ON THE COMMON FIXED POINT THEOREM OF ASYMPTOTIC MAPS AND ITS APPLICATIONS

TOSHIKAZU WATANABE

ABSTRACT. In this paper we consider an asymptotic version of α-ψ contractive mappings in vector metric spaces and give some fixed point theorems on this spaces.

1. INTRODUCTION

In [34], Samet and Vetro-Vetro intoroduced the notion of α-ψ contractive mapping and consider the fixed point theorems. In [38], we intoruduced the notion of α-ψ_n contractive mapping which is a generalization of the α-ψ contractive mapping and consider the fixed point theorem. That fixed point theorem includes the Caccioppoli's fixed point theorem, and α-ψ_n contractive mapping includes the (c)-comparison operator. On the other hand for the mappings of the metric space and vector metric spaces, several authors study common fixed theorem. For instance in [4, 15], Altun and Cevik introduce a vector metric spaces and proved Banach contraction theorem. In [33], Rahimi generalized fixed point theorem and they prove common fixed point theorems for four mappings in ordered vector metric spaces. They also extend and generalize well-known comparable results in the literature. In this article we consider the common fixed point theorem for the α-ψ and α-ψ_n contractive mappings under the ordered vector metric spaces settings. We also consider the two applications of common fixed point theorems. One is the implicit integral equations and the other is dynamic programings. For the implicit integral equations and common fixed point theorems, see [2, 18]. For the dynamic programings and common fixed point theorems, see [7, 8, 10].

2. PRELIMINARIES

Throughout this paper we denote by $\mathbb{N}$ the set of all positive integers and $\mathbb{R}$ the real number. In this section we give sevral preliminaries.

Let E be a non-empty set. A relation $\leq$ on E is called:

(i) reflexive if $x \leq x$ for all $x \in E$
(ii) transitive if $x \leq y$ $y \leq z$ imply $x \leq z$
(iii) antisymmetric if $x \leq y$ and $y \leq x$ imply $x = y$
(iv) preorder if it is reflexive and transitive. A preorder is called a partial order if it is antisymmetric.
(v) translation invariant if $x \leq y$ implies $(x + z) \leq (y + z)$ for any $z \in E$
(vi) scale invariant if $x \leq y$ implies $\lambda x \leq \lambda y$ for any $\lambda > 0$. A preorder $\leq$ is called partial order or an order relation if it is antisymmetric.

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Given a partially ordered set $(E, \leq)$, that is, the set E equipped with a partial order $\leq$, the notation $x < y$ stands for $x \leq y$ and $x \neq y$. An order interval $[x, y]$ in E is the set $\{z \in E : x \leq z \leq y\}$. A real linear space E equipped with an order relation $\leq$ on E which is compatible with the algebraic structure of E is called an ordered linear space or ordered vector space. The ordered vector space E is called a Riesz space(vector lattice or linear lattice) if for every $x, y \in E$, there exist $x \wedge y = \inf\{x, y\}$ and $x \vee y = \sup\{x, y\}$. If we denote $x_+ = 0 \vee x$, $x_- = 0 \wedge (-x)$ and $|x| = x \vee (-x)$, then $x = x_+ - x_-$ and $|x| = x_+ + x_-$.

A closed convex set K in E is called closed convex cone, if for any $\lambda > 0$ and $x \in K$, $\lambda x \in K$. A convex cone K is called pointed if $K \cap (-K) = \{0\}$. From now on we shall call a closed convex pointed cone simply cone.

The cone $\{x \in E : x \geq 0\}$ of nonnegative elements in an ordered vector space E is denoted by E_+. E is said to be Archimedean if $a/n \downarrow 0$ holds for every $a \in E_+$.

Let K be a cone in an ordered vector space E. Denote $x \leq y$ if $y - x \in K$. Then $\leq$ defines a partial order on E called the order induced by K. Conversely, if $\leq$ is a partial order on E, then E is called ordered vector space and the set $K = \{x \in E : x \geq 0\}$ is a cone called the positive cone of E. In this case it is easy to see that $x < y$ if and only if $y - x \in E$. Note that if $a \leq ha$ where $a \in K$ and $h \in (0, 1)$, then $a = 0$.

Definition 1. *A sequence of vectors $\{x_n\}$ in an ordered vector space E is said to: (i) decrease to an element $x \in E$ if $x_{n+1} \leq x_n$ for every $n \in N$ (set of natural numbers) and $x = \inf\{x_n : n \in N\} = \wedge_{n \in N} x_n$. We denote it by $x_n \downarrow x$. (ii) increase to an element $x \in E$ if $x_n \leq x_{n+1}$ for every $n \in N$ and $x = \sup\{x_n : n \in N\} = \vee_{n \in N} x_n$. We denote it by $x_n \uparrow x$.*

Definition 2. *A sequence of vectors $\{x_n\}$ in an ordered vector space E is said to be order convergent to $x \in E$ if there exist sequences $\{y_n\}$ and $\{z_n\}$ in ordered vector space E such that $y_n \downarrow x$, $z_n \uparrow x$ and $z_n \leq x_n \leq y_n$. We denote this by $x = o\text{-}\lim_{n \to \infty} x_n$ or $x_n \to^o x$. If the sequence is order convergent, then its order limit is unique.*

Definition 3. *A sequence of vectors $\{x_n\}$ in an ordered vector space E is said to be order Cauchy sequence in E if the sequence $\{x_m - x_n\}$ in cone K is order convergent to* 0.

Definition 4. *Let E and F be two ordered vector spaces. A mapping $f : E \to F$ is called order continuous at x_0 in E if for any sequence $\{x_n\}$ in E such that $x = o\text{-}\lim_{n \to \infty} x_n$, we have $f(x) = o\text{-}\lim_{n \to \infty} f(x_n)$.*

Remark 5. *If $x_n \in K$ for every $n \in N$ and $x = o\text{-}\lim_{n \to \infty} x_n$, then $x \in K$. Also, if $x_n \in K$ for every $n \in N$ and $\{y_n\}$ is any sequence for which $y_n - x_n \in K$ with $0 = o\text{-}\lim_{n \to \infty} y_n$, then $0 = o\text{-}\lim_{n \to \infty} x_n$.*

Definition 6. *Let E be ordered vector space. A cone $K \subset E$ is called regular if every decreasing sequence of elements in K is convergent.*

Definition 7. *Riesz space E is complete if there exists* $\sup A$ *and* $\inf A$ *for each bounded countable subset A of E. For more details on Riesz space, order convergence, and order continuity, we refer to [1-3] and references mentioned therein.*

Definition 8. *Let E be a Riesz space. $f : E \to E$ sequenes such that $f(x) \leq f(y)$ whenever $x, y \in E$ and $x \leq y$, then f is said to be nondecreasing and $f(x) \geq f(y)$ whenever $x, y \in E$ and $x \geq y$, then f is said to be nonincreasing.*

Definition 9. *Let E be a Riesz space. The set $(UF)f = \{x \in E : x - f(x) \in K\}$ is called upper fixed point set of f, $(LF)f = \{x \in E : f(x) - x \in K\}$ is called lower fixed point set of f and $(F)f = \{x \in E : f(x) = x\}$ is called the set of all fixed points of f.*

Definition 10. *Let E be a Riesz space. The self map f on E is called: (i) dominated on E if $(UF)f = E$. (ii) dominating on E if $(LF)f = E$.*

Example 1. *Let $E = [0, 1]$ be endowed with the usual ordering. Let $f : E \to E$ be defined by $f(x) = x$. Then $(LF)f = E$.*

Definition 11. *Let E be a Riesz space. Two mappings $f, g : E \to E$ are said to be mutually dominated if $f(x) \in (UF)g$ and $g(x) \in (UF)f$ for all $x \in E$. That is, $f(x) \geq g(f(x))$ and $g(x) \geq f(g(x))$ for all $x \in E$.*

Definition 12. *Let E be a Riesz space. Two mappings $f, g : E \to E$ are said to be mutually dominating if $f(x) \in (LF)g$ and $gx \in (LF)f$ for all $x \in E$. That is, $f(x) \leq g(f(x))$ and $g(x) \leq f(g(x))$ for all $x \in E$. The following two examples show that there exist discontinuous and mutually dominating mappings which are not nondecreasing mappings*

The examples of mutually dominating maps which are not non decreasing maps and mutually dominating maps but not non decreasing, see [30, example 15,16].

Definition 13. *Let E be a Riesz space and K be its positive cone. A monotone increasing mapping $\psi_n : K \to K$ is called comparison operator if $\lim_{n\to\infty} \psi_n(t) = 0$ for each $t \in K$.*

Definition 14. *Let E be a Riesz space and $\{x_n\}$ be a sequence in E. If $o\text{-}\lim_{j\to\infty} x_j$ exists, then we say that the series $\sum_{n=1}^{\infty} x_n$ is order convergent.*

Next we give the definitions of α-ψ_n and α-ψ contractive for the mapping $T : X \to X$, see [34, 38]

Definition 15. [34] *Let (X, d) be a metric space. We say that mapping $T : X \to X$ is α-ψ contractive if there exist a mapping $\alpha : X \times X \to [0, 1)$ and a sequence of nondecreasing mappings ψ of E into itself such that the series $\sum_{n=1}^{\infty} \psi^n(t)$ converges for all $t > 0$ and for any $x, y \in X, n \in N$, we have*

$$\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y)).$$

Definition 16. [38] *Let (X, d) be a metric space. We say that the mapping $T : X \to X$ is α-ψ_n contractive if there exist a mapping $\alpha : X \times X \to [0, 1)$ and a sequence of nondecreasing mappings ψ_n of E into itself such that the series $\sum_{n=1}^{\infty} \psi_n(t)$ converges for all $t > 0$ and for any $x, y \in X, n \in N$, we have*

$$\alpha(x, y)d(T^n x, T^n y) \leq \psi_n(d(x, y)).$$

We also give the definiton of α-admissible mappping.

Definition 17. [34] *We say mapping f is α-admissible if*

$$\alpha(x, y) > 1 \text{ implies } \alpha(f(x), f(y)) > 1.$$

Next we consider the versions of α-ψ_n and α-ψ contractive for the mappings f and g.

Definition 18. *Let (X, d) be a metric space. We say that mappings $f, g : X \to X$ are said to mutually α-ψ contractive if there exist a mapping $\alpha : X \times X \to [0, 1)$ and a sequence of nondecreasing mappings ψ of E into itself such that the series $\sum_{n=1}^{\infty} \psi^n(t)$ converges for all $t > 0$ and for any $x, y \in X, n \in N$, we have*

$$\alpha(x, y)d(f(x), g(y)) \leq \psi(d(x, y)).$$

And we say that two mapppings $f, g : X \to X$ are said to mutually α-ψ_n contractive if there exist a mapping $\alpha : X \times X \to [0, 1)$ and sequence of nondecreasing mappings ψ_n of E into itself such that the series $\sum_{n=1}^{\infty} \psi_n(t)$ converges for all $t > 0$ and for any $x, y \in X, n \in N$, we have

$$\alpha(x, y)d(f^n(x), g^n(y)) \leq \psi_n(d(x, y)).$$

We also give the version of α-admissible mappping.

Definition 19. *We say that mappings $f, g : X \to X$ are said to mutually α-admissible if*

$$\alpha(x, y) \geq 1 \text{ imples } \alpha(f(x), g(y)) \geq 1 \text{ or } \alpha(g(x), f(y)) \geq 1.$$

Definition 20. *A pair of self-mappings (f, g) on a cone metric space (X, d) is said to be compatible if, for arbitrary sequence $\{x_n\} \subset X$, such that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n = t \in X$, and for arbitrary $c \in intP$, there exists $n_0 \in N$ such that $d(fg(x_n), gf(x_n))$ converges, whenever $n > n_0$.*

Definition 21. [4, 15] *Let X be a non-empty set and E be a Riesz space. The function $d : X \times X \to E$ is said to be a vector metric (or E-metric) if it is satisfying the following properties:*

(vm1) *d(x, y) = 0 if and only if $x = y$,*
(vm2) *d(x, y) d(x, z) + d(y, z)*

for all $x, y, z \in X$. Also the triple (X, d, E) (briefly X with the default parameters omitted) is said to be vector metric space. For arbitrary elements x, y, z, w of a vector metric space, the following statements are satised.

(i) $0 \in d(x, y)$;
(ii) $d(x, y) = d(y, x)$;
(iii) $|d(x, z) - d(y, z)| \leq d(x, y)$;
(iv) $|d(x, z) - d(y, w)| \leq d(x, y) + d(z, w)$.

Now we give some examples of vector metric spaces. It is well known that $\mathbb{R}^2$ is a Riesz space with coordinatwise ordering dened by

$$(x_1, y_1) \leq (x_2, y_2) \text{ if and only if } x_1 \leq x_2 \text{ and } y_1 \leq y_2$$

for $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$. Again $\mathbb{R}^2$ is a Riesz space with lexicographical ordering dened by

$$(x_1, y_1) \leq (x_2, y_2) \text{ if and only if } x_1 < x_2 \text{ or } x_1 = x_2, y_1 \leq y_2.$$

Note that $\mathbb{R}^2$ is Archimedean with coordinatwise ordering but not with lexicographical ordering.

Example 2. *(a) Let $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ defined by*

$$d((x_1, y_1), (x_2, y_2)) \leq (\alpha|x_1 - x_2|, \beta|y_1 - y_2|)$$

is a vector metric, where α, β are positive real numbers.
(b) Let $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$ dened by

$$d(x, y) = (\alpha|x - y|, \beta|x - y|)$$

is a vector metric, where $\alpha, \beta \geq 0$ and $\alpha + \beta > 0$.

Now we have the following common fixed point theorem which is an extention of [38, Theorem 1].

Theorem 22. *Let (X, d, E) be an E-complete vector metric space with E is Archimedean. Let $f, g : X \to X$ be α-ψ_n contractive mappings satisfying the following conditions;*

(i) *there exists $x_0 \in X$ such that*

$$\alpha(x_0, x_0) \geq 1,\ \alpha(x_0, g(x_0)) \geq 1 \text{ and } \alpha(g(x_0), f(x_0)) \geq 1.$$

(ii) *f or g is continuous.*
(iii) *For all $x, y \in X$, there exists $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$.*

Then, the common fixed point problem $CFP(f, g, E)$ has a solution.

Proof. By (i) there exists x_0 such that $\alpha(x_0, x_0) \geq 1$, $\alpha(x_0, g(x_0)) \geq 1$, and $\alpha(g(x_0), f(x_0)) \geq 1$. Then we define the sequences $x_1 = g(x_0)$, $x_2 = f(x_0)$, $x_3 = g(x_1) = g^2(x_0)$, $x_4 = f(x_2) = f^2(x_0)$, $x_{2n+1} = g(x_{2n-1})$, $x_{2n+2} = f(x_{2n})$. In this case $\alpha(x_1, x_2) = \alpha(g(x_0), f(x_0)) \geq 1$. Then we have

$$d(x_{2n+1}, x_{2n+2}) = d(g^n(x_1), f^n(x_2)) \leq \alpha(x_1, x_2)d(g^n(x_1), f^n(x_2)) \leq \psi_n(d(x_1, f(x_0)))$$

and

$$d(x_{2n}, x_{2n+1}) = d(f^n(x_0), g^n(x_1)) \leq \alpha(x_0, x_1)d(f^n(x_0), g^n(x_1)) \leq \psi_n(d(x_0, x_1)).$$

Since $d(x_0, x_1), d(x_1, f(x_0)) > 0$, there exists n_1 with even such that

$$\sum_{k=n_1/2}^{\infty} \psi_k(d(x_0, x_1) + d(x_1, f(x_0))) \to 0 \tag{1}$$

order converges, and there exists n_2 with odd such that

$$\sum_{k=(n_2-1)/2}^{\infty} \psi_k(d(x_0, x_1 + d(x_1, f(x_0))) \to 0 \tag{2}$$

order converges. In (1) and (2), the right hand order converges to 0 as take $n \geq \max\{ n_1, n_2\}$ and $n \to \infty$. Then for any $m > n \geq n_0 = \min(n_1, n_2)$, if n and m are even, then we have

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{k=n/2}^{m/2-1} \psi_k(d(x_0, x_1) + d(x_1, f(x_0)) \\ &\leq \sum_{k=n/2}^{\infty} \psi_k(d(x_0, x_1) + d(x_1, f(x_0)) \end{aligned}$$

if n and m are odd, then we have

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{k=(n-1)/2}^{(m-1)/2-1} \psi_k(d(x_0, x_1) + d(x_1, f(x_0)) \\ &\leq \sum_{k=(n-1)/2}^{\infty} \psi_k(d(x_0, x_1) + d(x_1, f(x_0)) \end{aligned}$$

if n is even and m is odd, then we have

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{k=n/2}^{(m-1)/2-1} \psi_k(d(x_0, x_1) + d(x_1, f(x_0)) \\ &\leq \sum_{k=n/2}^{\infty} \psi_k(d(x_0, x_1) + d(x_1, f(x_0)) \end{aligned}$$

if n is odd and m is even, then we have

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{k=(n-1)/2}^{m/2-1} \psi_k(d(x_0, x_1) + d(x_1, f(x_0)) \\ &\leq \sum_{k=(n-1)/2}^{\infty} \psi_k(d(x_0, x_1) + d(x_1, f(x_0)). \end{aligned}$$

Then all cases $d(x_n, x_m)$ order onverge to 0 as $(m >)n \to \infty$. Thus $\{x_n\}$ is Cauchy. Since X is complete, we have $z \in X$ such that $x_n \to z$ as $n \to \infty$. Then $f(x_{2n}) \to z$ and $g(x_{2n+1}) \to z$. as $n \to \infty$. Since f is continous, we have $f(x_{2n}) \to f(z)$. In this case we have

$$\begin{aligned} d(f(z), z) &\leq d(f(z), f(x_{2n}))) + d(f(x_{2n}), g(x_{2n+1}))) + d(g(x_{2n+1}), z) \\ &= d(f(z), f(x_{2n}))) + d(f^n(x_0), g^n(x_1)) + d(g(x_{2n+1}), z) \\ &\leq d(f(z), f(x_{2n}))) + \alpha(x_0, x_1)d(f^n(x_0), g^n(x_1))) + d(g(x_{2n+1}), z) \\ &\leq d(f(z), f(x_{2n})) + \psi_n(d(x_0, x_1)) + d(g(x_{2n+1}), z). \end{aligned}$$

Since $d(x_0, x_1) > 0$, we have $\psi_n(d(x_0, x_1)) \to^o 0$ as $n \to \infty$. Then we have $f(z) = z$. Let $d(g(z), z) > 0$. Then we have

$$\begin{aligned} d(g(z), z) =& d(g(z), f(z)) \\ &\leq \alpha(z, z)d(f^n(z), g^n(z)) \\ &\leq \psi_n(d(f(z), g(z))) < d(g(z), f(z)) = d(g(z), z) \end{aligned}$$

which is contardiction. Thus we have $d(g(z), z) = 0$. So we have $g(z) = z$. Hence, we proved that f and g have a common fixed point $z \in X$.

Next we consider the uniquness of common fixed points Suppose that u and v are two different common fixed points of f and g. From (iv), there exists $z \in X$ such that

(3) $$\alpha(u, z) \geq 1 \text{ and } \alpha(v, z) \geq 1.$$

Define the sequence $\{z_n\}$ in X by $z_0 = z$ and $z_{n+1} = g(z_n)$ for all $n = 0, 1, 2, \ldots$. Then we have

$$d(u, z_{n+1}) = d(f^n(u), g^n(z)) \leq \alpha(u, z) d(f^n(u), g^n(z)) \leq \psi_n(d(u, z)).$$

Then we have $d(u, g^n(z)) \leq \psi_n(d(u, z)$. Similarly $d(v, g^n(z)) \leq \psi_n(d(v, z)$. Since $\psi_n(d(u, z)) \to^o 0$ and $\psi_n(d(v, z)) \to^o 0$ as $n \to \infty$. Then we have $g^n(z) \to u$ and $g^n(z) \to v$ as $n \to \infty$. Then the uniqueness of the limit gives us $u = v$. □

Next we consider the following common fiexed point Theorem, whih is a version of [38, Theorem 2]. mwhih

Theorem 23. *Let (X, d, E) be an E-complete vector metric space with E is Archimedean. Let $f, g : X \to X$ be mappings satisfying the following conditions;*

(i) *fg and gf are α-ψ_n contractive mappings;*
(ii) *f and g are mutually α-admissible;*
(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, x_0) \geq 1$ and $\alpha(x_0, g(x_0)) \geq 1$;*
(iv) *If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in N$ and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that*

$$\alpha(x_{n_k}, x) \geq 1 \text{ for all } k \in N.$$

(v) *$\psi_1(t) < t$ for all $t > 0$.*
(vi) *For all $x, y \in X$, there exists $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$.*

Then, the common fixed point problem $CFP(fg, gf, E)$ has a solution.

Proof. We define the sequence $\{x_n\}$ with $x_1 = f(x_0)$, $x_2 = g(x_1) = gf(x_0)$, $x_3 = f(x_2) = fg(x_1)$, $x_4 = g(x_3) = gfgf(x_0)$, $x_{2n+1} = f(x_{2n}) = fg(x_{2n-1})$ and $x_{2n+2} = g(x_{2n+1}) = gf(x_{2n})$. Put $S = fg$ and $T = gf$. Then

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(fg(x_{2n-1}), gf(x_{2n})) \\ &= d(S^n(x_1), T^n(x_2)) \leq \alpha(x_1, x_2) d(S^n(x_1), T^n(x_2)) \\ &\leq \psi_n(d(x_1, x_2)) = \psi_n(d(x_1, g(x_1))) \end{aligned} \tag{4}$$

and

$$\begin{aligned} d(x_{2n}, x_{2n+1}) &= d(gf(x_{2n-2}), fg(x_{2n-1})) \\ &= d(T^n(x_0), S^n(x_1)) \leq \alpha(x_0, x_1) d(T^n(x_0), S^n(x_1)) \\ &\leq \psi_n(d(x_0, x_1)). \end{aligned} \tag{5}$$

Since $d(x_0, x_1), d(x_1, g(x_1)) > 0$, for the case of (4) if n is even, there exists n_1 suh that for any $n \geq n_1$,

$$\sum_{k=n/2}^{\infty} \psi_k(d(x_0, x_1)$$

for the case of (5) if n is even, there exists n_2 suh that for any $n \geq n_2$,

$$\sum_{k=n/2}^{\infty} \psi_k(d(x_1, g(x_1)),$$

for the case of (4) if n is odd, there exists n_3 suh that for any $n \geq n_3$,

$$\sum_{k=(n-1)/2+1}^{\infty} \psi_k(d(x_0, x_1),$$

for the case of (5) if n is odd, there exists n_4 suh that for any $n \geq n_4$,

$$\sum_{k=(n-1)/2}^{\infty} \psi_k(d(x_1, g(x_1)).$$

Then all the cases, order converges to 0 as $n \to \infty$.

Then for $m > n \geq n_0 = \min(n_1, n_2, n_3, n_4)$, if n and m are even, then we have

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{k=n/2}^{m/2-1} \psi_k(d(x_0, x_1) + \sum_{k=n/2}^{m/2-1} \psi_k(d(x_1, x_2))) \\ &\leq \sum_{k=n/2}^{\infty} \psi_k(d(x_0, x_1) + \sum_{k=n/2}^{\infty} \psi_k(d(x_1, g(x_1))), \end{aligned}$$

if n and m are odd, then

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{k=(n-1)/2+1}^{(m-1)/2} \psi_k(d(x_0, x_1) + \sum_{k=(n-1)/2}^{(m-1)/2-1} \psi_k(d(x_1, g(x_1))) \\ &\leq \sum_{k=(n-1)/2+1}^{\infty} \psi_k(d(x_0, x_1) + \sum_{k=(n-1)/2}^{\infty} \psi_k(d(x_1, g(x_1))), \end{aligned}$$

if n is even and m is odd, then

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{k=n/2}^{(m-1)/2+1} \psi_k(d(x_0, x_1) + \sum_{k=n/2}^{(m-1)/2-1} \psi_k(d(x_1, f(x_0))) \\ &\leq \sum_{k=n/2}^{\infty} \psi_k(d(x_0, x_1) + \sum_{k=n/2}^{\infty} \psi_k(d(x_1, g(x_1))), \end{aligned}$$

if n is odd and m is even, then

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{k=(n-1)/2+1}^{m/2-1} \psi_k(d(x_0, x_1) + \sum_{k=(n-1)/2}^{m/2-1} \psi_k(d(x_1, f(x_0))) \\ &\leq \sum_{k=(n-1)/2+1}^{\infty} \psi_k(d(x_0, x_1) + \sum_{k=(n-1)/2}^{\infty} \psi_k(d(x_1, g(x_1))). \end{aligned}$$

Then the sequence $\{x_n\}$ is Cauchy in X. Since X is complete, there exist $z \in X$ such that $x_n \to z$ as $n \to \infty$. Then $gf(x_{2n}) \to z$ and $fg(x_{2n+1}) \to z$. as $n \to \infty$. By (ii), we have $\alpha(x_0, x_1) \geq 1$ and $\alpha(x_0, x_1) \geq 1$. Since f, g are mutually α-admissible, we have $\alpha(f(x_0), g(x_1)) \geq 1$ and $\alpha(gf(x_0), fg(x_1)) \geq 1$. Then $\alpha(x_1, x_2) \geq 1$ and $\alpha(x_2, x_3) \geq 1$. Inductively, we have for any $n = 0, 1, 2, \ldots$, $\alpha(x_n, x_{n+1}) \geq 1$. Then by (iii), there exists a subsequence $\{x_{n_k}\}$ such that $\alpha(x_{n_k}, u) \geq 1$ for all $k \in N$.

Then we have

$$\begin{aligned}
d(fg(z),z) &\le d(fg(z),fg(x_{2n_k}))) + d(fg(x_{2n_k}),gf(x_{2n_k+1}))) + d(gf(x_{2n_k+1}),z)\\
&= d(fg(z),fg(x_{2n_k}))) + d((fg)^{n_k}(x_0),(gf)^{n_k}(x_1)) + d(gf(x_{2n_k+1}),z)\\
&\le \alpha(z,x_{n_k})d(fg(z),gf(x_{n_k}))) + \alpha(x_0,x_1)d((fg)^{n_k}(x_0),(gf)^{n_k}(x_1)))\\
&\quad + d(gf(x_{2n_k+1}),z)\\
&\le \psi_1(d(z,x_{n_k})) + \psi_n(d(x_0,x_1)) + d(gf(x_{2n+1}),z)\\
&< d(z,x_{n_k}) + \psi_n(d(x_0,x_1)) + d(gf(x_{2n_k+1}),z).
\end{aligned}$$

By condition (iii), let $k \to \infty$ we have $d(fg(z),z) \to^o 0$. By the proof of Theorem 22, $d(gf(z),z) \to^o 0$, $fg(z) = gf(z) = z$ and we also have the mappings fg and gf have the uniqune common fixed point. □

Theorem 24. *under the ondition of Theorem 23, we assume that there exists $r > 0$ such that*

$$d(f(x),g(y)) \le rd(fg(x),gf(y))$$

then f and g have unique common fixed point.

Proof. Let z be a common fixed point of fg and gf. Then we have

$$d(f(z),g(z)) \le d(fg(z),gf(z)) = 0$$

Thus $f(z) = g(z) = z$. □

Example 3. *Define $d(x,y) = (x-y)^2$ and also define $f(x) = 2x-1$ and $g(x) = x^2$. Then $d(fg(x),gf(x)) = 4(x-1)^4 = 4d(f(x),g(x)) > d(f(x),g(x))$.*

3. APPLICATIONS

We shall study sufficient condition for the existence of common solution of the following integral equations([1, 18]).

We consider the implicit integral equation

$$p(t,x(t)) = \int_0^1 q(t,s,x(s))ds, t,s \in [0,1], \tag{6}$$

where $x \in L^p[0,1]$, $1 < p < \infty$. Integral equations like (6) were introduced by Feckan [18] and could occur in the study of nonlinear boundary value problems of ordinary differential equations. For $E = \mathbb{R}$ and $X = L^1([0,1])$, its norm is defined by $\|x\| = \int_0^1 |x(t)|dt$ and we define $d : X \times X \to \mathbb{R}$ by

$$d(x,y) = \|x(t) - y(t)\|.$$

Then d is a E-metric on X. Suppose that the following conditions holds: Define $f(x(t)) = p(t,x(t))$ and $g(x(t)) = \int_0^1 q(t,s,x(s))ds$ for any $t,s \in [0,1]$. In this case we assume the following conditions;

(e-i) $p(t,x(t))$ is continuos with respect to x for any t.

(e-ii) For any $t \in [0,1)$, $n \in N$ and $x,y \in X$, there exists $\psi_n : [0,1] \to \mathbb{R}$ such that

$$\|f^n(x) - g^n(y)\| \le \psi_n(\|x-y\|)$$

and ψ_n satisfies $\psi_n(t) < t$ for any $t > 0$ and $n \in N$.

(e-iii) there exists x_0 such that $p(t, x_0(t)) \le \int_0^1 q(t, s, x_0(s))ds \le x_0(t)$ for all $t \in [0, 1]$.

Then, the implicit integral equation (6) has a solution in $L^1[0, 1]$.

Remark 25. *If $\psi_n(t) = r^n t$ where $0 \le r < 1$, then we assume that*

$$\int_0^1 \left| p\left(t, \int_0^1 q(t, s, x(s))ds\right) - \int_0^1 q(t, s, p(s, (gf)(y(s))))ds \right| dt$$
$$\le r(\|x - y\|),$$

then we have

$$\int_0^1 \left| p\left(t, \int_0^1 q(t, s, (fg)^{n-1}(x(s)))ds\right) - \int_0^1 q(t, s, p(s, (gf)^{n-1}(x(s))))ds \right| dt$$
$$\le r^n \|x - y\| = \psi_n(\|x - y\|)$$

holds

Proof. We take $f(x(t)) = p(t, x(t))$ and $g(x(t)) = \int_0^1 q(t, s, x(s))ds$. Let

$$\alpha(x, y) = \begin{cases} 1 \text{ if } x \ge y, \\ 0 \text{ if } x < y. \end{cases}$$

For any x, y take $z = \max\{x, y\}$, then $\alpha(z, x) \ge 1$ and $\alpha(z, y) \ge 1$. Then the condition of (iii) in Theorem 22 is satisfied. By the condition (e-i), we have

$$\begin{aligned} \alpha(x, y)d(f^n(x), g^n(y)) &\le d(f^n(x), g^n(y)) \\ &= \|f^n(x) - g^n(y)\| \\ &\le \psi_n(\|x - y\|) \\ &\le \psi_n(d(x, y)) \end{aligned}$$

Moreover we take $x_0 \in X$ satisfying the condition (e-iii). Then we have

$$x_0(t) - g(x_0(t)) = x_0(t) - \int_0^1 q(t, s, x_0(s))ds \ge 0.$$

$$f(x_0(t)) = p(t, x_0(t)) \le \int_0^1 q(s, x_0(s))ds = g(x_0(t)).$$

that is, $\alpha(x_0, g(x_0)) \ge 1$ and $\alpha(g(x_0), f(x_0)) \ge 1$. Also we have $\alpha(x_0(t), g(x_0(t))) \ge 1$. Since $x_0(t) - x_0(t) = 0$, we have $\alpha(x_0(t), x_0(t)) \ge 1$. Then the condition (ii) of Theorem 22 is satisfied. By using Theorem 22, 24, the common fixed point problem $CFP(f, g, K)$ has a solution which in turn solves the integral equation. □

Next we consider the application of the dynamic progarming. Let X and Y be Banach spaces, $S \subset X$ be the state space, $D \subset Y$ be the decision space and i_X be the identity mapping on X. $B(S)$ denotes the set of all bounded real-valued functions on S and and C be closed convex subset of $B(S)$. Moreover $d(f, g) = \sup_{x \in S} |f(x) - g(x)|$. It is clear that $(B(S), d)$ is a complete metric space. Since C is closed subset of $B(S)$, (C, d) is also a complete metric space. By means of Theorem 22, in this section we study the existence and uniqueness of common solution of the following system of functional equations arising in dynamic programming:

$$f_i(x) = \sup_{y\in D} u(x,y) + H_i(x,y,f_i(T(x,y))),\ \text{any } y \in S,\ i \in \{1,2\}, \tag{7}$$

where $u : S \times D \to \mathbb{R}$, $T : S \times D \to S$ and $H_i : S \times D \times R \to \mathbb{R}$ for $i \in \{1, 2\}$. Suppose that the following conditions are satisfied:

(Dp-1) H_i are bounded for $i \in 1, 2$.
(Dp-2) For any $t \in [0, 1)$, $n \in N$ and H_i, there exists $\psi_n : [0, 1) \to \mathbb{R}$ such that

$$|H_1(x,y,A_1^{n-1}g(T(x,y))) - H_2(x,z,A_2^{n-1}h(T(x,z)))|$$
$$< \psi_n(d(g,h))$$

and ψ_n satisfies $\psi_n(t) < t$ for any $t > 0$ and $n \in N$.
(Dp-3) There exists $h_0 \in C$ such that $Ah_0 \in C$ and $Th_0 \in C$.
(Dp-4) There exists some $A_i \in \{A_1, A_2\}$ such that for any sequence $\{h_n\}_{n\geq 1} \subset C$ and $h \in C$,

$$\lim_{n\to\infty} \sup_{x\in S} |h_n(x) - h(x)| = 0 \text{ imples } \lim_{n\to\infty} \sup_{x\in S} |A_i h_n(x) - A_i h(x)| = 0;$$

We consider the ommon solution of the system of functional equations (7) in C. We assume that the normed space X be a complete Banach space.

Theorem 26. *Let X be a complete Banach space. Let C be a closed subspave of X. Assume that conditions* (Dp-1)-(Dp-5) *are satisfied. Then the system of functional equations (7) have a unique common solution in C.*

Proof. Let

$$\alpha(x,y) = \begin{cases} 1 \text{ if } x,y \in C, \\ 0 \text{ otherwise.} \end{cases}$$

For any $g, h \in C$, $x \in S$ and $\varepsilon > 0$, there exist $y, z \in D$ such that

$$A_1^n g(x) < u(x,y) + H_1(x,y,A_1^{n-1}g(T(x,y))) + \varepsilon,$$
$$A_2^n h(x) < u(x,z) + H_2(x,z,A_2^{n-1}h(T(x,z))) + \varepsilon.$$

and

$$A_1^n g(x) - A_2^n h(x) < H_1(x,y,A_1^{n-1}g(T(x,y))) - H_2(x,z,A_2^{n-1}h(T(x,z))) + \varepsilon$$

Note that

$$A_1^n g(x) > u(x,y) + H_1(x,y,A_1^{n-1}g(T(x,y))),$$
$$A_2^n h(x) > u(x,z) + H_1(x,z,A_2^{n-1}h(T(x,z))),$$

then

$$A_1^n g(x) - A_2^n h(x) > -H_1(x,y,A_1^{n-1}g(T(x,y))) - H_2(x,z,A_2^{n-1}h(T(x,z))) - \varepsilon$$

By the condition (Dp-2) for any $g, h \in C$, $x \in S$ there exists ψ_n such that

$$
\begin{aligned}
&\alpha(g,h)d(A_1^n g, A_2^n h) \\
&= \|A_1^n g(x) - A_2^n h(x)\| \\
&< |H_1(x, y, A_1^{n-1} g(T(x,y))) - H_2(x, z, A_2^{n-1} h(T(x,z)))| + \varepsilon \\
&< \psi_n(d(g,h)) + \varepsilon
\end{aligned}
$$

Letting $\varepsilon \to 0$ in (3.8), we obtain that

$$\alpha(g,h)d(A_1^n g, A_2^n h) < \psi_n(d(g,h))$$

Thus A_1 and A_2 are mutually α-ψ_n contractive. Let $h, k \in C$. By the condition (Dp-3), there exists $h_0 \in C$ such that $Ah_0 \in C$ and $Th_0 \in C$. Then we have $\alpha(h_0(s), h_0(s)) \geq 1$, $\alpha(h_0(s), A_2 h_0(s)) \geq 1$ and $\alpha(A_1 h_0(s), A_2 h_0(s)) \geq 1$. Then the condition of (i) in Theorem 22 is satisfied. By the condition (Dp-5), the mappings A_i are continuous self mappings of C. then the condition (ii) in Theorem 22 is satisfied. For any $g, h \in C$ take $k = \max\{g, h\}$ then $k \in C$ such that $\alpha(g,k) \geq 1$ and $\alpha(h,k) \geq 1$. then the condition (iii) in Theorem 22 is satisfied. □

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(Toshikazu Watanabe) Tokyo University of Information Sciences 4-1 Onaridai, Wakaba-ku,Chiba, 265-8501 Japan

Email address: twatana@edu.tuis.ac.jp