Variational proof of the existence of periodic orbits in the spatial Hill problem and its constrained problems

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Abstract

The Hill problem models the motion of a particle near a planet. In this paper, we show the existence of symmetric periodic orbits in the spatial Hill problem by using the variational method. We also study the problem under a constraint on a prescribed plane and show the existence of periodic orbits in the problem. The obtained orbits are applicable to artificial satellites around the Earth.

1 Introduction and Main Results

By using a variational method, Chenciner and Montgomery showed the existence of a remarkable periodic orbit called the figure-eight orbit (see [3]). Their result has led to a lot of works on the n-body problem and a number of periodic orbits has been shown to exist. Compared with the n-body problem, there are few results on the the restricted three-body problem (R3BP) using the variational methods because the technical parts of the level estimates for the R3BP are more difficult. In [7], Moeckel showed the existence of the transit orbit in the R3BP for regions from around the Earth to around the Moon. The result in [8] yields the existence of orbits realizing symbolic sequences in the Sitnikov problem. Arioli et al showed the existence of periodic orbits revolving around Jupiter in [1]. Chen proved the existence of the orbits moving away from the center in [2]. The Hill problem models motion of an asteroid or artificial satellite close to the second primary in the R3BP.

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Particles around the Earth are the most affected by the gravitational force of the Earth. Thus the motion is modeled by the Kepler problem

$$\ddot{X} = -\frac{X}{(X^2+Y^2+Z^2)^{3/2}}$$
$$\ddot{Y} = -\frac{Y}{(X^2+Y^2+Z^2)^{3/2}}$$
$$\ddot{Z} = -\frac{Z}{(X^2+Y^2+Z^2)^{3/2}}.$$

Other forces which affect the particles are the gravitational force by the Sun, the Coriolis and the centrifugal force due to the Earth revolution. The model which involve these force is the spatial Hill problem:

$$\begin{aligned}\ddot{X} &= 2\dot{Y} + 3X - \frac{X}{(X^2+Y^2+Z^2)^{3/2}}\\ \ddot{Y} &= -2\dot{X} - \frac{Y}{(X^2+Y^2+Z^2)^{3/2}}\\ \ddot{Z} &= -Z - \frac{Z}{(X^2+Y^2+Z^2)^{3/2}}.\end{aligned} \tag{1}$$

The Hill problem is more accurate than the Kepler problem for particles around the Earth like artificial satellites(see [5, 9] for more detail). This problem has been studied to design orbits of space probes. See [6] for example.

In this paper, we show the existence of several symmetric periodic orbits in the Hill problem. Let

$$L_{X+} = \{(X,0,0) \mid X>0\}, \qquad L_{X-} = \{(X,0,0) \mid X<0\},$$
$$L_{Y+} = \{(0,Y,0) \mid Y>0\}, \qquad L_{Y-} = \{(0,Y,0) \mid Y<0\},$$
$$P_{XZ} = \{(X,0,Z) \mid (X,Z) \in \mathbb{R}\backslash\{(0,0)\}\}, \quad P_{YZ} = \{(0,Y,Z) \mid (Y,Z) \in \mathbb{R}\backslash\{(0,0)\}\}.$$

Let $T_0 > 0$ be the constant determined by $\cos T_0 = T_0$, which is approximately 0.739.

Theorem 1 *For the spatial Hill problem* (1)*, the followings hold.*

(i). *For each $0 < T < 1$, there is a $2T$-periodic orbit satisfying $\boldsymbol{q}(0) \in L_{X+}, \boldsymbol{q}(T) \in L_{X-}$.*

(ii). *For each $0 < T < 1$, there is a $4T$-periodic orbit satisfying $\boldsymbol{q}(0) \in L_{X+}, \boldsymbol{q}(T) \in L_{Y+}$.*

(iii). *For each $0 < T < T_0$, there is a $4T$-periodic orbit satisfying $\boldsymbol{q}(0) \in L_{X+}, \boldsymbol{q}(T) \in L_{Y-}$.*

(iv). *For each $0 < T < T_0$, there is a $4T$-periodic orbit satisfying $\boldsymbol{q}(0) \in L_{X+}, \boldsymbol{q}(T) \in P_{YZ}$.*

(v). *For each* $0 < T < 1$, *there is a* $2T$*-periodic orbit satisfying* $\boldsymbol{q}(0) \in L_{Y+}, \boldsymbol{q}(T) \in L_{Y-}$.

(vi). *For each* $0 < T < T_0$, *there is a* $4T$*-periodic orbit satisfying* $\boldsymbol{q}(0) \in L_{Y+}, \boldsymbol{q}(T) \in P_{XZ}$.

See figure 1.

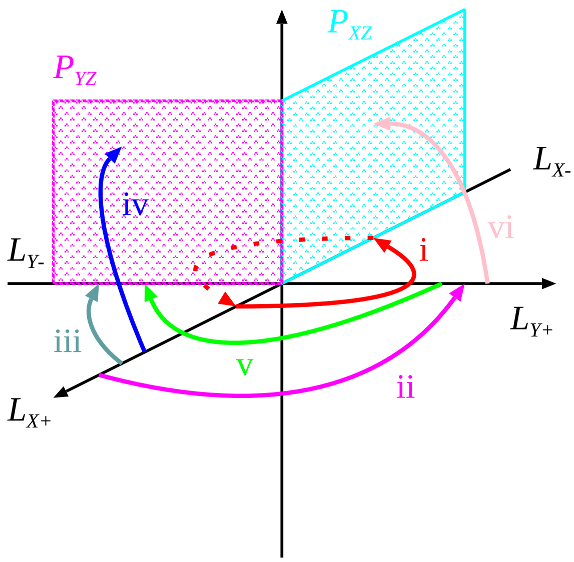

Figure 1: Boundary conditions

To prove this theorem, we use a variational method. The Lagrangian for the Hill problem (1) is

$$\mathcal{L} = \frac{\dot{X}^2}{2} + \frac{\dot{Y}^2}{2} + \frac{\dot{Z}^2}{2} + X\dot{Y} - Y\dot{X} + \frac{3X^2}{2} - \frac{Z^2}{2} + \frac{1}{\sqrt{X^2+Y^2+Z^2}}.$$

The Hill problem is equivalent to the variational problem with respect to the action functional

$$\mathcal{A}_T = \int_0^T \mathcal{L} dt.$$

The result of this paper is organized as follows. Next section, we show the coercivity condition for the existence of a minimizer under the boundary conditions corresponding to each orbit in Theorem 1. We also show that the obtained minimizers have no collision by applying our previous result. In Section 3, we state the reversibility of the Hill problem and show that the obtained minimizers are periodic orbits. In the viewpoint of an application to the trajectory design for artificial satellites, we need orbits on a prescribed plane. For example, geosynchronous satellites move directly above the Earth's equator. We also prove the existence of several periodic orbits in the constrained problem. Section 4 is devoted to the study of the existence of periodic orbits of the holonomic constraint system on a prescribed plane. Last section, we show the numerical solutions.

2 Coercivity and the existence of minimizers

For subsets $D_1, D_2 \subset \mathbb{R}^3$, let

$$\Omega(D_1, D_2; T) = \{\gamma \in H^1([0,T], \mathbb{R}^3 \backslash \{\mathbf{0}\}) \mid \gamma(0) \in D_1, \gamma(T) \in D_2\}.$$

Here H^1 denotes the Sobolev space. By taking $L_{X+}, \ldots, P_{YZ}$ in Section 1 as D_1 and D_2, we will show the existence of a minimizer of $\mathcal{A}_T|_{\Omega(D_1,D_2;T)}$. We call the functional $\mathcal{A}_T|_{\Omega(D_1,D_2;T)}$ coercive if $\mathcal{A}_T|_{\Omega(D_1,D_2;T)}(\gamma) \to \infty$ as $\|\gamma\|_{H^1} \to \infty (\gamma \in \Omega(D_1, D_2; T))$. It is well-known that there is a minimizer of $\mathcal{A}_T|_{\Omega(D_1,D_2;T)}$ on $\overline{\Omega(D_1, D_2; T)(\gamma)}$ if the functional is coercive.

By changing variables

$$X = (\cos t)x + (\sin t)y, Y = -(\sin t)x + (\cos t)y, Z = z,$$

the Lagrangian becomes

$$\begin{aligned}\mathcal{L}_{\mathrm{rot}} = &\frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \\ &+ \frac{3(\cos^2 t)x^2}{2} - \frac{3(\cos^2 t)y^2}{2} + 3\cos(t)\sin(t)xy - \frac{x^2}{2} + y^2 \\ &- \frac{z^2}{2} + \frac{1}{\sqrt{x^2+y^2+z^2}}.\end{aligned}$$

We estimate the terms on its second line by using the polar coordinate $(x, y) = r(\cos\theta, \sin\theta)$:

$$\begin{aligned}&\frac{3(\cos^2 t)x^2}{2} - \frac{3(\cos^2 t)y^2}{2} + 3\cos(t)\sin(t)xy - \frac{x^2}{2} + y^2 \\ &= \frac{3r^2\cos(2t-2\theta)}{4} + \frac{1}{4}r^2 \geq -\frac{1}{2}r^2 = -\frac{1}{2}(x^2+y^2).\end{aligned}$$

Therefore, we get

$$\mathcal{L}_{\mathrm{rot}} \geq \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{1}{2}x^2 - \frac{1}{2}y^2 - \frac{1}{2}z^2 + \frac{1}{\sqrt{x^2+y^2+z^2}} =: \tilde{\mathcal{L}}.$$

Define

$$\tilde{\mathcal{A}_T} = \int_0^T \tilde{\mathcal{L}} dt.$$

If $\tilde{A}_T|_{\Omega(D_1,D_2;T)}$ is coercive, so is $\mathcal{A}_T|_{\Omega(D_1,D_2;T)}$.

Let

$$(x(0), y(0)) \cdot (x(T), y(T)) = |(x(0), y(0))||(x(T), y(T))| \cos\rho.$$

For example, in the case of $D_1 = L_{X+}, D_2 = L_{X-}$ which corresponds to (i) in Theorem 1, the boundary condition is represented by

$$\begin{aligned}
&x(0) > 0, y(0) = z(0) = 0\\
&(\cos T)x(T) + (\sin T)y(T) < 0, -(\sin T)x(T) + (\cos T)y(T) = 0, z(T) = 0\\
&(x(T), y(T), z(T)) \in \{\xi(\cos(T+\pi), \sin(T+\pi), 0) \mid \xi > 0\}.
\end{aligned}$$

Hence, $\rho = T + \pi$.

Note that

$$\int_0^T |\dot{\boldsymbol{x}}|^2 dt \geq \frac{1}{T}\left(\int_0^T |\dot{\boldsymbol{x}}| dt\right)^2.$$

Let

$$r_{\max} = \max_{t\in[0,T]} |\boldsymbol{x}(t)|.$$

If $|\rho| < \pi/2$,

$$\tilde{\mathcal{A}_T} \geq \frac{1}{2T}(r_{\max}^2 \sin^2\rho) - \frac{1}{2}Tr_{\max}^2 = \frac{r_{\max}^2}{2}(\frac{\sin^2\rho}{T} - T).$$

If $\pi/2 < |\rho| < \pi$,

$$\tilde{\mathcal{A}_T} \geq \frac{1}{2T}(r_{\max}^2) - \frac{1}{2}Tr_{\max}^2 = \frac{r_{\max}^2}{2}(\frac{1}{T} - T).$$

In the case that

$$\|\boldsymbol{x}\|_{H^1} = (\|\dot{\boldsymbol{x}}\|_{L^2}^2 + \|\boldsymbol{x}\|_{L^2}^2)^{1/2} \to \infty$$

and that $\|\boldsymbol{x}\|_{L^2}^2 < \infty$, $\tilde{\mathcal{A}_T}$ diverges to infinity since

$$\tilde{\mathcal{A}_T} \geq \frac{1}{2}\|\dot{\boldsymbol{x}}\|_{L^2}^2 - \frac{1}{2}\|\boldsymbol{x}\|_{L^2}^2.$$

In the case of $\|\boldsymbol{x}\|_{L^2}^2 \to \infty$, $r_{\max} \to \infty$, and hence $\mathcal{A}_T$ diverges if $T < |\sin\rho|(|\rho| < \pi/2)$ or $T < 1(|\rho| > \pi/2)$. Now we adapt these computations to our setting in Theorem 1.

(i). $L_{X+} \to L_{X-}$: since $\rho = \pi + T$, $\mathcal{A}_T$ is coercive if $0 < T < 1$;

(ii). $L_{X+} \to L_{Y+}$: since $\rho = \pi/2 + T$, $\mathcal{A}_T$ is coercive if $0 < T < 1$;

(iii). $L_{X+} \to L_{Y-}$: since $\rho = \pi/2 - T$, $\mathcal{A}_T$ is coercive if $0 < T < \sin(\pi/2 - T) = \cos T$;

(iv). $L_{X+} \to P_{YZ}$: since $\rho = \pi/2 - T$, $\mathcal{A}_T$ is coercive if $0 < T < \sin(\pi/2 - T) = \cos T$;

(v). $L_{Y+} \to L_{Y-}$:since $\rho = \pi + T$, $\mathcal{A}_T$ is coercive if $0 < T < 1$;

(vi). $L_{Y+} \to P_{XZ}$:since $\rho = \pi/2 - T$, $\mathcal{A}_T$ is coercive if $0 < T < \sin(\pi/2 - T) = \cos T$.

The structure of the collision singularity $(X, Y, Z) = (0, 0, 0)$ is essential same as ones of the restricted three-body problem. We [4] established a method to avoid the collision singularities in the restricted three-body problem. We can apply the method to the Hill problem and show that the obtained minimizers have no collision.

3 Reversibility

Consider ordinary differential equations:

$$\dot{\boldsymbol{x}} = F(\boldsymbol{x}) \qquad (\boldsymbol{x} \in \mathbb{R}^n) \tag{2}$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth function.

Definition 1 (Reversible) *Let R be an linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$. If $F(\boldsymbol{x})$ satisfies*

$$F(R\boldsymbol{x}) + RF(\boldsymbol{x}) = \boldsymbol{0},$$

then (2) is said to be reversible with respect to R.

With a simple calculation, we get the following proposition:

Proposition 1 *In a reversible system with respect to R, if $\boldsymbol{x}(t)$ is a solution of Eq. (2), so is $R\boldsymbol{x}(-t)$.*

We define

$$\mathrm{Fix}(R) = \{\boldsymbol{x} \in \mathbb{R}^n \mid R\boldsymbol{x} = \boldsymbol{x}\}.$$

It is easy to see that, for a solution $\boldsymbol{x}(t)$ of (2) and a real value $s \in \mathbb{R}$, $\boldsymbol{x}(s) \in \mathrm{Fix}(R)$ is satisfied if and only if $\boldsymbol{x}(s + t) = R\boldsymbol{x}(s - t)$.

By letting $(V_X, V_Y, V_Z) = (\dot{X}, \dot{Y}, \dot{Z})$, we rewrite the Hill problem (1) as the first order differential equations:

$$\begin{aligned}
\dot{X} &= V_X \\
\dot{Y} &= V_Y \\
\dot{Z} &= V_Z \\
\dot{V}_X &= 2V_Y + 3X - \frac{X}{(X^2 + Y^2 + Z^2)^{3/2}} \\
\dot{V}_Y &= -2V_X - \frac{Y}{(X^2 + Y^2 + Z^2)^{3/2}} \\
\dot{V}_Z &= -Z - \frac{Z}{(X^2 + Y^2 + Z^2)^{3/2}}.
\end{aligned}$$

This system is reversible with respect to the following four linear maps:

$$\begin{aligned} R_1 &= \mathrm{diag}(1, -1, 1, -1, 1, -1) \\ R_2 &= \mathrm{diag}(1, -1, -1, -1, 1, 1) \\ R_3 &= \mathrm{diag}(-1, 1, 1, 1, -1, -1) \\ R_4 &= \mathrm{diag}(-1, 1, -1, 1, -1, 1). \end{aligned}$$

For those linear maps,

$$\begin{aligned} \mathrm{Fix}(R_1) &= \{{}^t(X, Y, Z, V_X, V_Y, V_Z) \mid Y = V_X = V_Z = 0\} \\ \mathrm{Fix}(R_2) &= \{{}^t(X, Y, Z, V_X, V_Y, V_Z) \mid Y = Z = V_X = 0\} \\ \mathrm{Fix}(R_3) &= \{{}^t(X, Y, Z, V_X, V_Y, V_Z) \mid X = V_Y = V_Z = 0\} \\ \mathrm{Fix}(R_4) &= \{{}^t(X, Y, Z, V_X, V_Y, V_Z) \mid X = Z = V_Y = 0\}. \end{aligned}$$

Consider the case (i). From the first variational formula, $\frac{\partial L}{\partial \dot{\boldsymbol{x}}}(0) \cdot L_{X+} = \frac{\partial L}{\partial \dot{\boldsymbol{x}}}(T) \cdot L_{X-} = 0$ Since

$$\frac{\partial L}{\partial \dot{\boldsymbol{x}}} = (\dot{X} - Y, \dot{Y} + X, \dot{Z}),$$

$$\dot{X}(0) = 0, \dot{X}(T) = 0$$

Therefore, $\begin{pmatrix} \boldsymbol{x}(0) \\ \dot{\boldsymbol{x}}(0) \end{pmatrix}, \begin{pmatrix} \boldsymbol{x}(T) \\ \dot{\boldsymbol{x}}(T) \end{pmatrix} \in \mathrm{Fix}(R_3)$. We have

$$\begin{pmatrix} \boldsymbol{x}(t) \\ \dot{\boldsymbol{x}}(t) \end{pmatrix} = R_3 \begin{pmatrix} \boldsymbol{x}(-t) \\ \dot{\boldsymbol{x}}(-t) \end{pmatrix}, \begin{pmatrix} \boldsymbol{x}(T+t) \\ \dot{\boldsymbol{x}}(T+t) \end{pmatrix} = R_3 \begin{pmatrix} \boldsymbol{x}(T-t) \\ \dot{\boldsymbol{x}}(T-t) \end{pmatrix}.$$

Therefore, we get

$$\begin{aligned} \begin{pmatrix} \boldsymbol{x}(t+2T) \\ \dot{\boldsymbol{x}}(t+2T) \end{pmatrix} &= \begin{pmatrix} \boldsymbol{x}(T+(t+T)) \\ \dot{\boldsymbol{x}}(T+(t+T)) \end{pmatrix} = R_3 \begin{pmatrix} \boldsymbol{x}(T-(t+T)) \\ \dot{\boldsymbol{x}}(T-(t+T)) \end{pmatrix} \\ &= R_3 \begin{pmatrix} \boldsymbol{x}(-t) \\ \dot{\boldsymbol{x}}(-t) \end{pmatrix} = \begin{pmatrix} \boldsymbol{x}(t) \\ \dot{\boldsymbol{x}}(t) \end{pmatrix}. \end{aligned}$$

Hence the obtained orbit is $2T$-periodic. The other cases (ii)-(vi) are similar.

4 Holonomic constraint

In the point of view of an application to orbits of artificial satellites, we need orbits on a prescribed plane. A prescribed plane is not invariant under the flow of the Hill problem in general. Hence we constraint the system to a prescribed plane with an external force like a jet by an artificial satellite.

Let $\boldsymbol{c} = (c_1, c_2, c_3)$ be a unit vector and consider the plane perpendicular to $\boldsymbol{c}$ passing the origin(Figure 2). The holonomic system is represented by the

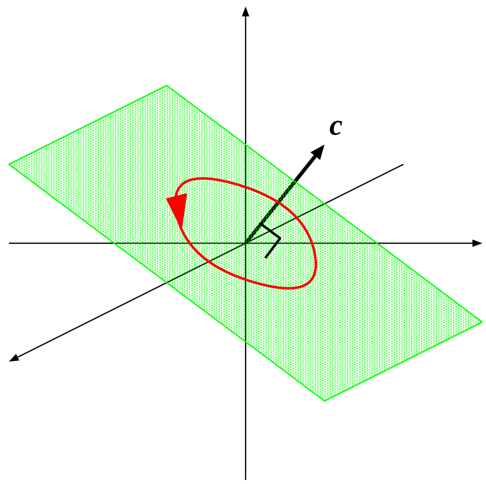

Figure 2: Holonomic constraints

Lagrangian system with the Lagrangian

$$\bar{\mathcal{L}} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + c_3(x\dot{y} - y\dot{x}) + \lambda_1 x^2 + \lambda_2 y^2 + \frac{1}{\sqrt{x^2 + y^2}}$$

where $\lambda_1 \leq 0 \leq \lambda_2$ are the constants determined by

$$\lambda_1 + \lambda_2 = -3c_1^2 + c_3^2 + 2, \quad \lambda_1 \lambda_2 = -\frac{3}{4}c_2^2.$$

The equations are

$$\begin{aligned} \ddot{x} &= 2c_3\dot{y} + 2\lambda_1 x - \frac{x}{(x^2 + y^2)^{3/2}} \\ \ddot{y} &= -2c_3\dot{x} + 2\lambda_2 y - \frac{y}{(x^2 + y^2)^{3/2}}. \end{aligned} \tag{3}$$

Let

$$\begin{aligned} l_{X+} &= \{(X, 0) \mid X > 0\} \\ l_{X-} &= \{(X, 0) \mid X < 0\} \\ l_{Y+} &= \{(0, Y) \mid Y > 0\} \\ l_{Y-} &= \{(0, Y) \mid Y < 0\}. \end{aligned}$$

See Figure 3.

Theorem 2 *For the holonomic system* (3)*, the followings hold:*

(i). *For each* $0 < T < \min\{\pi/2, 1/(c_3^2 - 2\lambda_1)\}$*, there is a* $2T$*-periodic orbit satisfying* $\boldsymbol{q}(0) \in l_{X+}, \boldsymbol{q}(T) \in l_{X-}$.

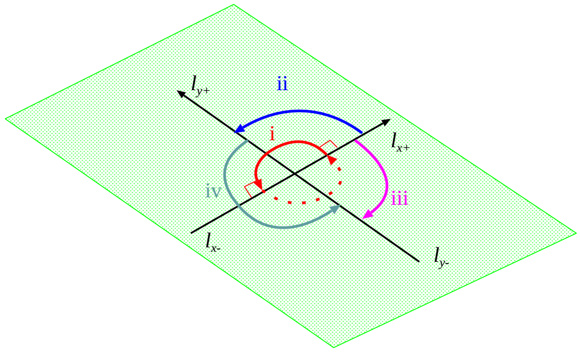

Figure 3: Boundary conditions in the constrained problem

(ii). *For each* $0 < T < \min\{\pi/2, 1/(c_3^2 - 2\lambda_1)\}$*, there is a* $4T$*-periodic orbit satisfying* $\boldsymbol{q}(0) \in l_{X+}, \boldsymbol{q}(T) \in l_{Y+}$.

(iii). *For each* $0 < T < \min\{\pi, T_1\}$*, there is a* $4T$*-periodic orbit* $\boldsymbol{q}(0) \in l_{X+}, \boldsymbol{q}(T) \in l_{Y-}$.

(iv). *For each* $0 < T < \min\{\pi/2, 1/(c_3^2 - 2\lambda_1)\}$*, there is a* $2T$*-periodic orbit satisfying* $\boldsymbol{q}(0) \in l_{Y+}, \boldsymbol{q}(T) \in l_{Y-}$.

The proof is similar as one for Theorem 1.

To apply these orbits for artificial satellites, we need to control it for $\boldsymbol{c}$-direction. But this must be less costly than we use the orbit of the Kepler problem.

5 Numerical computation

We numerically found the periodic solutions. In order to obtain those, we consider the Fourier series of the solutions and compute the Fourier coefficient by using the steepest descent method. Let

$$\boldsymbol{c} = (0.6490, 0.6490, 0.3971).$$

That is the direction of the earth's axis and the obtained periodic orbits are over the equator. Weather satellites move over the equator and its period is one day which is $T = 0.0172$ in our setting. Numerical solutions in Figure 5 corresponds to ones of Theorem 2 (i). The existence guarantees for small $T > 0$ theoretically, but numerical solutions for large $T > 0$ are obtained.

Quasi-zenith satellites are used for GPS system. The orbits are on the plane inclined at an angle of 45° with the earth's axis and their period are one day. In this case,

$$\boldsymbol{c} = (0.2603, 0.2603, 0.9298).$$

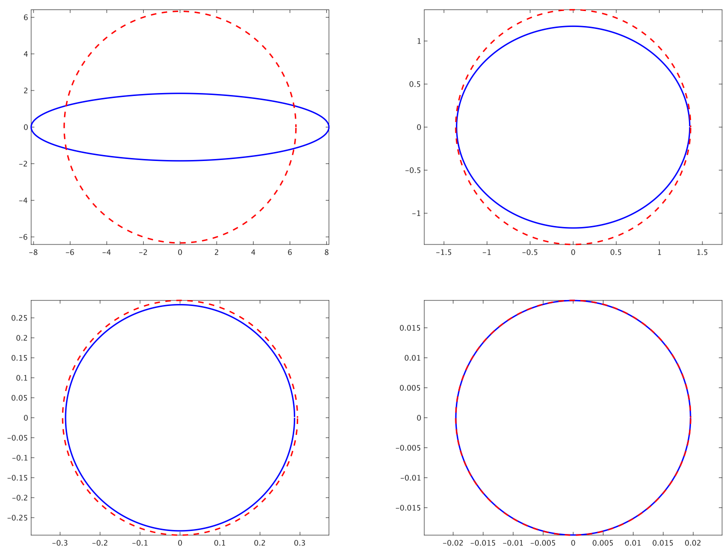

Figure 4: Numerical orbits for $\boldsymbol{c} = (0.6490, 0.6490, 0.3971)$

$T = 100$ (upper left), $T = 10$ (upper right), $T = 1$ (lower left) and $T = 0.0172$ (lower right)

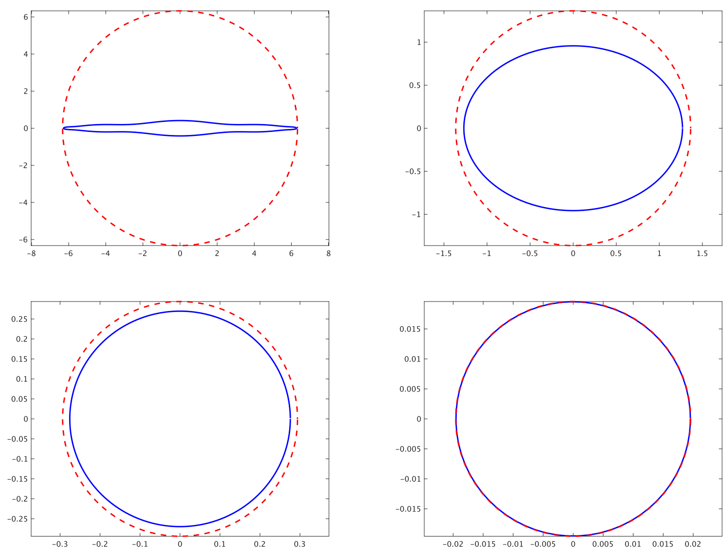

Figure 5: Numerical orbits for $\boldsymbol{c} = (0.2603, 0.2603, 0.9298)$

$T = 100$ (upper left), $T = 10$ (upper right), $T = 1$ (lower left) and $T = 0.0172$ (lower right)

Acknowledgement

Y. K. was supported by JSPS KAKENHI Grant Number JP20J21214. M. S. is supported by JST, PRESTO Grant Number JP1159274 and JSPS KAKENHI Grant Number JP18K03366.

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