RECENT PROGRESSES ON GENUS ONE EXTENSIONS OF MIXED TATE MOTIVES OVER Z

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Abstract. In this survey article, we give an overview of recent progress of construction of genus one extension of the category of mixed Tate motives over $\mathbf{Z}$ by Brown [6] and Hain-Matsumoto [18].

1. Introduction

The construction of (sub)categories of mixed motives satisfying the following conditions is one of important open problems in Arithmetic geometry: (i) This is a $\mathbf{Q}$-linear Tannakian category and a universal cohomology theory of varieties over $\mathbf{Q}$. (ii) The extension classes of simple objects can be computed by algebraic cycles. (iii) Its Tannakian fundamental group can compute explicitly.

The category of mixed Tate motives over a number field is one of the few subcategories of mixed motives that satisfy all of the above conditions. As an application of the existence of such a category, Goncharov and Terasoma proved independently that Zagier's conjectural dimension of the space of multiple zeta values gives an upper bound ([8], [29]). It is natural, therefore, to ask about possible extensions of this category.

Problem 1.1. Let $\mathsf{MTM}(\mathbf{Z})$ be the category of mixed Tate motives over $\mathbf{Z}$. Find a nice extension of $\mathsf{MTM}(\mathbf{Z})$.

The aim of this article is to give an overview of a recent attempts to construct of a nice category of mixed motives containing $\mathsf{MTM}(\mathbf{Z})$ by Francis Brown ([6]) , Richard Hain and Makoto Matsumoto ([18]).

Notation. For a field k, $\mathrm{Vec}_k^{\mathrm{fin}}$ denotes the category of finite dimensional k-vector spaces. For an abstract group Γ (resp. a pro-algebraic group $\mathcal{G}$ over k), $\mathrm{Rep}_k(\Gamma)$ (resp. $\mathrm{Rep}_k(\mathcal{G})$) denotes the category of representation of π (resp. algebraic representations of $\mathcal{G}$) on finite dimensional k-vector spaces.

2. Relative pro-unipotent completion

Our basic tool to construct subcategories of mixed motives is the relative pro-unipotent completion of a topological fundamental group. We recall this notion briefly.

Let k be a field of characteristic zero and let S be a reductive algebraic group over k. Let π be an abstract group and let

$$\rho_0\colon \pi \to S(k)$$

be a group homomorphism whose image is Zariski dense. A *relative unipotent lift of* ρ_0 is a tuple (G, pr, ρ_G) where:

- G is an algebraic group over k.

- pr: $G \to S$ is a surjective homomorphism whose kernel is unipotent.
- $\rho_G\colon \pi \to G(k)$ is a group homomorphism such that the composition $\mathrm{pr} \circ \rho_G$ is equal to ρ_0 and that the image of ρ_G is Zariski dense.

Definition 2.1. The *relative pro-unipotent completion* of π with respect to ρ_0 is a pro-algebraic group over k defined to be

$$\varprojlim_{\rho_G\colon \pi \to G(k)} G.$$

Here, ρ_G runs over relative unipotent lifts of ρ_0. This pro-algebraic group is denoted by $\pi(\rho_0)$ in this article. When $S = \mathrm{Spec}(k)$ and ρ_0 is the trivial representation, the relative pro-unipotent completion with respect to ρ_0 is called the *pro-unipotent completion of* π and this group is denoted by π^{un}/k or π^{un} simply.

Example 2.2. Let π be a finitely generated group. Then, the ring $\mathcal{O}(\pi^{\mathrm{un}}/k)$ of regular functions on π^{un}/k has the following explicit description ([12, Proposition 3.222]):

$$\mathcal{O}(\pi^{\mathrm{un}}/k) \cong \varinjlim_{n \geq 0} \mathrm{Hom}_{\mathbf{Z}}\left(\mathbf{Z}[\pi]/I^{n+1}, k\right).$$

Here, $\mathbf{Z}[\pi]$ is the group ring of π and I denotes its augmentation ideal. The existence of the natural isomorphism above follows from Proposition 2.4 below. This isomorphism is not only an isomorphism of k-vector spaces but also of commutative Hopf k-algebras. Here, the multiplication (resp. coproduct) of the right-hand side is the induced map by the diagonal map $\pi \to \pi \times \pi$ (resp. the multiplication $\pi \times \pi \to \pi$). In particular, if π is a free group of rank r, then $\mathcal{O}(\pi^{\mathrm{un}})$ is naturally isomorphic to a non-commutative polynomial ring in r-variables with the shuffle product and the concatenation coproduct.

By definition, we have the canonical representation

$$\rho_{\mathrm{univ}}\colon \pi \to \pi(\rho_0)(k),$$

which has a universal property for relative (pro-)unipotent lifts of ρ_0. Then, we have the induced natural functor between k-linear Tannakian categories

$$\mathrm{Rep}_k(\pi(\rho_0)) \to \mathrm{Rep}_k(\pi) \tag{2.1}$$

by ρ^{univ}, which is fully-faithful by the Zariski density of the image of ρ_{univ}. An object V of $\mathrm{Rep}_k(\pi)$ is said to be *relatively unipotent with respect to* ρ_0 if its Jordan-Hölder component extends to an algebraic representation of S via ρ_0.

Example 2.3. When ρ_0 is the trivial character, a relatively unipotent representation of π is nothing but a unipotent representation of π in the usual sense.

Proposition 2.4. *The essential image of (2.1) coincides with the full-subcategory of* $\mathrm{Rep}_k(\pi)$ *consisting of relatively unipotent representations with respect to* ρ_0*. In other words,* $\pi(\rho_0)$ *is canonically isomorphic to the Tannakian fundamental group of the category of relatively unipotent representations over* k *of* π *with respect to* ρ_0*.*

Proof. It is sufficient to show the essential surjectivity of the functor (2.1). Let (V, ρ) be a relatively unipotent representation of π on a finite dimensional k-vector space V. Let G be the Zariski closure of $\rho(\pi)$ in $\underline{\mathrm{Aut}}(V) \cong \mathrm{GL}_{N,k}$. As usual, we equip G with the reduced scheme structure. Then, it is easily checked that G forms a closed subgroup of $\underline{\mathrm{Aut}}(V)$. Moreover, there is a unique isomorphism $G/G^{\mathrm{un}} \xrightarrow{\sim} S$ compatible with representations of

π, where G^{un} is the unipotent radical of G. Therefore, by the definition of $\pi(\rho_0)$, there is a natural homomorphism $\mathrm{pr}\colon \pi(\rho_0) \twoheadrightarrow G$ and $\rho\circ\mathrm{pr}$ coincides with ρ_{univ}. This implies the essential surjectivity of (2.1). □

By definition, there exists a short exact sequence of pro-algebraic groups

$$1 \to \pi(\rho_0)^{\mathrm{un}} \to \pi(\rho_0) \to S \to 1,$$

where $\pi(\rho_0)^{\mathrm{un}}$ is the pro-unipotent radical of $\pi(\rho_0)$. For a pro-algebraic group $\mathcal{G}$ over k and a finite dimensional algebraic representation V, $H^i(\mathcal{G}, V)$ is defined by

$$H^i(\mathcal{G}, V) = \mathrm{Ext}^i_{\mathrm{Rep}_k(\mathcal{G})}(k, V),$$

where k is the trivial representation of $\mathcal{G}$, and $H^i(\mathcal{G})$ is defined to be $H^i(\mathcal{G}, k)$. To compute topological generators and relations of the Lie algebra of $\pi(\rho_0)^{\mathrm{un}}$, the following proposition is useful:

Proposition 2.5 ([17, Lemma 5.1]). *Let $G = S \ltimes U$ be a pro-algebraic group over k with reductive S and pro-unipotent U. Then, for any i, we have a natural isomorphism*

$$\mathrm{H}^i(U) \cong \bigoplus_{\lambda\in\Lambda} \mathrm{Ext}^i_{\mathrm{Rep}_k(G)}(k, V_\lambda) \otimes_k V_\lambda^\vee$$

of S-modules. Here, Λ is the set of isomorphism classes of irreducible representations of S and V_λ is a corresponding irreducible representation to λ.

Example 2.6. Let π be a free group of rank r and let π^{un} denote the pro-unipotent completion of π over k. Since π^{un} is pro-unipotent, this group can be reconstructed by its Lie algebra. Hence, to determine the isomorphism class of π^{un}, it suffices to compute the topological generators and primitive relations of $\mathrm{Lie}(\pi^{\mathrm{un}})$. Recall that $\mathrm{Lie}(\pi^{\mathrm{un}})$ is topologically generated by a topological basis of $H_1(\mathrm{Lie}(\pi^{\mathrm{un}}))$ and the set of primitive relations is given by $H_2(\mathrm{Lie}(\pi^{\mathrm{un}}))$ (cf. [18, Section 18]) . According to Proposition 2.5 and [18, Proposition 10.1], we have

$$H^i_{\mathrm{cts}}(\mathrm{Lie}(\pi^{\mathrm{un}})) \cong H^i(\pi^{\mathrm{un}}) \cong \begin{cases} k & i = 0, \\ \mathrm{Hom}_{\mathrm{Grp}}(\pi, k) & i = 1, \\ 0 & i = 2. \end{cases}$$

Here, $H^i_{\mathrm{cts}}(\mathrm{Lie}(\pi^{\mathrm{un}}))$ is the continuous cohomology group of the topological Lie algebra $\mathrm{Lie}(\pi^{\mathrm{un}})$ ([17, Subsection 5.1]). Since $\mathrm{Hom}_{\mathrm{Grp}}(\pi, k)$ is a k-vector space of rank r, we conclude that $\mathrm{Lie}(\pi^{\mathrm{un}})$ is isomorphic to the topological Lie algebra

$$\varprojlim_n \mathrm{Lie}_k(x_1, \ldots, x_r)/\Gamma^n\mathrm{Lie}_k(x_1, \ldots, x_r).$$

Here, $\mathrm{Lie}_k(x_1, \ldots, x_r)$ is the free Lie algebra over k of rank r and $\Gamma^n\mathrm{Lie}_k(x_1, \ldots, x_r)$ is the central descending series defined by

$$\begin{aligned} \Gamma^1\mathrm{Lie}_k(x_1, \ldots, x_r) &= \mathrm{Lie}_k(x_1, \ldots, x_r), \\ \Gamma^{i+1}\mathrm{Lie}_k(x_1, \ldots, x_r) &= [\mathrm{Lie}_k(x_1, \ldots, x_r), \Gamma^i\mathrm{Lie}_k(x_1, \ldots, x_r)]. \end{aligned}$$

3. Mixed Tate motives over Z

In this section, we recall basic facts about the category $\mathsf{MTM}(\mathbf{Z})$ of mixed Tate motives over $\mathbf{Z}$. Then, we recall Brown's fundamental theorem which is the basis for the idea of extending $\mathsf{MTM}(\mathbf{Z})$ to genus one world.

It is not the aim to state precise construction of this category. However, for the reader's convenience, we give a rough recipe of the construction of $\mathsf{MTM}(\mathbf{Z})$ with references.

(Step1) Construct the category $\mathsf{DMM}_{\mathrm{gm}}(\mathbf{Q})$ of Voevodsky's derived category of mixed motives over $\mathbf{Q}$ (cf. [30], [24], [2]).

(Step2) Define the full triangulated subcategory $\mathsf{DMTM}(\mathbf{Q})$ of $\mathsf{DMM}_{\mathrm{gm}}(\mathbf{Q})$ to be the smallest triangulated subcategory stable under extensions and containing $\mathbf{Q}(n)$.

(Step3) Show that there exists a natural truncated structure on $\mathsf{DMTM}(\mathbf{Q})$ by using Borel's computation ([4, Proposition 12.2]) of higher K-group of $\mathbf{Q}$ (see [22]).

(Step4) Define $\mathsf{MTM}(\mathbf{Q})$ to be the heart in the sense of Beilinson-Bernstein-Deligne ([3, Définition 1.3.1]) of $\mathsf{DMTM}(\mathbf{Q})$ with respect to the natural truncated structure.

(Step5) Define $\mathsf{MTM}(\mathbf{Z})$ to be the full-subcategory of $\mathsf{MTM}(\mathbf{Q})$ consisting of objects which are "unramified everywhere" (see [8, 1.7]).

Note that, by construction, $\mathsf{MTM}(\mathbf{Z})$ and $\mathsf{MTM}(\mathbf{Q})$ are $\mathbf{Q}$-linear abelian categories with a natural $\otimes$-structure. Moreover, it is known that they are Tannakian. For a smooth variety X over $\mathbf{Q}$ with a stratification $X \supset X_1 \supset \cdots \supset X_0 = \emptyset$ such that $X_i \setminus X_{i+1} = \coprod \mathbf{A}^{l_i}$, an object $h^n(X)(r)$ of $\mathsf{MTM}(\mathbf{Q})$ is defined for any $n, r \in \mathbf{Z}$. We call such an X a variety of mixed Tate type in this article.

3.1. **Properties $\mathsf{MTM}(\mathbf{Z})$.** We recall basic properties of $\mathsf{MTM}(\mathbf{Z})$ and $\mathsf{MTM}(\mathbf{Q})$. Let $\mathcal{R}^H_{\mathbf{Q}}$ be the category of the Hodge components of system of realizations over $\mathbf{Q}$ ([7, 1.4], [8, 2.13]). An object of $\mathcal{R}^H_{\mathbf{Q}}$ consists of tuple $H = (H_{\mathrm{B}}, H_{\mathrm{dR}}, \mathrm{comp}_{\mathrm{dR,B}})$ where:

- H_{B} is an object of $\mathrm{Vec}^{\mathrm{fin}}_{\mathbf{Q}}$ with an increasing filtration $W_\bullet H_{\mathrm{B}}$ and an $\mathbf{Q}$-linear endomorphism F_∞ such that $F_\infty^2 = \mathrm{id}$.
- H_{dR} is an object of $\mathrm{Vec}^{\mathrm{fin}}_{\mathbf{Q}}$ with an increasing filtration $W_\bullet H_{\mathrm{dR}}$ and a decreasing filtration $F^\bullet H_{\mathrm{dR}}$.
- $\mathrm{comp}_{\mathrm{dR,B}}$ is an isomorphism of underlying $\mathbf{C}$-vector spaces
$$\mathrm{comp}_{\mathrm{dR,B}} \colon H_{\mathrm{B}} \otimes_{\mathbf{Q}} \mathbf{C} \xrightarrow{\sim} H_{\mathrm{dR}} \otimes_{\mathbf{Q}} \mathbf{C},$$
which preserves the filtrations $W_\bullet$ on the both-hand sides.

They satisfies the following conditions:

- A bi-filtered module $(H_{\mathrm{B}}, W_\bullet H_{\mathrm{B}}, \mathrm{comp}_{\mathrm{dR,B}}^{-1}(F^\bullet H_{\mathrm{dR}} \otimes_{\mathbf{Q}} \mathbf{C}))$ is a $\mathbf{Q}$-mixed Hodge structure.
- Under the comparison isomorphism, we have $c_{\mathrm{dR}} = c_{\mathrm{B}} F_\infty$, where c_* is the complex conjugation with respect to the $\mathbf{R}$-structure $H_* \otimes_{\mathbf{Q}} \mathbf{R}$.

Example 3.1. Let X be a smooth variety over $\mathbf{Q}$. Then,
$$H_{\mathrm{B}} := H^n(X(\mathbf{C}), \mathbf{Q}(i)), \quad H_{\mathrm{dR}} := H^n_{\mathrm{dR}}(X/\mathbf{Q})(i)$$
forms a part of an object of $\mathcal{R}^H_{\mathbf{Q}}$ ([28, Theorem 4.2], [19, Proposition 3.1.16]). The symbol $H^n(X)(i)$ denotes the corresponding object of $\mathcal{R}^H_{\mathbf{Q}}$.

Let $\omega_* \colon \mathcal{R}^H_{\mathbf{Q}} \to \mathrm{Vec}^{\mathrm{fin}}_{\mathbf{Q}}$ be the functor defined by $\omega_*(H) = H_*$. Similar to the usual mixed Hodge structure, $\mathcal{R}^H_{\mathbf{Q}}$ is a $\mathbf{Q}$-linear Tannakian category and ω_* is a fiber functor.

Let C be a smooth algebraic curve over $\mathbf{Q}$. The, the symbol $\mathcal{R}^H_C$ denotes the category of the Hodge components of system of realizations over C. For the precise definition, see [7, 1.21]. Roughly speaking, an object of $\mathcal{R}^H_C$ consists of a tuple $\mathcal{F} = (\mathcal{F}_{\mathrm{B}}, \mathcal{F}_{\mathrm{dR}}, \mathrm{comp}_{\mathrm{dR,B}})$ where:

- $\mathcal{F}_{\mathrm{B}}$ is a $\mathbf{Q}$-local system over $C(\mathbf{C})$ with an increasing filtration $W_\bullet\mathcal{F}_{\mathrm{B}}$, which is functorial in the algebraic closure $\mathbf{C}$ of $\mathbf{R}$.
- $\mathcal{F}_{\mathrm{dR}} = (\mathcal{F}_{\mathrm{dR}}, \nabla)$ is a flat connection over C regular at infinity with two filtrations $F^\bullet\mathcal{F}_{\mathrm{dR}}$ and $W_\bullet\mathcal{F}_{\mathrm{dR}}$.
- $\mathrm{comp}_{\mathrm{dR,B}}$ is an isomorphism
$$\mathcal{F}_{\mathrm{B}} \otimes_{\mathbf{Q}} \mathbf{C} \xrightarrow{\sim} (\mathcal{F}_{\mathrm{dR},\mathbf{C}})^{\nabla=0}$$
of $\mathbf{C}$-local systems such that $(\mathcal{F}_{\mathrm{B}}, W_\bullet, \mathrm{comp}^*_{\mathrm{dR,B}}F^\bullet)$ forms an admissible variation of mixed Hodge structures ([28, Definition 14.49]) and that functorial in $\mathbf{C}$.

The basic properties of $\mathsf{MTM}(\mathbf{Z})$ is as follows:

THEOREM 3.2. *There exists a functor*
$$R_{\mathcal{H}}\colon \mathsf{MTM}(\mathbf{Q}) \to \mathcal{R}^H_{\mathbf{Q}},$$
which is called the Hodge realization functor *satisfying the following conditions:*

(1) *This functor is faithful exact $\otimes$-functor (*[8, 2.9, 2.11]*). Let $\omega_*\colon \mathsf{MTM}(\mathbf{Q}) \to \mathrm{Vec}^{\mathrm{fin}}_{\mathbf{Q}}$ denote the composition of $R_{\mathcal{H}}$ with ω_* by abuse of notation*[1]*.*

(2) *For a variety X over $\mathbf{Q}$ of mixed Tate type, we have a natural isomorphism*
$$R_{\mathcal{H}}(h^n(X)(i)) \cong H^n(X)(i).$$

(3) *(Structure of Tannakian π_1) We have a natural isomorphism of pro-algebraic groups*
$$\pi_1(\mathsf{MTM}(\mathbf{Z}), \omega_{\mathrm{dR}}) = \mathbf{G}_m \ltimes U^{\mathrm{dR}}_{\mathsf{MTM}}$$
over $\mathbf{Q}$, where $U^{\mathrm{dR}}_{\mathsf{MTM}}$ is the pro-unipotent radical of $\pi_1(\mathsf{MTM}(\mathbf{Z}), \omega_{\mathrm{dR}})$. Let $\mathrm{Lie}(U^{\mathrm{dR}}_{\mathsf{MTM}})_l$ be the subspace of $\mathrm{Lie}(U^{\mathrm{dR}}_{\mathsf{MTM}})$ on which $\mathbf{G}_m$ acts via the lth power of the standard character. Then, we have a natural isomorphism
$$\mathrm{GrLie}(U^{\mathrm{dR}}_{\mathsf{MTM}}) := \bigoplus_{l \in \mathbf{Z}} \mathrm{Lie}(U^{\mathrm{dR}}_{\mathsf{MTM}})_l \cong \mathrm{Lie}(\sigma_3, \sigma_5, \sigma_7, \sigma_9, \cdots)$$
([8, 2.4]*). Here, the right-hand side is the free graded Lie algebra over $\mathbf{Q}$ generated by homogeneous elements σ_{2k+1} with* $\deg(\sigma_{2k+1}) = 2k+1$.

(4) *The Hodge realization functor $R_{\mathcal{H}}$ is* fully-faithful and its essential image is closed under subobjects *(*[8, Proposition 2.14]*).*

3.2. **Brown's structure theorem.** For a pair (g, n) of non-negative integers, let $\mathscr{M}_{g,n}$ be the moduli stack of n-marked genus g curves over $\mathbf{Z}$ ([9], [21]).

EXAMPLE 3.3. The stack $\mathscr{M}_{0,4}$ is a smooth scheme over $\mathbf{Z}$. Explicitly, we have a natural identification
$$\mathscr{M}_{0,4} = \mathbf{P}^1 \setminus \{0, 1, \infty\}.$$
More generally, when $g = 0, n \geq 3$, $\mathscr{M}_{0,n}$ is isomorphic to $(\mathbf{P}^1 \setminus \{0,1,\infty\})^{n-3} \setminus \cup_{i<j}\Delta_{ij}$, where Δ_{ij} is the locus defined by $x_i = x_j$.

[1] Of course, they are fiber functors of $\mathsf{MTM}(\mathbf{Q})$ and $\mathsf{MTM}(\mathbf{Z})$.

Let $\Pi^{\mathrm{B}}_{0,4} := \pi_1(\mathscr{M}_{0,4}(\mathbf{C}); \overrightarrow{01})^{\mathrm{un}}$. As $\mathscr{M}_{0,4} = \mathbf{P}^1 \setminus \{0, 1, \infty\}$, its topological fundamental group $\pi_1(\mathscr{M}_{0,4}(\mathbf{C}); \overrightarrow{01})$ is a free group of rank two so that $\mathcal{O}(\Pi^{\mathrm{B}}_{0,4}) = \mathbf{Q}\langle x, y\rangle$. Let $\mathcal{C}_{\mathrm{dR}}(\mathscr{M}_{0,4})$ denote the category of unipotent flat connections over $\mathscr{M}_{0,4}/\mathbf{Q}$, which is a $\mathbf{Q}$-linear neutral Tannakian category (cf. [7, 10.26]). Let $\Pi^{\mathrm{dR}}_{0,4}$ be the Tannakian fundamental group of $\mathcal{C}_{\mathrm{dR}}(\mathscr{M}_{0,4})$ with the base point $\overrightarrow{01}$ (cf. [7, 15.28-15.36], [10, Subsection 1.1]). It is known that there is a natural isomorphism

$$\mathrm{Hom}_{\mathbf{Q}}(\mathcal{O}(\Pi^{\mathrm{dR}}_{0,4}), \mathbf{Q}) \cong \mathbf{Q}\langle\langle e_0, e_1\rangle\rangle,$$

where $\mathbf{Q}\langle\langle e_0, e_1\rangle\rangle$ is the ring of non-commutative formal power series with variables e_0, e_1. We sometimes identify e_i with the one form $\frac{dt}{t-i}$ on $\mathscr{M}_{0,4}/\mathbf{Q}$. Then, we have a map

$$\pi_1(\mathscr{M}_{0,4}(\mathbf{C}), \overrightarrow{01}) \to \mathbf{C}\langle\langle e_0, e_1\rangle\rangle; \quad \gamma \mapsto \sum_{b\colon \text{words in } e_0, e_1} \left(\int_\gamma \omega_b\right) b, \tag{3.1}$$

where ω_b is the corresponding sequence of $\frac{dt}{t-i}$, $i = 0, 1$ to b and $\int_\gamma \omega_b$ is the regularized iterated integrals (cf. [7, 15.53], [23, Section 8]).

Theorem 3.4. (1) *([7, 12.16, 15.50-15.53]) The map (3.1) induces an isomorphism*

$$\mathrm{comp}_{\mathrm{dR,B}}\colon \mathcal{O}(\Pi^{\mathrm{B}}_{0,4}) \otimes_{\mathbf{Q}} \mathbf{C} \xrightarrow{\sim} \mathcal{O}(\Pi^{\mathrm{dR}}_{0,4}) \otimes_{\mathbf{Q}} \mathbf{C}$$

of commutative Hopf algebras.

(2) *The triple $\mathcal{O}(\Pi^{\mathcal{H}}_{0,4}) := (\mathcal{O}(\Pi^{\mathrm{B}}_{0,4}), \mathcal{O}(\Pi^{\mathrm{dR}}_{0,4}), \mathrm{comp}_{\mathrm{dR,B}})$ forms a part of a Hopf algebra object of* $\mathrm{Ind}(\mathcal{R}^H_{\mathbf{Q}})$[2].

(3) *([8, Théorème 4.4]) There exists a Hopf algebra object $\mathcal{O}(\Pi^{\mathrm{mot}}_{0,4})$ of* $\mathrm{Ind}(\mathsf{MTM}(\mathbf{Z}))$ *with a natural isomorphism*

$$R_{\mathcal{H}}(\mathcal{O}(\Pi^{\mathrm{mot}}_{0,4})) \cong \mathcal{O}(\Pi^{\mathcal{H}}_{0,4})$$

of Hopf algebra objects of $\mathrm{Ind}(\mathcal{R}^H_{\mathbf{Q}})$.

Remark 3.5. For a k-linear neutral Tannakian category $\mathcal{T}$, the category $\mathrm{Aff.Sch}_{\mathcal{T}}$ of affine schemes in $\mathcal{T}$ in the sense of Deligne ([7, §5]) is defined as follows: Let $\mathrm{Alg}_{\mathcal{T}}$ denote the category of algebra objects of $\mathrm{Ind}(\mathcal{T})$. Then, $\mathrm{Aff.Sch}_{\mathcal{T}}$ is defined to be the *opposite category* of $\mathrm{Alg}_{\mathcal{T}}$. By definition, any fiber functor $\omega\colon \mathcal{T} \to \mathrm{Vec}^{\mathrm{fin}}_k$ induces an equivalence of categories

$$\mathrm{Aff.Sch}_{\mathcal{T}} \xrightarrow{\sim} \{\text{Affine schemes } /k \text{ equipped with algebraic actions of } \pi_1(\mathcal{T}, \omega)\}.$$

Let $\Pi^{\mathrm{mot}}_{0,4}$ denote the object of $\mathrm{Aff.Sch}_{\mathsf{MTM}(\mathbf{Z})}$ corresponding to $\mathcal{O}(\Pi^{\mathrm{mot}}_{0,4})$. This affine scheme in $\mathsf{MTM}(\mathbf{Z})$ is called the *motivic fundamental group* of $\mathscr{M}_{0,4}$ (with the base point $\overrightarrow{01}$).

Definition 3.6. Let V be an object of $\mathrm{Ind}(\mathcal{R}^H_{\mathbf{Q}})$. The full subcategory of $\mathcal{R}^H_{\mathbf{Q}}$ *generated by* V is the full subcategory of $\mathcal{R}^H_{\mathbf{Q}}$ whose objects is isomorphic to a sub-quotient of $\bigoplus_{n\geq 0} V^{\otimes n}$ or its dual.

Famous theorem of Brown states that the motivic fundamental group of $\mathscr{M}_{0,4}$ generates $\mathsf{MTM}(\mathbf{Z})$, namely:

[2] $\mathrm{Ind}(\mathcal{A})$ means the ind-category of $\mathcal{A}$.

THEOREM 3.7 ([5]). *Let* $\mathsf{MTM}(\mathbf{Z})'$ *denote the Tannakian full-subcategory of* $\mathcal{R}^{H}_{\mathbf{Q}}$ *generated by* $\mathcal{O}(\Pi^{\mathcal{H}}_{0,4})$. *Then,* $R_{\mathcal{H}}$ *induces an equivalence*

$$\mathsf{MTM}(\mathbf{Z}) \xrightarrow{\sim} \mathsf{MTM}(\mathbf{Z})'.$$

Sketch of the proof. Let $\mathcal{Z}^{\mathfrak{m}}$ be the space of motivic MZVs ([5, Subsection 2.2]). Then, we have a non-canonical injection

$$\mathcal{Z}^{\mathfrak{m}} \hookrightarrow \mathcal{O}(U^{\mathrm{dR}}_{\mathsf{MTM}}) \otimes_{\mathbf{Q}} \mathbf{Q}[x]$$

of graded algebras (this gives an upper bound of the space of MZVs proved by Goncharov and Terasoma). Brown proved the linearly independence of $\{\zeta^{\mathfrak{m}}(k_1, \ldots, k_d) \mid k_i = 2, 3\}$ over $\mathbf{Q}$. Then, by the dimension counting, we conclude that the injection above is an isomorphism. This implies that the action of $\pi_1(\mathsf{MTM}(\mathbf{Z}), \omega_{\mathrm{dR}})$ on $\Pi^{\mathrm{dR}}_{0,4} = \omega_{\mathrm{dR}}(\Pi^{\mathrm{mot}}_{0,4})$ is *faithful*. Then, conclusion of the theorem follows by a formal argument. $\square$

By Brown's theorem, we are led to the second definition of $\mathsf{MTM}(\mathbf{Z})$:

DEFINITION 3.8 (Quick "definition" of $\mathsf{MTM}(\mathbf{Z})$). The category of $\mathsf{MTM}(\mathbf{Z})$ is defined to be the full-subcategory of $\mathcal{R}^{H}_{\mathbf{Q}}$ generated by $\mathcal{O}(\Pi^{\mathcal{H}}_{0,4})$.

REMARK 3.9. Of course, this quick "definition" is not so useful. For example, it is very difficult to determine the structure of its Tannakian fundamental group without the original definition of $\mathsf{MTM}(\mathbf{Z})$ and Brown's theorem (this is needed to use Borel's computation). However, this "definition" has the advantage that similar definitions can be easily made. This is discussed in the next section.

4. Mixed modular motives over Z

Let's begin our exploration of the extension of $\mathsf{MTM}(\mathbf{Z})$ into the world of genus one. An idea to construct a natural extension of $\mathsf{MTM}(\mathbf{Z})$ is

$$\textbf{replace } \mathscr{M}_{0,4} \textbf{ by } \mathscr{M}_{1,1},$$

where $\mathscr{M}_{1,1}$ = the moduli of elliptic curves. Let $\overline{\mathscr{M}_{1,1}}$ be the smooth compactification of $\mathscr{M}_{1,1}$ and let

$$\mathrm{Spec}(\mathbf{Z}[\![q]\!]) \to \overline{\mathscr{M}_{1,1}}$$

be the classifying morphism defined by the Tate generalized elliptic curve ([20, (8.4)]). Then, this morphism defines a point ∞ of $\overline{\mathscr{M}_{1,1}}$ and a non-zero tangent vector $v = \frac{d}{dq}$ at ∞. By abuse of notation, we use the same v for the base points defined by v ([7, §15]).

4.1. Definition.

Recall that the pro-unipotent group $\Pi^{\mathrm{B}}_{0,4}$ is defined to be the pro-unipotent completion of $\pi_1(\mathscr{M}_{0,4}(\mathbf{C}), \overrightarrow{01})$. The group $\Pi^{\mathrm{B}}_{1,1}$ is constructed by a similar way. Note that we have

$$\pi_1(\mathscr{M}_{1,1}(\mathbf{C}), v) \cong \mathrm{SL}_2(\mathbf{Z})$$

(cf. [14, Subsection 3.5]). Let std: $\mathrm{SL}_2(\mathbf{Z}) \to \mathrm{SL}_2(\mathbf{Q})$ be the standard representation. We regard this as a representation of $\pi_1(\mathscr{M}_{1,1}(\mathbf{C}), v)$ by the natural isomorphism above. The pro-algebraic group $\Pi^{\mathrm{B}}_{1,1}$ is defined to be the relative pro-unipotent completion of $\pi_1(\mathscr{M}_{1,1}(\mathbf{C}), v)$ with respect to the standard representation. By definition, we have

$$\Pi^{\mathrm{B}}_{1,1} = \varprojlim_{(G,\rho)} G,$$

where $\rho\colon \pi_1(\mathscr{M}_{1,1}(\mathbf{C}), v) \to G(\mathbf{Q})$ runs over relative unipotent lifts of the standard representation of $\pi_1(\mathscr{M}_{1,1}(\mathbf{C}), v)$.

Remark 4.1. It seems that to take the relative pro-unipotent completion with respect to std is very natural. What happens if we take a pro-unipotent completion? It is well-known that $\mathrm{SL}_2(\mathbf{Z})$ is generated by two elements

$$S := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad T := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

(cf. [27, Subsection 1.5]) and it is easily checked the relations

$$S^2 = (ST)^3 = -E_2$$

hold. Hence, $\mathrm{SL}_2(\mathbf{Z})^{\mathrm{ab}}$ is an abelian group of order 12 so that $\mathrm{SL}_2(\mathbf{Z})$ has no non-trivial unipotent representation on a finite dimensional $\mathbf{Q}$-vector space. Hence, $\mathrm{SL}_2(\mathbf{Z})^{\mathrm{un}}$ is the trivial group and there is nothing to interest. This triviality is also deduced by the fact that there is no non-zero modular form of full-level of weight two.

Before to define a de Rham analogue of $\Pi^{\mathrm{dR}}_{0,4}$, we give a geometric interpretation of $\Pi^{\mathrm{B}}_{1,1}$. Recall that $\mathrm{Rep}_{\mathbf{Q}}(\pi_1(\mathscr{M}_{1,1}(\mathbf{C}), v))$ is equivalent to the category of $\mathbf{Q}$-local systems over the orbifold $\mathscr{M}_{1,1}(\mathbf{C})$ (cf. [14, Subsection 3.3]). On the other hand, by Proposition 2.4, $\mathrm{Rep}_{\mathbf{Q}}(\Pi^{\mathrm{B}}_{1,1})$ is equivalent to the category of relatively unipotent representations with respect to std. Since any irreducible algebraic representation of $\mathrm{SL}_{2,\mathbf{Q}}$ is isomorphic to $\mathrm{Sym}^n(\mathrm{std})$ for some n (cf. [16, Section 10]), $\mathrm{Rep}_{\mathbf{Q}}(\Pi^{\mathrm{B}}_{1,1})$ is naturally equivalent to the full subcategory of $\mathbf{Q}$-local systems over $\mathscr{M}_{1,1}(\mathbf{C})$ whose Jordan-Hölder component is isomorphic to $\mathrm{Sym}^n(\mathcal{V}_{\mathrm{B}})$ for some n, where $\mathcal{V}_{\mathrm{B}}$ is the $\mathbf{Q}$-local system over $\mathscr{M}_{1,1}(\mathbf{C})$ corresponding to the standard representation. A model of $\mathcal{V}_{\mathrm{B}}$ can be taken as follows. Let $\pi\colon \mathscr{E} \to \mathscr{M}_{1,1}$ be the universal elliptic curve over $\mathscr{M}_{1,1}$ and let $R^1\pi_*(\mathbf{Q})$ be the first higher direct image of the constant sheaf $\mathbf{Q}$ on $\mathscr{E}(\mathbf{C})$, which is a family of the first cohomology groups of elliptic curves with coefficients in $\mathbf{Q}$. Then, the fiber of $R^1\pi_*(\mathbf{Q})$ at v is canonically isomorphic to the standard representation of $\mathrm{SL}_2(\mathbf{Z})$ ([16, Section 9]). Therefore, $\mathcal{V}_{\mathrm{B}}$ *can be taken* as

$$\mathcal{V}_{\mathrm{B}} = R^1\pi_*(\mathbf{Q}).$$

Let us define a de Rham analogue. Define the coherent sheaf $\mathcal{V}_{\mathrm{dR}}$ on $\mathscr{M}_{1,1}$ by

$$\mathcal{V}_{\mathrm{dR}} := R^1\pi_*\Omega^{\bullet}_{\mathscr{E}/\mathscr{M}_{1,1}},$$

where $\Omega^i_{\mathscr{E}/\mathscr{M}_{1,1}}$ is the sheaf of ith differential forms on $\mathscr{E}$ relative to $\mathscr{M}_{1,1}$. Note that $\mathcal{V}_{\mathrm{dR}}$ is a family of the first algebraic de Rham cohomology groups of elliptic curves. This coherent sheaf is equipped with the Gauss-Manin connection[3] which is flat. Let $\mathcal{C}_{\mathrm{dR}}(\mathscr{M}_{1,1})$ be the category of flat connections with regular singularities at infinity whose Jordan-Hölder component is isomorphic to the flat connection $\mathrm{Sym}^i(\mathcal{V}_{\mathrm{dR}})$. Then, we can easily check that this category is a $\mathbf{Q}$-linear Tannakian category and v defines a fiber functor of this Tannakian category. Then, $\Pi^{\mathrm{dR}}_{1,1}$ is defined by

$$\Pi^{\mathrm{dR}}_{1,1} = \pi_1(\mathcal{C}_{\mathrm{dR}}(\mathscr{M}_{1,1}), v).$$

Similar to the $\mathscr{M}_{0,4}$-case, the Riemann-Hilbert correspondence induces a natural comparison isomorphism

$$\mathrm{comp}_{\mathrm{dR,B}}\colon \mathcal{O}(\Pi^{\mathrm{B}}_{1,1}) \otimes_{\mathbf{Q}} \mathbf{C} \xrightarrow{\sim} \mathcal{O}(\Pi^{\mathrm{dR}}_{1,1}) \otimes_{\mathbf{Q}} \mathbf{C}$$

[3]This flat connection is canonically isomorphic to the dual of $\mathcal{H}$ defined in [16, Section 9].

of Hopf $\mathbf{C}$-algebras.

THEOREM 4.2 ([16], [6, Subsection 13.2]). *The triple* $(\mathcal{O}(\Pi_{1,1}^{\mathrm{B}}), \mathcal{O}_{1,1}^{\mathrm{dR}}, \mathrm{comp}_{\mathrm{dR,B}})$ *forms a part of a Hopf algebra object* $\mathcal{O}(\Pi_{1,1}^{\mathcal{H}})$ *of* $\mathrm{Ind}(\mathcal{R}_{\mathbf{Q}}^{H})$.

Let $\Pi_{1,1}^{\mathcal{H}}$ denote the corresponding group object of $\mathrm{Aff.Sch}_{\mathcal{R}_{\mathbf{Q}}^{H}}$. Then, we can define a genus one analogue of $\mathsf{MTM}(\mathbf{Z})$ by mimicking the quick "definition" of $\mathsf{MTM}(\mathbf{Z})$:

DEFINITION 4.3 (cf. [6]). The category $\mathsf{MMM}(\mathbf{Z})$ is defined to be the Tannakian full-subcategory of $\mathcal{R}_{\mathbf{Q}}^{H}$ generated by $\mathcal{O}(\Pi_{1,1}^{\mathcal{H}})$.

REMARK 4.4. (1) This category is the same as $\mathcal{H}_{\mathcal{M}_{1,1}}$ in [6].
(2) It seems that $\Pi_{1,1}^{\mathcal{H}}$ can be constructed geometrically and that this is a realization of a certain ind-mixed motive at least in the sense of Nori (cf. [19]). This problem is still open.

We see two typical examples of objects in $\mathsf{MMM}(\mathbf{Z})$.

EXAMPLE 4.5. Let $V = (\mathcal{V}_{\mathrm{B},v}, \mathcal{V}_{\mathrm{dR},v}, \mathrm{comp}_{\mathrm{dR,B}})$ be the fiber of the variation of MHS $(\mathcal{V}_{\mathrm{B}}, \mathcal{V}_{\mathrm{dR}}, \mathrm{comp}_{\mathrm{dR,B}})$ at v. Then, by [26, Theorem 6.16], this admits a limit mixed Hodge structure which is isomorphic to $\mathbf{Q} \oplus \mathbf{Q}(-1)$ (cf. [23, Example 7.8]). Let $\Pi_{1,1}^{\mathcal{H},\mathrm{un}}$ be the closed subgroup of $\Pi_{1,1}^{\mathcal{H}}$ whose underlying group is the pro-unipotent radical of $\Pi_{1,1}^{\mathrm{B}}$. We will compute the structure of this pro-unipotent radical in Proposition 4.8 below. Then, we have a natural isomorphism

$$H^1(\Pi_{1,1}^{\mathcal{H},\mathrm{un}}) = \bigoplus_{k\geq 2} H^1(\mathrm{SL}_2(\mathbf{Z}), \mathrm{Sym}^{k-2}(V)) \otimes_{\mathbf{Q}} \mathrm{Sym}^{k-2}(V)^{\vee}$$

in $\mathcal{R}_{\mathbf{Q}}^{H}$, where the Hodge structure on $H^1(\mathrm{SL}_2(\mathbf{Z}), \mathrm{Sym}^{k-2}(V))$ is defined by the Eichler-Shimura isomorphism ([27, Chapter 8], [31, Section 12, Section 14]). Thus, for a Hecke eigen modular form f of full-level, the associated MHS H_f is an object of $\mathsf{MMM}(\mathbf{Z}) \otimes \overline{\mathbf{Q}}$.

EXAMPLE 4.6 ([18, Example 6.8]). For an elliptic curve E with the origin O, $E^{\times}$ denotes $E \setminus \{O\}$. Let $\pi_1^{\mathrm{un}}(E^{\times})$ be the pro-unipotent fundamental group of $E^{\times}$. Then, the family of Lie algebras

$$\left\{ \mathrm{Lie}(\pi_1^{\mathrm{un}}(\mathscr{E}_x^{\times})) \;\middle|\; x \in \mathscr{M}_{1,1} \right\}$$

forms a pro-local system over $\mathscr{M}_{1,1}$. Its fiber at v is an object of MMM.

Since $\mathrm{Lie}(\pi_1^{\mathrm{un}}(\mathscr{E}_v^{\times}, w))$ contains $\mathrm{Lie}(\Pi_{0,4}^{\mathrm{B}})$ as a sub pro-mixed Hodge structures (cf. [15, Section 18], [18, Section 28]), the category $\mathsf{MMM}(\mathbf{Z})$ is certainly an extension of $\mathsf{MTM}(\mathbf{Z})$. Namely:

PROPOSITION 4.7 ([6, Theorem 14.5]). *The category* $\mathsf{MMM}(\mathbf{Z})$ *contains* $\mathsf{MTM}(\mathbf{Z})$ *as a Tannakian full subcategory.*

4.2. Group structure of $\Pi_{1,1}^{\mathrm{B}}$.

Let us return to the determination of the group structure of $\Pi_{1,1}^{\mathrm{B}}$. By definition, we have

$$1 \to \Pi_{1,1}^{\mathrm{B,un}} \to \Pi_{1,1}^{\mathrm{B}} \to \mathrm{SL}_{2,\mathbf{Q}} \to 1, \tag{4.1}$$

where $\Pi_{1,1}^{\mathrm{B,un}}$ is the pro-unipotent radical of $\Pi_{1,1}^{\mathrm{B}}$. According to Proposition 2.5, we have an isomorphism

$$H^i(\Pi_{1,1}^{\mathrm{B,un}}) = \bigoplus_{k\geq 2} H^i(\Pi_{1,1}^{\mathrm{B}}, \mathrm{Sym}^{k-2}(V)) \otimes_{\mathbf{Q}} \mathrm{Sym}^{k-2}(V)^{\vee}$$

of $\mathrm{SL}_{2,\mathbf{Q}}$-modules. According to [18, Proposition 10.1], the natural homomorphism

$$H^i(\Pi^{\mathrm{B}}_{1,1}, \mathrm{Sym}^{k-2}(V)) \to H^i(\mathrm{SL}_2(\mathbf{Z}), \mathrm{Sym}^{k-2}(V))$$

induced by ρ_{univ} is isomorphism if $i \leq 1$ and injective if $i = 2$. Since $\mathrm{SL}_2(\mathbf{Z})$ contains a free group of finite rank as a finite index subgroup, the cohomology groups above vanish when $i \geq 2$. Thus, we have the following proposition:

Proposition 4.8. *The pro-Lie algebra* $\mathrm{Lie}(\Pi^{\mathrm{B,un}}_{1,1})$ *is topologically generated by a basis of*

$$\bigoplus_{k\geq 2} H^1(\mathrm{SL}_2(\mathbf{Z}), \mathrm{Sym}^{k-2}(V))^\vee \otimes_{\mathbf{Q}} \mathrm{Sym}^{k-2}(V) \tag{4.2}$$

freely.

We have an isomorphism of $\mathbf{C}$-vector spaces

$$M_k(\mathrm{SL}_2(\mathbf{Z})) \oplus \overline{S_k(\mathrm{SL}_2(\mathbf{Z}))} \xrightarrow{\sim} H^1(\mathrm{SL}_2(\mathbf{Z}), \mathrm{Sym}^{k-2}(V)) \otimes_{\mathbf{Q}} \mathbf{C},$$

where $M_k(\mathrm{SL}_2(\mathbf{Z})), S_k(\mathrm{SL}_2(\mathbf{Z}))$ denote the space of full-level modular forms and cuspforms of weight k, respectively. Therefore, $\mathrm{Lie}(\Pi^{\mathrm{B,un}}_{1,1})/\overline{\mathbf{Q}}$ is topologically generated freely by elements

$$e_f X^i Y^j,\ e'_f X^i Y^j,\ e_g X^i Y^j,$$

where f (resp. g) is a full-level normalized Hecke eigen cuspform (resp. Eisenstein series) of weight k and $i + j = k - 2$.

4.3. Zeta and modular generators of $\mathrm{Lie}(U^{\mathrm{dR}}_{\mathsf{MMM}})$.

Let $\pi_1(\mathsf{MMM}(\mathbf{Z}), \omega_{\mathrm{dR}})$ be the Tannakian fundamental group of $\mathsf{MMM}(\mathbf{Z})$ and let $U^{\mathrm{dR}}_{\mathsf{MMM}}$ be its pro-unipotent radical. According to Proposition 2.5, to determine the generators of this pro-unipotent group, we need to compute $\mathrm{Ext}^1_{\mathsf{MMM}(\mathbf{Z})}(\mathbf{Q}, H)$ for all simple object H. This is generally very hard task, however, Brown proved that this extension group is non-zero when $H = \mathbf{Q}(2n+1), H_f(d)$ with $n \geq 1,\ d \geq \mathrm{wt}(f)$. As a consequence, he had found a part of generators of $\mathrm{Lie}(U^{\mathrm{dR}}_{\mathsf{MMM}})$. Moreover, he proved that there is no non-trivial relation between those generators:

Theorem 4.9 ([6, Theorem 21.2]). *Let $\mathcal{B}$ denote the set of normalized Hecke eigen cuspforms of full-levels. Then, there exists a system of elements*

$$\{\sigma_{2n+1},\ \sigma'_f(d),\ \sigma''_f(d) \in \mathrm{Lie}(U^{\mathrm{dR}}_{\mathsf{MMM}}) \mid n \in \mathbf{Z}_{\geq 1},\ f \in \mathcal{B},\ d \geq \mathrm{wt}(f)\},$$

which generates a free *Lie subalgebra of* $\mathrm{Lie}(U^{\mathrm{dR}}_{\mathsf{MMM}})$.

See [6, Subsection 17.1] for a conjecture about topological generators and relations of $\mathrm{Lie}(U^{\mathrm{dR}}_{\mathsf{MMM}})$ based on an analogue of the Beilinson conjecture.

4.4. An analogous category $\mathsf{MMM}(\mathscr{M}_{1,1})$.

By the Tannakian duality, the fiber functor ω_{dR} of $\mathsf{MMM}(\mathbf{Z})$ induces an equivalence $\omega_{\mathrm{dR}}\colon \mathsf{MMM}(\mathbf{Z}) \cong \mathrm{Rep}_{\mathbf{Q}}(\pi_1(\mathsf{MMM}(\mathbf{Z}), \omega_{\mathrm{dR}}))$. On the other hand, we have a canonical action $\Pi^{\mathrm{dR}}_{1,1} \curvearrowleft \pi_1(\mathsf{MMM}(\mathbf{Z}), \omega_{\mathrm{dR}})$ by the definition of $\mathsf{MMM}(\mathbf{Z})$. It is natural to consider the representation of $\Pi^{\mathrm{dR}}_{1,1}$, not only $\pi_1(\mathsf{MMM}(\mathbf{Z}), \omega_{\mathrm{dR}})$.

Definition 4.10. The category $\mathsf{MMM}(\mathscr{M}_{1,1})$ is defined to be the category of algebraic representations of $\pi_1(\mathsf{MMM}(\mathbf{Z}), \omega_{\mathrm{dR}}) \ltimes \Pi^{\mathrm{dR}}_{1,1}$ on finite dimensional $\mathbf{Q}$-vector spaces:

$$\mathsf{MMM}(\mathscr{M}_{1,1}) := \mathrm{Rep}_{\mathbf{Q}}(\pi_1(\mathsf{MMM}(\mathbf{Z}), \omega_{\mathrm{dR}}) \ltimes \Pi^{\mathrm{dR}}_{1,1}).$$

This category is conjecturally equivalent to a full subcategory of motivic sheaves over $\mathscr{M}_{1,1}$. Here, we mean the category of motivic sheaves is the essential image of the realization functor from the category of motivic local systems over $\mathscr{M}_{1,1}$ in the sense of Arapura ([1]) in the category of system of realizations ([7, 1.21]). A motivic sheaf $\mathcal{F}$ is in $\mathsf{MMM}(\mathscr{M}_{1,1})$, then $\mathrm{Gr}^W_\bullet \mathcal{F} \cong \bigoplus_i \mathrm{Sym}^i(\mathcal{V}) \otimes M_i$ ($M_i \in \mathsf{MMM}(\mathbf{Z})$).

We have a sequence of Tannakian categories:

$$\mathsf{MTM}(\mathbf{Z}) \subset \mathsf{MMM}(\mathbf{Z}) \subset \mathsf{MMM}(\mathscr{M}_{1,1})$$

Problem 4.11. They are natural extensions of $\mathsf{MTM}(\mathbf{Z})$, but still huge (e.g. generators of π_1 is still unknown). Is there an "easier" intermediate category?

One of a solution is to take a " mixed Tate quotient". This will be done in the next section.

5. Mixed elliptic motives

Mixed elliptic motives was defined by Hain and Matsumoto in [18]. In this section, we give a brief review of their results. First, we give a group theoretic definition of the category of mixed elliptic motives over $\mathscr{M}_{1,1}$. Then, we see Hain-Matsumoto's original definition. One of main results of [18] is partial determination of the structure of the Tannakian fundamental group of this category. We state their results and give a sketch of the proof.

Remark 5.1. In [18, Definition 6.1], Hain and Matsumoto defined three categories of universal mixed elliptic motives over $\mathscr{M}_{1,1}$, $\mathscr{M}_{1,\vec{1}}$, and over $\mathscr{M}_{1,2}$. We only consider the category of the universal mixed elliptic motives over $\mathscr{M}_{1,1}$ for simplicity.

5.1. Group theoretical definition.

Let $\Pi_{1,1}^{\mathbf{Eis}}$ denote the **maximal mixed Tate quotient** of $\Pi_{1,1}^{\mathrm{dR}}$. That is, $\Pi_{1,1}^{\mathbf{Eis}}$ is a quotient pro-algebraic group of $\Pi_{1,1}^{\mathrm{dR}}$ satisfying the following properties:

- The kernel of the canonical projection $\mathrm{pr}\colon \Pi_{1,1}^{\mathrm{dR}} \twoheadrightarrow \Pi_{1,1}^{\mathbf{Eis}}$ is stable under the action of $\pi_1(\mathcal{R}_{\mathbf{Q}}^H, \omega_{\mathrm{dR}})$ so that the group $\pi_1(\mathcal{R}_{\mathbf{Q}}^H, \omega_{\mathrm{dR}})$ acts on $\Pi_{1,1}^{\mathbf{Eis}}$ naturally.
- The action of $\pi_1(\mathcal{R}_{\mathbf{Q}}^H, \omega_{\mathrm{dR}})$ on $\Pi_{1,1}^{\mathbf{Eis}}$ factors through the natural surjective homomorphism $\pi_1(\mathcal{R}_{\mathbf{Q}}^H, \omega_{\mathrm{dR}}) \twoheadrightarrow \pi_1(\mathsf{MTM}(\mathbf{Z}), \omega_{\mathrm{dR}})$.
- For any morphism $f\colon \Pi_{1,1}^{\mathrm{dR}} \twoheadrightarrow G$ satisfying the properties above, there exists a unique homomorphism $g\colon \Pi_{1,1}^{\mathbf{Eis}} \twoheadrightarrow G$ satisfying $f = g \circ \mathrm{pr}$.

Definition 5.2 ([18]). The category $\mathsf{MEM} = \mathsf{MEM}_{1,1}$ of **universal mixed elliptic motives** is defined by

$$\mathsf{MEM} = \mathrm{Rep}_{\mathbf{Q}}(\pi_1(\mathsf{MTM}(\mathbf{Z}), \omega_{\mathrm{dR}}) \ltimes \Pi_{1,1}^{\mathbf{Eis}}).$$

The following diagram is a relation of Tannakian fundamental groups that appear in this article:

$$\begin{tikzcd}
 & \pi_1(\mathsf{MMM}(\mathbf{Z}), \omega_{\mathrm{dR}}) \arrow[dl, two heads] & \\
\pi_1(\mathsf{MTM}(\mathbf{Z}), \omega_{\mathrm{dR}}) & & \pi_1(\mathsf{MMM}(\mathscr{M}_{1,1}), \omega_{\mathrm{dR}}) \arrow[ul, two heads] \arrow[dl, two heads] \\
 & \pi_1(\mathsf{MEM}, \omega_{\mathrm{dR}}) \arrow[ul, two heads] &
\end{tikzcd}$$

Here, $\pi_1(\mathsf{MEM}, \omega_{\mathrm{dR}})$ denotes the Tannakian fundamental group of MEM with the base point defined by the forgetful functor. By Tannakian duality, we have the following fully-faithful functors of Tannakian categories:

$$\begin{tikzcd}
 & \mathsf{MMM}(\mathbf{Z}) \arrow[dr, hook] & \\
\mathsf{MTM}(\mathbf{Z}) \arrow[ur, hook] \arrow[dr, hook] & & \mathsf{MMM}(\mathscr{M}_{1,1}) \\
 & \mathsf{MEM} \arrow[ur, hook] &
\end{tikzcd}$$

Note that $\Pi_{1,1}^{\mathbf{Eis}}$ is isomorphic to the de Rham realization of an affine group scheme $\Pi_{1,1}^{\mathbf{Eis},\mathrm{mot}}$ in $\mathsf{MTM}(\mathbf{Z})$. Let $\Pi_{1,1}^{\mathbf{Eis},\mathrm{B}}$ denote the Betti realization of $\Pi_{1,1}^{\mathbf{Eis},\mathrm{mot}}$. Then, it is easily checked that the category MEM is equivalent to the category $\mathrm{Rep}_{\mathbf{Q}}(\pi_1(\mathsf{MTM}(\mathbf{Z}), \omega_{\mathrm{B}}) \ltimes \Pi_{1,1}^{\mathbf{Eis},\mathrm{B}})$.

5.2. **Original (geometric) definition.** We see an original (geometric) definition of MEM due to Hain-Matsumoto here. Let $\mathcal{R}_{\mathscr{M}_{1,1}}^{H+\ell}$ be the category of the Hodge and ℓ-adic components of system of realizations over $\mathscr{M}_{1,1}$ in the sense of [7, 1.21] (cf. [8, 2.15]), where ℓ runs over all prime numbers. A *universal mixed elliptic motive in the original sense* ([18, Definition 6.1]) is a tuple $(\mathcal{F}, H, f)$ where:

(1) $\mathcal{F}$ be an object of $\mathcal{R}_{\mathscr{M}_{1,1}}^{H+\ell}$ such that

$$\mathrm{Gr}_n^W \mathcal{F} \cong \oplus_i \mathrm{Sym}^{n-2i}(\mathcal{V})(i)^{\oplus r_i}.$$

(2) H is an object of $\mathsf{MTM}(\mathbf{Z})$ equipped with an increasing filtration $W_\bullet H$, which does not have to match the original filtration on H as an object of $\mathsf{MTM}(\mathbf{Z})$.

(3) $f\colon \mathcal{F}_v \xrightarrow{\sim} R(H)$ is an isomorphism of objects of $\mathcal{R}_{\mathrm{Spec}(\mathbf{Z})}^{H+\ell}$ preserving $W_\bullet$. Here, the Hodge component of $\mathcal{F}_v$ is equipped with the limit mixed Hodge structure.

Hence, each universal mixed elliptic motive in the original sense is an object of $\mathcal{R}_{\mathscr{M}_{1,1}}^{H+\ell}$ which is a **successive extension of** $\mathrm{Sym}^m(\mathcal{V})(r)$.

Proposition 5.3. *The category of universal mixed elliptic motives in the original sense is naturally equivalent to* MEM.

Lemma 5.4. *Let $\mathcal{R}_{\mathscr{M}_{1,1}}^{H}(\mathcal{V})$ be the full-subcategory of $\mathcal{R}_{\mathscr{M}_{1,1}}^{H}$ consisting of objects whose Jordan-Hölder component is isomorphic to* $\mathrm{Sym}^n(\mathcal{V}) \otimes H$ *for some non-negative integer*

n and $H \in \mathrm{Obj}(\mathcal{R}^H_{\mathbf{Q}})$. Let $v\colon \mathcal{R}^H_{\mathscr{M}_{1,1}}(\mathcal{V}) \to \mathcal{R}^H_{\mathbf{Q}}$ be the functor defined by taking the fiber at v. Then, we have a natural isomorphism

$$\pi_1(\mathcal{R}^H_{\mathscr{M}_{1,1}}(\mathcal{V}), \omega_{\mathrm{dR}} \circ v) \cong \pi_1(\mathcal{R}^H_{\mathbf{Q}}, \omega_{\mathrm{dR}}) \ltimes \Pi^{\mathrm{dR}}_{1,1}.$$

Proof. Note that the category $\mathrm{Rep}_{\mathbf{Q}}(\pi_1(\mathcal{R}^H_{\mathbf{Q}}, \omega_{\mathrm{dR}}) \ltimes \Pi^{\mathrm{dR}}_{1,1})$ is equivalent to the category of V of objects in $\mathcal{R}^H_{\mathbf{Q}}$ equipped with the coaction of $\mathcal{O}(\Pi^{\mathcal{H}}_{1,1})$. Therefore, to prove the lemma, it suffices to show that the functor

$$\omega_{\mathrm{B}} \circ v\colon \mathcal{R}^H_{\mathscr{M}_{1,1}}(\mathcal{V}) \to \mathrm{Rep}_{\mathbf{Q}}(\pi_1(\mathcal{R}^H_{\mathbf{Q}}, \omega_{\mathrm{B}}) \ltimes \Pi^{\mathrm{B}}_{1,1}) \tag{5.1}$$

induced by $\omega_{\mathrm{B}} \circ v$ is an equivalence of Tannakian categories. Let us construct a quasi-inverse of the functor above.

Let $\mathsf{HRep}(\Pi^{\mathrm{B}}_{1,1})$ denote the category of Hodge representation of $\Pi^{\mathrm{B}}_{1,1}$ over $\mathbf{Q}$ in the sense of [13, Section 4], namely, this is the category of representations of $\pi_1(\mathsf{MHS}_{\mathbf{Q}}, \omega_{\mathrm{B}}) \ltimes \Pi^{\mathrm{B}}_{1,1}$. Then, according to [13, Theorem 5.1, Subsection 5.5], the functor $\omega_{\mathrm{B}} \circ v$ induces an equivalence of Tannakian categories

$$\mathsf{MHS}(\mathscr{M}_{1,1}, \mathcal{V}) \xrightarrow{\sim} \mathsf{HRep}(\Pi^{\mathrm{B}}_{1,1}), \tag{5.2}$$

where $\mathsf{MHS}(\mathscr{M}_{1,1}, \mathcal{V})$ is the category of admissible variations of MHSs over $\mathscr{M}_{1,1}$ whose Jordan-Hölder component is isomorphic to $\mathrm{Sym}^n(\mathcal{V}) \otimes H$ with $H \in \mathsf{MHS}_{\mathbf{Q}}$.

Then, a quasi-inverse of (5.1) is constructed as follows. Let H_{B} be a given representation of $\pi_1(\mathcal{R}^H_{\mathbf{Q}}, \omega_{\mathrm{B}}) \ltimes \Pi^{\mathrm{B}}_{1,1}$ and let H_{dR} be the corresponding representation of $\pi_1(\mathcal{R}^H_{\mathbf{Q}}, \omega_{\mathrm{dR}}) \ltimes \Pi^{\mathrm{dR}}_{1,1}$. Define a pair $\mathcal{F} = (\mathcal{F}_{\mathrm{B}}, \mathcal{F}_{\mathrm{dR}})$ to be the $\mathbf{Q}$-local system over $\mathscr{M}_{1,1,\mathrm{an}}$ and flat connection over $\mathscr{M}_{1,1}$ by representations H_{B} of $\Pi^{\mathrm{B}}_{1,1}$ and H_{dR} of $\Pi^{\mathrm{dR}}_{1,1}$, respectively. Then, $(\mathcal{F}_{\mathrm{B}}, \mathcal{F}_{\mathrm{dR},\mathbf{C}})$ forms an admissible variation of MHSs by the result of Hain above. Moreover, the lowest weight subbundle of $\mathcal{F}_{\mathrm{dR},\mathbf{C}}$ is a direct factor of the vector bundle associated with $H^0(\Pi^{\mathrm{dR,un}}_{1,1}, H_{\mathrm{dR}}) \otimes_{\mathbf{Q}} \mathbf{C}$ and this factorization is automatically defined over $\mathbf{Q}$. Hence, by the inductive argument on the length of the weight filtrations, we conclude that $W_\bullet \mathcal{F}_{\mathrm{dR},\mathbf{C}}$ descends to the filtration on $\mathcal{F}_{\mathrm{dR}}$. Then, the datum defined above forms an object of $\mathcal{R}^H_{\mathscr{M}_{1,1}}(\mathcal{V})$. The construction is obviously functorial in H_{B} and we can easily check that this defines a quasi-inverse of (5.1). $\square$

Sketch of the proof of Proposition 5.3. Let $(\mathcal{F}, H, f)$ be a universal mixed elliptic motive over $\mathscr{M}_{1,1}$ in the original sense. Then, by Lemma 5.4, $R_{\mathrm{dR}}(H)$ defines a mixed elliptic motive in our sense. Hence, this correspondence defines functor from the original category of MEMs to the category of our MEMs. The quasi-inverse is constructed as follows. Let H be our mixed elliptic motive. Then, by fixing equivalences in the proof of Lemma 5.4, we have an object $\mathcal{F}_{\mathcal{H}}$ of $\mathcal{R}^H_{\mathscr{M}_{1,1}}(\mathcal{V})$ corresponding to H. Then, for a prime number ℓ, the smooth $\mathbf{Q}_\ell$-sheaf $\mathcal{F}_\ell$ is defined to be the corresponding one to the representation

$$\pi_1^{\text{ét}}(\mathscr{M}_{1,1}/\mathbf{Q}, v) \cong \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \ltimes \widehat{\mathrm{SL}_2(\mathbf{Z})} \to \pi_1(\mathsf{MTM}(\mathbf{Z}), \omega_{\mathrm{B}})(\mathbf{Q}_\ell) \ltimes \Pi^{\mathbf{Eis},\mathrm{B}}_{1,1}(\mathbf{Q}_\ell).$$

The comparison between $\mathcal{F}_\ell$ and $\mathcal{F}_{\mathrm{B}}$ is the induced isomorphism by $\pi_1(\mathscr{M}_{1,1}(\mathbf{C}), v)^{\widehat{}} \xrightarrow{\sim} \pi_1^{\text{ét}}(\mathscr{M}_{1,1}/\overline{\mathbf{Q}}, v)$. We take f as the canonical isomorphism between $\mathcal{F}_v$ and $R(H)$. According to [18, Remark 6.2], $W_\bullet H$ is recovered by the action of $\pi_1(\mathscr{M}_{1,1}(\mathbf{C}), v)$ on $R_{\mathrm{B}}(H)$ via $\pi_1(\mathscr{M}_{1,1}(\mathbf{C}), v) \to \Pi^{\mathbf{Eis},\mathrm{B}}_{1,1}(\mathbf{Q})$. This defines filtrations $W_\bullet$ on $\mathcal{F}_\ell$. We leave to the leader to show that this is a quasi-inverse of the natural functor defined by v. $\square$

From now, we identify those two categories. Then, for each object H of MEM in our sense, two weight filtrations $W_\bullet H$ and $M_\bullet H$ are equipped. The first filtration is the fiber of global filtration $W_\bullet\mathcal{F}$ and the second is the weigh filtration as an object of $\mathcal{R}^H_{\mathbf{Q}}$.

5.3. **Structure of** $\pi_1(\mathsf{MEM},\omega_{\mathrm{dR}})$**.** Let $U^{\mathrm{dR}}_{\mathsf{MEM}}$ be the pro-unipotent radical of $\pi_1(\mathsf{MEM},\omega_{\mathrm{dR}})$. Then, by definition, we have a short exact sequence

$$1\to U^{\mathrm{dR}}_{\mathsf{MEM}}\to\pi_1(\mathsf{MEM},\omega_{\mathrm{dR}})\to\mathrm{GL}_{2,\mathbf{Q}}\to 1 \tag{5.3}$$

of pro-algebraic groups over $\mathbf{Q}$. Therefore, to compute topological generators of $\mathrm{Lie}(U^{\mathrm{dR}}_{\mathsf{MEM}})$, it is sufficient to compute extension groups $\mathrm{Ext}^1_{\mathsf{MEM}}(\mathbf{Q},\mathrm{Sym}^i(\mathcal{V})(r))$ for each non-negative integer i and an integer r (Proposition 2.5).

Theorem 5.5 ([18, Theorem 15.1]). *We have*

$$\mathrm{Ext}^1_{\mathsf{MEM}}(\mathbf{Q},\mathrm{Sym}^i(\mathcal{V})(r))^\vee=\begin{cases}\mathbf{Q}z_r, & i=0,\ r\geq 3,\textit{odd},\\ \mathbf{Q}e_i, & i\geq 1,\textit{even},\ r=i+1,\\ 0 & \textit{otherwise}.\end{cases}$$

In particular, $\mathrm{Lie}(U^{\mathrm{dR}}_{\mathsf{MEM}})$ *has a topological generators*

$$z_{2r+1},\quad e_0^ie_{2k+2}\quad (r\geq 1,\ k\geq 1,\ 0\leq i\leq 2k).$$

Next, let us consider the relations of $U^{\mathrm{dR}}_{\mathsf{MEM}}$. According to [18, Proposition B.1], there exists a natural splitting of $W_\bullet H$ and $M_\bullet H$ functorial in $H\in\mathrm{Obj}(\mathsf{MEM})$. This splitting gives a splitting of (5.3) and each W-graded piece of H is stable under the action of $\mathrm{GL}_{2,\mathbf{Q}}$. Then, $\mathrm{Lie}(U^{\mathrm{dR}}_{\mathsf{MEM}})$ is equipped with pro bi-graded Lie algebra structure ([18, Subsection 19.2]). Let $\mathrm{GrLie}(U^{\mathrm{dR}}_{\mathsf{MEM}})$ be the associated bi-graded Lie algebra over $\mathbf{Q}$. Since $\mathrm{Lie}(U^{\mathrm{dR}}_{\mathsf{MEM}})$ is recovered by $\mathrm{GrLie}(U^{\mathrm{dR}}_{\mathsf{MEM}})$, to determine the structure of $\mathrm{Lie}(U^{\mathrm{dR}}_{\mathsf{MEM}})$, it suffices to determine the structure of $\mathrm{GrLie}(U^{\mathrm{dR}}_{\mathsf{MEM}})$. Let $\mathfrak{f}$ be the free Lie algebra generated by symbols $\boldsymbol{z}_{2r+1},\quad \boldsymbol{e}_0^i\boldsymbol{e}_{2k+2}\quad (r\geq 1,\ k\geq 1,\ 0\leq i\leq 2k)$. There exists a natural action of $\mathrm{GL}_{2,\mathbf{Q}}$ on $\mathfrak{f}$ by identifying this Lie algebra with the free Lie algebra

$$\mathrm{Lie}\left(\bigoplus_{k\geq 2,r\in\mathbf{Z}}\mathrm{Ext}^1_{\mathsf{MEM}}(\mathbf{Q},\mathrm{Sym}^{k-2}(\mathcal{V})(r))^\vee\otimes_{\mathbf{Q}}\mathrm{Sym}^{k-2}(V)(r)\right).$$

Here, we take $\boldsymbol{e}_{2k+2}$ is an invariant vector under the action of $T\in\mathrm{SL}_2(\mathbf{Z})$. Then, by Theorem 5.5, we have a $\mathrm{GL}_{2,\mathbf{Q}}$-equivariant surjective homomorphism

$$\mathfrak{f}\twoheadrightarrow\mathrm{GrLie}(U^{\mathrm{dR}}_{\mathsf{MEM}}),\quad \boldsymbol{z}_{2r+1}\mapsto z_{2r+1},\ \ \boldsymbol{e}_0^i\boldsymbol{e}_{2k+2}\mapsto e_0^ie_{2k+2}. \tag{5.4}$$

Note that $\mathfrak{r}$ is contained in $[\mathfrak{f},\mathfrak{f}]$ by Theorem 5.5. We mean a *relation* of $\mathrm{GrLie}(U^{\mathrm{dR}}_{\mathsf{MEM}})$ an element of the kernel $\mathfrak{r}$ of (5.4). Let $\mathfrak{f}_g$ be the Lie subalgebra of $\mathfrak{f}$ generated by $\{\boldsymbol{e}^i\boldsymbol{e}_{2k+2}\}_{k\geq 1,\ 0\leq i\leq 2k}$ so that the image of $\mathfrak{f}_g$ under (5.4) is $\mathrm{GrLie}(\Pi^{\mathbf{Eis},\mathrm{un}}_{1,1})$. A *geometric relation* means an element of $\mathfrak{r}_g:=\mathfrak{r}\cap\mathfrak{f}_g$. In this article, a highest weight vector of a $\mathrm{GL}_{2,\mathbf{Q}}$-module V is an element of $\{v\in V\mid Tv=v\}$ and V^{hwt} denotes the space of highest weight vectors. Since any irreducible algebraic representation of $\mathrm{GL}_{2,\mathbf{Q}}$ is generated by its highest weight vectors, $\mathfrak{r}$ (resp. $\mathfrak{r}_g$) is determined by $\mathfrak{r}^{\mathrm{hwt}}$ (resp. $\mathfrak{r}_g^{\mathrm{hwt}}$).

Let $\Gamma^i\mathfrak{f}$ be the central descending series defined by $\Gamma^{i+1}\mathfrak{f}=[\mathfrak{f},\Gamma^i\mathfrak{f}]$, $\mathfrak{f}=\Gamma^1\mathfrak{f}$ and let us consider the natural mapping

$$\Gamma^2\mathfrak{f}\twoheadrightarrow\mathrm{Gr}^2_\Gamma\mathfrak{f}:=\Gamma^2\mathfrak{f}/\Gamma^3\mathfrak{f}. \tag{5.5}$$

A relation $x \in \mathfrak{r}$ of $\mathrm{GrLie}(U^{\mathrm{dR}}_{\mathsf{MEM}})$ is called **quadratic** if the image of x under (5.5) does not zero, namely, the leading term of x is quadratic. To determine the image of $\mathfrak{r}$ under (5.5), it is sufficient to determine the image of $\mathfrak{r}^{\mathrm{hwt}}$ in $(\mathrm{Gr}^2_\Gamma \mathfrak{f})^{\mathrm{hwt}}$ under (5.5). The set $(\mathrm{Gr}^2_\Gamma \mathfrak{f}_g)^{\mathrm{hwt}}$ of highest weights vectors are described as follows:

LEMMA 5.6 ([25, Proposition 4.1], [18, Proposition 24.2]). *For non-negative integers a, b, d satisfying $d \geq 2,\ 2\min\{a,b\} \geq d-2$, define an element $\boldsymbol{w}^d_{a,b} \in \mathrm{Gr}^2_\Gamma \mathfrak{f}_g$ by*

$$\boldsymbol{w}^d_{a,b} = \sum_{i+j=d-2,\ i,j\geq 0} (-1)^i \binom{d-2}{i} (2a-i)!(2b-j)![\boldsymbol{e}_0^i \boldsymbol{e}_{2a+2}, \boldsymbol{e}_0^j \boldsymbol{e}_{2b+2}].$$

Then, the set

$$\{\boldsymbol{w}^d_{a,b} \mid a,b,d \in \mathbf{Z},\ d \geq 2,\ 2\min\{a,b\} \geq d-2\}$$

is a basis of $(\mathrm{Gr}^2_\Gamma \mathfrak{f}_g)^{\mathrm{hwt}}$.

Before to state their result, we recall period polynomials defined by modular forms briefly. For an even positive integer k greater than two, let V_k be the space of homogeneous polynomials in x, y of degree $k-2$ over $\mathbf{Q}$. Then, the group $\mathrm{GL}_2(\mathbf{Q})$ acts on V_k by

$$f(x,y)|_\gamma = f(ax+by, cx+dy), \quad \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbf{Q}).$$

The subspace W_k of V_k is defined by

$$W_k = \{f \in V_k \mid f|_{1+S} = f|_{1+TS+(TS)^2} = 0\}$$

([6, Subsection 7.3]). It is easily checked that $\varepsilon = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ preserves the subspace W_k. Let $W_k^\pm$ denote the ± 1-eigen spaces of ε and $f^\pm$ denotes the projection of $f \in W_k$ to $W_k^\pm$ by a natural projection. We call elements of $W_k^+ \otimes_{\mathbf{Q}} \mathbf{C}$ (resp. $W_k^- \otimes_{\mathbf{Q}} \mathbf{C}$) an even (resp. odd) period polynomials. Note that the space W_k is closely related to the cohomology group of $\mathrm{SL}_2(\mathbf{Z})$. Let $Z^1_{\mathrm{cusp}}(\mathrm{SL}_2(\mathbf{Z}), V_k)$ be the set of inhomogeneous one cocycles of $\mathrm{SL}_2(\mathbf{Z})$ ([6, (7.3)]) coefficients in V_k satisfying $c(T) = 0$. Then, we have

$$Z^1_{\mathrm{cusp}}(\mathrm{SL}_2(\mathbf{Z}), V_k) \xrightarrow{\sim} W_k; \quad c \mapsto c(S)$$

([6, (7.4)]). Elements of the image of coboundary one cocycles under the isomorphism above are called *coboundary period polynomials*. The period polynomial $r_f \in W_k \otimes_{\mathbf{Q}} \mathbf{C}$ associated with a cuspform f of weight k is defined by the above correspondence. Explicitly, this is constructed as follows: For a modular form f of weight k, put $\omega_f = (2\pi\sqrt{-1})^{k-1} f(\tau)(x-\tau y)^{k-2} d\tau$, which defines an element of $H^0_{\mathrm{dR}}(\mathscr{M}_{1,1}, \mathrm{Sym}^{k-2}(\mathcal{V}_{\mathrm{dR}}))$. When f is a cuspform, the period polynomial r_f is defined by

$$r_f = \int_0^{\sqrt{-1}\infty} \omega_f,$$

where $\int_a^b$ denotes the integration along the geodesic path from a to b on $\mathfrak{H} \coprod \mathbf{P}^1(\mathbf{Q})$.

One of the main results of Hain-Matsumoto's paper is as follows:

THEOREM 5.7 ([18, Theorem 25.1]). *The image of $\mathfrak{r}_g^{\mathrm{hwt}}$ under (5.5) is given by*

$$\left\{ \sum_{a,b,d} \alpha^d_{a,b} \boldsymbol{w}^d_{a,b} \in (\mathrm{Gr}^2_\Gamma \mathfrak{f}_g)^{\mathrm{hwt}} \;\middle|\; \forall d, \sum_{a+b=k} \alpha^d_{a,b} x^{2a-d+2} y^{2b-d+2} = r_f^{\mathrm{sgn}((-1)^d)}, \exists f \in S_{2k-2d+6}(\mathrm{SL}_2(\mathbf{Z})) \right\}.$$

By specializing $d=2$, we have the following very simple assertion:

COROLLARY 5.8. *Let ξ be an element of $\mathfrak{r}_g^{\mathrm{hwt}}$. Then, a congruence*

$$\xi \equiv \sum_{a+b=k,\ a,b\geq 0} c_a[\boldsymbol{e}_{2a+2}, \boldsymbol{e}_{2b+2}] \pmod{[[\mathfrak{f},\mathfrak{f}],\mathfrak{f}]}$$

holds if and only if $\sum_{a+b=k,a,b\geq 0} c_a x^{2a}y^{2b} = r_f^+$ for a full-level cuspform f of weight $2k+2$.

We have seen that cuspforms produces geometric quadratic relations. How about coboundary period polynomials? The answer is that they produce relations between z_{2k+1}s and $e_0^i e_{2k+2}$s:

THEOREM 5.9 ([18, Theoerm 25.1]). *For all $m \geq 2, k \geq 1$, there exists an element $\xi(m,k) \in \mathfrak{r}$ satisfying the following congruence relation:*

$$\begin{aligned}&\xi(m,k) \equiv [\boldsymbol{z}_{2m-1}, \boldsymbol{e}_{2k+2}]\\ &-\frac{(2m-2)!}{(2m+2k)!}\binom{2k+2}{2}\frac{B_{2m+2k}}{B_{2k+2}}\sum_{i+j=2m-2}(-1)^i\frac{(2k+i)!}{i!}[\boldsymbol{e}_0^i\boldsymbol{e}_{2m}, \boldsymbol{e}_0^j\boldsymbol{e}_{2m+2k}] \pmod{\Gamma^3\mathfrak{f}}.\end{aligned}$$

Here, B_n is the nth Bernoulli number.

The summarizing table of the results above is as follows:

TABLE 1. Table of quadratic relations

	z_{2r+1}	$e_0^i e_{2k+2}$
z_{2r+1}	Non	"coboundary period polynomial"
$e_0^i e_{2k+2}$	"coboundary period polynomial"	cuspforms

Under the natural surjection $\mathrm{Lie}(U^{\mathrm{dR}}_{\mathsf{MEM}}) \twoheadrightarrow \mathrm{Lie}(U^{\mathrm{dR}}_{\mathsf{MTM}})$, z_{2r+1} maps to the free generator σ_{2r+1}. Hence, there is no relation between z_{2r+1}s.

Sketch of the proof of Theorem 5.7. According to Pollack's computation ([25, Theorem 3]), there is no non-trivial quadratic geometric relation coming from cuspforms. The converse inclusion relation follows from:

- Explicit computations of period computations arising from two Eisenstein series ([6, Theorem 9.2]).
- Relate Brown's computation to cup products of $\{e_0^i e_{2k+2}\}_{i,k}$ by using the Beilinson-Deligne cohomology theory for affine group schemes in $\mathsf{MHS}_{\mathbf{Q}}$ ([13, Section 8, Section 10]).

See [18, Proof of Theorem 25.1] for more details. □

Is there a relation that is not a quadratic relation? The conjecture is:

CONJECTURE 5.10 (cf. [18, Corollary 25.4]). *Every non-trivial primitive relation of* $\mathrm{GrLie}(U^{\mathrm{dR}}_{\mathsf{MEM}})$ *is a quadratic relation.*

This is true if an analogue of the Beilinson conjecture ([18, Conjecture 17.1 (i)]).

6. **Problems**

In this section, we collect problems, which is not solved satisfactory to the best of the author's knowledge.

6.1. **Elliptic analogue of Brown's theorem.** The representation $\pi_1(\mathsf{MTM}(\mathbf{Z})) \to \mathrm{Aut}(\Pi_{0,4})$ is the induced representation by the splitting of

$$1 \to \Pi_{0,4} \to \pi_1(\mathsf{MTM}(\mathscr{M}_{0,4})) \to \pi_1(\mathsf{MTM}(\mathscr{M}_{0,3}))) \to 1$$

(note that $\mathscr{M}_{0,3} = \mathrm{Spec}(\mathbf{Z})$). Genus one analogue of the sequence is

$$1 \to \pi_1^{\mathrm{un}}(\mathscr{E}_v^{\times}, w) \to \pi_1(\mathsf{MEM}(\mathscr{M}_{1,2})) \to \pi_1(\mathsf{MEM}(\mathscr{M}_{1,1}))) \to 1,$$

and the induced representation is the **monodromy representation** ($\mathsf{MEM}(\mathscr{M}_{1,1}) = \mathsf{MEM}$). Therefore, a naive analogous question is as follows:

Problem 6.1 ([18, Question 26.2])**.** Is the monodromy representation

$$\rho\colon \pi_1(\mathsf{MEM}) \to \mathrm{Aut}(\pi_1^{\mathrm{un}}(\mathscr{E}_v^{\times}, w))$$

injective?

6.2. **Analogue of the Beilinson conjecture.** The Hodge realization functor defines the regulator

$$\mathrm{reg}^2_{\mathcal{H}}\colon \mathrm{Ext}^2_{\mathsf{MEM}}(\mathbf{Q}, \mathrm{Sym}^{k-2}(\mathcal{V})(r)) \to H^2_{\mathcal{D}}(\mathscr{M}_{1,1}/\mathbf{R}, \mathrm{Sym}^{k-2}(\mathcal{V})_{\mathbf{R}}(r)).$$

Conjecture 6.2 (HM20, Conjecture 17.1 (i)). *The regulators* $\mathrm{reg}^2_{\mathcal{H}} \otimes \mathbf{R}$ *are isomorphisms for all* k, r.

If the conjecture is true, then we can compute the second cohomology group of $U^{\mathrm{dR}}_{\mathsf{MEM}}$. Since the set of relations can be determined by the second cohomology group of $U^{\mathrm{dR}}_{\mathsf{MEM}}$, we can know the explicit structure of $\pi_1(\mathsf{MEM})$ if the conjecture above is positive.

Note that, to show Brown's theorem, we need to know the explicit structure of $\pi_1(\mathsf{MTM})$. Thus, to attack the elliptic analogue of Brown's theorem according to his method, the first difficulty seems to be to determine the explicit structure of $\pi_1(\mathsf{MEM})$. Then, it is natural to ask the following question:

Problem 6.3. Can we prove the elliptic analogue of the Brown's theorem assuming the conjecture above?

6.3. **Higher level case.** Let $\mathsf{MEM}_1(N)$ denote the universal mixed elliptic motives over the modular curve $Y_1(N)$.

Problem 6.4. Compute quadratic relations of generators of $\mathrm{Lie}(U_{\mathsf{MEM}_1(N)})$.

One of difficult points is to compute cup products of Eisenstein symbols *explicitly* (The paper [11] is a work of this type).

Problem 6.5. What is the meaning of $W_\bullet \mathrm{Lie}(U_{\mathsf{MEM}_1(N)}) \cap \mathrm{Lie}(U_{\mathsf{MTM}(\mathbf{Z}[1/N])})$? (This is closely related to the depth when $N = 1$. See [18, Part 4].)

Problem 6.6. Consider similar problems for the modular curve $Y(N)/\mathbf{Z}[\mu_N, 1/N]$.

6.4. **Problems on $\Pi_{1,1}^{\mathcal{H}}$.** One step extensions of objects of $\mathsf{MMM}(\mathbf{Z})^{\mathsf{ss}}$ appearing in $\mathcal{O}(\Pi_{1,1}^{\mathcal{H}})$ was studied by Brown in [6] partially.

Problem 6.7. Study the two step extensions in $\mathcal{O}(\Pi_{1,1}^{\mathcal{H}})$.

Problem 6.8. Replace the base point $\frac{d}{dq}$ by a CM elliptic curve. What will happen?

After the replacement of the base point, then it seems that $\mathcal{O}(\Pi_{1,1}^{\mathcal{H}})$ has a geometric description (cf. [8, Proposition 3.4]).

Problem 6.9. Find an **explicit** description of $\mathcal{O}(\Pi_{1,1}^{\mathcal{H}})$ by relative cohomology groups of open Kuga-Sato varieties.

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