Generalized cone-continuity of set-valued maps with scalarization (Study on Nonlinear Analysis and Convex Analysis)

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Dedicated to the memory of Professor Kazimierz Goebel

1 Introduction

A set-valued map is a map that associates with a set depending on each element. Considering a singleton set, a set-valued mapping can be launched as a single-valued mapping. Because the concept of set-valued mappings includes single-valued maps, set-valued optimization presents a significant generalization along with unification of scalar and vector optimization problems.

On the other hand, transforming vectors or sets into real number, in other words scalarization, processes a quintessential methodology solving optimization problems with vector-valued or set-valued maps. One of challenging scalarizing functions ensues sublinear scalarization introduced by Tammer (see [2], [3], [4] and [5])

$$h_C(v; d) := \inf \{t \in \mathbb{R} \mid v \in td - C\}$$

where C is a convex cone in a real topological vector space and $d \in C$.

In general, composition is an operator analyzed a function from the results of another function. Several mathematical properties of each nested function are usually preserved by a composite operation. A continuous map composition, for example, is continuous on topological spaces. Under certain assumptions, we may characterize solutions for multicriteria questions using scalarization based on this attribute. This prompts us to explore how composite functions involving

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a set-valued map and a scalarization function convey continuity of primary set-valued mappings via various scalarization for sets.

Continuity notions of set-valued maps are significant in many branches of mathematics, including nonlinear functional analysis, optimization theory, and convex analysis. Cone-continuity is studied in several direction (see [16] and [9]).

Recently, Ike, Liu, Ogata and Tanaka [6] show certain results on the inheritance property of some kinds of continuity of set-valued maps via scalarization functions for sets: if a set-valued map has a kind of continuity (lower continuity or upper continuity; see [4]) then the composition of its set-valued map and a certain scalarization function assures a similar semicontinuity to that of its scalarization function defined on the family of nonempty subsets of a real topological vector space. Their results are generalizations of results in earlier study by Kuwano, Tanaka and Yamada [15].

The aim of this paper is to propose some idea how to obtain and apply the generalization in [1] by Dechboon and Tanaka of the inheritance property which is introduced by [6].

2 Basic Notations

Throughout the paper, let X be a topological space and Y a real topological vector space. Let θ_Y be the zero vector in Y and $\mathcal{P}(Y)$ denote the set of all nonempty subsets of Y. The topological interior, topological closure, convex hull, and complement of a set $A \in \mathcal{P}(Y)$ are denoted by $\operatorname{int} A$, $\operatorname{cl} A$, $\operatorname{co} A$, and A^{c}, respectively. Furthermore, we assume that C is a convex cone in Y with $\operatorname{int} C \neq \emptyset$ and $\theta_Y \in C$. Then, $C + C = C$ holds, and $\operatorname{int} C$ and $\operatorname{cl} C$ are also convex cones. Accordingly, we can define a preoder $\leq_C$ on Y induced by C as follows:

$$\text{for } y_1, y_2 \in Y, y_1 \leq_C y_2 \overset{\text{def}}{\iff} y_2 - y_1 \in C.$$

This preorder is compatible with the linear structure of Y:

$$\text{for all } y_1, y_2, y_3 \in Y, \quad y_1 \leq_C y_2 \implies y_1 + y_3 \leq_C y_2 + y_3; \tag{1}$$

$$\text{for all } y_1, y_2 \in Y \text{ and } \ t > 0, \quad y_1 \leq_C y_2 \implies t y_1 \leq_C t y_2. \tag{2}$$

When C is pointed (i.e., $C \cap (-C) = \{\theta_Y\}$), $\leq_C$ is antisymmetric and then a partial order.

Proposition 1. *Let C, C' be convex cones in Y and $d \in Y$. Assume that $C + (0, +\infty)d \subset C'$. Then, for any $v_1, v_2 \in Y$ and $t, t' \in \mathbb{R}$ with $t > t'$,*

$$v_1 + td \leq_C v_2 \implies v_1 + t'd \leq_{C'} v_2.$$

As generalizations of partial orderings for vectors, we give a definition of certain binary relations between sets in Y, called set relations. This is a modified version of the original one proposed in [12].

Definition 2 (set relations, [12])**.** For $A, B \in \mathcal{P}(Y)$, we define the following eight types of binary relations on $\mathcal{P}(Y)$.

(i) $A \leq_C^{(1)} B \overset{\text{def}}{\iff} \forall a \in A,\ \forall b \in B,\ a \leq_C b \iff A \subset \bigcap_{b \in B}(b - C) \iff B \subset \bigcap_{a \in A}(a + C)$;

(ii) $A \leq_C^{(2L)} B \overset{\text{def}}{\iff} \exists a \in A \text{ s.t. } \forall b \in B,\ a \leq_C b \iff A \cap \left(\bigcap_{b \in B}(b - C)\right) \neq \emptyset$;

(iii) $A \leq_C^{(2U)} B \overset{\text{def}}{\iff} \exists b \in B \text{ s.t. } \forall a \in A,\ a \leq_C b \iff \left(\bigcap_{a \in A}(a + C)\right) \cap B \neq \emptyset$;

(iv) $A \leq_C^{(2)} B \overset{\text{def}}{\iff} A \leq_C^{(2L)} B \text{ and } A \leq_C^{(2U)} B \iff A \cap \left(\bigcap_{b \in B}(b - C)\right) \neq \emptyset$ and $\left(\bigcap_{a \in A}(a + C)\right) \cap B \neq \emptyset$;

(v) $A \leq_C^{(3L)} B \overset{\text{def}}{\iff} \forall b \in B,\ \exists a \in A \text{ s.t. } a \leq_C b \iff B \subset A + C$;

(vi) $A \leq_C^{(3U)} B \overset{\text{def}}{\iff} \forall a \in A,\ \exists b \in B \text{ s.t. } a \leq_C b \iff A \subset B - C$;

(vii) $A \leq_C^{(3)} B \overset{\text{def}}{\iff} A \leq_C^{(3L)} B \text{ and } A \leq_C^{(3U)} B \iff B \subset A + C \text{ and } A \subset B - C$;

(viii) $A \leq_C^{(4)} B \overset{\text{def}}{\iff} \exists a \in A,\ \exists b \in B \text{ s.t. } a \leq_C b \iff A \cap (B - C) \neq \emptyset \iff (A + C) \cap B \neq \emptyset$.

In the above definition, the letters L and U stand for "lower" and "upper," respectively. Each relation $\leq_C^{(j)}$ is transitive for $j = 1, 2L, 2U, 3L, 3U$ and not transitive for $j = 4$. Since $\theta_Y \in C$, $\leq_C^{(j)}$ is reflexive for $j = 3L, 3U, 4$ and hence a preorder for $j = 3L, 3U$. Besides, for each $j = 1, 2L, 2U, 3L, 3U, 4$, the relation $\leq_C^{(j)}$ satisfies certain similar properties to conditions (1) and (2) for all $A, B \in \mathcal{P}(Y)$,

(i) $A \leq_C^{(j)} B \implies A + y \leq_C^{(j)} B + y$ for $y \in Y$;

(ii) $A \leq_C^{(j)} B \implies tA \leq_C^{(j)} tB$ for $t > 0$.

Also, we easily obtain the following implications:

$$\begin{cases} A \leq_C^{(1)} B \implies A \leq_C^{(2L)} B \implies A \leq_C^{(3L)} B \implies A \leq_C^{(4)} B; \\ A \leq_C^{(1)} B \implies A \leq_C^{(2U)} B \implies A \leq_C^{(3U)} B \implies A \leq_C^{(4)} B; \\ A \leq_C^{(1)} B \implies A \leq_C^{(2)} B \implies A \leq_C^{(3)} B \implies A \leq_C^{(4)} B \end{cases} \tag{3}$$

for $A, B \in \mathcal{P}(Y)$.

Proposition 3 ([6]). *Let C' and C be two nonempty convex cones in Y and $d \in Y$. Assume that $C' + (0, +\infty)d \subset C$. Then, for each $j = 1, 2L, 3L, 2U, 3U, 4$, any $A, B \in \mathcal{P}(Y)$, $s, s' \in \mathbb{R}$ with $s' < s$ and $t, t' \in \mathbb{R}$ with $t < t'$,*

$$\begin{aligned} & A \leq_{C'}^{(j)} B + s'd \implies A \leq_{C}^{(j)} B + sd, \\ \text{and} \quad & A + t'd \leq_{C'}^{(j)} B \implies A + td \leq_{C}^{(j)} B. \end{aligned}$$

3 Unification of Scalarizing Functions

Now, we recall the scalarization scheme [13] for sets in a real vector space related to the set relations, which are certain generalizations as unification of several nonlinear scalarizations proposed in [5].

Definition 4 ([7, 13]). For each $j = 1, 2L, 3L, 2U, 3U, 4$, we define

$$I_C^{(j)}(A; V, d) := \inf \left\{ t \in \mathbb{R} \,\middle|\, A \leq_C^{(j)} (V + td) \right\}, \tag{4}$$

$$S_C^{(j)}(A; V, d) := \sup \left\{ t \in \mathbb{R} \,\middle|\, (V + td) \leq_C^{(j)} A \right\}, \tag{5}$$

for any $A, V \in \mathcal{P}(Y)$ and $d \in Y$.

The idea of these scalarization functions is introduced in [13], which originates from the idea of Gerstewitz's (Tammer's) sublinear scalarizing functional in [2]; see [4, 7]. This type of scalarization measures how far a given reference set needs to be moved toward a specific direction to fulfill each set relation between a target set and its moved reference set. Note that V and d in (4) and (5) are index parameters for scalarization which play key roles as a reference set and a reference direction, respectively.

Proposition 5 ([7]). *Let C be a convex cone in V. The following inequalities hold between each scalarizing function for sets:*

$$\begin{gathered} I_C^{(4)}(A; W, d) \leq I_C^{(3L)}(A; W, d) \leq I_C^{(2L)}(A; W, d) \leq I_C^{(1)}(A; W, d); \\ I_C^{(4)}(A; W, d) \leq I_C^{(3U)}(A; W, d) \leq I_C^{(2U)}(A; W, d) \leq I_C^{(1)}(A; W, d); \\ I_C^{(4)}(A; W, d) \leq I_C^{(3)}(A; W, d) \leq I_C^{(2)}(A; W, d) \leq I_C^{(1)}(A; W, d); \\ S_C^{(1)}(A; W, d) \leq S_C^{(2L)}(A; W, d) \leq S_C^{(3L)}(A; W, d) \leq S_C^{(4)}(A; W, d); \\ S_C^{(1)}(A; W, d) \leq S_C^{(2U)}(A; W, d) \leq S_C^{(3U)}(A; W, d) \leq S_C^{(4)}(A; W, d); \\ S_C^{(1)}(A; W, d) \leq S_C^{(2)}(A; W, d) \leq S_C^{(3)}(A; W, d) \leq S_C^{(4)}(A; W, d) \end{gathered}$$

for $A, W \in \mathcal{P}(V) \setminus \{\emptyset\}$ and $d \in C$.

Proposition 6 ([7]). *Let C be a convex cone in V. There are certain relations among the scalarizations of types $(2L)$, $(2U)$, (2) as well as $(3L)$, $(3U)$, (3):*

(i) $I_C^{(2)}(A;W,d) = \max\left\{I_C^{(2L)}(A;W,d), I_C^{(2U)}(A;W,d)\right\}$;

(ii) $I_C^{(3)}(A;W,d) = \max\left\{I_C^{(3L)}(A;W,d), I_C^{(3U)}(A;W,d)\right\}$;

(iii) $S_C^{(2)}(A;W,d) = \min\left\{S_C^{(2L)}(A;W,d), S_C^{(2U)}(A;W,d)\right\}$;

(iv) $S_C^{(3)}(A;W,d) = \min\left\{S_C^{(3L)}(A;W,d), S_C^{(3U)}(A;W,d)\right\}$

for $A, W \in \mathcal{P}(V) \setminus \{\emptyset\}$ *and* $d \in C$.

Proposition 7 ([6])**.** *Let* $A, V \in \mathcal{P}(Y)$ *and* $d \in Y$. *Then the following statements hold*

$$\begin{aligned}
-I_C^{(1)}(-A;-V,d) &= S_C^{(1)}(A;V,d),\\
-I_C^{(2L)}(-A;-V,d) &= S_C^{(2U)}(A;V,d),\\
-I_C^{(3L)}(-A;-V,d) &= S_C^{(3U)}(A;V,d),\\
-I_C^{(2U)}(-A;-V,d) &= S_C^{(2L)}(A;V,d),\\
-I_C^{(3U)}(-A;-V,d) &= S_C^{(3L)}(A;V,d),\\
-I_C^{(4)}(-A;-V,d) &= S_C^{(4)}(A;V,d).
\end{aligned}$$

For each j without $j = 4$, scalarizing functions $I_C^{(j)}(\cdot\,;W,d)$ and $S_C^{(j)}(\cdot\,;W,d)$ with a nonempty reference set W and a direction d have the following monotonicity with respect to $\leq_C^{(j)}$, which is referred to as "j-monotonicity" in [10]:

$$\begin{cases} A \leq_C^{(j)} B \implies I_C^{(j)}(A;W,d) \leq I_C^{(j)}(B;W,d);\\ A \leq_C^{(j)} B \implies S_C^{(j)}(A;W,d) \leq S_C^{(j)}(B;W,d). \end{cases} \tag{6}$$

4 Generalized cone-continuity

Let $\mathcal{N}(x)$ and $\preccurlyeq$ be a neighborhood system of a point $x \in X$ and a binary relation on $\mathcal{P}(Y)$, respectively.

Definition 8 (Definition 12 in [1])**.** Let $F : X \to \mathcal{P}(Y)$, $x_0 \in X$, $\preccurlyeq$ a binary relation on $\mathcal{P}(Y)$ and $C \subset Y$ a convex cone. We say that F is $(\preccurlyeq, C)$-continuous at x_0 if

$$\forall W \subset Y, W \text{ open }, W \preccurlyeq F(x_0), \exists V \in \mathcal{N}_X(x_0) \text{ s.t. } W + C \preccurlyeq F(x), \forall x \in V.$$

As special cases, for $A, B \in \mathcal{P}(Y)$, we consider binary relations $\operatorname{int} A \cap B \neq \emptyset$ and $B \subset \operatorname{int} A$ by $A \preccurlyeq_1 B$ and $A \preccurlyeq_2 B$, respectively. Accordingly, $(\preccurlyeq_1, C)$-continuity and $(\preccurlyeq_2, C)$-continuity coincide with "C-lower continuity" and "C-upper continuity" for set-valued maps, respectively. Indeed, $F : X \to \mathcal{P}(Y)$ is $(\preccurlyeq_1, C)$-continuous at x_0 if and only if

$$\forall W \subset Y, W \text{ open }, W \cap F(x_0) \neq \emptyset, \exists V \in \mathcal{N}_X(x_0) \text{ s.t. } (W + C) \cap F(x) \neq \emptyset, \forall x \in V,$$

that is, F is C-lower continuous at x_0. Similarly, F is $(\preccurlyeq_2, C)$-continuous at x_0 if and only if

$$\forall W \subset Y, W \text{ open }, F(x_0) \subset W, \exists V \in \mathcal{N}_X(x_0) \text{ s.t. } F(x) \subset W + C, \forall x \in V,$$

that is, F is C-upper continuous at x_0; see Definition 2.5.16 of [4].

Remark 9. If $C = \{0\}$ then $(\preccurlyeq, C)$-continuity for set-valued maps becomes $\preccurlyeq$-continuity in Definition 3.2 in [6]. Moreover, $\preccurlyeq_1$-continuity and $\preccurlyeq_2$-continuity coincide with the classical notions of lower continuity and upper continuity for set-valued maps, respectively.

Definition 10 (Definition 14 in [1])**.** Let $\varphi : \mathcal{P}(Y) \to \mathbb{R} \cup \{\pm\infty\}$, $A_0 \in \mathcal{P}(Y)$, $\preccurlyeq$ a binary relation on $\mathcal{P}(Y)$, and C a convex cone in Y with $C \neq Y$. Then, we say that φ is

(i) $(\preccurlyeq, C)$-lower semicontinuous at A_0 if $\forall r < \varphi(A_0), \exists W \in \mathcal{P}(Y)$, W open, s.t. $W \preccurlyeq A_0$ and $r < \varphi(A), \forall A \in U(W + C, \preccurlyeq)$;

(ii) $(\preccurlyeq, C)$-upper semicontinuous at A_0 if $\forall r > \varphi(A_0), \exists W \in \mathcal{P}(Y)$, W open, s.t. $W \preccurlyeq A_0$ and $r > \varphi(A), \forall A \in U(W + C, \preccurlyeq)$,

where $U(V, \preccurlyeq) := \{A \in \mathcal{P}(Y) \,|V \preccurlyeq A\}$.

Remark 11. When $C = \{0\}$, $(\preccurlyeq, C)$-lower and $(\preccurlyeq, C)$-upper semicontinuities are coincident with $\preccurlyeq$-lower and $\preccurlyeq$-upper semicontinuities, respectively, which are introduced in Definition 3.3 of [6]. In Definition 10, we adopt that if $\varphi(A_0) = -\infty$ (resp. $+\infty$) then φ is $(\preccurlyeq, C)$-lower (resp. upper) semicontinuous at A_0.

Therefore, we can easily show the following results as generalizations of Theorems 3.1 and 3.2 in [6].

Theorem 12 (Theorem 16 in [1])**.** *Let $F : X \to \mathcal{P}(Y)$, $\varphi : \mathcal{P}(Y) \to \mathbb{R} \cup \{\pm\infty\}$, $x_0 \in X$, $\preccurlyeq$ a binary relation on $\mathcal{P}(Y)$, and $C \subset Y$ a convex cone. If F is $(\preccurlyeq, C)$-continuous at x_0 and φ is $(\preccurlyeq, C)$-lower semicontinuous at $F(x_0)$, then $\varphi \circ F$ is lower semicontinuous at x_0.*

Theorem 13 (Theorem 17 in [1])**.** *Let $F : X \to \mathcal{P}(Y)$, $\varphi : \mathcal{P}(Y) \to \mathbb{R} \cup \{\pm\infty\}$, $x_0 \in X$, $\preccurlyeq$ a binary relation on $\mathcal{P}(Y)$, and $C \subset Y$ a convex cone. If F is $(\preccurlyeq, C)$-continuous at x_0 and φ is $(\preccurlyeq, C)$-upper semicontinuous at $F(x_0)$, then $\varphi \circ F$ is upper semicontinuous at x_0.*

5 Continuity of Scalarization and Consequences

Proposition 14. *Let $A_0, V \in \mathcal{P}(Y)$, C' and C two nonempty convex cones in Y such that $C' \subset C \neq Y$, $d \in \operatorname{int} C$. Then, the following statements hold:*

(i) $I_C^{(j)}(\cdot;V,d)$ *is* $(\preccurlyeq_1, C')$*-lower semicontinuous at* A_0 *for* $j = 1, 3U$.

(ii) $S_C^{(j)}(\cdot;V,d)$ *is* $(\preccurlyeq_1, C')$*-upper semicontinuous at* A_0 *for* $j = 1, 3L$.

(iii) $I_C^{(j)}(\cdot;V,d)$ *is* $(\preccurlyeq_1, -C')$*-upper semicontinuous at* A_0 *for* $j = 2L, 4$.

(iv) $S_C^{(j)}(\cdot;V,d)$ *is* $(\preccurlyeq_1, -C')$*-lower semicontinuous at* A_0 *for* $j = 2U, 4$.

(v) $I_C^{(j)}(\cdot;V,d)$ *is* $(\preccurlyeq_2, -C')$*-upper semicontinuous at* A_0 *for* $j = 1, 3U$.

(vi) $S_C^{(j)}(\cdot;V,d)$ *is* $(\preccurlyeq_2, -C')$*-lower semicontinuous at* A_0 *for* $j = 1, 3L$.

(vii) $I_C^{(j)}(\cdot;V,d)$ *is* $(\preccurlyeq_2, C')$*-lower semicontinuous at* A_0 *for* $j = 2L, 4$.

(viii) $S_C^{(j)}(\cdot;V,d)$ *is* $(\preccurlyeq_2, C')$*-upper semicontinuous at* A_0 *for* $j = 2U, 4$.

The following examples determine that $I_C^{(j)}(\cdot;V,d)$ and $S_C^{(j)}(\cdot;V,d)$ do not satisfy continuity for some j.

Example 15. Let $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $A_0 = \{(x,y) : x+y = 12 \text{ and } 2 \leq x \leq 10\}$, $V = \{(x,y) : x+y = 6 \text{ and } 1 \leq x \leq 5\}$ and $d = (1,1)$.

Then $I_C^{(3L)}(\cdot;V,d)$ is not $(\preccurlyeq_1, C')$-lower semicontinuous at A_0. Additionally, this example can illustrate that $I_C^{(3L)}(\cdot;V,d)$ is not $(\preccurlyeq_1, -C')$-lower semicontinuous at A_0. Besides, $S_C^{(3U)}(\cdot;-V,d)$ is neither $(\preccurlyeq_1, C')$-upper semicontinuous nor $(\preccurlyeq_1, -C')$-upper semicontinuous at $-A_0$.

Moreover, we have $I_C^{(3L)}(\cdot;V,d)$ is not $(\preccurlyeq_1, C')$-upper semicontinuous at A_0. Additionally, this example can illustrate that $I_C^{(3L)}(\cdot;V,d)$ is not $(\preccurlyeq_1, -C')$-upper semicontinuous at A_0. Besides, $S_C^{(3U)}(\cdot;-V,d)$ is neither $(\preccurlyeq_1, C')$-lower semicontinuous nor $(\preccurlyeq_1, -C')$-lower semicontinuous at $-A_0$.

Example 16. Let $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $A_0 = \{(x,y) : x+y = 12 \text{ and } 4 \leq x \leq 8\}$, $V = \{(x,y) : x+y = 6 \text{ and } 1 \leq x \leq 5\}$ and $d = (1,1)$.

Therefore $I_C^{(2U)}(\cdot;V,d)$ is not $(\preccurlyeq_1, C')$-lower semicontinuous at A_0. Additionally, this example can illustrate that $I_C^{(2U)}(\cdot;V,d)$ is not $(\preccurlyeq_1, -C')$-lower semicontinuous at A_0. Besides, $S_C^{(2L)}(\cdot;-V,d)$ is neither $(\preccurlyeq_1, C')$-upper semicontinuous nor $(\preccurlyeq_1, -C')$-upper semicontinuous at $-A_0$.

Example 17. Let $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $A_0 = \{(x,y) : x+y = 12 \text{ and } 2 \leq x \leq 5\}$, $V = \{(x,y) : x+y = 6 \text{ and } 1 \leq x \leq 5\}$ and $d = (1,1)$. Therefore $I_C^{(2U)}(\cdot;V,d)$ is not $(\preccurlyeq_1, C')$-upper semicontinuous at A_0. Additionally, this example can illustrate that $I_C^{(2U)}(\cdot;V,d)$ is not $(\preccurlyeq_1, -C')$-upper semicontinuous at A_0. Besides, $S_C^{(2L)}(\cdot;-V,d)$ is neither $(\preccurlyeq_1, C')$-lower semicontinuous nor $(\preccurlyeq_1, -C')$-lower semicontinuous at $-A_0$.

For $I_C^{(2U)}(\cdot;V,d), S_C^{(2L)}(\cdot;V,d), I_C^{(3L)}(\cdot;V,d)$ and $S_C^{(3U)}(\cdot;V,d)$, the continuity properties are shown using the compactness assumptions.

Proposition 18. *Let $A_0, V \in \mathcal{P}(Y)$, C' and C two nonempty convex cones such that $C' \subset C \neq Y$ and $d \in \operatorname{int} C$. Assume that A_0 and V are compact. Then, the following statements hold:*

$I_C^{(2U)}(\cdot;V,d)$ *is $(\preccurlyeq_2, -C')$-upper semicontinuous at A_0.*

$S_C^{(2L)}(\cdot;V,d)$ *is $(\preccurlyeq_2, -C')$-lower semicontinuous at A_0.*

$I_C^{(3L)}(\cdot;V,d)$ *is $(\preccurlyeq_2, C')$-lower semicontinuous at A_0.*

$S_C^{(3U)}(\cdot;V,d)$ *is $(\preccurlyeq_2, C')$-upper semicontinuous at A_0.*

By Theorems 12 and 13, the following results are obtained.

Theorem 19. *Let $F : X \to \mathcal{P}(Y)$, $x_0 \in X$, $V \in \mathcal{P}(Y)$, C' and C two nonempty convex cones in Y such that $C' \subset C \neq Y$ and $d \in \operatorname{int} C$. The following statements hold:*

(a) *If F is $(\preccurlyeq_1, C')$-continuous at x_0, then*
   (i) $I_C^{(j)}(F(\cdot);V,d)$ *is lower semicontinuous at x_0, for all $j = 1, 3U$,*
   (ii) $S_C^{(j)}(F(\cdot);V,d)$ *is upper semicontinuous at x_0, for all $j = 1, 3L$.*

(b) *If F is $(\preccurlyeq_1, -C')$-continuous at x_0, then*
   (i) $I_C^{(j)}(F(\cdot);V,d)$ *is upper semicontinuous at x_0, for all $j = 2L, 4$,*
   (ii) $S_C^{(j)}(F(\cdot);V,d)$ *is lower semicontinuous at x_0, for all $j = 2U, 4$.*

(c) *If F is $(\preccurlyeq_2, -C')$-continuous at x_0, then*
   (i) $I_C^{(j)}(F(\cdot);V,d)$ *is upper semicontinuous at x_0, for all $j = 1, 3U$,*
   (ii) $S_C^{(j)}(F(\cdot);V,d)$ *is lower semicontinuous at x_0, for all $j = 1, 3L$.*

(d) *If F is $(\preccurlyeq_2, C')$-continuous at x_0, then*
   (i) $I_C^{(j)}(F(\cdot);V,d)$ *is lower semicontinuous at x_0, for all $j = 2L, 4$,*
   (ii) $S_C^{(j)}(F(\cdot);V,d)$ *is upper semicontinuous at x_0, for all $j = 2U, 4$.*

Moreover, by Proposition 18, the consequent result can be implied by assuming compactness.

Theorem 20. *Let $F : X \to \mathcal{P}(Y)$, $x_0 \in X$, $V \in \mathcal{P}(Y)$, C' and C two nonempty convex cones such that $C' \subset C \neq Y$ and $d \in \operatorname{int} C$. Assume that $F(x_0)$ and V are compact.*

(a) *If F is $(\preccurlyeq_2, -C')$-continuous at x_0, then*

(i) $I_C^{(2U)}(F(\cdot);V,d)$ *is upper semicontinuous at* x_0,

(ii) $S_C^{(2L)}(F(\cdot);V,d)$ *is lower semicontinuous at* x_0,

(b) *If* F *is* $(\preccurlyeq_2, C')$*-continuous at* x_0*, then*

(i) $I_C^{(3L)}(F(\cdot);V,d)$ *is lower semicontinuous at* x_0,

(ii) $S_C^{(3U)}(F(\cdot);V,d)$ *is upper semicontinuous at* x_0.

Finally, from the Theorems 19 and 20 together with Remark 11, we can summarize the previous results as follows.

Table 1: Continuity properties of the composite functions.

F	$(\preccurlyeq_1, C')$ -conti.	$(\preccurlyeq_2, C')$ -conti.	$(\preccurlyeq_1, -C')$ -conti.	$(\preccurlyeq_2, -C')$ -conti.	l.c. $(C' = \{0\})$	u.c. $(C' = \{0\})$
$I_C^{(1)} \circ F$	l.s.c.	-	-	u.s.c.	l.s.c.	u.s.c.
$I_C^{(2L)} \circ F$	-	l.s.c.	u.s.c.	-	u.s.c.	l.s.c.
$I_C^{(3L)} \circ F$	-	l.s.c. $(*)$	-	-	-	l.s.c. $(*)$
$I_C^{(2U)} \circ F$	-	-	-	u.s.c. $(*)$	-	u.s.c. $(*)$
$I_C^{(3U)} \circ F$	l.s.c.	-	-	u.s.c.	l.s.c.	u.s.c.
$I_C^{(4)} \circ F$	-	l.s.c.	u.s.c.	-	u.s.c.	l.s.c.
$S_C^{(1)} \circ F$	u.s.c.	-	-	l.s.c.	u.s.c.	l.s.c.
$S_C^{(2L)} \circ F$	-	-	-	l.s.c. $(*)$	-	l.s.c. $(*)$
$S_C^{(3L)} \circ F$	u.s.c.	-	-	l.s.c.	u.s.c.	l.s.c.
$S_C^{(2U)} \circ F$	-	u.s.c.	l.s.c.	-	l.s.c.	u.s.c.
$S_C^{(3U)} \circ F$	-	u.s.c. $(*)$	-	-	-	u.s.c. $(*)$
$S_C^{(4)} \circ F$	-	u.s.c.	l.s.c.	-	l.s.c.	u.s.c.

where $(*)$ means the compactness assumptions are required and "l.c." and "u.c." denote lower continuity and upper continuity of F, respectively.

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