On dynamical systems of finite-dimensional compact metric spaces and finite-to-one zero-dimensional covers

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1 Introduction

In this note, we study the existence of finite-to-one zero-dimensional covers of dynamical systems.

A pair (X, f) is called a *dynamical system* if X is compact metric space (= compactum) and $f : X \to X$ is a map on X. A dynamical system $(Z, \tilde{f})$ *covers* (X, f) *via* a map $p : Z \to X$ provided that p is an onto map and the following diagram is commutative, i.e., $p\tilde{f} = fp$.

$$\begin{array}{ccc} Z & \xrightarrow{\tilde{f}} & Z \\ {\scriptstyle p}\downarrow & & {\scriptstyle p}\downarrow \\ X & \xrightarrow{f} & X \end{array}$$

We call the map $p : Z \to X$ a *factor mapping*. If Z is zero-dimensional, then we say that the dynamical system $(Z, \tilde{f})$ is a *zero-dimensional cover* of (X, f). Moreover, if the factor mapping is a finite-to-one map, then we say that the dynamical system $(Z, \tilde{f})$ is a *finite-to-one zero-dimensional cover* of (X, f).

The (symbolic) dynamical systems on Cantor sets have been studied by many mathmaticians and also the stronge relations between Markov partitions and symbolic dynamics have been studied (e.g., see [1], [3], [4], [5], [11], [16, Proposition 3.19] and [18]). In [1], Anderson proved that for any dynamical system (X, f), there exists a zero-dimensional cover $(Z, \tilde{f})$ of (X, f), and moreover in [4, Theorem A.1] Boyle, Fiebig and Fiebig proved that any dynamical system (X, f) has a zero-dimensional cover $(Z, \tilde{f})$ such that the topological entropy $h(f)$ of f is equal to $h(\tilde{f})$, where the factor mappings are not necessarily finite-to-one. In topology, there is a classical theorem by Hurewicz [8] that any compactum X is at most n-dimensional if and only if there is a zero-dimensional compactum Z with an onto map $p : Z \to X$ whose fibers have cardinality at most $n + 1$. In the theory of dynamical systems, we have the related general problem (e.g., see[3], [4], [10] and [15]):

Problem 1.1. *What kinds of dynamical systems can be covered by zero-dimensional dynamical systems via finite-to-one maps?*

The motivation for this problem comes from (symbolic) dynamics on Cantor sets. To study dynamical properties of the original dynamics (X, f), the finiteness of the fibers of the factor mapping may be very impotant and so, in this note we focus on the finiteness of fibers of factor mappings. Related to Problem 1.1, first Kulesza [15] proved the following significant theorem:

Theorem 1.2. (Kulesza [15]) *For each homeomorphism f on an n-dimensional compactum X with zero-dimensional set $P(f)$ of periodic points, there is a zero-dimensional cover $(Z, \tilde{f})$ of (X, f) via an at most $(n+1)^n$-to-one map such that $\tilde{f} : Z \to Z$ is a homeomorphism.*

Kulesza also showed that Problem 1.1 needs the assumption $\dim P(f) \leq 0$ (See the proof of Example 2.2 and Remark 2.3 of [15]).

In [10] Ikegami, Kato and Ueda improved the theorem of Kulesza as follows: The condition of at most $(n+1)^n$-to-one map can be strengthened to the condition of at most 2^n-to-one map.

The aim of this note is to give a partial answer to Problem 1.1. For the proofs, we need more general and careful arguments than the arguments of [7], [9] and [10].

2 Preliminaries

In this note, all spaces are separable metric spaces and maps are continuous function. Let $\mathbb{N}$ be the natural numbers, $\mathbb{Z}$ the set of integers, $\mathbb{Z}_+$ the set of all nonnegative integers, i.e., $\mathbb{Z}_+ = \{0, 1, 2, ...\}$ and $\mathbb{R}$ be the real line. Let X be a space with subset K. Then $\mathrm{cl}(K)$ and $\mathrm{int}(K)$ denote its *closure* and *interior*, respectively. In addition, $\mathrm{bd}(K)$ is the boundary of K. If X is a space then $A \subset X$ is called a G_δ-set of X if there are open subsets $U_n \subset X$, $n \in \mathbb{N}$, such that $A = \bigcap_{n=1}^{\infty} U_n$. The complement of a G_δ-set is called an F_σ-set and is characterized by being the union of a countable collection of closed subsets of X. A subset B of a space X is *residual* in X if B contains a dense G_δ-set of X. For a space X, $\dim X$ means the topological (covering) dimension of X (e.g., see [6]). For a collection $\mathcal{C}$ of subsets of X, we put

$$\mathrm{ord}(\mathcal{C}) = \sup\{\mathrm{ord}_x \mathcal{C} \mid x \in X\},$$

where $\mathrm{ord}_x \mathcal{C}$ is the number of menbers of $\mathcal{C}$ which contains x. A closed set K in X is *regular closed* in X if $\mathrm{cl}(\mathrm{int}(K)) = K$. A collection $\mathcal{C}$ of regular closed sets in X is called a *regular closed partition* of X provided that $\mathcal{C}$ is a finite family, $\bigcup \mathcal{C} = X$ and $C \cap C' = \mathrm{bd}(C) \cap \mathrm{bd}(C')$ for each $C, C' \in \mathcal{C}$ with $C \neq C'$. For regular closed partitions $\mathcal{A}$ and $\mathcal{B}$ of X, $\mathcal{A}@\mathcal{B}$ denotes the regular closed partition

$$\{\mathrm{cl}(\mathrm{int}(A) \cap \mathrm{int}(B)) \mid A \in \mathcal{A} \text{ and } B \in \mathcal{B}\}$$

of X. Then we have the following proposition.

Proposition 2.1. *For regular closed partitions $\mathcal{A}$ and B of X, $\mathcal{A}@\mathcal{B}$ is a regular closed partition of X.*

A collection $\{A_\lambda\}_{\lambda\in\Lambda}$ of subsets of X is called a *swelling* of a collection $\{B_\lambda\}_{\lambda\in\Lambda}$ of subsets of X provided that $B_\lambda \subset A_\lambda$ for each $\lambda \in \Lambda$, and if for any $m \in \mathbb{N}$ and $\lambda_1, ..., \lambda_m \in \Lambda$, we have

$$\bigcap_{i=1}^{m} A_{\lambda_i} \neq \emptyset \text{ if and only if } \bigcap_{i=1}^{m} B_{\lambda_i} \neq \emptyset.$$

Conversely, a family $\{B_\lambda\}_{\lambda\in\Lambda}$ of subsets of X is called a *shrinking* of a cover $\{A_\lambda\}_{\lambda\in\Lambda}$ of X if $\{B_\lambda\}_{\lambda\in\Lambda}$ is a cover of X and $B_\lambda \subset A_\lambda$ for each $\lambda \in \Lambda$.

Let X and Y be compacta. A map $f : X \to Y$ is *zero-dimensional* if $\dim f^{-1}(y) \leq 0$ for each $y \in Y$. A map $f : X \to Y$ is *zero-dimensional preserving map* if for any zero-dimensional closed subset D of X, $\dim f(D) \leq 0$. A map $f : X \to Y$ is *two-sided zero-dimensional* if f is zero-dimensional and zero-dimensional preserving. A map $f : X \to Y$ is *semi-open* (*quasi-open*) if for any nonempty open set U of X, $f(U)$ contains a nonempty open set of Y. An onto map $p : X \to Y$ is *at most k-to-one* ($k \in \mathbb{N}$) if for any $y \in Y$, $|p^{-1}(y)| \leq k$.

For a map $f : X \to X$, a subset A of X is *f-invariant* if $f(A) \subset A$. We define set

$$O(f) = \{f^p(x) \mid p \in \mathbb{Z}_+\}$$

which denotes the (positive) orbit of $x \in X$. We define the *eventual orbit* of $x \in X$:

$$EO(x) = \{z \in X \mid \text{there exist } i, j \in \mathbb{Z}_+ \text{ such that } f^i(x) = f^j(z)\}$$
$$= \{z \in X \mid \text{there exist } j \in \mathbb{Z}_+ \text{ such that } f^j(z) \in O(x)\}.$$

Let $P(f)$ be the set of all periodic points of f, i.e.

$$P(f) = \{x \in X \mid f^j(x) = x \text{ for some } j \in \mathbb{N}\}.$$

A point $x \in X$ is *eventually periodic* if there is some $p \in \mathbb{Z}_+$ such that $f^p(x) \in P(f)$. Let $EP(f)$ be the set of all eventually periodic points of f;

$$EP(f) = \bigcup_{p=0}^{\infty} f^{-p}(P(f)).$$

Let X be a compactum and $\mathcal{U}, \mathcal{V}$ be two covers of X. Put

$$\mathcal{U} \vee \mathcal{V} = \{U \cap V \mid U \in \mathcal{U},\ V \in \mathcal{V}\}.$$

The quantity $N(\mathcal{U})$ denotes minimal cardinality of subcovers of $\mathcal{U}$. Let $f : X \to X$ be a map and let $\mathcal{U}$ be an open cover of X. Put

$$h(f,\mathcal{U}) = \lim_{n\to\infty} \frac{\log N(\mathcal{U} \vee f^{-1}(\mathcal{U}) \vee \cdots \vee f^{-n+1}(\mathcal{U}))}{n}.$$

The *topological entropy of* f, denoted by $h(f)$, is the supremum of $h(f,\mathcal{U})$ for all open covers $\mathcal{U}$ of X.

3 Zero-dimensional covers

In this section, we study zero-dimensional covers of some dynamical systems. Theorem 3.18 is a main theorem. We need the following well-known results of dimension theory (e.g., see [6], [19], [20]).

Proposition 3.1. ([6, Theorem 1.5.3 and 1.5.11]).

(1) If $\{F_i \mid i \in \mathbb{N}\}$ *sequence of closed subsets of a separable metric space* X *with* $\dim F_i \leq n$*, then*

$$\dim \left(\bigcup_{i=1}^{\infty} F_i \right) \leq n.$$

(2) If M *is subset of a separable metric space* X *with* $\dim M \leq n$*, then there is a* G_δ*-set* M^* *such that* $M \subset M^*$ *and* $\dim M^* \leq n$*.*

By Proposition 3.1, we obtain the following proposition.

Proposition 3.2. *If X is a separable metric space with $\dim X \leq n$ $(1 \leq n < \infty)$ and E is an F_σ-set of X with $\dim E \leq n-1$, then there exists a zero-dimensional F_σ-set Z of X such that*

$$Z \cap E = \emptyset, \ \ \dim(X - Z) \leq n - 1.$$

Proposition 3.3. ([6, Theorem 1.5.13]). *Let M be a subset of a separable metric space X and $\dim M \leq n$. For any disjoint closed subsets A, B of X, there exists a partition L between A and B such that $\dim(M \cap L) \leq n-1$. In particular, if $\dim M \leq 0$, there exists a partition L between A and B such that $M \cap L = \emptyset$,*

Theorem 3.4. (Hurewicz's theorem [6, Theorem 4.3.4]). *If $f : X \to Y$ is a closed map between separable metric spaces and $k \geq 0$ such that $\dim f^{-1}(y) \leq k$ for each $y \in Y$, then $\dim X \leq \dim Y + k$.*

We say a collection $\mathcal{G}$ of subsets of a compactum X with $\dim X = n < \infty$ is *in general position* provided that if $\mathcal{S} \subset \mathcal{G}$ and $1 \leq |\mathcal{S}| \leq n+1$, then $\dim(\bigcap \mathcal{S}) \leq n - |\mathcal{S}|$, where $\bigcap \mathcal{S} = \bigcap \{S \mid S \in \mathcal{S}\}$ and $|\mathcal{S}|$ denotes the cardinality of $\mathcal{S}$.

Lemma 3.5. ([10, Lemma 3.2]). *Let X be a compactum with $\dim X = n < \infty$. Suppose that for any $j \in \mathbb{N}$, $\mathcal{G}(j)$ is a finite collection of F_σ-sets of X and $\mathcal{G}(j)$ is in general position. Then there is a zero-dimensional F_σ-set Z of X such that if A is a subset of X with $A \cap Z = \emptyset$, then $\mathcal{G}(j) \cup \{A\}$ is in general position for each $j \in \mathbb{N}$.*

Lemma 3.6. ([10, Lemma 3.3]). *Let $\mathcal{C} = \{C_i \mid 0 \leq i \leq m\}$ be a finite open cover of a compactum X with $\dim X = n < \infty$, and $\mathcal{B} = \{B_i \mid 0 \leq i \leq m\}$ be a closed shrinking of $\mathcal{C}$. Suppose that O is an open set of X, Z is an at most zero-dimensional F_σ-set of O, and for each $j \in \mathbb{N}$, $\mathcal{G}(j)$ is a finite collection of F_σ-subsets of O such that each $\mathcal{G}(j)$ is in general position. Then there is an open shrinking $\mathcal{C}' = \{C'_i \mid 0 \leq i \leq m\}$ of $\mathcal{C}$ such that for each $0 \leq i \leq m$,*

(1) $B_i \subset C'_i \subset C_i$,
(2) $C'_i = C_i$ *if* $\mathrm{bd}(C_i) \cap O = \emptyset$,
(3) $C'_i - O = C_i - O$,
(4) $\mathrm{bd}(C'_i) - O \subset \mathrm{bd}(C_i) - O$,
(5) $\mathrm{bd}(C'_i) \cap Z = \emptyset$,
(6) $\mathcal{G}(j) \cup \{\mathrm{bd}(C'_i) \cap O \mid 0 \leq i \leq m\}$ *is in general position for any* $j \in \mathbb{N}$.

Lemma 3.7. *Suppose that $f : X \to X$ is a map of a compactum X. If $x \notin P(f)$, then $f^{-i}(x) \cap f^{-j}(x) = \emptyset = f^{-i}(x) \cap f^j(\{x\})$ for any $i, j \geq 0$ with $i \neq j$. Moreover if $x \notin EP(f)$, $f^i(\{x\}) \cap f^j(\{x\}) = \emptyset$ for any $i, j \in \mathbb{Z}$ with $i \neq j$.*

By Proposition 3.1, we obtain the following lemma.

Lemma 3.8. *Suppose that $f : X \to X$ is a zero-dimensional map of a compactum X. Then $EP(f)$ is an F_σ-set of X with $\dim EP(f) = \dim P(f)$.*

By Hurewicz's theorem, Proposition 3.1 and Proposition 3.2, we obtain the following proposition.

Proposition 3.9. *If $f : X \to X$ is a two-sided zero-dimensional onto map of compactum X, then for any closed subset A of X, $\dim A = \dim f(A) = \dim f^{-1}(A)$. Moreover, if A is an F_σ-set of X, $\dim A = \dim f(A) = \dim f^{-1}(A)$.*

By Proposition 3.2, we obtain the following proposition.

Proposition 3.10. *Let $f : X \to X$ be a two-sided zero-dimensional map of a compactum X and $i_j \in \mathbb{Z}_+$ $(j = 0, 1, ..., k)$. Suppose that M_{i_j} $(j = 0, 1, ..., k)$ are F_σ-sets of X and A, B are disjoint closed subsets of X. Then there exists a partition L between A and B in X such that*

$$\dim(M_{i_j} \cap f^{-i_j}(L)) \leq \dim M_{i_j} - 1$$

for each j.

Lemma 3.11. (cf. [10, Lemma 3.4]). *Let $f : X \to X$ be a two-sided zero-dimensional map of a compactum X such that $\dim X = n < \infty$ and $\dim P(f) \leq 0$. Let F be an F_σ-set of X with $\dim F \leq 0$. Suppose that $\mathcal{C} = \{C_i \mid 0 \leq i \leq M\}$ is a finite open cover of X and let $\mathcal{B} = \{B_i \mid 0 \leq i \leq M\}$ be a closed shrinking of $\mathcal{C}$. Then for each $k = 0, 1, 2, ...$, there is an open shrinking $\mathcal{C}'(k) = \{C'_i \mid 0 \leq i \leq M\}$ of $\mathcal{C}$ such that for each $0 \leq i \leq M$,*

(1) $B_i \subset C'_i \subset C_i$,

(2) $\{f^{-p}(\mathrm{bd}(C'_i)) \mid 0 \leq i \leq M,\ p = 0, 1, ..., k\}$ is in general position,

(3) $\mathrm{bd}(C'_i) \cap (EP(f) \cup F) = \emptyset$ for each i.

Lemma 3.12. (cf. [10, Lemma 3.5]). *Suppose that $f : X \to X$ is a two-sided zero-dimensional map of a compactum X such that $\dim X = n < \infty$ and with $\dim P(f) \leq 0$. Let F be an F_σ-set of X with $\dim F \leq 0$. Then, for* *each $j \in \mathbb{N}$, there is a finite open cover $\mathcal{C}(j) = \{C(j)_i \mid 1 \leq i \leq m_j\}$ of X such that*

(1) $\mathrm{mesh}(\mathcal{C}(j)) < 1/j$,

(2) $\mathrm{ord}\,\mathcal{G} \leq n$, *where* $\mathcal{G} = \{f^{-p}(\mathrm{bd}(C(j)_i)) \mid 1 \leq i \leq m_j,\ j \in \mathbb{N},\ p \in \mathbb{Z}_+\}$,

(3) $F \cap L = \emptyset$, *where* $L = \bigcup\{\mathrm{bd}(C(j)_i) \mid 1 \leq i \leq m_j,\ j \in \mathbb{N}\}$.

Lemma 3.13. *Let $f : X \to X$ be a map of a compactum X and let H be a subset of X. Suppose that for $j \in \mathbb{N}$, $\mathcal{C}(j) = \{C(j)_i \mid 1 \leq i \leq m_j\}$ is a finite open cover of X such that* $\mathrm{mesh}(\mathcal{C}(j)) < 1/j$, $H \cap \bigcup \mathcal{G} = \emptyset$ *and* $\mathrm{ord}\,\mathcal{G} \leq n$, *where $\mathcal{G} = \{f^{-p}(\mathrm{bd}(C(j)_i)) \mid 1 \leq i \leq m_j,\ j \in \mathbb{N},\ p \in \mathbb{Z}_+\}$. Then, for $j \in \mathbb{N}$ there is a finite regular closed partition $\mathcal{D}(j)$ of X such that the following properties hold;*

(1) $\mathrm{mesh}(\mathcal{D}(j)) \leq 1/j$,

(2) $\mathcal{D}(j+1)$ is a refinement of $\mathcal{D}(j)$,

(3) $\prod_{p=0}^{\infty} \mathrm{ord}_{f^p(x)}\,\mathcal{D}(j) \leq 2^n$ *for each $x \in X$, and*

(4) if $x \in H$, then $\prod_{p=0}^{\infty} \mathrm{ord}_{f^p(x)}\,\mathcal{D}(j) = 1$.

Let $Y_k = \{1, 2, ..., k\}$ ($k \in \mathbb{N}$) be the discrete space having k-elements and let $Y_k^{\mathbb{Z}_+} = \Pi_0^{\infty} Y_k$ be the product space. Then the shift map $\sigma : Y_k^{\mathbb{Z}_+} \to Y_k^{\mathbb{Z}_+}$ is definied by $\sigma(x)_i = x_{i+1}$ for $x = (x_0, x_1, x_2, ...) \in Y_k^{\mathbb{Z}_+}$.

Lemma 3.14. *Let $f : X \to X$ be a map of a compactum X and let H be a subset of X. Suppose that there is $m \in \mathbb{N}$ and a sequence of finite regular closed partitions $\mathcal{D}(j)$ ($j \in \mathbb{N}$) of X such that*

(1) $\mathrm{mesh}\,\mathcal{D}(j) \leq 1/j$,

(2) $\mathcal{D}(j+1)$ is a refinement of $\mathcal{D}(j)$,

(3) $\prod_{p=0}^{\infty} \mathrm{ord}_{f^p(x)}\,\mathcal{D}(j) \leq m$ *for each $x \in X$, and*

(4) $H \cap D = \emptyset$, where $D = \bigcup\{f^{-p}(\mathrm{bd}(d)) \mid d \in \mathcal{D}(j),\ j \in \mathbb{N},\ p \in \mathbb{Z}_+\}$, *i.e, if $x \in H$,*

$$\prod_{p=0}^{\infty} \mathrm{ord}_{f^p(x)}\,\mathcal{D}(j) = 1.$$

Then there is a zero-dimensional cover $(Z, \tilde{f})$ of (X, f) via an at most m-to-one map $p : Z \to X$ such that $|p^{-1}(x)| = 1$ for $x \in H$. Moreover, if X is perfect, then Z can be taken as a Cantor set C.

By use of the above results, we will prove the following theorem.

Theorem 3.15. *Suppose that $f : X \to X$ is a two-sided zero-dimensional map of a compactum X such that* $\dim X = n < \infty$*. If* $\dim P(f) \leq 0$*, then there exist a dense G_δ-set of X and a zero-dimensional cover $(Z, \tilde{f})$ of (X, f) via an at most 2^n-to-one onto map p such that $P(f) \subset H$ and $|p^{-1}(x)| = 1$ for $x \in H$. Moreover, if X is perfect, then Z can be chosen as a Cantor set. In particular, $h(f) = h(\tilde{f})$.*

A map $f : X \to X$ of a compactum X is *positively expansive* if there is $\varepsilon > 0$ such that for any $x, y \in X$ with $x \neq y$, there is $k \in \mathbb{Z}_+$ such that $d(f^k(x), f^k(y)) \geq \varepsilon$. A map $f : X \to X$ of a compactum X is *positively continuum-wise expansive* if there is $\varepsilon > 0$ such that for any nondegenerate subcontinuum A of X, there is a $k \in \mathbb{Z}_+$ such that $\mathrm{diam}(f^k(A)) \geq \varepsilon$ (see [12]). Such an $\varepsilon > 0$ is called an *expansive constant* for f.

For a map $f : X \to X$, we consider the following subset of X;

$$I_0(f) = \bigcup \{M \mid M \text{ is a zero-dimensional } f\text{-invariant closed set of } X\}.$$

Proposition 3.16. (cf. [13, Proposition 2.5]). *Let $f : X \to X$ be a positively continuum-wise expansive map of a compact metric space X. Then $I_0(f)$ is a zero-dimensional F_σ-set of X. In particular,* $\dim P(f) \leq 0$*.*

Recall $Y_k = \{1, 2, ..., k\}$ and the shift map $\sigma : Y_k^{\mathbb{Z}_+} \to Y_k^{\mathbb{Z}_+}$ defined by $\sigma(x)_j = x_{j+1}$.

By Lemma 3.12 and proof of Lemma 3.13, we have the following theorem which is a more precise result than [16, Proposition 20].

Theorem 3.17. *Let $f : X \to X$ be a positively expansive map of a compactum X with* $\dim X = n < \infty$*. Then there exist $k \in \mathbb{N}$ and a closed σ-invariant set Σ of $\sigma : Y_k^{\mathbb{Z}_+} \to Y_k^{\mathbb{Z}_+}$ such that (Σ, σ) is a zero-dimensional cover (= symbolic extension) of (X, f) via an at most 2^n-to-one map $p : \Sigma \to X$ satisfying that $|p^{-1}(x)| = 1$ for any $x \in I_0(f)$.*

$$\begin{array}{ccc} \Sigma & \xrightarrow{\ \sigma\ } & \Sigma \\ {\scriptstyle p}\big\downarrow & & {\scriptstyle p}\big\downarrow \\ X & \xrightarrow{\ f\ } & X \end{array}$$

For a map $f : X \to X$ on a compactum X, let

$$D_+(f) = \{x \in X \mid \dim f^{-1}(x) \geq 1\}.$$

The following main theorem is a generalization of Theorem 3.15.

Theorem 3.18. (a generalization of Theorem 3.15). *Let $f : X \to X$ be a map on an n-dimensional compactum X ($n < \infty$). Suppose that f is a zero-dimension preserving map,* $\dim D_+(f) \le 0$ *and* $\dim EP(f) \le 0$. *Then there exist a dense G_δ-set H of X and a zero-dimensional cover $(Z, \tilde{f})$ of (X, f) via an at most 2^n-to-one onto map p such that $EP(f) \subset H$ and $|p^{-1}(x)| = 1$ for $x \in H$. Moreover, if X is perfect, then Z can be chosen as a Cantor set. In particular, $h(f) = h(\tilde{f})$.*

Let G be a graph (= compact connected 1-dimensional polyhedron). A map $f : G \to G$ is *piece-wise monotone* (with respect to some triangulation K) if for any edge E of K (i.e., $E \in K^1$), the restriction $f|E : E \to G$ of f to the edge E is injective.

Lemma 3.19. *Suppose that $f : X \to X$ is a semi-open map of a compactum X and $\{\mathcal{C}(j) \mid j \in \mathbb{N}\}$ is a sequence of finite regular closed partitions of X such that*

(i) there is $m \in \mathbb{N}$ such that $\mathrm{ord}(\mathcal{C}(j)) \le m$ *for each $j \in \mathbb{N}$,*

(ii) $\mathcal{C}(j+1)$ refines $f^{-1}(\mathcal{C}(j))$ @ $\mathcal{C}(j)$,

(iii) $\lim_{j\to\infty} \mathrm{mesh}\, \mathcal{C}(j) = 0$.

Then there is a zero-dimensional cover $(Z, \tilde{f})$ of (X, f) via an at most m-to-one map $p : Z \to X$. Moreover, if X is perfect, then Z can be taken as a Cantor set.

By Lemma 3.19, we obtain the following theorem.

Theorem 3.20. *If $f : G \to G$ is a piece-wise monotone map on a graph G, then there is a zero-dimensional cover $(C, \tilde{f})$ of (G, f) via an at most 2-to-one map, where C is a Cantor set.*

4 Zero-dimensional decompositions of dynamical systems.

In dimension theory, the following decomposition theorem is well-known [6, Theorem 1.5.8]: A separable metric space X is $\dim X \le n$ ($n \in \mathbb{Z}_+$) if and only if X can be represented as the union of $n+1$ subspaces $Z_0, Z_1, ..., Z_n$ of X such that $\dim Z_i \le 0$ for each $i = 0, 1, ..., n$. In this section, we study the similar dynamical decomposition theorems of two-sided zero-dimensional maps (cf. [7]). We consider bright spaces and dark spaces of maps except n

times, and by use of these spaces we prove some dynamical decomposition theorems of spaces related to given maps.

Let $f : X \to X$ be a map. A subset Z of X is a *bright space* of f except n times ($n \in \mathbb{Z}_+$) if for any $x \in X$,

$$|\{p \in \mathbb{Z}_+ \mid f^p(x) \notin Z\}| \leq n.$$

Also we say that $L = X - Z$ is a *dark space* of f except n times. For each $z \in X$, put

$$t(z) = |\{p \in \mathbb{Z}_+ \mid f^p(z) \in L\}|.$$

Also we put

$$T(x) = \max\{t(z) \mid z \in EO(x)\}$$

for each $x \in X$. For a dark space L of f except n times and $0 \leq j \leq n$, we put

$$A_f(L, j) = \{x \in X \mid T(x) = j\}.$$

By Proposition 3.2 and Lemma 3.12, we have the following theorem which is an extension of [7, Theorem 2.4].

Theorem 4.1. *Suppose that $f : X \to X$ is a two-sided zero-dimensional map of a compactum X with $\dim X = n < \infty$. Then there is a bright space Z of f except n times such that Z is a zero-dimensional dense G_δ-set of X and the dark space $L = X - Z$ of f is an $(n-1)$-dimensional F_σ-set of X if and only if $\dim P(f) \leq 0$.*

The following theorem is an extension of [7, Corollary 2.5].

Corollary 4.2. *Suppose that X is a compactum with $\dim X = n < \infty$ and $f : X \to X$ is a two-sided zero-dimensional onto map. Then there exists a zero-dimensional G_δ-dense set Z of X such that for any $n+1$ integers $k_0 < k_1 < \cdots < k_n$ ($k_i \in \mathbb{Z}$),*

$$X = f^{k_0}(Z) \cup f^{k_1}(Z) \cup \cdots \cup f^{k_n}(Z)$$

if and only if $\dim P(f) \leq 0$.

By use of F_σ-dark spaces, we have the following decomposition theorem which is an extension of [7, Theorem 2.6].

Theorem 4.3. *Suppose that X is a compactum with* $\dim X = n$ $(< \infty)$ *and* $f : X \to X$ *is a two-sided zero-dimensional map on X with* $\dim P(f) \leq 0$*. If L is a dark space of f except n times such that L is an F_σ-set of X and* $\dim(X - L) \leq 0$*, then* $\dim A_f(L, j) = 0$ *for each* $j = 0, 1, 2, ..., n$*. In particular, we have the f-invariant zero-dimensional decomposition of X related to the dark space L:*

$$X = A_f(L, 0) \cup A_f(L, 1) \cup \cdots \cup A_f(L, n).$$

Lemma 4.4. ([12, Lemma 5.6]). *Suppose that $f : X \to X$ is a positively continuum-wise expansive map on a compactum X. Then there exists a $\delta > 0$ satisfying the condition: for any $\gamma > 0$ there is $N \in \mathbb{N}$ such that if A is a subcontinuum of X with* $\operatorname{diam} A \geq \gamma$*, then* $\operatorname{diam} f^n(A) \geq \delta$ *for all $n \geq N$.*

By Lemma 3.12, Lemma 4.4 and the proof of Theorem 4.3, we have the following decomposition theorem which is an extension of [7, Theorem 2.8].

Theorem 4.5. *Suppose that X is a compactum with* $\dim X = n$ $(< \infty)$ *and $f : X \to X$ is a positively expansive map. Then there exists a compact $(n-1)$-dimensional dark space L of f except n times such that* $\dim A_f(L, j) = 0$ *for each $j = 0, 1, 2, ..., n$. In particular, there is the f-invariant zero-dimensional decomposition of X related to the compact dark space L:*

$$X = A_f(L, 0) \cup A_f(L, 1) \cup \cdots \cup A_f(L, n).$$

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