Gibonacci Optimization

— duality —

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Abstract

We show that a *parametric linear system of equations* plays a fundamental part in establishing a mutual relation between minimization problem (primal) and maximization problem (dual). The system is of $2n$-equation on $2n$-variable, called *zero-minimum condition*. It yields a couple of second-order finite (n-) linear difference equation on n-variable, which constitute the respective *optimal conditions*. The respective equations have a mimimum solution for primal and a maximum one for dual. Both the optimal solutions are expressed in terms of *Gibonacci* sequence, which is a parametric generalization of the *Fibonacci* one. Either solution is characterized by the backward Gibonacci sequence and its complementary – *Hibonacci* sequence –.

1 Introduction

Recenly a new duality for quadratic optimization has been extensively developed by Iwamoto, Kimura, Fujita and Kira [12–25]. They have given several kinds of duality through some methods. These supply related dualities and associated dual problems for the classical optimization problems by Bellman and others [1–7,26], [9,11,28,29]. The duality and its approach are characterized by — Fibonacci [8,10,27,30] and complementarity —, respectively.

This paper enhances the Fibonacci duality through a parametric linear system of equations. The Fibonacci duality is expanded to *Gibonacci* one. The complementarity is replaced by a pair of linear equations — an equality condition —. This is called a *zero-minimum condition* for a $2n$-variable parametric minimization problem.

Section 2 gives a $2n$-variable parametric minimization problem, where a parameter λ ranges over $(0, \infty)$. The objective function turns out to be nonnegative. It attains zero iff a linear system of $2n$-equations on $2n$-variables has a solution. Section 3 presents a pair of λ-parametric minimization problem and λ-parametric maximization problem for $\lambda > 0$. Section 4 discusses a new duality — Gibonacci duality — . This covers Fibonacci duality. The principal idear is based upon the complementarity.

2 Complementary approach

This section specifies a $2n$-variable minimization problem. Throughout the section, let $c \in R^1$ and $\lambda > 0$ be given constants.

An original problem is a $2n$-variable (x, μ) with a parameter λ and a fixed initial value $x_0 = c$:

$$
\mathrm{Q} \qquad \begin{aligned} \text{minimize} \quad & -2\lambda x_0\mu_1 + \sum_{k=1}^{n-1} \big[(x_{k-1} - x_k)^2 + x_k^2 + \lambda^2\mu_k^2 + (\mu_k - \mu_{k+1})^2 \\ & \qquad\qquad + 2(\lambda - 1)x_k(\mu_k - \mu_{k+1})\big] \\ & \qquad + (x_{n-1} - x_n)^2 + x_n^2 + (\lambda^2 + 1)\mu_n^2 + 2(\lambda - 1)x_n\mu_n \\ \text{subject to} \quad & \text{(i)} \;\; x \in R^n,\; x_0 = c, \quad \text{(ii)} \;\; \mu \in R^n. \end{aligned}
$$

Let us define the objective function by $h : R^n \times R^n \to R^1$

$$
\begin{aligned} h(x, \mu) \;=\; & -2\lambda c\mu_1 + \sum_{k=1}^{n-1} \big[(x_{k-1} - x_k)^2 + x_k^2 + \lambda^2\mu_k^2 + (\mu_k - \mu_{k+1})^2 \\ & \qquad + 2(\lambda - 1)x_k(\mu_k - \mu_{k+1})\big] \\ & + (x_{n-1} - x_n)^2 + x_n^2 + (\lambda^2 + 1)\mu_n^2 + 2(\lambda - 1)x_n\mu_n. \end{aligned}
$$

We have an *evaluation* as follows.

Lemma 1 *Let $(x,\ \mu)$ be feasible. Then it holds that*

$$
h(x, \mu) \;\geq\; 0. \tag{1}
$$

The sign of equality holds iff

$$
\text{(Zm)} \qquad \begin{aligned} c - x_1 &= \lambda\mu_1, & x_1 &= \mu_1 - \mu_2 & & \\ x_{k-1} - x_k &= \lambda\mu_k, & x_k &= \mu_k - \mu_{k+1} & & 2 \leq k \leq n-1 \\ x_{n-1} - x_n &= \lambda\mu_n, & x_n &= \mu_n & & \end{aligned}
$$

holds.

This is a linear system of $2n$-equation on $2n$-variable $(x,\ \mu)$. We call (Zm) a *zero-minimum condition*.

Proof. First we present an identity, which plays a fundamental role in analyzing the pair. Let $x = \{x_k\}^n$, $\mu = \{\mu_k\}_1^n$ be any two sequences of real number with $x_0 = c$. Then an identity

$$
\text{(C)} \quad c\mu_1 \;=\; \sum_{k=1}^{n-1}[(x_{k-1} - x_k)\mu_k + x_k(\mu_k - \mu_{k+1})] + (x_{n-1} - x_n)\mu_n + x_n\mu_n
$$

holds true. This identity is called *complementary*. The complementary identity implies that

$$
\text{(QI)}\quad
\begin{aligned}
&-2\lambda x_0\mu_1+\sum_{k=1}^{n-1}\left[(x_{k-1}-x_k)^2+x_k^2+\lambda^2\mu_k^2+(\mu_k-\mu_{k+1})^2+2(\lambda-1)\lambda x_k(\mu_k-\mu_{k+1})\right]\\
&\qquad\qquad+(x_{n-1}-x_n)^2+x_n^2+(\lambda^2+1)\mu_n^2+2(\lambda-1)\lambda x_n\mu_n\\
&=\sum_{k=1}^{n-1}\left[(x_{k-1}-x_k-\lambda\mu_k)^2+(x_k-\mu_k+\mu_{k+1})^2\right]\\
&\qquad\qquad+(x_{n-1}-x_n-\lambda\mu_n)^2+(x_n-\mu_n)^2.
\end{aligned}
$$

This is an identity on $R^n\times R^n$, which is called *quadratic*. Hence we have an inequality

$$h(x,\mu)\ \geq\ 0.$$

The sign of equality holds iff (Zm) holds. Thus the inequality (2) with zero-minimum condition is shown. □

The objective function is also expressed as follows.

Lemma 2 *Let* $(x,\,\mu)$ *be feasible. Then it holds that*

$$
\begin{aligned}
h(x,\mu)\ =\ -2c\mu_1+\sum_{k=1}^{n-1}&\left[(x_{k-1}-x_k)^2+x_k^2+\lambda^2\mu_k^2+(\mu_k-\mu_{k+1})^2+2(1-\lambda)(x_{k-1}-x_k)\mu_k\right]\\
&+(x_{n-1}-x_n)^2+x_n^2+(\lambda^2+1)\mu_n^2+2(1-\lambda)(x_{n-1}-x_n)\mu_n.
\end{aligned}
$$

Lemma 3 *Let*

$$\gamma:=2+\lambda,\quad \xi:=1+\lambda\quad(\lambda\neq 0).$$

Then the zero-minimum condition (Zm) *yields a pair of linear systems of* n*-equation on* n*-variable:*

Case $n=1$

$$\text{(EQ)}\qquad c\ =\ \xi x_1\qquad c\ =\ \xi\mu_1.$$

Case $n=2$

$$
\text{(EQ)}\qquad
\begin{aligned}
c\ &=\ \gamma x_1-x_2 & c\ &=\ \xi\mu_1-\mu_2\\
x_1\ &=\ \xi x_2 & \mu_1\ &=\ \gamma\mu_2.
\end{aligned}
$$

Case $n\geq 3$

$$\text{(EQ)}\qquad \begin{aligned} c &= \gamma x_1 - x_2 & c &= \xi\mu_1 - \mu_2 & \\ x_{k-1} &= \gamma x_k - x_{k+1} & \mu_{k-1} &= \gamma\mu_k - \mu_{k+1} & 2 \le k \le n-1 \\ x_{n-1} &= \xi x_n & \mu_{n-1} &= \gamma\mu_n. & \end{aligned}$$

Conversely the pair (EQ) *yields* (Zm) *under the condition that either system has a unique solution. This condition is assured by the nonsingularity of the relevant* $n \times n$ *martices* A_n, B_n *i.e.,* [1]

$$|A_n| \neq 0, \ |B_n| \neq 0.$$

The pair (EQ) is divided into two linear systems:

$$(\text{EQ}_x)\qquad \begin{aligned} c &= x_0 & \\ x_{k-1} &= \gamma x_k - x_{k+1} & 1 \le k \le n-1 \\ x_{n-1} &= \xi x_n & \end{aligned}$$

and

$$(\text{EQ}_\mu)\qquad \begin{aligned} c &= \xi\mu_1 - \mu_2 & \\ \mu_{k-1} &= \gamma\mu_k - \mu_{k+1} & 2 \le k \le n-1 \\ \mu_{n-1} &= \gamma\mu_n & \end{aligned}$$

Now we have the ojective function

$$\begin{aligned} h(x,\mu) \ =\ & -2\lambda x_0\mu_1 + \sum_{k=1}^{n-1}\left[(x_{k-1}-x_k)^2 + x_k^2 + \lambda^2\mu_k^2 + (\mu_k - \mu_{k+1})^2 + 2(\lambda-1)x_k(\mu_k - \mu_{k+1})\right] \\ & + (x_{n-1}-x_n)^2 + x_n^2 + (\lambda^2+1)\mu_n^2 + 2(\lambda-1)x_n\mu_n \quad (x_0 = c). \end{aligned}$$

A *triple zero property* holds as follows.

Lemma 4 *Let a feasible* (x, μ) *satisfy* (Zm_n)*. Then it holds that*

$$\text{(tZ)}\qquad \begin{aligned} & h(x,\mu) \\ =\ & -c(c-x_1) + \sum_{k=1}^{n}\left[(x_{k-1}-x_k)^2 + \lambda x_k^2\right] \\ =\ & -\lambda c\mu_1 + \sum_{k=1}^{n-1}\left[\lambda^2\mu_k^2 + \lambda(\mu_k - \mu_{k+1})^2\right] + (\lambda^2+\lambda)\mu_n^2 \\ =\ & 0. \end{aligned}$$

[1] It holds that $|A_n| = |B_n|$.

3 Case $\lambda > 0$

Consider the Case $\lambda > 0$. We define

$$\gamma := 2 + \lambda\,(> 2), \quad \xi := 1 + \lambda\,(> 1).$$

Now let us solve a pair of linear systems of (finite) difference equations

$$(\mathrm{EQ}_x) \qquad \begin{aligned} c &= x_0 \\ x_{k-1} &= \gamma x_k - x_{k+1} \qquad 1 \le k \le n-1 \\ x_{n-1} &= \xi x_n \end{aligned}$$

and

$$(\mathrm{EQ}_\mu) \qquad \begin{aligned} c &= \xi \mu_1 - \mu_2 \\ \mu_{k-1} &= \gamma \mu_k - \mu_{k+1} \qquad 2 \le k \le n-1 \\ \mu_{n-1} &= \gamma \mu_n. \end{aligned}$$

We consider a second-order linear difference equation

$$x_{n+2} - \gamma x_{n+1} + x_n \;=\; 0, \quad x_0 = 0,\ x_1 = 1. \tag{2}$$

Lemma 5 *The equation (2) has a unique solution*

$$x_n \;=\; \frac{\beta^n - \alpha^n}{\beta - \alpha} \tag{3}$$

where $\alpha\,(<)\,\beta$ are the two positive solution

$$\alpha \;=\; \frac{\gamma - \sqrt{\gamma^2 - 4}}{2}, \quad \beta \;=\; \frac{\gamma + \sqrt{\gamma^2 - 4}}{2} \tag{4}$$

to the characteristic equation

$$t^2 - \gamma t + 1 \;=\; 0. \tag{5}$$

We note that

$$\alpha + \beta \;=\; \gamma, \quad \alpha\beta \;=\; 1$$
$$0 < \alpha < 1 < \beta < \infty.$$

Definition 1 *Let us define the sequence $\{G_n\}$ by*

$$G_n \;=\; \frac{\beta^n - \alpha^n}{\beta - \alpha}. \tag{6}$$

We call $\{G_n\}$ a *two-step Gibonacci* sequence. The reason is that $G_n = F_{2n}$ for $\gamma = 3$, where $\{F_n\}$ is the *Fibonacci* sequence. Thus $\{G_k\}$ satisfies a second-order linear difference equation

$$G_{k+1} \;=\; \gamma G_k - G_{k-1}, \quad G_1 = 1, \; G_0 = 0. \tag{7}$$

This has a unique solution (6).

Lemma 6 *The system* (EQ_x) *has a unique solution*

$$x_k \;=\; c\,\frac{\xi G_{n-k} - G_{n-1-k}}{\xi G_n - G_{n-1}} \qquad 0 \le k \le n$$

, while the system (EQ_μ) *has a unique solution*

$$\mu_k \;=\; c\,\frac{G_{n+1-k}}{\xi G_n - G_{n-1}} \qquad 1 \le k \le n.$$

That is

$$\begin{aligned} &(x_1,\; x_2,\; \ldots,\; x_k,\; \ldots,\; x_{n-1},\; x_n) \\ =\; &\frac{c}{H_n}(H_{n-1},\; H_{n-2},\; \ldots, H_{n-k},\; \ldots,\; H_1,\; H_0), \\ &(\mu_1,\; \mu_2,\; \ldots,\; \mu_k,\; \ldots,\; \mu_{n-1},\; \mu_n) \\ =\; &\frac{c}{H_n}(G_n,\; G_{n-1},\; \ldots, G_{n+1-k},\; \ldots,\; G_2,\; G_1) \end{aligned}$$

where

$$H_n \;\;:=\;\; \xi G_n - G_{n-1}. \tag{8}$$

The sequence $\{H_n\}$ is called *Hibonacci*. Then it holds that

$$\lambda G_n \;=\; H_n - H_{n-1}, \quad H_n \;=\; G_{n+1} - G_n, \quad H_0 = G_1. \tag{9}$$

The Hibonacci sequence $\{H_k\}$ satisfies the second-order linear difference equation

$$H_{k+1} \;=\; \gamma H_k - H_{k-1}, \quad H_1 = \xi, \; H_0 = 1. \tag{10}$$

This has a unique solution

$$H_k \;=\; \frac{\xi(\beta^k - \alpha^k) - (\beta^{k-1} - \alpha^{k-1})}{\beta - \alpha}. \tag{11}$$

Theorem 1 *The zero-minimum condition* (Zm) *has a unique solution* (x, μ)*;*

$$\begin{aligned} x &= (x_1,\ x_2,\ \ldots,\ x_k,\ \ldots,\ x_{n-1},\ x_n) \\ &= \frac{c}{H_n}(H_{n-1},\ H_{n-2},\ \ldots, H_{n-k},\ \ldots,\ H_1,\ H_0), \end{aligned} \tag{12}$$

$$\begin{aligned} \mu &= (\mu_1,\ \mu_2,\ \ldots,\ \mu_k,\ \ldots,\ \mu_{n-1},\ \mu_n) \\ &= \frac{c}{H_n}(G_n,\ G_{n-1},\ \ldots, G_{n+1-k},\ \ldots,\ G_2,\ G_1) \end{aligned} \tag{13}$$

where

$$G_n = \frac{\beta^n - \alpha^n}{\beta - \alpha}, \quad H_n = \xi G_n - G_{n-1}.$$

Hence Q *attainas the zero minimum at* (x, μ)*.*

We have defined the objective function $h : R^n \times R^n \to R^1$ by

$$\begin{aligned} h(x,\mu) = -2\lambda c\mu_1 + \sum_{k=1}^{n-1} \big[(x_{k-1} - x_k)^2 + x_k^2 + \lambda^2\mu_k^2 + (\mu_k - \mu_{k+1})^2 \\ + 2(\lambda - 1)x_k(\mu_k - \mu_{k+1})\big] \\ + (x_{n-1} - x_n)^2 + x_n^2 + (\lambda^2 + 1)\mu_n^2 + 2(\lambda - 1)x_n\mu_n. \end{aligned}$$

Then (QI) is summarized as follows.

Corollary 1 *It holds that*

$$\begin{aligned} &\text{(i)} \quad h(x,\mu) \geq 0 \quad \forall (x,\mu) \in R^n \times R^n \\ &\text{(ii)} \quad h(x,\mu) = 0 \iff (x,\mu) \ \textit{satisfies}\ \text{(EQ)}. \end{aligned}$$

The objective function $h(x,\mu)$ attains the zero-minimum. From Lemma 4 (Triple Zero), we have a *triple zero property* for the solution.

Corollary 2 *Let* (x,μ) *be the solution given in (12), (13). Then it holds that*

$$\text{(tZ)} \quad \begin{aligned} & h(x,\mu) \\ &= -c(c - x_1) + \sum_{k=1}^{n} \big[(x_{k-1} - x_k)^2 + \lambda x_k^2\big] \\ &= -\lambda c\mu_1 + \sum_{k=1}^{n-1} \big[\lambda^2\mu_k^2 + \lambda(\mu_k - \mu_{k+1})^2\big] + (\lambda^2 + \lambda)\mu_n^2 \\ &= 0. \end{aligned}$$

Here we define two functions $f,\ g: R^n \to R^1$ by

$$\begin{aligned} f(x) &= \sum_{k=1}^{n} \left[(x_{k-1} - x_k)^2 + \lambda x_k^2 \right] \\ g(\mu) &= 2\lambda c \mu_1 - \sum_{k=1}^{n-1} \left[\lambda^2 \mu_k^2 + \lambda(\mu_k - \mu_{k+1})^2 \right] - (\lambda^2 + \lambda)\mu_n^2. \end{aligned}$$

Note that $f(x)$ is convex and $g(\mu)$ is concave. We consider a pair of minimization problem and maximization problem

$$\begin{array}{lllll} \mathrm{P} & \text{minimize} & f(x) & \text{subject to} & x \in R^n \\ \mathrm{D} & \text{Maximize} & g(\mu) & \text{subject to} & \mu \in R^n. \end{array}$$

4 Gibonacci Duality

Let any $\lambda > 0$ be given. Then we consider a pair of minimization (primal) problem and maximization (dual) problem.

4.1 Primal and dual

The pair is

$$\mathrm{P} \quad \begin{array}{l} \text{minimize} \quad \sum_{k=1}^{n} \left[(x_{k-1} - x_k)^2 + \lambda x_k^2 \right] \\ \text{subject to} \quad \text{(i)} \ \ x \in R^n, \ x_0 = c \end{array}$$

$$\mathrm{D} \quad \begin{array}{l} \text{Maximize} \ \ 2\lambda c \mu_1 - \sum_{k=1}^{n-1} \left[\lambda^2 \mu_k^2 + \lambda(\mu_k - \mu_{k+1})^2 \right] - (\lambda^2 + \lambda)\mu_n^2 \\ \text{subject to} \quad \text{(i)} \ \ \mu \in R^n. \end{array}$$

Then both P and D are dual to each other. An equality condition is

$$\text{(EC)} \qquad \begin{array}{rcllcll} c - x_1 &=& \lambda\mu_1 & x_1 &=& \mu_1 - \mu_2 & \\ x_{k-1} - x_k &=& \lambda\mu_k & x_k &=& \mu_k - \mu_{k+1} & k = 2, 3, \ldots, n-1 \\ x_{n-1} - x_n &=& \lambda\mu_n & x_n &=& \mu_n. & \end{array}$$

The primal P attains a minimum $m = \left(1 - \dfrac{H_{n-1}}{H_n}\right) c^2$ at $x = (x_1, x_2, \ldots, x_n)$, while the dual D does a maximum $M = \lambda \dfrac{G_n}{H_n} c^2$ at $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$:

$$x_k = c\,\frac{H_{n-k}}{H_n}, \qquad \mu_k = c\,\frac{G_{n+1-k}}{H_n} \tag{14}$$

that is

$$x = (x_1,\, x_2,\, \ldots, x_k, \ldots, x_n) \;=\; \frac{c}{H_n}(H_{n-1},\; H_{n-2},\; \ldots, H_{n-k},\; \ldots,\; H_0)$$
$$\mu = (\mu_1,\, \mu_2,\, \ldots, \mu_k, \ldots, \mu_n) \;=\; \frac{c}{H_n}(G_n,\; G_{n-1},\; \ldots, G_{n+1-k},\; \ldots,\; G_1) \tag{15}$$

where

$$G_n \;=\; \frac{\beta^n - \alpha^n}{\beta - \alpha}, \quad H_n \;=\; \xi G_n - G_{n-1}$$
$$\alpha \;=\; \frac{\gamma - \sqrt{\gamma^2 - 4}}{2}, \quad \beta \;=\; \frac{\gamma + \sqrt{\gamma^2 - 4}}{2} \tag{16}$$
$$\gamma \;=\; 2 + \lambda, \quad \xi \;=\; 1 + \lambda.$$

Thus

$$\lambda G_n \;=\; H_n - H_{n-1}, \quad H_0 \;=\; G_1$$
$$\alpha \;=\; \frac{\lambda + 2 - \sqrt{\lambda^2 + 4\lambda}}{2}, \quad \beta \;=\; \frac{\lambda + 2 + \sqrt{\lambda^2 + 4\lambda}}{2}. \tag{17}$$

Hence the the optimum point $(x,\, \mu)$ satisfies (EC) and the optimum values are same $m = M$.

4.1.1 Solution method

We note that the objective function

$$f(x) \;=\; \sum_{k=1}^{n} \left[(x_{k-1} - x_k)^2 + \lambda x_k^2\right] \quad (\, x_0 = c\,)$$

is convex. The first-order partial derivative $f_k(x) := \dfrac{\partial f}{\partial x_k}(x)$ is

$$\begin{aligned}
\frac{1}{2} f_1(x) \;&=\; -(c - x_1) + \lambda x_1 + (x_1 - x_2) \\
&=\; -(x_2 - \gamma x_1 + c) \quad \big(\,\gamma := 2 + \lambda\,\big) \\
\frac{1}{2} f_k(x) \;&=\; -(x_{k-1} - x_k) + \lambda x_k + (x_k - x_{k+1}) \\
&=\; -(x_{k+1} - \gamma x_k + x_{k-1}) \qquad 2 \le k \le n-1 \\
\frac{1}{2} f_n(x) \;&=\; -(x_{n-1} - x_n) + \lambda x_n \\
&=\; -(-\xi x_n + x_{n-1}) \quad \big(\,\xi := 1 + \lambda\,\big).
\end{aligned}$$

Furthermore an identity

$$f(x) \;=\; c(c - x_1) + \frac{1}{2} \sum_{k=1}^{n} x_k f_k(x) \tag{18}$$

holds true.

A minimum point x satisfies the first-order condition $f_k(x) = 0 \ \ 1 \leq k \leq n$, which is

$$
(\mathrm{EQ}_x) \qquad
\begin{aligned}
c &= x_0 \\
x_{k-1} &= \gamma x_k - x_{k+1} \qquad 1 \leq k \leq n-1 \\
x_{n-1} &= \xi x_n .
\end{aligned}
$$

As was shown in Lemma 6, this has a unique solution

$$
x = (x_1,\, x_2,\, \ldots, x_k, \ldots, x_n) \;=\; \frac{c}{H_n}(H_{n-1},\; H_{n-2},\; \ldots, H_{n-k},\; \ldots,\; H_0).
$$

Then the identity claims that

$$
f(x) \;=\; c(c - x_1) \;=\; \big(1 - \frac{H_{n-1}}{H_n}\big)\, c^2 .
$$

Second we solve D. The objective function

$$
g(\mu) \;=\; 2\lambda c\mu_1 - \sum_{k=1}^{n-1} \left[\lambda^2\mu_k^2 + \lambda(\mu_k - \mu_{k+1})^2\right] - (\lambda^2 + \lambda)\mu_n^2
$$

is concave. The first-order partial derivative $g_k(\mu) := \dfrac{\partial g}{\partial \mu_k}(\mu)$ is

$$
\begin{aligned}
\frac{1}{2\lambda} g_1(\mu) &= c - \lambda\mu_1 - (\mu_1 - \mu_2) \\
&= \mu_2 - \xi\mu_1 + c \quad (\,\xi := 1 + \lambda\,) \\
\frac{1}{2\lambda} g_k(\mu) &= (\mu_{k-1} - \mu_k) - \lambda\mu_k - (\mu_k - \mu_{k+1}) \\
&= \mu_{k+1} - \gamma\mu_k + \mu_{k-1} \qquad 2 \leq k \leq n-1 \quad (\,\gamma := 2 + \lambda\,) \\
\frac{1}{2\lambda} g_n(\mu) &= (\mu_{n-1} - \mu_n) - (\lambda + 1)\mu_n \\
&= -\gamma\mu_n + \mu_{n-1} .
\end{aligned}
$$

Furthermore an identity

$$
g(\mu) \;=\; \lambda c\mu_1 + \frac{1}{2}\sum_{k=1}^{n} \mu_k g_k(\mu) \tag{19}
$$

holds true.

A maximum point μ satisfies the first-order condition $g_k(\mu) = 0 \ \ 1 \leq k \leq n$, which is

$$
(\mathrm{EQ}_\mu) \qquad
\begin{aligned}
c &= \xi\mu_1 - \mu_2 \\
\mu_{k-1} &= \gamma\mu_k - \mu_{k+1} \qquad 2 \leq k \leq n-1 \\
\mu_{n-1} &= \gamma\mu_n .
\end{aligned}
$$

As was shown in Lemma 6, this has a unique solution

$$\mu = (\mu_1, \mu_2, \ldots, \mu_k, \ldots, \mu_n) \;=\; \frac{c}{H_n}(G_n,\; G_{n-1},\; \ldots, G_{n+1-k},\; \ldots,\; G_1).$$

Then the identity claims that

$$g(\mu) \;=\; \lambda c \mu_1 \;=\; \lambda \frac{G_n}{H_n} c^2.$$

Thus D ha s the desired maximum solution.

4.1.2 Derivation P $\Longleftrightarrow$ D

Let x be feasible for P. Then for any μ we have

$$\begin{aligned}
&\sum_{k=1}^{n} \left[(x_{k-1} - x_k)^2 + \lambda x_k^2\right] \\
=\;& (c - x_1)^2 - 2\lambda\mu_1(c - x_1) + \lambda x_1^2 + 2\lambda\mu_1(c - x_1) \\
&+ \sum_{n=2}^{n} \left[(x_{k-1} - x_k)^2 - 2\lambda\mu_k(x_{k-1} - x_k) + \lambda x_k^2 + 2\lambda\mu_k(x_{k-1} - x_k)\right] \\
=\;& 2\lambda c\mu_1 + (c - x_1 - \lambda\mu_1)^2 - \lambda^2\mu_1^2 + \lambda\left\{x_1^2 - 2(\mu_1 - \mu_2)x_1\right\} \\
&+ \sum_{n=2}^{n-1} \left[(x_{k-1} - x_k - \lambda\mu_k)^2 - \lambda^2\mu_k^2 + \lambda x_k^2 - 2\lambda(\mu_k - \mu_{k+1})x_k\right] \\
&\quad + \left[(x_{n-1} - x_n)^2 - 2\mu_k\lambda(x_{n-1} - x_n) + \lambda x_n^2 - 2\lambda\mu_n x_n\right] \\
=\;& 2\lambda c\mu_1 + (c - x_1 - \lambda\mu_1)^2 - \lambda^2\mu_1^2 + \lambda\left\{x_1 - (\mu_1 - \mu_2)\right\}^2 - \lambda(\mu_1 - \mu_2)^2\Big] \\
&+ \sum_{n=2}^{n-1} \left[(x_{k-1} - x_k - \lambda\mu_k)^2 - \lambda^2\mu_k^2 + \lambda\left\{x_k - (\mu_k - \mu_{k+1})\right\}^2 - \lambda(\mu_k - \mu_{k+1})^2\right] \\
&\quad + (x_{n-1} - x_n - \lambda\mu_n)^2 - \lambda^2\mu_n^2 + \lambda(x_n - \mu_n)^2 - \lambda\mu_n^2 \\
\geq\;& 2\lambda c\mu_1 - \sum_{k=1}^{n-1} \left[\lambda^2\mu_k^2 + \lambda(\mu_k - \mu_{k+1})^2\right] - (\lambda^2 + \lambda)\mu_n^2.
\end{aligned}$$

The equality holds iff (EC) holds.

Conversely, D $\Longrightarrow$ P is shown as follows. Let μ be feasible for P. Then for any x we have

$$2\lambda c\mu_1 - \sum_{k=1}^{n-1} \left[\lambda^2\mu_k^2 + \lambda(\mu_k - \mu_{k+1})^2\right] - (\lambda^2 + \lambda)\mu_n^2 \;\leq\; \sum_{k=1}^{n} \left[(x_{k-1} - x_k)^2 + \lambda x_k^2\right].$$

The equality holds iff (EC) holds. □

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