



TITLE:

Freak wave in a two-dimensional directional wavefield with bottom topography change. Part 1. Normal incidence wave

AUTHOR(S):

Lyu, Zuorui; Mori, Nobuhito; Kashima, Hiroaki

CITATION:

Lyu, Zuorui ...[et al]. Freak wave in a two-dimensional directional wavefield with bottom topography change. Part 1. Normal incidence wave. *Journal of Fluid Mechanics* 2023, 959: A19.

ISSUE DATE:

2023-03-25

URL:

<http://hdl.handle.net/2433/283113>

RIGHT:

This article has been published in a revised form in *Journal of Fluid Mechanics* <https://doi.org/10.1017/jfm.2023.73>. This version is free to view and download for private research and study only. Not for re-distribution or re-use. © The Author(s), 2023. Published by Cambridge University Press; The full-text file will be made open to the public on 17 September 2023 in accordance with publisher's 'Terms and Conditions for Self-Archiving'; This is not the published version. Please cite only the published version. この論文は出版社版ではありません。引用の際には出版社版をご確認ください。

Freak Wave in a Two-Dimensional Directional Wavefield with Bottom Topography Change: Part I. Normal Incident Wave

Zuorui Lyu¹, Nobuhito Mori^{2,3*} and Hiroaki Kashima⁴

¹Coastal Engineering Laboratory, Dept. of Civil Eng., School of Eng.,

The University of Tokyo, Bunkyo-ku, Tokyo, 113-8656, Japan

²Disaster Prevention Research Institute, Kyoto University, Gokasho, Uji,

Kyoto, 611-0011, Japan

³Swansea University, Bay Campus, Skewen, Swansea SA1 8EN, United

Kingdom

⁴Coastal and Ocean Engineering Department, Port and Airport Research

Institute, 3-1-1 Nagase, Yokosuka, Kanagawa, 239-0826, Japan

* Corresponding author

Email address: mori@oceanwave.jp

Tel.: +81 0774-38-4146. Fax: +81 0774-38-4321.

Abstract

In the propagation and evolution of sea waves, previous studies pointed out that the occurrence of the freak wave height is significantly related to the quasi-resonant four-wave interaction in the modulated waves. From the numerical-experimental study over an uneven bottom, the nonlinear effect caused by the bathymetry change also contributes to the occurrence

of extreme events in the unidirectional waves. To comprehensively analyze the two-dimensional wave field, this study develops an evolution model for a directional random wavefield based on the depth-modified Nonlinear Schrödinger equation, which considers the nonlinear resonant interactions and the wave shoaling the shallow water. Through Monte Carlo simulation, we discuss the directional effect on the four-wave interaction in the wave train and the maximum wave height distribution from deep to shallow water with a slow varying slope. The numerical result indicates that the directional spreading has a dispersion effect on the occurrence of the freak wave height. In shallow water environment, this effect becomes weak, and the bottom topography change is the main influencing factor in the wave evolution.

Key words

Freak wave; depth-modified NLS equation; two-dimensional wavefield; directional random wave; wave shoaling

1 Introduction

With the development of the technology in observation and recording, rogue/ freak/ extreme wave (from now on called “freak wave”) has been generally recognized as a kind of marine disaster instead of a small probability event in special circumstances. The concept of the freak wave was first put up by Draper (1965), and its occurrence behaves as the wave height abnormally exceeds the significant wave height by factor 2. Under different conditions, the occurrence probability of the freak wave clearly varies, which inspires us to think that the influence mechanism in wave height distribution is more complicated than generally expected.

Researches to date indicate that the generation mechanism of the freak wave can be divided into internal and external factors. For the internal factor, the nonlinear interaction plays an important role in the occurrence of freak waves. Through a nonlinear wave evolution model, Benjamin (1967) indicated that modulational instability, a phenomenon leading to the energy concentration in a narrow-spectrum wave train, will make the waveform of surface gravity waves significantly unstable in deep-water. After the '90s, it is considered an important reason for causing freak waves to behave as the four-wave interactions in the nonlinear wave model.

In the study of Janssen (2003), the instability in Benjamin (1967) is actually an example of a non-resonant four-wave interaction in which the carrier wave is phase-locked with the sidebands. Based on the four-wave interactions and random wave statistics, a complete prediction model of freak waves in deep-water is given by Janssen (2003) and Mori and Janssen (2006) and is verified by the wave tank experiments (e.g., Mori et al., 2007; Kashima and Mori, 2019). When it comes to the external factors, the bathymetry effect, the interaction with current, wind force, and others contribute to the evolution of nonlinear wave to varying degrees. The bathymetry effect on the water depth and local topography change have been known as significant influencing factors in the second-order nonlinear wave evolution.

For a more general explanation of freak wave occurrence, we need to consider the synthetic effect of internal and external factors. For a uniform unidirectional wave train, the modulational instability has a critical water depth at $kh = 1.363$ from Benjamin (1967). When $kh < 1.363$, the modulational instability disappears, which implies the occurrence of the freak wave at relatively shallow water may decrease due to the attenuation of four-wave interactions. However, when it comes to a directional wavefield without limiting to collinear instabilities, there is no critical water depth such as $kh = 1.363$ due to the transversal disturbance (Benney and Roskes, 1969; McLean et al., 1981). If we think that the four-wave interactions play an important role in the occurrence of the freak wave, the variation of water depth seems to be a non-negligible influencing factor. From the observation record, such as the World Ocean and the coast in Nikolkina and Didenkulova (2011), the freak wave occurs in deep-water and finite and shallow water. Therefore, further investigation of the evolution of the four-wave interactions in the nonlinear modulated wave with the variation of bottom topography is necessary to study freak wave behaviors both offshore and onshore.

For a weakly-dispersive long-wave train, its propagating process can be summarized by a partial differential equation about time and space. The Nonlinear Schrödinger (NLS) equation (e.g., Zakharov, 1968; Davey and Stewartson, 1974), which can reflect the nonlinear effect caused by modulation interactions, is widely used to describe the amplitude evolution. From its standard format, a constant depth can be extended to a modified form with varying depth considering spatial inhomogeneity. For example, Djordjevic and Redekopp (1978) derived a solution for an envelope-hole soliton moving over an uneven bottom and gave a depth-modified

NLS (dNLS) equation with a slope effect. Variation of the depth in Liu and Dingemans (1989) was divided into different scales, and then they gave evolution equations for modulated wave groups over an uneven bottom in different types. The evolution of modulated wave train over an uneven bottom can also refer to Peregrine (1983), Turpin et al. (1983), and Mei and Benmoussa (1984).

Several numerical studies of the different nonlinear evolution equations have been given, such as the NLS-like equation (Zeng and Trulsen, 2012; Lyu et al., 2021; Kimmoun et al., 2021), the KdV equation (Sergeeva et al., 2011; Majda et al., 2019), Boussinesq equations (Gramstad et al., 2013; Kashima et al., 2014; Zhang et al., 2019) and other nonlinear methods (Viotti and Dias, 2014; Zheng et al., 2020; Lawrence et al., 2021; etc.). From the research mentioned above, a similar conclusion can be derived that the increase of bottom slope angle will give rise to the inhomogeneity of wavefield and lead to the increase of the kurtosis of surface elevation in very shallow water, which equals to the increase of exceeding occurrence probability of extreme wave height $P_m(H_{\max})$. The abrupt slope change, like the demarcation point between the sloping section (deep to shallow) and the flat bottom, gives a significant increase of $P_m(H_{\max})$ as well as the skewness μ_3 of the surface elevation, which suggests the bathymetry effect is more reflected by the second-order effect. If we compare the numerical results in flat bottoms with different water depths, $P_m(H_{\max})$ decreases from deep to medium water, as well as the kurtosis μ_4 of surface elevation, and it corresponds to Benjamin (1967)'s result of the evolution of modulational instability: in shallow water $kh \leq 1.363$, modulational instability disappears and the μ_4 shows an inverse effect on the unidirectional wave train in deep-water case. The variation of $P_m(H_{\max})$ with depth change can also be referred to Mendes et al. (2022), in which the second-order theory provides a physical explanation about wave statistics.

A similar process in shallow water can be verified in experiments. Physical modeling experiments can provide a reference for the steeper slope case and more variety of bottom topography as Trulsen et al. (2012), Kashima et al. (2014), Ma et al. (2014), Bolles et al. (2019), Zhang et al. (2019), Kashima and Mori (2019), Trulsen et al. (2020) and Lawrence et al. (2021, 2022). The value of kurtosis μ_4 and skewness μ_3 show a local maximum near the edge between the sloping bottom and flat bottom (deep to shallow). As the slope angle extremely becomes steep, this local maximum reaches its peak at the abrupt depth transitions in shallow

water (Zheng et al., 2020; Li et al., 2021).

It should be pointed out that most of the numerical and experimental studies to date concentrate on the unidirectional wave train. The wave behaviors in the natural state are a two-dimensional (2D) hydrodynamic problem. The stability of deep-water random waves in 2D space can be referred to as Alber and Saffman (1978). For long-crested irregular waves propagating over 2D bathymetry, Lawrence et al. (2022) discussed the statistical properties of the surface elevation and the velocity field through the numerical simulation by the high-order spectral method. Besides, a 2D wavefield model can provide the directional wave spreading due to wind or current effect and the dispersion on one more horizontal dimension. Recent works show that the four-wave interaction decreases due to the directional dispersion effects. The maximum wave height in a directional wavefield decreases with the unidirectional wave in the numerical simulation through a modified form of NLS equation in Gramstad and Trulsen (2007). The enhancement of kurtosis is significantly suppressed by the increase of directional spread in the directional wave in Waseda (2006), Waseda et al. (2009), Onorato et al. (2009a), and Onorato et al. (2009b). Based on the contribution from the directional bandwidth in the directional spectrum, Mori et al. (2011) gave the theoretical estimation of kurtosis for directional sea states, and the fourth-order cumulant and directional spreading can predict the occurrence probability of freak waves. However, a comprehensive discussion about the bathymetry effect on the 2D modulated wave is still to be supplemented.

To develop the freak wave analysis in 2D wave fields, this study investigates the directional nonlinear modulated wave evolution in an uneven bottom as an expansion work of Lyu et al. (2021). The dNLS equation in 2D form for a slow-varying bottom is applied to establish the numerical model. In a 2D wave basin, we consider the directional dispersion effect as part of the initial value problem and give the discussion to compare the roles in the wave evolution from different parameters such as the slope angle, the water depth, and directional spreading. Random wave statistics and a Monte-Carlo simulation are conducted to give the evolution of nonlinear effect and the distribution of extreme events in two-dimensional space-time (2D+T). Section 2 gives the derivation of the theoretical model and its numerical solving. Section 3 gives the computation result and the discussion, and they are summarized in Section 4. This article concentrates on the case when the wave normally incidents with the contour line

of the bottom, so we do not consider the wave refraction. The oblique wave case will be given in the later work, **Part II**.

2 Methodology

2.1 2D dNLS equation for an uneven bottom

For a 2D flow field, we continue to assume the flow is irrotational, inviscid, and incompressible with a free water surface. A coordinate system (x, y, z) is defined with origin O , as shown in **Figure 1**. Plane Oxy is defined along the quiescent water surface, and z axis is defined vertically upward direction, opposite to gravity acceleration g . An incident directional random wave train comes from an external field, and its principal wave direction is along the x axis. The bottom $z = -h(x, y)$ mainly varies in the principal direction in the region between dashed lines A and B. Velocity potential Φ and free surface elevation η are defined as

$$\Phi = \Phi(x, y, z, t), \quad \eta = \eta(x, y, t). \quad (1)$$

where t represents time.

In the entire flow field, Φ is a solution of the Laplace equation to satisfy continuity:

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0. \quad (2)$$

On the boundary of the free surface $z = \eta(x, y, t)$, Φ and η satisfy the kinematic boundary condition (i.e., free surface equation) and the dynamic boundary condition (i.e., Bernoulli equation):

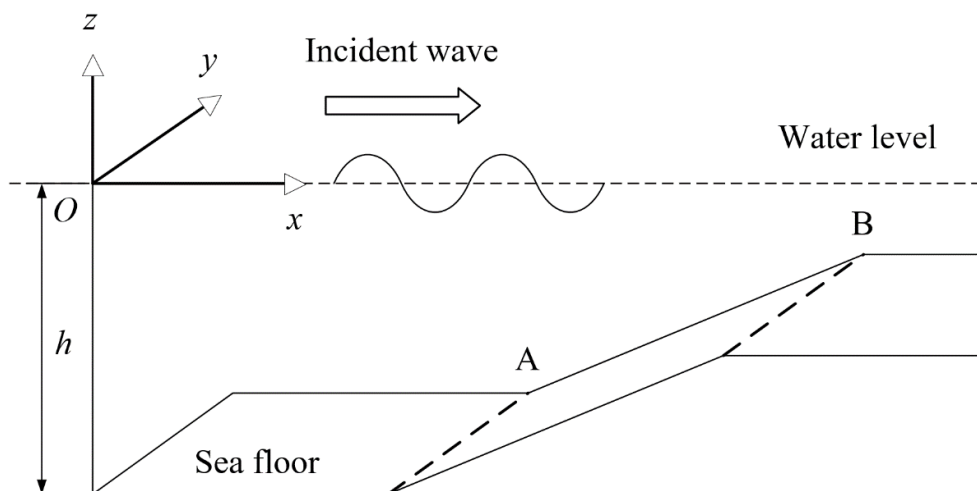


Figure 1 Sketch of the wave propagating over an uneven bottom in a 2D wavefield

$$\frac{\partial \Phi}{\partial z} = \frac{\partial \eta}{\partial t} + \frac{\partial \Phi}{\partial x} \frac{\partial \eta}{\partial x} + \frac{\partial \Phi}{\partial y} \frac{\partial \eta}{\partial y}, \quad z = \eta, \quad (3)$$

$$2 \frac{\partial \Phi}{\partial t} + 2g\eta + \left(\frac{\partial \Phi}{\partial x}\right)^2 + \left(\frac{\partial \Phi}{\partial y}\right)^2 + \left(\frac{\partial \Phi}{\partial z}\right)^2 = 0, \quad z = \eta. \quad (4)$$

At the bottom of the flow field, Φ satisfies the no-flux boundary along the seafloor. If the water depth h is constant at a flat bottom $z = -h$, Φ satisfies the flat bottom equation:

$$\frac{\partial \Phi}{\partial z} = 0, \quad z = -h. \quad (5)$$

If we assume, the bottom is uneven, and water depth varies at $z = -h(x, y)$, Φ satisfies the uneven bottom equation:

$$\frac{\partial \Phi}{\partial z} + \frac{\partial h}{\partial x} \frac{\partial \Phi}{\partial x} + \frac{\partial h}{\partial y} \frac{\partial \Phi}{\partial y} = 0, \quad z = -h(x, y). \quad (6)$$

Based on the periodicity of the time and space in the propagation of gravity waves, wave frequency ω and wave number k satisfy the linear dispersion relation:

$$\omega = \sqrt{gk\sigma}, \quad (7)$$

where $\sigma = \tanh kh$. For a medium that has no temporal variation, carrier wave frequency $\omega = \omega_0$, where subscript 0 represents the constant angular frequency independent of the variable bathymetry. For a flat bottom with a constant water depth h , carrier wave number $k = k_0$ is also constant as ω ; for an uneven bottom, wave number k will be changed because of spatial inhomogeneity due to bottom topography. The change in wave dispersion will also be reflected in the group speed c_g :

$$c_g = \frac{g}{2\omega_0} [\sigma + kh(1 - \sigma^2)]. \quad (8)$$

In other words, k and c_g are functions of h .

For a weakly nonlinear wave train, the modulation parameter comes from the contribution from the small perturbation in high-order harmonic, so we further expand the velocity potential Φ and free surface elevation η into harmonic functions. In this research, we assume the modulation caused by the nonlinear effect and the depth variations are in the same order of magnitude, referring to Liu and Dingemans (1989). Therefore, we make this small parameter equal to wave steepness ε , and expand Φ and η in the form of:

$$\Phi(x, y, z, t) = \sum_{n=1}^{\infty} \varepsilon^n \left[\sum_{m=-n}^n \Phi_{nm}(x, y, z, t) E^m \right], \quad (9)$$

$$\eta(x, y, t) = \sum_{n=1}^{\infty} \varepsilon^n \left[\sum_{m=-n}^n \eta_{nm}(x, y, t) E^m \right], \quad (10)$$

$$E = \exp \left\{ i \left[\int^x k(x) dx - \omega_0 t \right] \right\}, \quad (11)$$

where E represents the harmonic functions, and the complex conjugates part satisfy $\Phi_{n,-m} = \bar{\Phi}_{nm}$, $\eta_{n,-m} = \bar{\eta}_{nm}$. We take $n \leq 3$ in the derivation since ε is very small.

With the expansion of Φ and η to the third-order of ε , the method of multiple scales introduced in Davey and Stewartson (1974) is applied to give the solution at different orders and harmonic. The details in this process are similar to Hasimoto and Ono (1972). To simplify the problem, we suppose the water depth h varies slowly. Additionally, we want to concentrate on the variation of depth in the wave propagating direction, so we assume the magnitude of the gradient of depth change satisfies $h'(x) \sim O(\varepsilon^2)$ and $h'(y) \sim O(\varepsilon^3)$. Considering the expansion form in Eq. (9) and (10), the effect of bottom topography change is only reflected in the third-order $O(\varepsilon^3)$ and $h'(y) \sim O(\varepsilon^3)$ is equivalent to $h'(y) = 0$. As for the dispersion relation between wavenumber and frequency, the carrier $\omega = \omega_0$ is still constant since there is no temporal variation, but carrier wave number k changes. Based on the above inference, we can get $k = k(x)$ and $c_g = c_g(x)$ on the principal wave direction, and we can introduce a specific variable substitution of t and x , referring to Djordjevic and Redekopp (1978):

$$\tau = \varepsilon \left[\int^x \frac{dx}{c_g(x)} - t \right], \quad \xi = \varepsilon^2 x, \quad \zeta = \varepsilon y. \quad (12)$$

We apply Eq. (12) to transfer (x, y, t) to (ξ, ζ, τ) and take Eq. (9) – (11) into Eq. (1) – (6), then the evolution equation of envelope $\bar{A}(\xi, \zeta, \tau)$ for a very mild slope can be given in the form of:

$$i\beta_h \bar{A} + i \frac{\partial \bar{A}}{\partial \xi} + \beta_t \frac{\partial^2 \bar{A}}{\partial \tau^2} + \beta_y \frac{\partial^2 \bar{A}}{\partial \zeta^2} = \beta_n |\bar{A}|^2 \bar{A}, \quad (13)$$

where

$$\beta_h = \frac{(1 - \sigma^2)(1 - kh\sigma)}{\sigma + kh(1 - \sigma^2)} \frac{d(kh)}{d\xi} = \frac{1}{2c_g} \frac{d(c_g)}{d\xi}, \quad (14-1)$$

$$\beta_t = -\frac{1}{2\omega_0 c_g} \left[1 - \frac{gh}{c_g^2} (1 - \sigma^2)(1 - kh\sigma) \right], \quad (14-2)$$

$$\beta_y = \frac{1}{2k} \frac{\partial \omega}{\partial k} \equiv \frac{c_g}{2k}, \quad (14-3)$$

$$\beta_n = k^2 \omega_0 \left[\frac{1}{16} (9 - 10\sigma^2 + 9\sigma^4) - \frac{1}{2 \sinh^2 2kh} \right]$$

$$+ \left[\frac{\omega_0^3}{g} \frac{1}{2g} (\sigma^2 - 1) + \frac{k}{c_g} \right] \left(\frac{c_g^2}{c_g^2 - gh} \right) \left[\frac{g^2 k}{2\omega_0 c_g} + \frac{\omega_0^2}{4\sinh(kh)^2} \right]. \quad (14 - 4)$$

β_h represents the contribution from topography change, which is proportional to the derivative of wave number with respect to coordinate ξ . When the topography change becomes very small, $\beta_h \approx 0$ and Eq. (13) is equivalent to the evolution equation in Davey and Stewartson (1974). β_t and β_y give the local curvature based on the linear dispersion relation for carrier wave period and lateral component of wave number, respectively. β_n represents the contribution from nonlinear effect from the quasi-resonant four-wave interaction.

2.2 Numerical solution and freak wave estimation

In the 1D problem of the unidirectional wave train, as Lyu et al. (2021), the evolution equation of \bar{A} is numerically solved in a spatial step by pseudo-spectral method. Based on the periodicity of wave envelop on time and space, the partial differential term in evolution equation can be transformed into a more concise form by using Fourier transform. In the 1D problem, we can simplify the partial differential term with respect to time and express it in the frequency domain. In the 2D problem, the variation in the lateral direction requires the consideration of one more dimension. Based on the periodicity of time and space for a 2D wavefield, the 2D Fourier transform is applied to transform Eq. (13) into an ordinary differential equation:

$$\frac{d\bar{A}}{d\xi} = -i\beta_n |\bar{A}|^2 \bar{A} - i\beta_t \omega_\tau^2 \bar{A} - i\beta_y k_\zeta^2 \bar{A} - \beta_h \bar{A}, \quad (15)$$

where we take the Fourier transformation F_τ with respect to τ on time and F_ζ with respect to ζ on lateral spatial coordinate, respectively, and ω_τ and k_ζ represent corresponding variables about the frequency and the lateral wave number in the Fourier transform:

$$\dot{A}(\omega_\tau, \xi, \zeta) = F_\tau[\bar{A}(\tau, \xi, \zeta)], \quad \ddot{A}(\omega_\tau, \xi, k_\zeta) = F_\zeta[\dot{A}(\omega_\tau, \xi, \zeta)]. \quad (16)$$

Through the spatial evolution of ξ in Eq. (15), the wave envelope can be numerically simulated from an initial condition at $\xi = \xi_0$. We assume the initial Fourier amplitude $\dot{A}(\omega_\tau, \xi_0, k_\zeta)$ satisfies the 2D Gaussian shape directional spectral:

$$\begin{aligned} \dot{A}(\omega_\tau, \xi_0, k_\zeta) &= \dot{A}(\omega_\tau, \xi_0, \theta_\zeta) \\ &= \frac{a}{2\pi\sigma_\omega\sigma_\theta} \exp \left\{ -\frac{1}{2} \left[\left(\frac{\omega_\tau - \omega_0}{\sigma_\omega} \right)^2 + \left(\frac{\theta_\zeta - \theta_0}{\sigma_\theta} \right)^2 \right] + i\psi \right\}, \quad (17) \end{aligned}$$

where a represents the standard deviation of envelop and $\varepsilon = ka$, $\theta_\zeta = \arctan\left(\frac{k_\zeta}{k_0}\right)$ represents the direction of a single wave with the lateral wave number k_ζ , k_0 and ω_0 are carrier wave number and frequency, σ_θ is dimensionless directional width, ψ is the random phase uniformly distributed at $[0, 2\pi]$. σ_ω is frequency spectral width, and we will also use its dimensionless form $\sigma_s = \frac{\sigma_\omega}{\omega_0}$ in the following discussion. θ_0 is the principal wave direction, and here we set θ_0 is fixed at $\theta_0 = 0$. When $\theta_0 \neq 0$, the incident wave is oblique with the contour line of topography, and wave refraction will occur due to the inhomogeneity of dispersion relation on the wave front line. This part of the discussion will be given in **Part II**.

In this problem, the initial spectral is decided by the contribution from both the temporal and spatial frequency, and \bar{A} distributed in a 2D plane about frequency ω_τ and k_ζ (or θ_ζ). Referring to the 1D problem as Janssen (2003), the Benjamin–Feir index (BFI) is defined as the ratio between wave steepness ε and dimensionless spectral bandwidth σ_s :

$$BFI = \frac{\sqrt{2}\varepsilon}{\sigma_s}. \quad (18)$$

In this study, we hope to separate the effect from different contributions in the temporal and spatial part and concentrate more on the directional dispersion effect, so we continue to use Eq. (18) as the definition of BFI in the initial condition. The parameter σ_θ will be taken as an individual parameter in the spectral bandwidth, and σ_s is set as constant (i.e., σ_ω is constant in Eq. (17)).

On principal wave propagation direction, we can solve the wave envelope \bar{A} at each spatial step on x (or ξ) through Eq. (15), and give the wave surface elevation η by Eq. (19):

$$\eta(x, y, t) = \varepsilon \operatorname{Re} \left[\frac{1}{2} \bar{A}(x, y, t) E \right] + \varepsilon^2 \operatorname{Re} \left[\frac{k \cosh kh}{8 \sinh^3 kh} (2 \cosh^2 kh + 1) \bar{A}^2(x, y, t) E^2 \right]. \quad (19)$$

Similar to Lyu et al. (2021), we integrate $\eta(y, t)$ in Eq. (19) at each step from offshore to onshore, assuming periodic boundary conditions in time, and construct the surface elevation in a discrete 2D+T form. Eq. (19) considers the contribution from first-order to second-order and second harmonic. The progressive wave is established on the periodicity only in $(\int^x k(x) dx - \omega_0 t)$, because the contribution of $k_{\zeta 0} = 0$ on the component of lateral length y of carrier propagating direction.

Because of the very small probability of freak wave occurrence, an ensemble simulation

size is necessary to provide a more reliable prediction. In this study, we continue to estimate the nonlinear interaction in wave evolution by the fourth-order moment kurtosis μ_4 and the third-order moment skewness μ_3 , and their variation will be compared with the wave height distribution directly obtained from the 2D+T surface elevation data in a large-size Monte Carlo simulation.

μ_4 and μ_3 of a fixed point (x_*, y_*) in the wavefield are derived from the discrete surface elevation $\eta(x_*, y_*, t)$ in time series:

$$\mu_4 = \frac{\text{Ex}(\eta_i - \bar{\eta})^4}{\eta_{\text{rms}}^4}, \quad \mu_3 = \frac{\text{Ex}(\eta_i - \bar{\eta})^3}{\eta_{\text{rms}}^3}, \quad (20)$$

where Ex represents expected value, i represents the index number of the data sample, $\bar{\eta}$ is the mean value and η_{rms} is the root mean square value of η . For a wave train in the Gaussian process (i.e., linear random waves), $\mu_4 = 3$ and $\mu_3 = 0$. The theoretical value of μ_4 can be changed with different nonlinear processes or hypotheses. For a narrowband second-order nonlinear wave train, the Stokes wave model gives contribution from bound waves (Longuet-Higgins, 1963). Thus, the values of μ_4 and μ_3 estimated inbound wave (marked as μ_4^b , μ_3^b) are related to the wave steepness ε :

$$\mu_4^b = 3 + 24\varepsilon^2, \quad \mu_3^b = 3\varepsilon. \quad (21)$$

In Janssen (2003) and Mori and Janssen (2006), μ_4 (marked as μ_4^*) can change the quasi-resonant and non-resonant interactions than Eq. (21). It is parameterized by the fourth-order cumulant κ_{40} , which is proportional to the square of BFI defined by Janssen (2003):

$$\mu_4^* = \kappa_{40} + 3, \quad \kappa_{40} = \frac{\pi}{\sqrt{3}} \text{BFI}^2. \quad (22)$$

Based on the contribution from the quasi-resonant four-wave interactions in Eq. (22) on wave height H , Mori and Janssen (2006) gave the exceeding probability $P(H)$:

$$P(H) = e^{-\frac{H^2}{8}} \left[1 + \frac{\kappa_{40}}{384} (H^4 - 16H^2) \right], \quad (23)$$

and exceeding probability $P_m(H_{\text{max}})$ of maximum wave height H_{max} :

$$P_m(H_{\text{max}}) = 1 - \exp \left\{ -N_0 e^{-\frac{H_{\text{max}}^2}{8}} \left[1 + \frac{\kappa_{40}}{384} (H_{\text{max}}^4 - 16H_{\text{max}}^2) \right] \right\}, \quad (24)$$

where N_0 represents the number of waves in a wave train. Eq. (24) is well validated by the μ_4 in wave tank experiment from Mori et al. (2007) and Kashima and Mori (2019). With the consideration of the directional effect, Mori et al. (2011) gave the estimation of maximum κ_{40}

with the directional spread σ_θ by:

$$\kappa_{40} = \frac{\pi}{\sqrt{3}} \text{BFl}^2 \left(\frac{\alpha}{\sigma_\theta} \right), \quad (25)$$

where α is an empirical coefficient, and $\alpha = 0.0276$ from their asymptotic analysis in Monte Carlo simulation.

3 Numerical results

3.1 Model setup

With the 2D Gaussian shape directional spectral in Eq. (17) and inverse Fourier transform, we give the initial envelope $\bar{A}(\tau, \xi_0, \zeta)$ in the matrix of time series and spatial distribution on the lateral direction. As a pseudo-spectral method, discrete Fourier transform requires a sufficient length of the target variable to ensure the result has enough information in the evolution. On the other hand, Fourier transforms for a 2D matrix require a large amount of time in calculation compared with the 1D model, which is a critical problem since we apply the Monte Carlo method simulation.

From the Eq. (17), the shape of the initial directional spectrum is decided by the dimensionless spread σ_θ and the frequency spectral width σ_ω . In Lyu et al. (2021), the effect from σ_ω has been discussed from the unidirectional wave train with different initial BFI. For a 2D wavefield, the dimensionless spread σ_θ in different sea states vary from 0.15 to 0.5 from the research on observation and wave age (Ewans, 1998; Forristall and Ewans, 1998; Banner and Young, 1994). Yuen and Lake (1982) indicated a limitation of the 2D NLS-like wave model that the instability of the wave train continually increases in a certain interval, which reflects in our numerical model that the result cannot reach convergence in Monte Carlo simulation when $\sigma_\theta \leq 0.25$. Therefore, we set the $\sigma_\theta = 0.3, 0.4, 0.5$ in the following comparison for different directional spreading.

To ensure the accuracy and computational efficiency, we make the calculation step is constant at $d\xi = 2 \times 10^{-5} L_0$ where L_0 is carrier wavelength. For the initial condition, carrier angular frequency $\omega = 2.5 \text{s}^{-1}$, sampling time $dt = 0.1 \text{s}$, time length $L_T = 40T_0$, where T_0 is wave period; carrier lateral wave number $k_{\zeta 0} = 0$, sampling distance $d_y = 0.5L_0$. To ensure

that the initial condition given by Eq. (17) converges to a Gaussian process, we generate the initial Fourier amplitude by a sufficiently large number ($N = \frac{L_T}{dt} = 1000$) of component waves with different frequency ω_τ uniformly distributed around the carrier frequency. The kurtosis μ_4 and skewness μ_3 of the initial surface elevation $\eta(x_0, y, t)$ satisfy the Gaussian distribution that $\mu_4 = 3$ and $\mu_3 = 0$. The initial water depth starts at a medium depth $kh = 5$, wave steepness $\varepsilon = 0.1$. The width of the 2D computational domain $L_y = 30L_0$ to reflect the propagation of directional waves (the analysis and discussion to decide the appropriate set of d_y and L_y is given in the **supplementary materials**). For different forms of bottom topography, the length of the computational domain on the principal direction varies from $30L_0$ to $150L_0$, and the calculation stops at the shallow water $kh = 1.1$, where wave steepness $\varepsilon = 0.1343$. This study assumes the water surface basically maintains the form of a mechanical wave in the evolution, so the wave-breaking case is not taken into consideration. Therefore, our simulation stops before the wave steepness reaches the threshold value of breaking in very shallow water.

Mei and Benmoussa (1984) introduced a normalization to make all parameters dimensionless in the realization process. We apply a different normalization for the variable $(\bar{A}, \xi, \zeta, k_\zeta, \omega_\tau, \tau)$ in programming as follows:

$$\begin{aligned} A' &= \frac{2\pi}{L_0} \bar{A}, & \xi' &= \frac{2\pi}{L_0} \xi, & \zeta' &= \frac{2\pi}{\varepsilon L_y} \zeta, \\ k_\zeta' &= \varepsilon \frac{L_y}{2\pi} k_\zeta, & \omega' &= \varepsilon \frac{L_T}{2\pi} \omega_\tau, & \tau' &= \frac{2\pi}{\varepsilon L_T} \tau. \end{aligned} \quad (26)$$

3.2 Evolution of modulated wave over flat bottoms

Firstly, we consider the wave evolution over a flat bottom to investigate the effect of initial conditions as reference runs for uneven bottom cases. If we set $\beta_h = 0$, Eq. (13) is equivalent to the 2D NLS equation in Davey and Stewartson (1974), and the evolution process is constructed by spatial step on the principal wave direction. The surface elevation is given in discrete 2D+T form. In **Figure 2**, we give the transient surface elevation η at $t = 40T_0$ from three random samples with different directional spread σ_θ and initial BFI = 1 ($\sigma_s = 0.14$) at water depth $kh = 5$. The viewing angle is vertical to the horizontal plane, and we use the color gradient to represent the elevation. The 2D irregular waves propagate from the left to the right

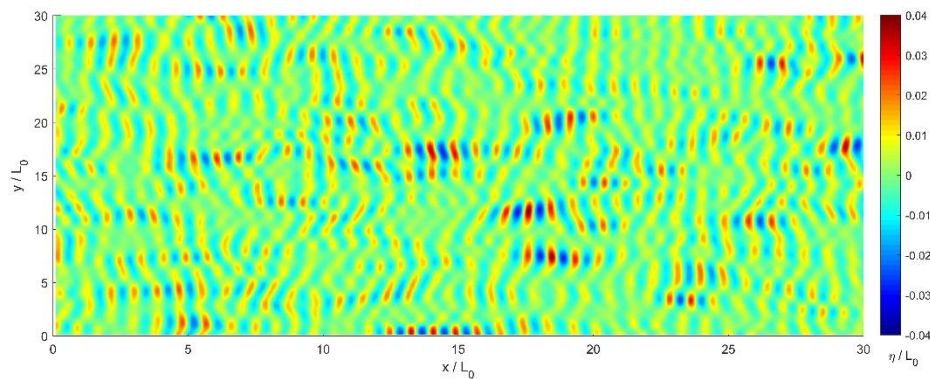
side, and the wavefront lines can figure out the multi-directions. As the σ_θ increases from 0.3 to 0.5, the propagation becomes more divergent in different directions, which makes the wavefront appear to be more discontinuous.

The initial condition with a random phase in Eq. (17) helps generate an irregular wave train, and a Monte Carlo simulation is conducted. For a domain in strict-sense stationary (SSS), the statistical properties keep invariant to any shift at any order, and the surface elevation for 2D irregular wave at a fixed point will be generally closed to a zero-mean SSS process when the Monte Carlo simulation for random wave phase has enough ensemble size. In this study, we set the ensemble size is 300 and give the average value of kurtosis μ_4 and skewness μ_3 in time series on each space node. The deciding process of the ensemble size can also be referred to the **supplementary materials**.

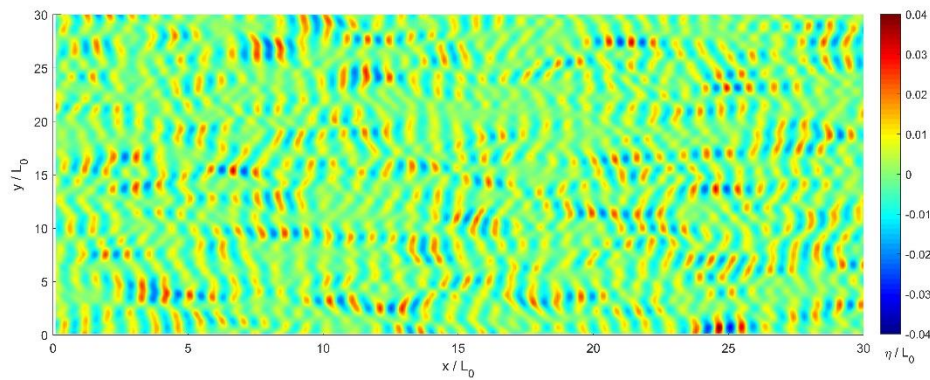
The result in 2D form can be modified into 1D form by being averaged again on one space dimension because the statistical significance is uniformly distributed in this 2D domain. To show the wave evolution process, we take the averaged value on the lateral direction of the principal wave direction and give the Monte Carlo result of the unidirectional wave train and the 2D wavefield with initial BFI = 0.5 ($\sigma_s = 0.28$) for a flat bottom $kh = 7$ to discuss the effect from the directional spread σ_θ in **Figure 3**. The result in the unidirectional wave train can be regarded as $\sigma_\theta = 0$. As the σ_θ increases, both kurtosis μ_4 and skewness μ_3 decrease, which implies the increase of directional spreading disperses the nonlinear interactions in the wave train.

For the 2D propagation in a directional wavefield, both short-time and long-time behavior of κ_{40} for a narrowband wave train is related to the directional width and a frequency width (Janssen and Bidlot, 2009). We are also interested in the kurtosis distribution at the intermediate stages in freak wave forecasting. Mori et al. (2011) conducted an asymptotic analysis of κ_{40} , BFI and σ_θ by numerical simulation, and gave Eq. (25) with empirical coefficient α . **Table 1** shows the ensemble-averaged result of the expected maximum κ_{40} and mean κ_{40} at $x \in [20L_0, 30L_0]$ from Monte Carlo simulation of the wave model in this study, and gives the empirical coefficient α_{\max} and α_{mean} by Eq. (25) for maximum κ_{40} and mean κ_{40} , respectively. Due to different calculation conditions, we give different empirical coefficients with Mori et al. (2011) ($\alpha = 0.0276$), and the expected maximum κ_{40} is significantly larger

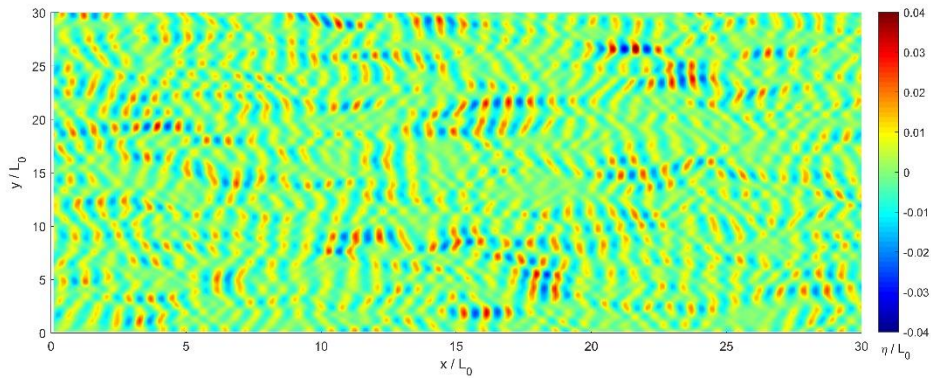
than their prediction. Nevertheless, the result from $kh = 7$ shows that the increase of σ_θ and the decrease of BFI leads to the decrease of both maximum and mean κ_{40} , but the change in empirical coefficient α represents the κ_{40} is not strictly inversely proportional to σ_θ as Eq.(25), and the case from different initial BFI will ask for a different α . The α_{mean} for mean κ_{40} at $kh = 7$ is better than α_{max} to describe the kurtosis of the wavefield, and its value is around 0.09~0.14. When the water depth becomes shallow ($kh = 5, 3, 1.1$), Eq.(25) is no longer applicable and α_{mean} and α_{max} significantly decrease. For $kh = 1.1$, the increase of σ_θ leads to the increase of κ_{40} and α_{mean} seems anomalous. The behavior of κ_{40} further indicates that the occurrence of extreme wave height in medium and shallow water cannot only be predicted by the four-wave interaction as in deep-water. The contribution from water depth becomes important, especially in shallow water, and the simulation over a changing depth may reveal this process more effectively.



(a) $\sigma_\theta = 0.3$



(b) $\sigma_\theta = 0.4$



(c) $\sigma_\theta = 0.5$

Figure 2 Transient surface elevation η at $t = 40T_0$ from different directional spread σ_θ
with initial BFI = 1, $kh = 5$

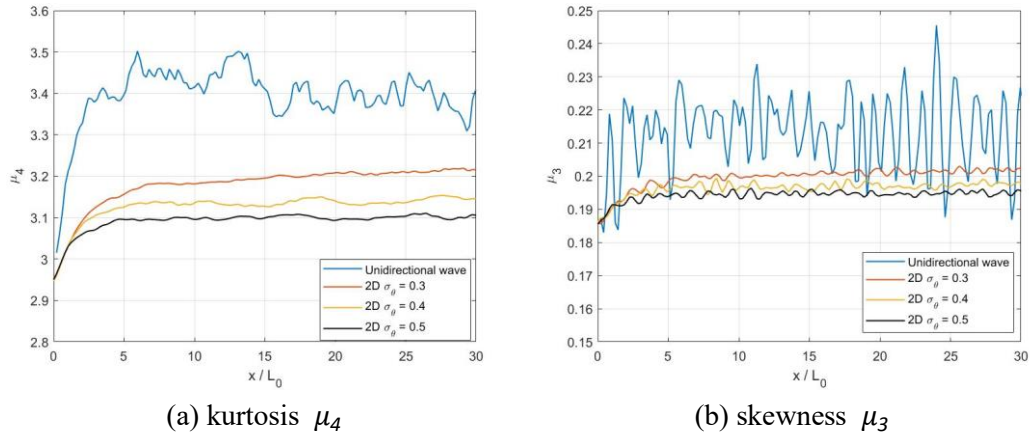


Figure 3 Spatial evolution of kurtosis μ_4 and skewness μ_3 of surface elevation from directional spread σ_θ with initial BFI = 0.5, $kh = 7$

(1D ($\sigma_\theta = 0$): blue. 2D: red: $\sigma_\theta = 0.3$; yellow: $\sigma_\theta = 0.4$; black: $\sigma_\theta = 0.5$)

Table 1 The ensemble-averaged κ_{40} dependence on BFI and σ_θ at different kh

kh	σ_θ	Initial BFI	Max κ_{40}	Mean κ_{40}	α_{\max}	α_{mean}
7.0	0.3	0.5	6.329	0.180	4.187	0.119
7.0	0.4	0.5	4.555	0.127	4.018	0.112
7.0	0.5	0.5	4.206	0.092	4.638	0.101
7.0	0.6	0.5	3.841	0.064	5.082	0.085
7.0	0.3	0.4	4.441	0.136	4.591	0.141
7.0	0.5	0.4	3.725	0.083	6.418	0.143
5.0	0.3	0.4	3.976	0.117	4.110	0.121
5.0	0.5	0.4	3.368	0.070	5.803	0.121
3.0	0.3	0.4	3.148	0.065	3.254	0.067
3.0	0.5	0.4	2.948	0.038	5.079	0.066
1.1	0.3	0.4	2.294	0.059	2.371	0.061
1.1	0.5	0.4	2.754	0.113	4.745	0.195

3.3 Evolution of modulated wave over uneven bottoms

This section conducts the Monte Carlo simulation of the 2D wave model for uneven bottoms. The variation in the bottom topography brings about the spatial inhomogeneity in the dispersion, which reflects in both second-order and third-order effects. Our numerical model by Eq. (13) assumes the depth mildly changes in the principal wave direction, and the bottom topography does not vary in its lateral direction. Therefore, the statistical properties keep stationary on the y-axis but vary on the x-axis.

As a simulation of the bottom topography from offshore to onshore, we set the depth slowly decreases from a medium water depth to shallow water. In previous research (e.g., Kashima and Mori, 2019; Li et al., 2021; Lyu et al., 2021), the abrupt change in depth or slope will lead to a significant local peak in μ_3 and μ_4 due to the aftereffect. Usually, the water depth decreases in a smoother way in natural seabed. On the other hand, the local peak caused by this unusual topography is so significant that it may cover the difference caused by the directional effect. To make our simulation closer to the natural seabed, we adjust the sloping region in **Figure 1** between A and B to become smoother and continuously decrease in **Figure 4**. The sloping region is divided into 2 parts by the dividing line C at $kh = 1.2$: the depth linearly decreases with a slope γ_s from $kh = 5$ to $kh = 1.2$; the depth decreases with a decaying slope $\gamma_s' = \gamma_s \left(\frac{h}{1.55}\right)^{40}$ from $kh = 1.2$ to $kh = 1.1$. In the shallow water area, the derivative of the decreasing depth is approximately continuous, and we set $\gamma_s = 0.05, 0.02, 0.01$ to ensure the depth changes very mild under the assumption $h'(x) \sim O(\varepsilon^2)$.

Figure 5 to 6 shows the averaged values of kurtosis and skewness from the Monte Carlo simulation of the bottom type in **Figure 4**. In **Figure 5**, we give the averaged kurtosis μ_4 in 2D form at the different directional spread σ_θ and slope angle γ_s with initial BFI = 0.4 ($\sigma_s = 0.35$). Comparing the result from different σ_θ in **Figure 5 (a), (b), and (c)**, we find the μ_4 decreases in the deep-water depth but increases in the shallow water when the initial directional spreading σ_θ increases from 0.3 to 0.5, which shows the same phenomenon results in a flat bottom in **3.1**. In the medium water depth region between locations A and C, μ_4 decreases as the decrease of water depth, and it rebounds at the end of the constant sloping region (location C) where $kh = 1.2$. In the region between C and B, where the slope angle mildly decreases, μ_4

decreases and becomes stable at the same level as the final flat bottom in shallow water $kh = 1.1$. The evolution of μ_4 indicates that the directional dispersion effect decreases the occurrence probability of freak waves in deep and medium water but increases it in shallow water. As the wave propagates from the medium water to shallow water, the wave evolution is significantly affected by the bottom topography, and the 1D result in **Figure 5 (d)** clearly gives the rebound of μ_4 due to the slope angle. To further study the effect from the bottom topography, we give the mean μ_4 at $\gamma_s = 0.02$ and $\gamma_s = 0.01$ with initial BFI = 0.4 and $\sigma_\theta = 0.3$ in **Figure 5 (e)** and **Figure 5 (f)**. The result shows the rebound of μ_4 decreases as the bottom change become milder, and **Figure 5 (g)** provides the variation of μ_4 on the principal wave direction in 1D. Comparing the μ_4 over uneven bottoms between the unidirectional wave train and the 2D wavefield, we find the slope angle similarly affects the wave evolution. However, its contribution is more significant in 2D due to the dispersion effect on the four-wave interaction, which implies the second-order effect plays a more important role in a directional 2D wavefield.

In **Figure 6**, we give skewness μ_3 in the same form with μ_4 in **Figure 5** at the same condition. Different from μ_4 , μ_3 does not influence directional spread from **Figure 6 (a)**, **(b)**, **(c)**, and **(d)** due to the unchanging second-order nonlinear interaction. When the bottom topography changes, **Figure 6 (e)**, **(f)**, and **(g)** show the μ_3 increases as the water depth become shallow. This process will slow down if the slope angle becomes mild, but it only means μ_3 is determined by the water depth kh and the change from slope angle γ_s has little influence. Different from the unidirectional wave train, μ_3 in the 2D wavefield is basically only determined by the variation in dispersion due to depth change, and hardly affected by the local bathymetry effect. When the wave trains propagate into shallow water, the μ_3 increases with the increase of wave steepness ε .

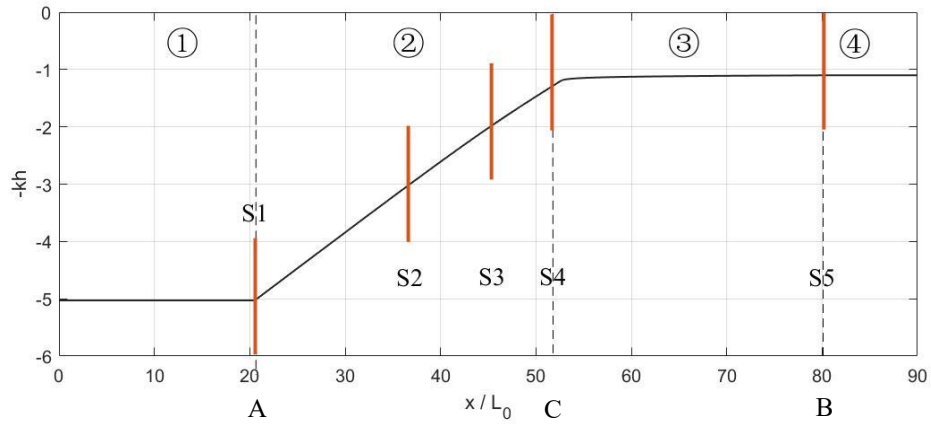


Figure 4 The variation of water depth kh on the bottom topography with $\gamma_s = 0.02$

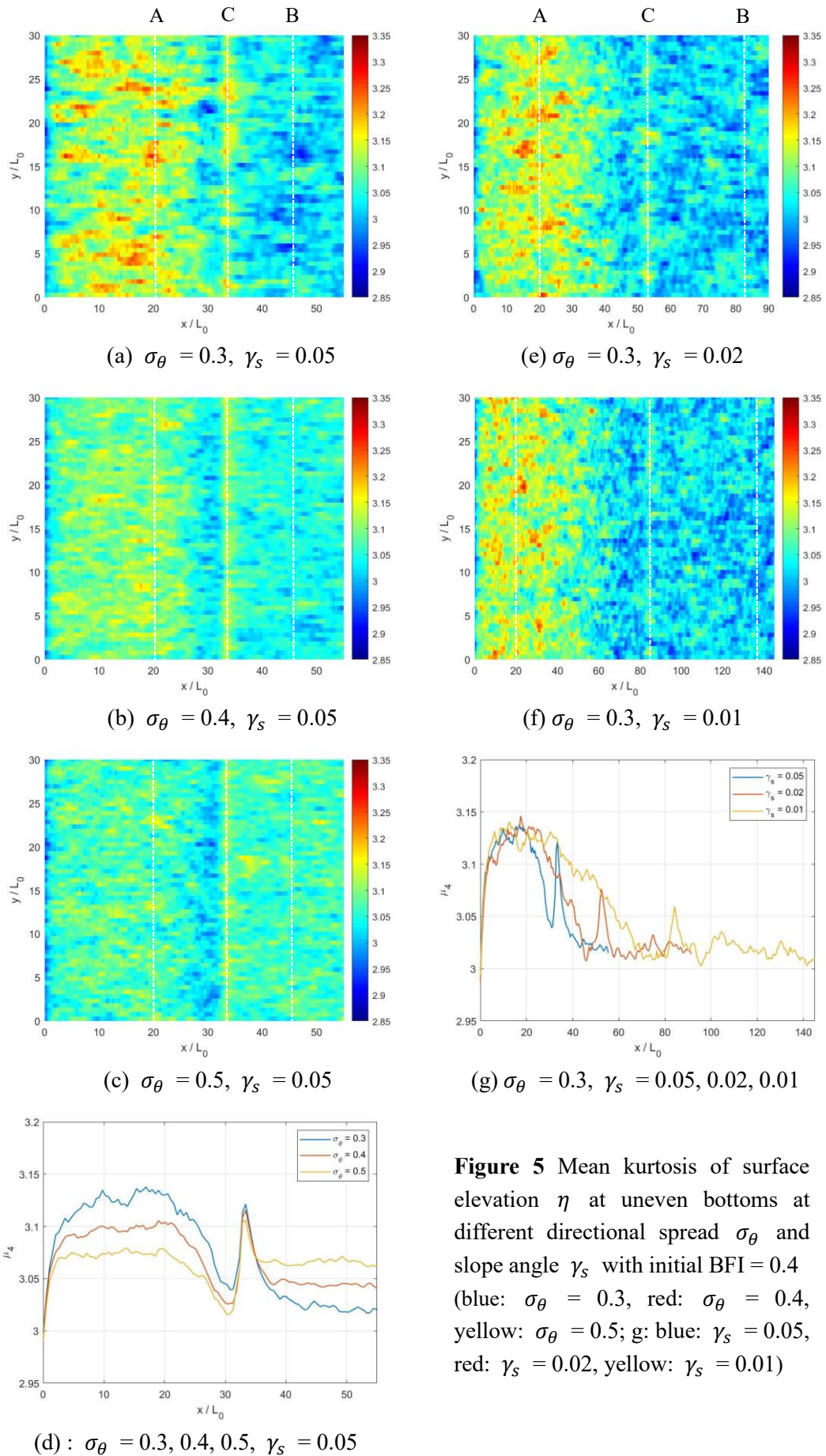


Figure 5 Mean kurtosis of surface elevation η at uneven bottoms at different directional spread σ_{θ} and slope angle γ_s with initial BFI = 0.4 (blue: $\sigma_{\theta} = 0.3$, red: $\sigma_{\theta} = 0.4$, yellow: $\sigma_{\theta} = 0.5$; g: blue: $\gamma_s = 0.05$, red: $\gamma_s = 0.02$, yellow: $\gamma_s = 0.01$)

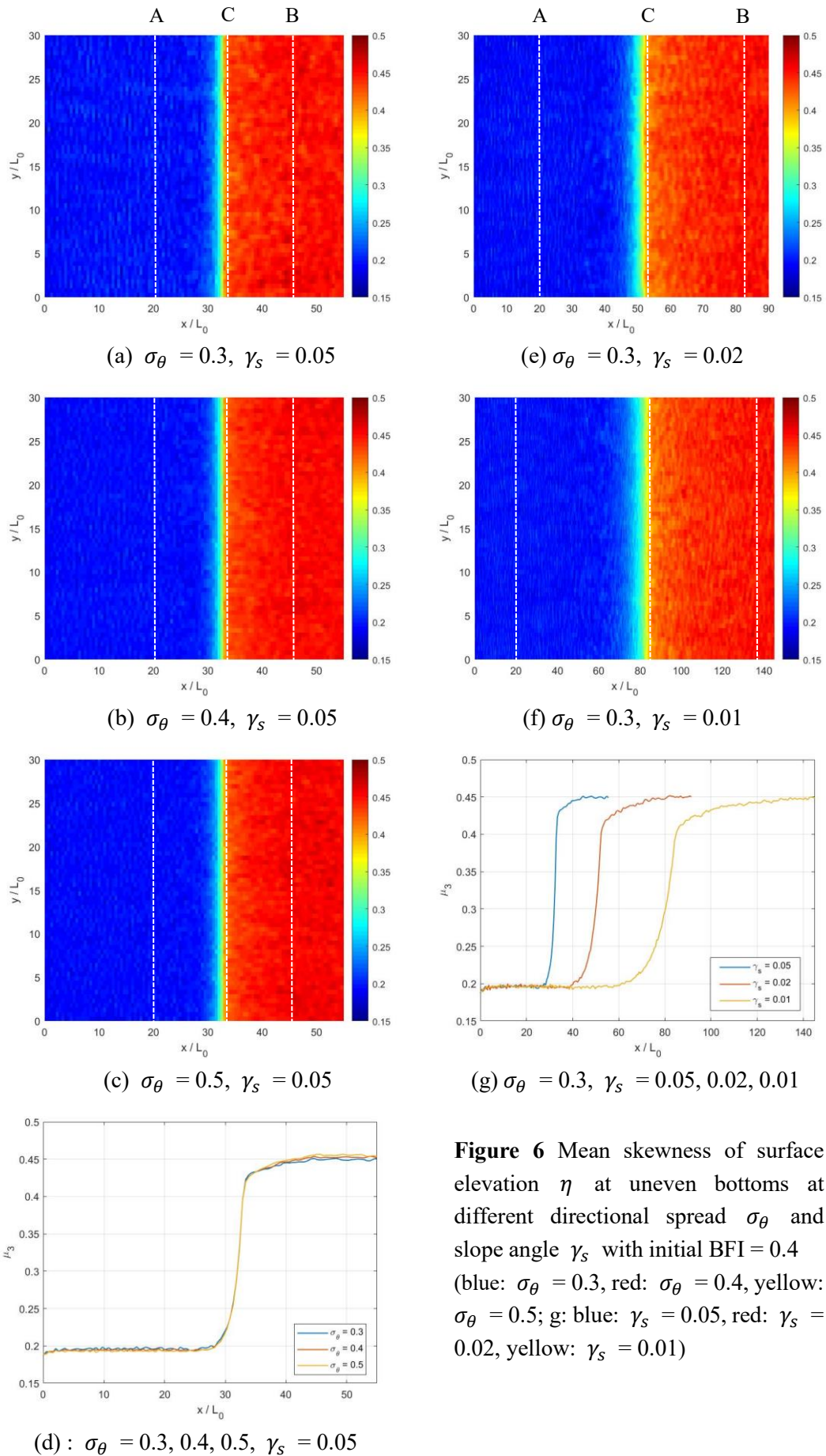


Figure 6 Mean skewness of surface elevation η at uneven bottoms at different directional spread σ_θ and slope angle γ_s with initial BFI = 0.4 (blue: $\sigma_\theta = 0.3$, red: $\sigma_\theta = 0.4$, yellow: $\sigma_\theta = 0.5$; g: blue: $\gamma_s = 0.05$, red: $\gamma_s = 0.02$, yellow: $\gamma_s = 0.01$)

3.4 Quantitative analysis of the extreme wave height

In sections 3.2 and 3.3, the evolutions of the nonlinear resonant interactions have been discussed with the bathymetry effect on different water depths. μ_4 and μ_3 represent different nonlinear mechanisms, and they show different characteristics on the directional wavefield over an uneven bottom. However, the final result of the occurrence of extreme wave height is their combined effect, and we cannot simply give the estimation until giving a quantitative analysis of the distribution of surface elevation η .

In **Figure 7**, we give the expected maximum wave height H_{\max} and maximum wave crest η_{\max} on the principal wave direction. In the same form with μ_4 and μ_3 in **Figure 5** and **Figure 6**, H_{\max} and η_{\max} are counted from the time series sampling data at a fixed point in the 2D wavefield, and their ensemble-averaged value is given by Monte Carlo simulation. We make the wave height dimensionless by taking $H_{\max}/\eta_{\text{rms}}$ and $\eta_{\max}/\eta_{\text{rms}}$, and continue to write them as H_{\max} and η_{\max} for convenience. The result contains the wave train with different directional spreading $\sigma_\theta = 0.3, 0.4, 0.5$ over the bottom topography $\gamma_s = 0.05, 0.02, 0.01$, and the theoretical result from the linear model (Rayleigh distribution) and the numerical result from the second-order bound wave model in $\sigma_\theta = 0.3, \gamma_s = 0.05$ are given as a comparison. The Rayleigh distribution can be referred to as the standard linear narrow-banded wave theory in Goda (1970, 2000), in which the wave height H follows the Gaussian distribution and the exceeding probability $P_m(H_{\max})$ follows:

$$P_m(H_{\max}) = 1 - \exp\left(-N_0 e^{-\frac{H_{\max}^2}{8}}\right), \quad (27)$$

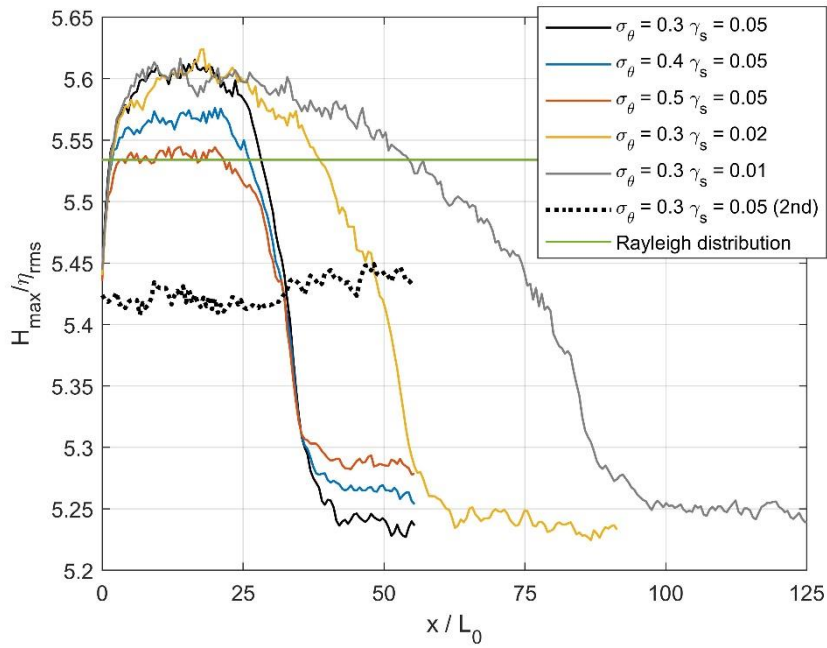
where the number of waves $N_0 = 40$ in our simulation. From the deep-water to shallow, H_{\max} in 2D model monotonically decreases as the water depth in comparison with the second-order model. Similar with μ_4 , The peak of H_{\max} is suppressed by the increase of σ_θ in deep-water but is enhanced in shallow water. However, there is no significant change in the slope. In **Figure 7 (b)**, the η_{\max} significantly increases in shallow water, and the local peak reflects the contribution from the slope. Even the bottom changes in a relatively continuous process, this local peak still occurs, and it is consistent with the variation of μ_4 . Different with H_{\max} in shallow water, the increase of η_{\max} is related to the μ_3 , in keeping with the general deformation of waves in the shoaling process. In Rayleigh distribution, H_{\max} and η_{\max} are

independent with the bathymetry. The linear model underestimates H_{\max} in deep-water and overestimates it in the shallow water, and significantly underestimates η_{\max} . As a result of the second-order bound wave model, H_{\max} hardly affected by the bottom topography and η_{\max} is determined by wave steepness. Comparing different models, the present model provides more consideration about the different effects of the four-wave interaction in various depths. Comparing the result in **Figure 5-7**, we find the rebound of μ_4 at location C does not reflect in the H_{\max} but shows in η_{\max} , which implies the wave height distribution at this area is more determined by the topography effect in second-order rather than the four-wave interaction in third-order.

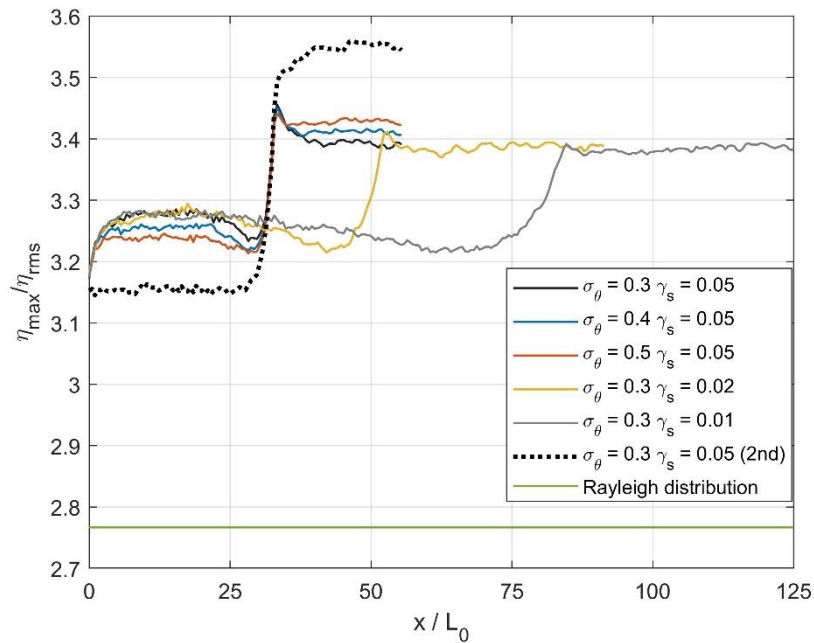
When concentrating on the freak wave problem, only the expected value is insufficient to estimate the extreme case. Therefore, we integrate the distribution of H_{\max} and η_{\max} at each section on the principal wave direction. If we follow the common definition of the freak wave as the wave height exceeds the significant wave height by two times, then the freak wave height is $H_{\max} > 8\eta_{\text{rms}}$ and corresponding wave crest is equivalent to $\eta_{\max} > 4\eta_{\text{rms}}$ under the standard linear narrow-banded wave theory from Goda (1970, 2000). **Figure 8** calculates the probabilities as mentioned above and gives the fit curves through the tenth-order polynomial. A little different from **Figures 5-7**, the horizontal axis is set to be the water depth kh in **Figure 8**, so we can more easily check the effect from different γ_s on the same depth. In **Figure 8 (a)**, the $P_m(H_{\max} > 8\eta_{\text{rms}})$ monotonically decreases with the water depth as expected H_{\max} , but more details can be given: in the relatively deep-water as $kh > 4$, the convergence of $P_m(H_{\max} > 8\eta_{\text{rms}})$ becomes worse, which implies variance of surface elevation significantly rises; for the shallow water $kh < 2$, the increase of slope angle γ_s enhances the probability of freak waves; the directional spreading σ_θ plays a more important role in deep-water, but bottom topography has a greater impact in shallow water. Compared with **Figure 8 (a)**, $P_m(\eta_{\max} > 4\eta_{\text{rms}})$ in **Figure 8 (b)** is much larger than $P_m(H_{\max} > 8\eta_{\text{rms}})$ in general, and it experiences a process of descending, ascending, and descending again with the decrease of kh . In the linear model, the probabilities in **Figure 8 (a)** and **(b)** are the same result, and their deviation with current models are corresponding to the evolution in **Figure 7**. The result from the second-order bound wave model continues to show a strong correlation with wave steepness ε , and it gives an abnormal enhancement in both $P_m(H_{\max} > 8\eta_{\text{rms}})$ and $P_m(\eta_{\max} > 4\eta_{\text{rms}})$.

in very shallow water after the slope section, which is not consistent with the natural state and experiments.

In **Figure 9** and **Figure 10**, we give the wave height distribution in Cumulative Distribution Function (CDF) in logarithmic coordinates at specific sections at different water depths. We choose five sections on the sloping region in **Figure 4**: S1 (dashed line A in previous figures) where $kh = 5$; S2 where $kh = 3$; S3 where $kh = 2$; S4 (dashed line C in previous figures) where $kh = 1.2$; S5 (dashed line B in previous figures) where $kh = 1.1$. **Figure 9** shows the CDF. of $P_m(H_{\max})$ from the same conditions as before and compared with Rayleigh distribution (i.e., linear distribution). As the water depth decreases from S1 to S5, the occurrence probability of H_{\max} decreases from the comparison with the Rayleigh distribution (the linear distribution model does not change with water depth). The nonlinear model gives a higher exceeding probability of extreme events than the linear distribution in deep-water (S1) but lower in shallow water (S4, S5). They have a similar prediction of wave heights distribution in medium water depth (S2, S3). In S1-S3, the increase of directional spread σ_θ leads to the decrease of the exceeding probability for $H_{\max}/\eta_{rms} > 6$. However, in shallow water depth (S4, S5), the effect of σ_θ is very limited or even becomes opposite. The effect from γ_s works mainly in the shallow region, and focuses on the distribution of larger values (i.e., the occurrence of “freak wave”). The result from the second-order bound wave model is hardly affected by the water depth from S1 to S4, but increases in very shallow water S5, especially for the occurrence of extreme value, and exceeds the result from the present model. In **Figure 10**, we give the CDF. of $P_m(\eta_{\max})$ in the same form. Basically, $P_m(\eta_{\max})$ has a similar variation with $P_m(H_{\max})$ under the effect from σ_θ and γ_s , but $P_m(\eta_{\max})$ markedly exceeds the Rayleigh distribution, which indicates that the wave deformation makes the wave crest exceeds the half of wave height due to the nonlinear effect. In shallow water, this deviation becomes more obvious from smaller value $H_{\max}/\eta_{rms} > 3$, even the extreme case decreases, which indicates the second-order nonlinear effect significantly rises due to the bottom topography change.



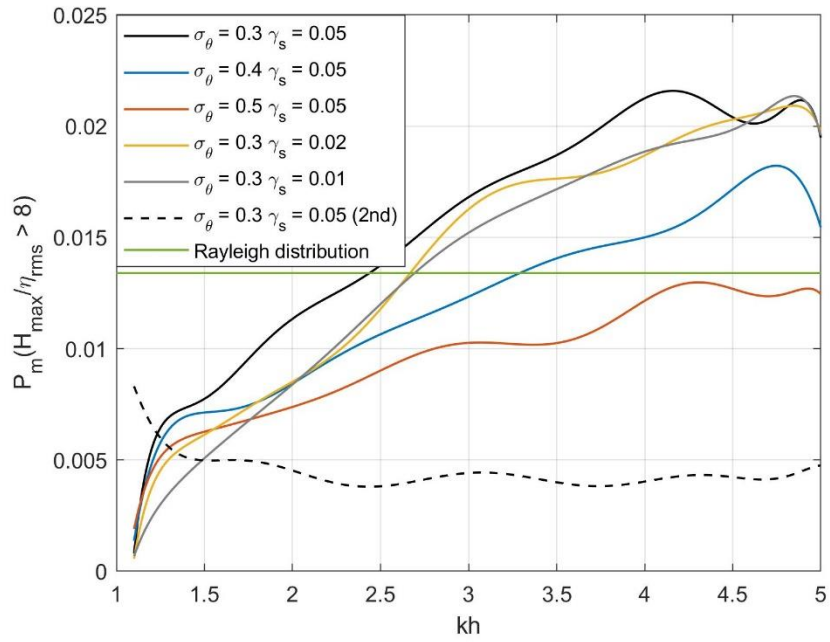
(a) maximum wave height H_{\max}



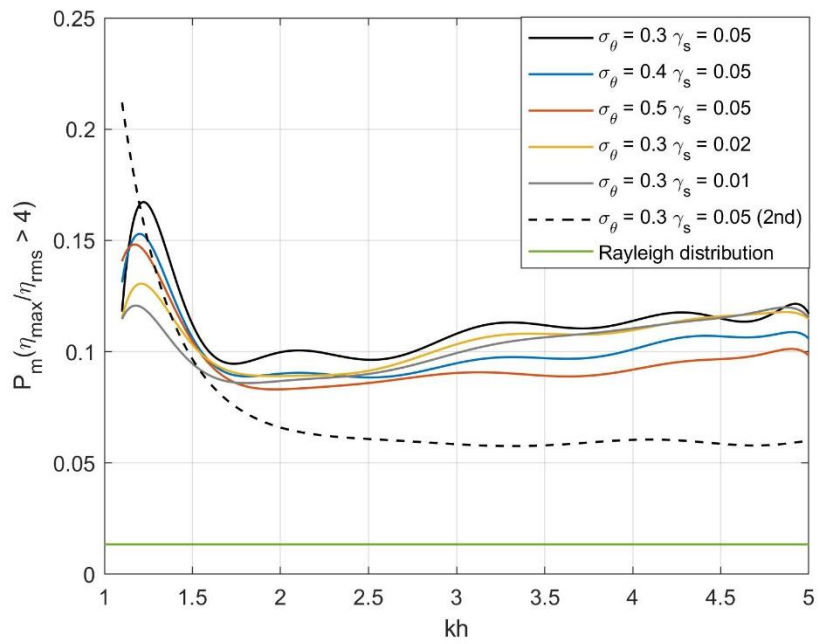
(b) maximum wave crest η_{\max}

Figure 7 Ensemble-averaged expected maximum wave height and free surface elevation distribution at initial BFI = 0.4 from different σ_θ and γ_s

(Present model: black: $\sigma_\theta = 0.3$ $\gamma_s = 0.05$; blue: $\sigma_\theta = 0.4$ $\gamma_s = 0.05$; red: $\sigma_\theta = 0.5$ $\gamma_s = 0.05$; yellow: $\sigma_\theta = 0.3$ $\gamma_s = 0.02$; gray: $\sigma_\theta = 0.3$ $\gamma_s = 0.01$. Second-order model: dotted: $\sigma_\theta = 0.3$ $\gamma_s = 0.05$. Rayleigh distribution: green line)



(a) probability of $H_{\max} > 8\eta_{rms}$

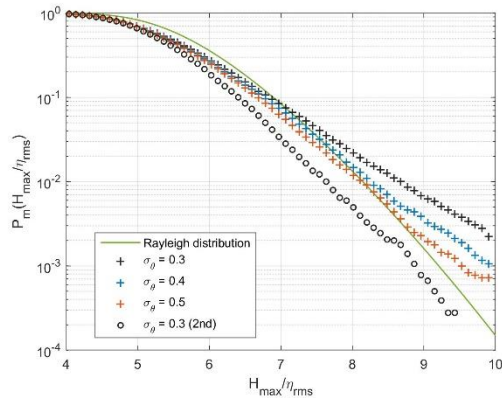


(b) probability of $\eta_{\max} > 4\eta_{rms}$

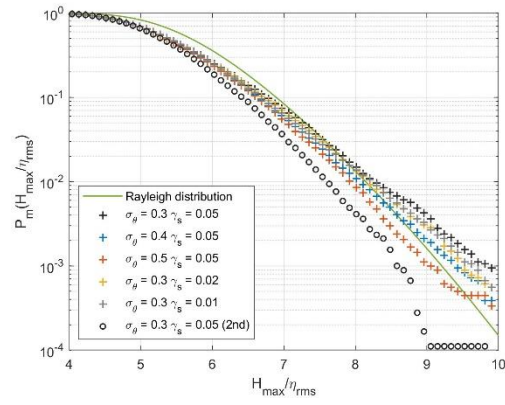
Figure 8 Occurrence probability of the freak wave in wave height and free surface elevation

distribution at initial BFI = 0.4 from different σ_θ and γ_s

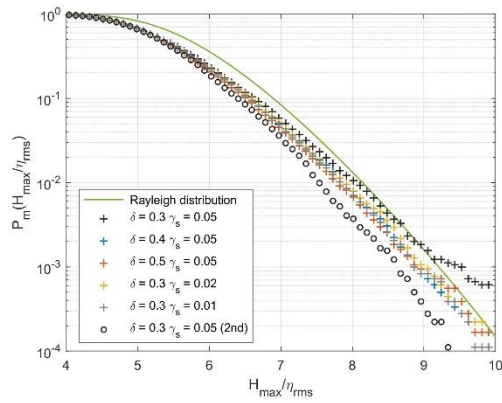
(Present model: black: $\sigma_\theta = 0.3$ $\gamma_s = 0.05$; blue: $\sigma_\theta = 0.4$ $\gamma_s = 0.05$; red: $\sigma_\theta = 0.5$ $\gamma_s = 0.05$; yellow: $\sigma_\theta = 0.3$ $\gamma_s = 0.02$; gray: $\sigma_\theta = 0.3$ $\gamma_s = 0.01$. Second-order model: dotted: $\sigma_\theta = 0.3$ $\gamma_s = 0.05$. Rayleigh distribution: green line)



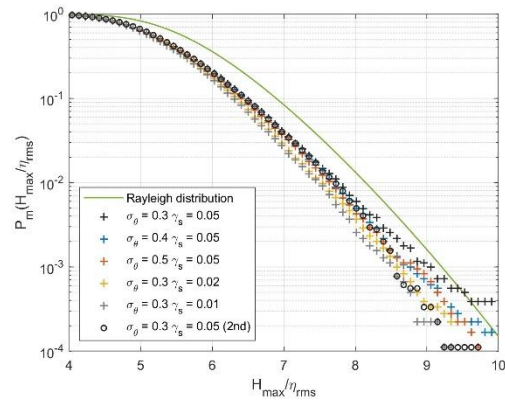
(a) S1 ($kh = 5$)



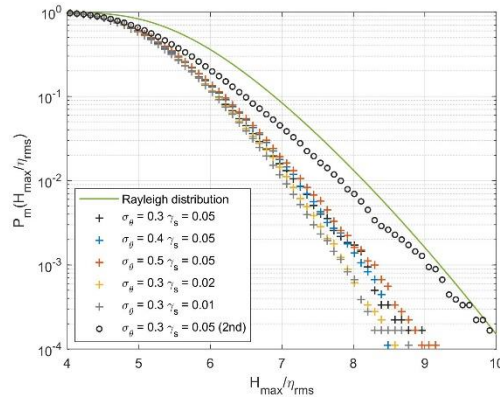
(b) S2 ($kh = 3$)



(c) S3 ($kh = 2$)



(d) S4 ($kh = 1.2$)



(e) S5 ($kh = 1.1$)

Figure 9 Exceedance probability of maximum wave height H_{\max} at initial BFI = 0.4 from different σ_{θ} and γ_s

(Present model: black cross: $\sigma_{\theta} = 0.3 \ \gamma_s = 0.05$; blue: $\sigma_{\theta} = 0.4 \ \gamma_s = 0.05$; red: $\sigma_{\theta} = 0.5 \ \gamma_s = 0.05$; yellow: $\sigma_{\theta} = 0.3 \ \gamma_s = 0.02$; gray: $\sigma_{\theta} = 0.3 \ \gamma_s = 0.01$. Second-order model: black circle: $\sigma_{\theta} = 0.3 \ \gamma_s = 0.05$. Rayleigh distribution: green line)

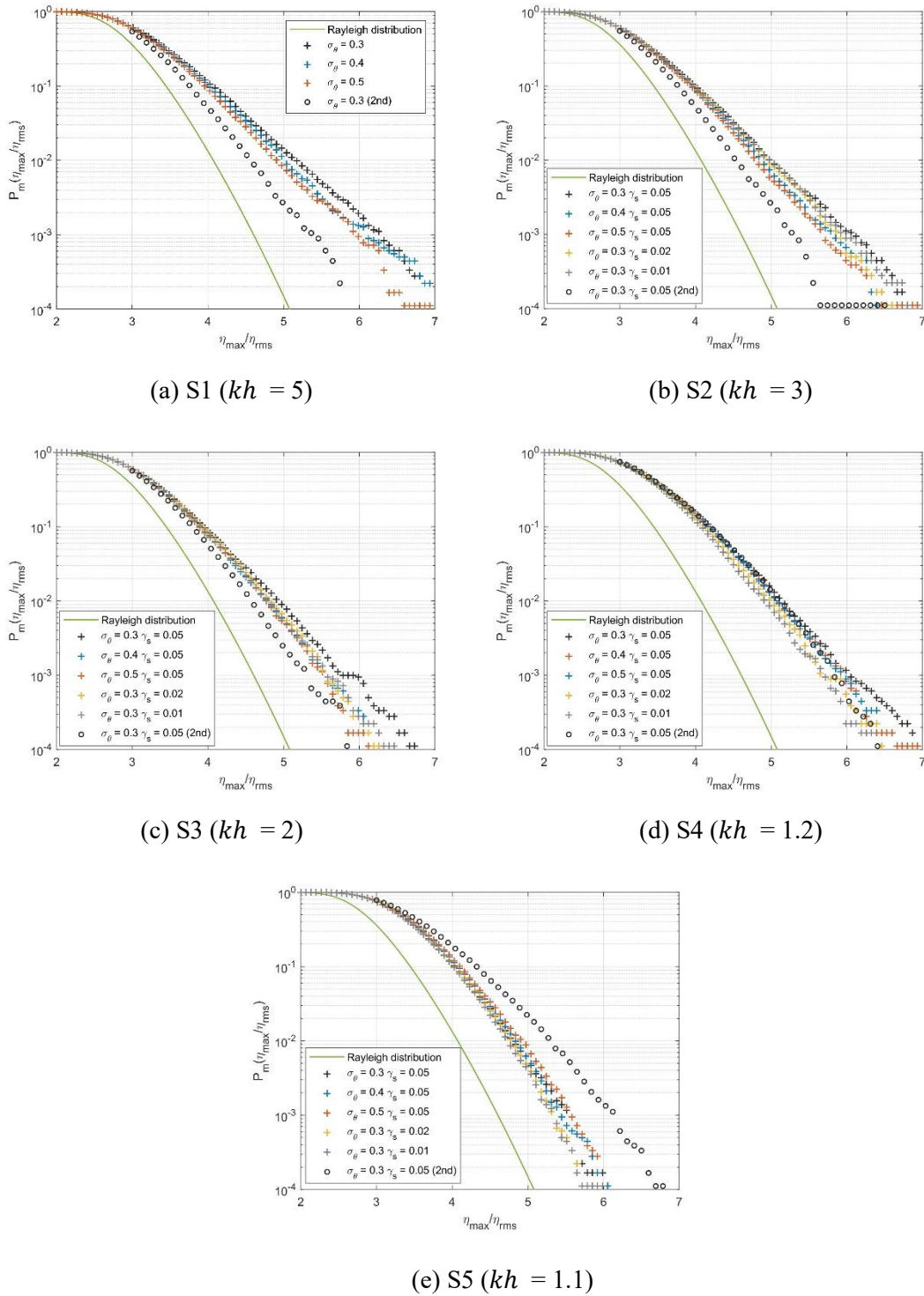


Figure 10 Exceedance probability of maximum wave crest η_{\max} at initial BFI = 0.4 from different σ_θ and γ_s

(Present model: black cross: $\sigma_\theta = 0.3 \ \gamma_s = 0.05$; blue: $\sigma_\theta = 0.4 \ \gamma_s = 0.05$; red: $\sigma_\theta = 0.5 \ \gamma_s = 0.05$; yellow: $\sigma_\theta = 0.3 \ \gamma_s = 0.02$; gray: $\sigma_\theta = 0.3 \ \gamma_s = 0.01$. Second-order model: black circle: $\sigma_\theta = 0.3 \ \gamma_s = 0.05$. Rayleigh distribution: green line)

4 Conclusion

Based on the 2D dNLS equation and pseudo-spectral method, we establish a third-order nonlinear model for the evolution of the directional wave train in a 2D wavefield for an uneven bottom. With the Monte Carlo simulation from random initial phase information, we summarize the nonlinear effect from four-wave interaction and spatial inhomogeneity and the dispersion from directional spreading in the wave evolution through the statistical features of random irregular waves.

To investigate the occurrence of the freak wave in different conditions, we discuss the contribution to the nonlinear interactions from different mechanisms and hypotheses, respectively, and compare the distribution of extreme wave height and crest to find the essential factor. The result indicates:

- A) Compared with the unidirectional wave, the directional spreading in the 2D wavefield significantly affects the wave train evolution and the occurrence of freak waves. The rise of the directional dispersion will make the kurtosis decrease in deep-water but increase in shallow water. Correspondingly, the directional spread contributes to the exceedance probability of maximum wave height and crest, the same as kurtosis.
- B) The directional dispersion effect has almost no effect on the skewness of surface elevation at second-order, and the wave steepness mainly determines the skewness in a 2D wavefield.
- C) In shallow water, a steep slope angle leads to the local peak of kurtosis due to wave shoaling. Correspondingly, it reflects in the increase of the exceedance probability of maximum wave height and crest.
- D) Regarding the degree of impact, the dispersion effect from directional spread mainly affects the wave evolution and the occurrence of the freak wave in deep-water. However, the bottom topography change becomes the major role in the medium and shallow water before the wave breaking.

The model allows a weakly oblique incident wave angle to the slope. The oblique wave case will be given near future.

It should be pointed out that this model still needs to be improved due to the following limitations: first, the wave breaking in shallow water depth is not taken into consideration;

second, the bottom topography is idealized, which ignores the variation on the lateral direction and restricts the slope in a relatively mild range about the steepness squared. Additionally, the wavefield in this study is sufficiently narrow banded spectra. To apply the results of this manuscript to field data, we also need to consider the contribution from the bandwidth of the spectrum to the distribution of wave height and crest height (e.g., Næss (1985)). Furthermore, we assume that there is no extra contribution to the wave evolution during its propagation processes like wind or current, so the initial conditions and bottom topography only decide the directional spreading.

Acknowledge

Z.L. wishes to thank MEXT. Scholarship of Japan and N.M. appreciates for Grants-in-Aid for Scientific Research KAKENHI (19H00782) by JSPS for providing necessary support.

Declaration of Interests

The authors report no conflict of interest.

Reference

- Alber I., and Saffman P.: Stability of random nonlinear deep-water waves with finite bandwidth spectra. *T.R.W., Defense and Space System Group Tech*, 1978, Rep. 31326-6035-RU-00, 89 pp.
- Banner M. L., Young I. R.: Modeling spectral dissipation in the evolution of wind waves, *Journal of physical oceanography*, 1994, 24(7): 1550-1571.
- Benjamin T.B.: Instability of periodic wavetrains in nonlinear dispersive systems. *Proceedings of the Royal Society of London, Series A, Mathematical and Physical Sciences*, 1967, 299(1456), pp. 59-76.
- Benney D. J., Roskes G. J.: Wave instabilities. *Studies in Applied Mathematics*, 1969, 48(4): 377-385.
- Bolles C. T., Speer K., Moore M. N. J.: Anomalous wave statistics induced by abrupt depth change, *Physical review fluids*, 2019, 4(1): 011801.
- Davey A., Stewartson K.: On three-dimensional packets of surface waves, *Proceedings of the Royal Society of London, A, Mathematical and Physical Sciences*, 1974, 338(1613), pp. 101-110.
- Djordjević V.D., Redekopp L.G.: On the development of packets of surface gravity waves moving over an uneven bottom, *Zeitschrift für angewandte Mathematik und Physik ZAMP*, 1978, 29(6), pp. 950-962.
- Draper L.: 'Freak' ocean waves. *Marine Observer*, 1965, 35, 193-195.
- Ewans K. C.: Observations of the directional spectrum of fetch-limited waves, *Journal of Physical Oceanography*, 1998, 28(3): 495-512.
- Forristall G. Z., Ewans K. C.: Worldwide measurements of directional wave spreading, *Journal of Atmospheric and oceanic technology*, 1998, 15(2): 440-469.
- Goda Y. Numerical experiments on wave statistics with spectral simulation, *Report Port Harbour Res. Inst.*, 1970, 9: 3-57.
- Goda Y.: Random Seas and Design of Maritime Structures, 2d ed. *World Scientific*, 2000, 464 pp.

Gramstad O., Trulsen K.: Influence of crest and group length on the occurrence of freak waves, *Journal of Fluid Mechanics*, 2007, 582: 463.

Gramstad O., Zeng H., Trulsen K., et al.: Freak waves in weakly nonlinear unidirectional wave trains over a sloping bottom in shallow water, *Physics of Fluids*, 2013, 25(12): 122103.

Hasimoto H., Ono H.: Nonlinear modulation of gravity waves, *Journal of the Physical Society of Japan*, 1972, 33(3): 805-811.

Janssen P.A.E.M.: Nonlinear four-wave interactions and freak waves, *Journal of Physical Oceanography*, 2003, 33(4), pp. 863-884.

Janssen P.A.E.M., Bidlot J. R.: On the extension of the freak wave warning system and its verification, *European Centre for Medium-Range Weather Forecasts*, 2009, Memo. 588, 42 pp.

Kashima H., Hirayama K., Mori N.: Estimation of freak wave occurrence from deep to shallow water regions, *Coastal Engineering Proceedings*, 2014, 1(34): 36.

Kashima H., Mori N.: Aftereffect of high-order nonlinearity on extreme wave occurrence from deep to intermediate water, *Coastal Engineering*, 2019, 153: 103559.

Kimmoun O, Hsu H C, Hoffmann N, et al.: Experiments on unidirectional and nonlinear wave group shoaling, *Ocean Dynamics*, 2021, 71(11): 1105-1112.

Lawrence C., Trulsen K., Gramstad O.: Statistical properties of wave kinematics in long-crested irregular waves propagating over non-uniform bathymetry, *Physics of Fluids*, 2021, 33(4): 046601.

Lawrence C., Trulsen K., Gramstad O.: Extreme wave statistics of surface elevation and velocity field of gravity waves over a two-dimensional bathymetry. *Journal of Fluid Mechanics*, 2022, 939, A41.

Li Y., Draycott S., Zheng Y., et al.: Why rogue waves occur atop abrupt depth transitions, *Journal of Fluid Mechanics*, 2021, 919.

Liu P.L.F., Dingemans M.W.: Derivation of the third-order evolution equations for weakly nonlinear water waves propagating over uneven bottoms, *Wave motion*, 1989, 11(1), pp. 41-64.

Longuet-Higgins M.: The effect of nonlinearities on statistical distributions in the theory of sea waves, *Journal of fluid mechanics*, 1963, 17(3): 459-480.

Lyu, Z., Mori, N., Kashima, H.: Freak wave in high-order weakly nonlinear wave evolution with bottom topography change, *Coastal Engineering*, 2021, 167: 103918.

- Ma Y., Dong G., Ma X.: Experimental study of statistics of random waves propagating over a bar, *Coastal Engineering Proceedings*, 2014 (34): 30-30.
- Majda A. J., Moore M. N. J., Qi D.: Statistical dynamical model to predict extreme events and anomalous features in shallow water waves with abrupt depth change, *Proceedings of the National Academy of Sciences*, 2019, 116(10): 3982-3987.
- McLean J. W., Ma Y. C., Martin D. U., Saffman, P. G. and Yuen H. C.: Three-dimensional instability of finite-amplitude water waves. *Physical Review Letters*, 1981, 46(13): 817-820.
- Mei C C., Benmoussa C.: Long waves induced by short-wave groups over an uneven bottom, *Journal of Fluid Mechanics*, 1984, 139: 219-235.
- Mendes S., Scotti A., Brunetti M. and Kasparian j.: Non-homogeneous analysis of rogue wave probability evolution over a shoal. *Journal of Fluid Mechanics*, 2022, 939, A25.
- Mori N., Janssen P.A.E.M.: On kurtosis and occurrence probability of freak waves, *Journal of Physical Oceanography*, 2006, 36(7), pp. 1471-1483.
- Mori N., Onorato M., Janssen P.A.E.M., Osborne A.R., Serio M., 2007. On the extreme statistics of long-crested deep-water waves: theory and experiments. *Journal of Geophysical Research: Oceans*, 2007, 112(C9).
- Mori N., Onorato M., Janssen P.A.E.M.: On the estimation of the kurtosis in directional sea states for freak wave forecasting. *Journal of Physical Oceanography*, 2011, 41 (8), 1484-1497.
- Næss, A.: The joint crossing frequency of stochastic processes and its application to wave theory, *Applied ocean research*, 1985, 7(1): 35-50.
- Nikolkina I., Didenkulova I.: Rogue waves in 2006–2010, *Natural hazards and Earth system sciences*, 2011, 11(11): 2913-2924.
- Onorato M., Cavaleri L., Fouques S., et al.: Statistical properties of mechanically generated surface gravity waves: a laboratory experiment in a three-dimensional wave basin, *Journal of Fluid Mechanics*, 2009a, 627, 235–257.
- Onorato M., Waseda T., Toffoli A., et al.: Statistical properties of directional ocean waves: the role of the modulational instability in the formation of extreme events, *Physical review letters*, 2009b, 102(11): 114502.
- Peregrine D. H.: Water waves, nonlinear Schrödinger equations and their solutions, *The ANZIAM Journal*, 1983, 25(1): 16-43.

- Sergeeva A., Pelinovsky E., Talipova T.: Nonlinear random wave field in shallow water: variable Korteweg-de Vries framework, *Natural Hazards and Earth System Sciences*, 2011, 11(2): 323-330.
- Trulsen K., Zeng H., Gramstad O.: Laboratory evidence of freak waves provoked by non-uniform bathymetry, *Physics of Fluids*, 2012, 24(9): 097101.
- Trulsen K., Raustøl A., Jorde S., & Rye L.: Extreme wave statistics of long-crested irregular waves over a shoal, *Journal of Fluid Mechanics*, 2020, 882, R2.
- Turpin F.-M., Benmoussa C., Mei C. C.: Effects of slowly varying depth and current on the evolution of a Stokes wavepacket, *Journal of Fluid Mechanics*, 1983, 132: 1-23.
- Viotti C., Dias F.: Extreme waves induced by strong depth transitions: Fully nonlinear results, *Physics of Fluids*, 2014, 26(5): 051705.
- Waseda T.: Impact of directionality on the extreme wave occurrence in a discrete random wave system, *Proceedings of 9th International Workshop on Wave Hindcasting and Forecasting*, 2006, Victoria, Canada, Environment Canada, P8.
- Waseda T., Kinoshita T., and Tamura H.: Evolution of a random directional wave and freak wave occurrence, *Journal of Physical Oceanography*, 2009, 39, 621–639.
- Yuen H., Lake B.: Nonlinear dynamics of deep-water gravity waves, *Advances in applied mechanics*, 1982, 22: 67-229.
- Zakharov V.E.: Stability of periodic waves of finite amplitude on the surface of a deep fluid, *Journal of Applied Mechanics and Technical Physics*, 1968, 9(2), pp. 190-194.
- Zeng H., Trulsen K.: Evolution of skewness and kurtosis of weakly nonlinear unidirectional waves over a sloping bottom, *Natural Hazards and Earth System Sciences*, 2012, 12(3), 631.
- Zhang J., Benoit M., Kimmoun O., et al.: Statistics of extreme waves in coastal waters: large scale experiments and advanced numerical simulations, *Fluids*, 2019, 4(2): 99.
- Zheng Y., Lin Z., Li Y., et al.: Fully nonlinear simulations of unidirectional extreme waves provoked by strong depth transitions: The effect of slope. *Physical Review Fluids*, 2020, 5(6): 064804.