_K^{\mathrm{ab}})^p$$

for the injective homomorphism [cf. Theorem 3.5, (i)] induced by the reciprocity map $r_K^p : K^\times \to (G_K^{\mathrm{ab}})^p$*. Then there exists a functorial group-theoretic algorithm*

$$G \quad \rightsquigarrow \quad \mu_{p^\infty}(G)$$

for constructing — from the topological group G *— a data* $\mu_{p^\infty}(G)$ *consisting of a subgroup of* $(G_K^{\mathrm{ab}})^p$ *that satisfies the following condition: if one applies this functorial group-theoretic algorithm to* $G = G_K$*, then*

$$\mu_{p^\infty}(G) \;=\; \hat{r}_K^p(\mu_{p^\infty}(K))$$

[cf. the discussion at the beginning of the proof of Proposition 3.7]. Moreover, by applying this group-theoretic algorithm to every normal open subgroup $H \subseteq G$*, one constructs the group [equipped with natural action of* G*]*

$$\mu_{p^\infty,s}(G) \;\stackrel{\mathrm{def}}{=}\; \varinjlim_{H \subseteq G} \; \mu_{p^\infty}(H)$$

— where the transition maps are induced by the transfers — that satisfies the following condition: if one applies this functorial group-theoretic algorithm to $G = G_K$*, then the [completed] reciprocity maps* $\{\hat{r}_L^p\}_{K \subseteq L}$ *— where* $K \subseteq L$ $(\subseteq K^{\mathrm{sep}})$ *ranges over the finite Galois extensions — induce a* G*-equivariant isomorphism*

$$\mu_{p^\infty}(K^{\mathrm{sep}}) \;\stackrel{\sim}{\to}\; \mu_{p^\infty,s}(G).$$

(iii) In the notation of (ii), suppose, moreover, that K *is complete. Let* $I \subseteq G$ *be a closed subgroup such that there exists an isomorphism* $G \stackrel{\sim}{\to} G_K$ *that induces an isomorphism* $I \stackrel{\sim}{\to} I_K$*. Then there exists a functorial group-theoretic algorithm*

$$I \;\subseteq\; G \quad \rightsquigarrow \quad (U_1(I \subseteq G), \quad \mathcal{I}(I \subseteq G) \;\subseteq\; \mathcal{K}(I \subseteq G))$$

for constructing — from the pair of topological groups $I \subseteq G$ *— a data* $(U_1(I \subseteq G), \mathcal{I}(I \subseteq G) \subseteq \mathcal{K}(I \subseteq G))$ *consisting of groups that satisfies the following condition: if one applies this functorial group-theoretic algorithm to* $(I \subseteq G) = (I_K \subseteq G_K)$*, then*

$$U_1(I \subseteq G) \;=\; \hat{r}_K^p(U_{1,K}),$$

and $\mathcal{I}(I \subseteq G) \subseteq \mathcal{K}(I \subseteq G))$ *is isomorphic to* $\mathcal{I}_K \subseteq K$*. Moreover, by applying this group-theoretic algorithm to every normal open subgroup* $H \subseteq G$ *[and the subgroup* $I \cap H \subseteq H$*], one constructs a data* $(U_{1,s}(I \subseteq G), \mathcal{I}_s(I \subseteq G) \subseteq \mathcal{K}_s(G))$ *consisting of the groups [equipped with natural actions of* G*]*

$$U_{1,s}(I \subseteq G) \;\stackrel{\mathrm{def}}{=}\; \varinjlim_{H \subseteq G} \; U_1(I \cap H \subseteq H), \quad \mathcal{I}_s(I \subseteq G) \;\stackrel{\mathrm{def}}{=}\; \varinjlim_{H \subseteq G} \; \mathcal{I}(I \cap H \subseteq H),$$

$$\mathcal{K}_s(I \subseteq G) \;\stackrel{\mathrm{def}}{=}\; \varinjlim_{H \subseteq G} \; \mathcal{K}(I \cap H \subseteq H)$$

— where the transition maps are induced by the transfers — that satisfies the following condition: if one applies this functorial group-theoretic algorithm to $(I \subseteq G) = (I_K \subseteq G_K)$*, then the [completed] reciprocity maps* $\{\hat{r}_L^p\}_{K \subseteq L}$ *— where* $K \subseteq L$ $(\subseteq K^{\mathrm{sep}})$ *ranges over the finite Galois extensions — induce a* G*-equivariant isomorphism*

$$U_{1,K^{\mathrm{sep}}} \;\stackrel{\sim}{\to}\; U_{1,s}(I \subseteq G),$$

and there exists a G*-equivariant isomorphism*

$$K^{\mathrm{sep}} \;\stackrel{\sim}{\to}\; \mathcal{K}_s(I \subseteq G)$$

that maps $\mathcal{I}_K$ *onto* $\mathcal{I}(I \subseteq G)$*.*

Proof. First, assertion (i) follows immediately from a similar argument to the argument applied in the proof of Theorem 4.7, (i).

Next, we verify assertion (ii). To verify assertion (ii), it suffices to give a group-theoretic characterization of the data

$$G_K \curvearrowright \mu_{p^\infty}(K^{\mathrm{sep}})$$

in terms of the underlying topological group structure of G_K that reflects the condition concerning $\hat{r}^p_K$. Let $K \subseteq L$ ($\subseteq K^{\mathrm{sep}}$) be a finite Galois extension. Note that since k is a strongly p-quasi-finite field, it holds that k_L is also a p-quasi-finite field. Then it follows immediately from Proposition 3.7 that $\mu_{p^\infty}(L)$ may be characterized as the tosion subgroup of $(G^p_L)^{\mathrm{ab}}$ [cf. (i)] via the reciprocity map. Thus, in light of the theory of transfer, we obtain a suitable group-theoretic characterization of the data $G_K \curvearrowright \mu_{p^\infty}(K^{\mathrm{sep}})$, as desired. This completes the proof of assertion (ii).

Next, we verify assertion (iii). To verify assertion (iii), it suffices to give a group-theoretic characterization of the multiplicative/additive groups

$$U_{1,K^{\mathrm{sep}}}, \quad \mathcal{I}_K \subseteq K^{\mathrm{sep}}$$

[equipped with the natural actions of G_K] in terms of the underlying topological group structures of the pair $I_K \subseteq G_K$ that reflects the condition concerning the reciprocity maps. On the other hand, since K is complete, we recall that the p-adic logarithm map determines isomorphisms

$$U^\mu_{K^{\mathrm{sep}}} \xrightarrow{\sim} K^{\mathrm{sep}}, \quad \frac{1}{2p} \cdot U^\mu_K = \frac{1}{2p} \cdot (U_{1,K^{\mathrm{sep}}})^{G_K} / \mu_{p^\infty}(K^{\mathrm{sep}})^{G_K} \xrightarrow{\sim} \mathcal{I}_K.$$

In particular, it suffices to give a group-theoretic characterization of the data

$$G_K \curvearrowright U_{1,K^{\mathrm{sep}}}$$

in terms of the underlying topological group structures of the pair $I_K \subseteq G_K$ that reflects the condition concerning the reciprocity maps.

Next, let $K \subseteq L$ ($\subseteq K^{\mathrm{sep}}$) be a finite Galois extension such that $\zeta_p \in L$ [cf. (ii)]. Then, for each positive integer m, we have the cup-product

$$H^1(G^p_L, \mu_{p^m}(K^{\mathrm{sep}})) \times H^1(G^p_L, \mathbb{Z}/p^m\mathbb{Z}) \longrightarrow H^2(G^p_L, \mu_{p^m}(K^{\mathrm{sep}})).$$

Again, we note that since k is a strongly p-quasi-finite field, it holds that k_L is also a p-quasi-finite field. In particular, in light of Kummer theory, Pontryagin duality, and [31], Chapter XII, §3, Theorem 2, we obtain a group-theoretic characterization of the image of the injection

$$L^\times/(L^\times)^{p^m} \hookrightarrow G^{\mathrm{ab}}_L/p^m G^{\mathrm{ab}}_L$$

[cf. Remark 2.13.1; Theorem 3.5, (i)]. Then, by varying m, we also obtain a group-theoretic characterization of $\mathrm{Im}(\hat{r}^p_L)$. Thus, since K is complete, we conclude from Theorem 2.16, together with Remark 1.8.3, that $\hat{r}^p_L(U_{1,L})$ may be characterized as

$$\mathrm{Im}(I_K \cap G_L = I_L \subseteq G_L \twoheadrightarrow (G^p_L)^{\mathrm{ab}}) \cap \mathrm{Im}(\hat{r}).$$

Finally, by applying the theory of transfer, we obtain a group-theoretic characterization of the data $G_K \curvearrowright U_{1,K^{\mathrm{sep}}}$, as desired. This completes the proof of assertion (iii), hence of Theorem 7.2. □

Next, we determine the structure of the group of G_K-isometries that appears in Mochizuki's inter-universal Teichmüller theory [cf. [24], Example 1.8, (iv)]. [Note that such a determination is closely related to the problem posed in Remark 6.7.1.] Let us recall the definition of the group of G_K-isometries [in a generalized situation]:

Definition 7.3. We shall write

$$\mathrm{Ism}(G_K)$$

for the group of G_K-equivariant automorphisms of $U^{\mu}_{K^{\mathrm{sep}}}$ that, for each finite separable extension $K \subseteq L$ ($\subseteq K^{\mathrm{sep}}$), *preserve* the "lattice" $U^{\mu}_L \subseteq (U^{\mu}_{K^{\mathrm{sep}}})^{G_L}$. We shall refer to each element in $\mathrm{Ism}(G_K)$ as a *G_K-isometry.* Note that U^{μ}_L admits a natural structure of $\mathbb{Z}_p$-module compatible with the natural action of G_K. In particular, there exists a natural injection

$$\mathbb{Z}_p^{\times} \hookrightarrow \mathrm{Ism}(G_K).$$

Remark 7.3.1. In the notation of Definition 7.3, suppose that k is finite. Then, for each separable extension $K \subseteq L$ ($\subseteq K^{\mathrm{sep}}$), it holds that

$$U^{\mu}_L \stackrel{\sim}{\to} \mathcal{O}_L^{\times}/\mu(L).$$

In particular, the above definition of $\mathrm{Ism}(G_K)$ is compatible with the definition of $\mathrm{Ism}(G_K)$ in [24], Example 1.8, (iv).

Theorem 7.4. *In the notation of Definition 7.3, suppose that G_k is not a pro-prime-to-p group. Then the natural injection*

$$\mathbb{Z}_p^{\times} \hookrightarrow \mathrm{Ism}(G_K)$$

is bijective.

Proof. First, by replacing K by a field extension of K obtained by adjoining suitable systems of p-power roots of liftings $\in \mathcal{O}_K$ of a p-basis of k, we may assume without loss of generality that k is perfect. On the other hand, since G_k is not a prime-to-p group, and $\mathrm{char}(k) = p$, a p-Sylow subgroup of G_k is a nontrivial free pro-p group. Then, by replacing K by a suitable unramified extension of K, we may assume without loss of generality that $G_k \cong \mathbb{Z}_p$. In particular, k is a strongly p-quasi-finite field.

Next, let $\sigma \in \mathrm{Ism}(G_K)$ be an element. For each finite Galois extension $K \subseteq L$ ($\subseteq K^{\mathrm{sep}}$), write $\sigma_L : U^{\mu}_L \stackrel{\sim}{\to} U^{\mu}_L$ for the automorphism induced by σ. Here, we note that the norm map $L^{\times} \to K^{\times}$ induces naturally a homomorphism

$$N^{\mu}_{L/K} : U^{\mu}_L \longrightarrow U^{\mu}_K.$$

Then since σ_L is $\mathrm{Gal}(L/K)$-equivariant, for each finite Galois extension $K \subseteq L$ ($\subseteq K^{\mathrm{sep}}$), it holds that

$$\sigma_K(\mathrm{Im}(N^{\mu}_{L/K})) = \mathrm{Im}(N^{\mu}_{L/K}).$$

Next, write

$$P^{\mu}_K \subseteq (G_K^{\text{ab-tor}})^p \stackrel{\mathrm{def}}{=} (G_K^{\mathrm{ab}})^p/(G_K^{\mathrm{ab}})^p_{\mathrm{tor}}$$

for the torsion-free $\mathbb{Z}_p$-submodule that arises as the image of $P_K \subseteq G_K$ via the natural surjection $G_K \twoheadrightarrow (G_K^{\text{ab-tor}})^p$. Then, in light of local class field theory [cf. Theorems 2.13; 2.16; 3.5, (iii)], together with the equalities $\{\sigma_K(\mathrm{Im}(N^{\mu}_{L/K})) = \mathrm{Im}(N^{\mu}_{L/K})\}_{K \subseteq L}$, the following hold:

- The reciprocity map induces an injective homomorphism $U^{\mu}_K \hookrightarrow P^{\mu}_K$ [cf. Proposition 3.7].
- σ_K induces an automorphism of P^{μ}_K that preserves arbitrary open subgroups of P^{μ}_K and is compatible with σ_K via the injection $U^{\mu}_K \hookrightarrow P^{\mu}_K$.

In particular, it follows from Proposition 6.4 that there exists a unique element $a_K \in \mathbb{Z}_p^\times$ such that $\sigma_K(x) = a_K \cdot x$ for every $x \in U_K^\mu$. Moreover, by applying the above argument to all normal open subgroups of G_K, for each finite Galois extension $K \subseteq L$ ($\subseteq K^{\mathrm{sep}}$), we obtain a unique element $a_L \in \mathbb{Z}_p^\times$ such that $\sigma_L(x) = a_L \cdot x$ for every $x \in U_L^\mu$. Note that the uniqueness of a_K implies that $a_K = a_L$. Thus, we conclude that the natural injection $\mathbb{Z}_p^\times \hookrightarrow \mathrm{Ism}(G_K)$ is bijective. This completes the proof of Theorem 7.4. □

Remark 7.4.1. In light of Theorem 7.4, it would be interesting to determine the structure of $\mathrm{Ism}(G_K)$ in the case where G_k is a pro-prime-to-p group. However, at the time of writing of the present paper, the authors have no meaningful insight on this question [even if we restrict our attention to the case where k is an algebraically closed field].

Remark 7.4.2. In the first paragraph of the proof of Theorem 7.4, we replaced K by an infinite algebraic extension field of K. Note that such an infinite algebraic extension field may not be complete even if we assume that K is complete. This is one of the main reasons why we need to discuss p-local class field theory in the general Henselian setting in §2 in the present paper [cf. [30], §1].

8 Neukirch-Uchida-type result for quasi-p-adic local fields with p-quasi-finite residues

Let p be a prime number. In the present section, by combining results obtained in the previous sections with Murotani's recent results on certain homomorphisms between the absolute Galois groups of mixed characteristic complete discrete valuation fields with perfect residues [cf. [26]], we prove a Neukirch-Uchida-type/an absolute version of the Grothendieck Conjecture-type result for quasi-p-adic local fields with p-quasi-finite residues [cf. Definitions 2.8, 8.10; Theorem 8.11 below]. This result may be regarded as a generalization of Mochizuki's result for p-adic local fields [cf. [19], Theorem; [22], Theorem 3.5, (ii); [22], Corollary 3.7].

Throughout the present section, we maintain the notation of §2. Moreover, we shall write $v_K : K^\times \twoheadrightarrow \mathbb{Z}$ for the discrete valuation on K. By a slight abuse of notation, we shall also write v_K for the unique extension of v_K to K^{sep}. On the other hand, in the case where $\mathrm{char}(K) = 0$, we shall write e for the absolute ramification index of K.

First, we begin by recalling some facts on Artin-Schreier equations.

Proposition 8.1. *Suppose that K is complete. Let n be a positive integer coprime to p; $u \in \mathcal{O}_K^\times$ the Teichmüller representative of an element $\in k^\times$; π_K a uniformizer of K. Write $K \subseteq L$ for the finite field extension of degree p associated to the Artin-Schreier equation*

$$y^p - y + \frac{1}{u \cdot \pi_K^n} \;=\; 0,$$

where y denotes an indeterminate. Fix a root $\lambda \in L$ of this equation and a pair of positive integers c, d such that c is odd, and $-nc + pd = 1$. Write

$$\pi_L \;\stackrel{\mathrm{def}}{=}\; -\lambda^c \cdot \pi_K^d \cdot u^{\frac{c}{p}} \;\in\; L.$$

Then the following hold:

(i) The field extension $K \subseteq L$ is a totally ramified extension of degree p. Moreover, π_L is a uniformizer of L such that $N_{L/K}(\pi_L) = \pi_K$.

(ii) Suppose that $n < \frac{pe}{p-1}$ in the case where $\mathrm{char}(K) = 0$. Then the totally ramified extension $K \subseteq L$ [cf. (i)] is a Galois extension. Moreover, in the notation of Proposition 2.4, it holds that

$$s \;=\; n, \quad \overline{\eta}^{p-1} \;=\; \overline{u}^{1-\frac{1}{p}},$$

where $\overline{u}$ denotes the image of u in $k^\times$.

Proof. First, we verify assertion (i). Observe that since $v_K(u) = 0$, the equality $\lambda^p - \lambda + \frac{1}{u \cdot \pi_K^n} = 0$ immediately implies that $v_K(\lambda) = -\frac{n}{p}$. Then since $v_K(u) = 0$, and $-nc + pd = 1$, it holds that

$$v_K(\pi_L) \;=\; v_K(-\lambda^c \cdot \pi_K^d \cdot u^{\frac{c}{p}}) \;=\; -\frac{nc}{p} + d \;=\; \frac{1}{p}.$$

In particular, π_L is a uniformizer of L. On the other hand, since c is odd, and $-nc+pd = 1$, it holds that

$$N_{L/K}(\pi_L) \;=\; N_{L/K}(-\lambda^c \cdot \pi_K^d \cdot u^{\frac{c}{p}}) \;=\; (-1)^p \cdot ((-1)^p \cdot \frac{1}{u \cdot \pi_K^n})^c \cdot \pi_K^{pd} \cdot u^c \;=\; \pi_K.$$

This completes the proof of assertion (i).

Next, we verify assertion (ii). In light of [3], Proposition 2.5, it suffices to verify that $\overline{\eta}^{p-1} \;=\; \overline{u}^{1-\frac{1}{p}}$. Let $\sigma \in \mathrm{Gal}(L/K)$ be a generator such that $\sigma(\lambda) - \lambda - 1 \in \mathfrak{m}_K$ [cf. [3], Proposition 2.5]. Then there exists an element $h \in \mathcal{O}_L$ such that $v_K(h) > v_K(\lambda^{-1})$, and

$$1 + \eta \cdot \pi_L^n \;=\; \frac{\sigma(\pi_L)}{\pi_L} \;=\; \left(\frac{\sigma(\lambda)}{\lambda}\right)^c \;=\; (1 + \lambda^{-1} + h)^c.$$

Note that since $v_K(h) > v_K(\lambda^{-1})$, this equality implies that $\eta \cdot \pi_L^n \equiv c\lambda^{-1} \pmod{\mathfrak{m}_L^{n+1}}$. In the remainder, for each $z \in \mathcal{O}_L$, write $\overline{z}$ for the image of z via the natural surjection $\mathcal{O}_L \twoheadrightarrow k$. Then we observe that the equality $\lambda^p - \lambda + \frac{1}{u \cdot \pi_K^n} = 0$ immediately implies that

$$\overline{\lambda^{pd} \cdot \pi_K^{nd} \cdot u^d} \;=\; \overline{\lambda^{pd} \cdot (u\pi_K^n)^d} \;=\; \overline{\left(\frac{\lambda^p}{\lambda - \lambda^p}\right)^d} \;=\; (-1)^d \;\in\; k^\times.$$

Thus, since c is coprime to p, and $-nc + pd = 1$, we conclude that

$$\overline{\eta} \;=\; \overline{\frac{c\lambda^{-1}}{\pi_L^n}} \;=\; \overline{\frac{(-1)^n \cdot c}{\lambda^{pd} \cdot \pi_K^{nd} \cdot u^{\frac{cn}{p}}}} \;=\; \overline{(-1)^{n-d} \cdot c \cdot u^{\frac{1}{p}}} \;\in\; k^\times,$$

hence that

$$\overline{\eta}^{p-1} \;=\; \overline{u}^{1-\frac{1}{p}}.$$

This completes the proof of assertion (ii), hence of Proposition 8.1. □

Next, we compute the intersection of certain subgroups of $K^\times$ associated to the images of norm maps.

Lemma 8.2. *Write $\wp : k \to k$ for the Artin-Schreier map, i.e., the $\mathbb{F}_p$-linear map defined by $\wp(x) = x^p - x$ for each $x \in k$. For each $x \in k^\times$, write $k_x \stackrel{\mathrm{def}}{=} x \cdot \mathrm{Im}(\wp) \subseteq k$. Suppose that $G_k^p \neq \{1\}$. Then it holds that*

$$\bigcap_{x \in k^\times} k_x \;=\; \{0\}.$$

Proof. Suppose that there exists a nonzero element $\in \bigcap_{x \in k^\times} k_x$. Then it follows immediately from the various definitions involved that $k^\times \subseteq \mathrm{Im}(\wp)$. On the other hand, since $G_k^p \neq \{1\}$, it holds that $\mathrm{Im}(\wp) \subseteq k$ is a proper $\mathbb{F}_p$-subspace. This is a contradition. Thus, we conclude that

$$\bigcap_{x \in k^\times} k_x \ = \ \{0\}.$$

This completes the proof of Lemma 8.2. □

Lemma 8.3. *Let n be a positive integer. In the case where* $\mathrm{char}(K) = 0$*, suppose that $n \leq \frac{pe}{p-1}$. Decompose n as a product $p^N \cdot n'$, where N is a nonnegative integer; n' a positive integer. Then it holds that*

$$U_{n',K}^{p^N} \ \subseteq \ U_{n,K}.$$

Moreover, if $n < \frac{pe}{p-1}$ in the case where $\mathrm{char}(K) = 0$*, then the p^N-th power map $U_{n',K} \to U_{n,K}$ induces the p^N-th power map $k = U_{n',K}/U_{n'+1,K} \to U_{n,K}/U_{n+1,K} = k$ on k, where the identifications $k = U_{n',K}/U_{n'+1,K}$ and $k = U_{n,K}/U_{n+1,K}$ are determined by a uniformizer of K.*

Proof. It follows immediately from the various definitions involved that we may assume without loss of generality that $N = 1$. Then Lemma 8.3 follows immediately from [3], Chapter I, (5.6), Proposition; [3], Chapter I, (5.7), Proposition. □

Proposition 8.4. *Let n be a positive integer. Suppose that $G_k^p \neq \{1\}$. In the case where* $\mathrm{char}(K) = 0$*, suppose, moreover, that $n < \frac{pe}{p-1}$. Then the equality*

$$U_{n+1,K} \ = \ \bigcap_{K \subseteq L} (U_{n+1,K} \cdot N_{L/K}(U_{n,L})) \cap U_{n,K}$$

holds, where $K \subseteq L$ $(\subseteq K^{\mathrm{sep}})$ ranges over the totally ramified cyclic extensions of degree p. In particular, the equality

$$U_{n+1,K} \ = \ \bigcap_{K \subseteq L} U_{n+1,K} \cdot N_{L/K}(U_{\psi_{L/K}(n),L})$$

holds, where $K \subseteq L$ $(\subseteq K^{\mathrm{sep}})$ ranges over the totally ramified cyclic extensions of degree p.

Proof. In light of Lemma 2.1, by replacing K by the p-adic completion of K, we may assume without loss of generality that K is complete. Fix a uniformizer π_K of K. Recall that π_K determines a natural identification of $U_{n,K}/U_{n+1,K}$ and k. Let $x \in k^\times$. In the remainder, by a slight abuse of notation, we also write x for the Teichmüller representative of x. Write $k_x \overset{\mathrm{def}}{=} x \cdot \mathrm{Im}(\wp) \subseteq k$. Then it follows immediately from Lemma 8.2 that it suffices to construct a totally ramified cyclic extension $K \subseteq L_x$ of degree p such that the image of $(U_{n+1,K} \cdot N_{L_x/K}(U_{n,L_x})) \cap U_{n,K}$ via the natural quotient $U_{n,K} \twoheadrightarrow k$ coincides with k_x $(\subseteq k)$.

First, suppose that n is coprime to p. Write $K \subseteq K_x$ for the totally ramified cyclic extension of degree p asssociated to the Artin-Schreier equation

$$y^p - y + \frac{1}{x \cdot \pi_K^n} \ = \ 0,$$

where y denotes an indeterminate [cf. Proposition 8.1, (ii)]. Fix a root λ of this equation and a pair of positive integers c, d such that c is odd, and $-nc + pd = 1$. Write $\pi_{K_x} \overset{\mathrm{def}}{=} -\lambda^c \cdot \pi_K^d \cdot x^{\frac{c}{p}}$. Then π_{K_x} is a uniformizer of K_x such that $N_{K_x/K}(\pi_{K_x}) = \pi_K$ [cf. Proposition 8.1, (i)]. By using π_{K_x}, we identify $U_{n,K_x}/U_{n+1,K_x}$ and k. Then it follows immediately from Proposition 2.4, (i), (ii), (iii), (iv), (v), together with Proposition 8.1, (ii), that

$$N_{K_x/K}(U_{n,K_x}) \subseteq U_{n,K}, \quad N_{K_x/K}(U_{n+1,K_x}) \subseteq U_{n+1,K},$$

and $N_{K_x/K}$ induces a homomorphism $k = U_{n,K_x}/U_{n+1,K_x} \to U_{n,K}/U_{n+1,K} = k$ that maps

$$z \mapsto z^p - x^{1-\frac{1}{p}} \cdot z = x \cdot ((x^{-\frac{1}{p}}z)^p - x^{-\frac{1}{p}}z).$$

In particular, the image of $(U_{n+1,K} \cdot N_{K_x/K}(U_{n,K_x})) \cap U_{n,K} = U_{n+1,K} \cdot N_{K_x/K}(U_{n,K_x})$ via the natural quotient $U_{n,K} \twoheadrightarrow k$ coincides with k_x ($\subseteq k$).

Next, we consider the general case. Decompose n as the product $p^N \cdot n'$, where N is a nonnegative integer; n' a positive integer coprime to p. Write $K \subseteq M_x$ for the totally ramified cyclic extension of degree p associated to the Artin-Schreier equation

$$y^p - y + \frac{1}{x^{\frac{1}{p^N}} \cdot \pi_K^{n'}} = 0.$$

Note that, in light of Lemma 8.3, together with the perfectness of k, we have the following commutative diagram:

$$\begin{array}{ccccccccc} k & = & U_{n',M_x}/U_{n'+1,M_x} & \xrightarrow{\sim} & (U_{n,M_x} \cap N^{-1}_{M_x/K}(U_{n,K}))/U_{n+1,M_x} & = & U_{n,M_x}/U_{n+1,M_x} & = & k \\ & & \downarrow & & \downarrow & & & & \\ k & = & U_{n',K}/U_{n'+1,K} & \xrightarrow{\sim} & U_{n,K}/U_{n+1,K} & = & k, & & \end{array}$$

where the horizontal arrows denote the isomorphisms induced by the p^N-th power maps; the vertical arrows denote the natural homomorphisms induced by the norm map $N_{M_x/K}$. Thus, it follows immediately from the discussion in the previous paragraph that the image of the right-hand vertical arrow [in the above commutative diagram] coincides with $x \cdot \mathrm{Im}(\wp)$. This completes the proof of Proposition 8.4. □

Here, we recall an elementary property of the p-adic logarithmic map associated to a mixed characteristic complete discrete valuation field of residue characteristic p.

Lemma 8.5. *Suppose that* $\mathrm{char}(K) = 0$, *and K is complete. Let n be a positive integer such that* $n > \frac{e}{p-1}$. *Then the p-adic logarithm map determines an isomorphism*

$$U_{n,K} \xrightarrow{\sim} \mathfrak{m}_K^n.$$

Proof. Lemma 8.5 follows immediately from a similar argument to the argument applied in the proof of [27], Chapter II, Proposition 5.5. □

Next, by applying the computation of the intersection of certain subgroups of $K^\times$ associated to the images of norm maps discussed above, in the case where k is p-quasi-finite, we investigate a relation between $U_{n,K}$ and $(G_K^{\mathrm{ab}})^n$ via the reciprocity map.

Definition 8.6. Suppose that k is a p-quasi-finite field of characteristic p. Then:

(i) We shall write

$$r_w : U_{1,K} \hookrightarrow (G_K^p)^{\mathrm{ab}}$$

for the injective homomorphism obtained by restricting the reciprocity map $r_K^p : K^\times \to (G_K^p)^{\mathrm{ab}}$ to $U_{1,K}$ [cf. Theorems 2.13; 3.5].

(ii) Let n be a positive integer. Then we shall write

$$r_w^n : U_{n,K}/U_{n+1,K} \longrightarrow (G_K^{\mathrm{ab}})^n/(G_K^{\mathrm{ab}})^{n+1}$$

for the natural homomorphism induced by r_w [cf. Theorem 2.16]. Here, we note that $(G_K^{\mathrm{ab}})^m = ((G_K^p)^{\mathrm{ab}})^m$ for each positive integer m.

Proposition 8.7. *Let n be a positive integer. Suppose that k is a p-quasi-finite field of characteristic p. In the case where* $\mathrm{char}(K) = 0$*, suppose, moreover, that $n < \frac{pe}{p-1}$. Then it holds that*

$$U_{n,K} = r_w^{-1}((G_K^{\mathrm{ab}})^n).$$

Proof. Note that there is nothing to prove in the case where $n = 1$. Then, in light of induction on n, it suffices to verify that, if $\mathrm{char}(K) = p$ (respectively, $\mathrm{char}(K) = 0$), then r_w^m is injective for each positive integer m (respectively, $m < \frac{pe}{p-1}$). Observe that, for each totally ramified cyclic extension $K \subseteq L$ of degree p and each positive integer m, it follows immediately from Proposition 2.15, (v), that r_w induces an isomorphism

$$U_{m,K}/U_{m+1,K} \cdot N_{L/K}(U_{\psi_{L/K}(m),L}) \xrightarrow{\sim} \mathrm{Gal}(L/K)^m/\mathrm{Gal}(L/K)^{m+1}.$$

Thus, by varying the totally ramified cyclic extensions $K \subseteq L$ ($\subseteq K^{\mathrm{sep}}$) of degree p, we conclude from Proposition 8.4 that r_w^m is injective. This completes the proof of Proposition 8.7. □

Next, in order to state our main theorem in the present section, we introduce some notions concerning mixed characteristic Henselian discrete valuation fields and their absolute Galois groups.

Definition 8.8 ([22], Definition 3.6, (i), (ii))**.** Suppose that K is a mixed characteristic complete discrete valuation field, and k is a strongly p-quasi-finite field of characteristic p. Then:

(i) For each open subgroup $H \subseteq G_K$, write $K \subseteq K_H$ for the corresponding finite field extension;

$$\mathrm{Tor}_p(H)$$

for the subgroup that arises as the image of U_{1,K_H} via the reciprocity map $K_H^\times \to (H^p)^{\mathrm{ab}}$ [cf. Theorem 2.13]. Note that $\mathrm{Tor}_p(H)$ may be reconstructed, in a purely group-theoretic way, from the pair $I_{K_H} \subseteq H$ [cf. Theorem 7.2, (iii)]. We shall refer to $\mathrm{Tor}_p(H)$ as the *p-toral portion* of H. In particular, by applying the p-adic logarithm $U_{1,K_H} \to K_H$, we obtain a natural isomorphism

$$\lambda_H : \mathrm{Tor}_p(H) \otimes \mathbb{Q}_p \xrightarrow{\sim} K_H.$$

(ii) Let I be a collection of open subgroups that form a basis of the topology of G_K. Then we shall refer to a collection $\{N_H\}_{H \in I}$ as a *uniformly p-toral neighborhood* of G_K [indexed by I] if there exist nonnegative integers a, b such that, for each $H \in I$, it holds that

$$N_H \subseteq \mathrm{Tor}_p(H) \otimes \mathbb{Q}_p$$

is a subgroup such that

$$p^a \cdot \mathcal{O}_{K_H} \subseteq \lambda_H(N_H) \subseteq p^{-b} \cdot \mathcal{O}_{K_H} \subseteq K_H.$$

Definition 8.9 ([22], Definitions 3.1, (iv); 3.6, (iii))**.** Let K_1, K_2 be mixed characteristic complete discrete valuation fields with strongly p-quasi-finite residue fields of characteristic p;

$$\phi : G_{K_1} \xrightarrow{\sim} G_{K_2}$$

an isomorphism of profinite groups. For $i = 1, 2$, write

$$C_i$$

for the underlying topological G_{K_i}-module of the p-adic completion of K_i^{sep};

$$\chi_i : G_{K_i} \longrightarrow \mathbb{Z}_p^\times$$

for the p-adic cyclotomic character associated to K_i. Then we shall say that:

(i) ϕ is *HT-type* if the topological G_{K_1}-module obtained by composing ϕ with the natural action of G_{K_2} on C_2 is isomorphic to the topological G_{K_1}-module C_1.

(ii) ϕ is *CHT-type* if ϕ is of HT-type, and $\chi_1 = \chi_2 \circ \phi$.

(iii) ϕ is *RF-preserving* [i.e., "ramification filtration preserving"] if, for each positive integer n, it holds that $\phi(G_{K_1}^n) = G_{K_2}^n$.

(iv) ϕ is *uniformly p-toral* if G_{K_1} admits a uniformly p-toral neighborhood $\{N_H\}_{H \in I}$ such that $\{N_{\phi(H)} \stackrel{\mathrm{def}}{=} \phi(N_H)\}_{\phi(H) \in I^\phi}$ — where I^ϕ denotes the collection of open subgroups $\{\phi(H) \subseteq G_{K_2}\}_{H \in I}$ — forms a uniformly p-toral neighborhood of G_{K_2}.

(v) ϕ is *geometric* if ϕ arises from an isomorphism $K_2 \xrightarrow{\sim} K_1$ of fields.

Definition 8.10 ([8], Definition 1.2, (ii))**.** We shall say that the Henselian discrete valuation field K is a *quasi-p-adic local field* if K is complete, of mixed characteristic, and the residue field k of K is algebraic over the prime field.

Remark 8.10.1. Suppose that K is a quasi-p-adic local field. Then:

(i) One may easily verify that G_k is isomorphic to a closed subgroup of $\widehat{\mathbb{Z}}$. In particular, if k is a p-quasi-finite field [or G_k is not a pro-prime-to-p group], then k is a strongly p-quasi-finite field automatically.

(ii) The closed subgroup $I_K \subseteq G_K$ may be reconstructed, in a purely group-theoretic way, from the underlying topological group structure of G_K [cf. Remark 1.7.1].

Finally, we prove our main theorem in the present section, i.e., a Neukirch-Uchida-type/an absolute version of the Grothendieck Conjecture-type result for quasi-p-adic local fields with p-quasi-finite residues.

Theorem 8.11. *Let K_1, K_2 be quasi-p-adic local fields with p-quasi-finite residue fields of characteristic p;*

$$\phi : G_{K_1} \xrightarrow{\sim} G_{K_2}$$

an isomorphism of profinite groups. Then the following conditions are equivalent:

(i) ϕ is HT-type.

(ii) ϕ *is CHT-type.*

(iii) ϕ *is RF-preserving.*

(iv) ϕ *is uniformly p-toral.*

(v) ϕ *is geometric.*

Proof. First, we note that the residue fields of K_1 and K_2 are strongly p-quasi-finite fields [cf. Remark 8.10.1, (i)]. Then the equivalence of conditions (i), (ii) follows immediately from Proposition 3.7, together with the various definitions involved. The equivalence of conditions (ii), (v) follows from [26], Corollary 1.10, together with Remark 8.10.1, (ii). Next, observe that it is immediate that condition (v) implies conditions (iii), (iv). On the other hand, it follows immediately from a similar argument to the argument applied in the proof of [22], Corollary 3.7, that condition (iv) implies condition (i). Finally, suppose that condition (iii) holds. For each $i \in \{1, 2\}$ and each open subgroup $H_i \subseteq G_{K_i}$, write $K_i \subseteq K_{H_i}$ for the finite field extension corresponding to $H_i \subseteq G_{K_i}$; e_{H_i} for the absolute ramification index of K_{H_i}. Let I_1 be a collection of open subgroups $H_1 \subseteq G_{K_1}$ such that $p-1$ divides e_{H_1}, and $e_{H_1} \geq 2$. Write I_2 for the collection $\{\phi(H_1) \subseteq G_{K_2}\}_{H_1 \in I_1}$ of open subgroups of G_{K_2}. Note that, for each $i \in \{1, 2\}$, I_i forms an open neighborhood of G_{K_i}. Moreover, we observe that, for any $H_2 \in I_2$, it holds that $p-1$ divides e_{H_2}, and $e_{H_2} \geq 2$ [cf. [25], Theorem A]. Next, for each $i \in \{1, 2\}$ and each $H_i \in I_i$, write

$$N_{H_i} \;\subseteq\; \mathrm{Tor}_p(H_i) \otimes \mathbb{Q}_p \;\stackrel{\sim}{\to}\; K_{H_i}$$

for the subgroup that arises as the image of

$$U_{1+\frac{e_{H_i}}{p-1}, K_{H_i}}$$

via the p-adic logarithm map. Here, we observe that since $e_{H_i} \geq 2$, it holds that $1 + \frac{e_{H_i}}{p-1} < \frac{pe_{H_i}}{p-1}$. Let $H_1 \in I_1$ be an element. Write $H_2 \stackrel{\mathrm{def}}{=} \phi(H_1)$. Then since condition (iii) holds, it follows immediately from Theorem 7.2, (iii); Proposition 8.7, together with [25], Theorem A, that ϕ induces an isomorphism

$$N_{H_1} \;\stackrel{\sim}{\to}\; N_{H_2}.$$

On the other hand, for each $i \in \{1, 2\}$, since $\frac{e_{H_i}}{p-1} < 1 + \frac{e_{H_i}}{p-1}$, it follows immediately from Lemma 8.5 that

$$p^2 \mathcal{O}_{K_{H_i}} \;\subseteq\; N_{H_i} \;=\; \mathfrak{m}_{K_{H_i}}^{1+\frac{e_{H_i}}{p-1}} \;\subseteq\; \mathcal{O}_{K_{H_i}}.$$

In particular, $\{N_{H_1}\}_{H_1 \in I_1}$ and $\{N_{H_2}\}_{H_2 \in I_2}$ form respective uniformly p-toral neighborhoods of G_{K_1}, G_{K_2}. Thus, we conclude that ϕ is uniformly p-toral, hence that condition (iv) holds. This completes the proof of Theorem 8.11. □

Remark 8.11.1. In the case where k is finite, Abrashkin proved a positive characteristic analogue of the equivalence of conditions (iii), (v) in Theorem 8.11 [cf. [1]]. Thus, in light of Theorem 8.11, it is natural to pose the following question:

> Question: Let K_1, K_2 be complete discrete valuation fields of characteristic p whose residue fields are p-quasi-finite fields and algebraic over the prime field; $\phi : G_{K_1} \stackrel{\sim}{\to} G_{K_2}$ an isomorphism of profinite groups that preserves the respective ramification filtrations. Then does ϕ arise from a [necessarily, unique — cf. [9], Theorem, (4)] field isomorphism $K_2 \stackrel{\sim}{\to} K_1$?

Acknowledgements

The authors would like to thank Professor Akio Tamagawa for his helpful comment that led us to obtain a simplified proof of Proposition 1.5, (iv). The first author was supported by JSPS KAKENHI Grant Number 20K14285. This research was also supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University, as well as by the International Research Center for Next Generation Geometry [a research center affiliated with the Research Institute for Mathematical Sciences].

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