SPHERICAL HARMONICS AND HARDY'S INEQUALITIES

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Abstract. We consider the derivative operators for radial direction and spherical direction. We also investigate the operator which takes the spherical average for functions. We reconfirm those properties with particular attention to orthogonality. As an application, the Hardy type inequality is presented with spherical derivatives in the framework of equalities. This clarifies the difference between contribution by radial and spherical derivatives in the improved Hardy inequality as well as nonexistence of nontrivial extremizers without compactness arguments.

This paper is based on the joint work with Neal Bez and Tohru Ozawa.

1. Preliminaries part 1

In this section we define and investigate some derivative operators in Euclian space. Although the facts are well-known, we here check the all lines for calclations and the orthogonality of those. We define the radial derivative and the spherical derivative

$$\partial_r = \frac{x}{|x|}\cdot\nabla = \sum_{j=1}^{n}\frac{x_j}{|x|}\partial_j,$$

$$L = \nabla - \frac{x}{|x|}\partial_r = \left(\partial_1 - \frac{x_1}{|x|}\partial_r, \dots, \partial_n - \frac{x_n}{|x|}\partial_r\right).$$

where $\partial_j = \partial/\partial x_j, j = 1, \dots, n$. We define

$$L_j = \partial_j - \frac{x_j}{|x|}\partial_r = \partial_j - \sum_{k=1}^{n}\frac{x_j x_k}{|x|^2}\partial_k, \quad j = 1, \dots, n,$$

to have $Lf = (L_1 f, \dots, L_n f)$. We use the polor coordinate $x = r\omega, r = |x|, \omega = x/|x|$ for $x \in \mathbb{R}^n\backslash\{0\}$. We may write $f(x) = f(r,\omega)$. If $f(r,\omega)$ depends on r only, that is $f(x) = f(r)$, we have

$$\partial_r f(x) = \sum_{j=1}^{n}\frac{x_j}{|x|}\partial_j f(|x|) = \sum_{j=1}^{n}\frac{x_j}{|x|}f'(r)\partial_j|x| = f'(r)\sum_{j=1}^{n}\frac{x_j^2}{|x|^2} = f'(r),$$

$$L_j f(x) = \left(\partial_j - \frac{x_j}{|x|}\partial_r\right) f(|x|) = f'(r)\frac{x_j}{|x|} - \frac{x_j}{|x|}f'(r) = 0.$$

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If $f(r,\omega)$ depends on ω only, that is $f(x)=f(\omega)=f\left(\frac{x}{|x|}\right)$, we have

$$\begin{aligned}
\partial_r f(x) &= \sum_{j=1}^{n}\frac{x_j}{|x|}\partial_j\left(f\left(\frac{x}{|x|}\right)\right)=\sum_{j=1}^{n}\frac{x_j}{|x|}\sum_{k=1}^{n}(\partial_k f)\left(\frac{x}{|x|}\right)\partial_j\frac{x_k}{|x|}\\
&= \sum_{j=1}^{n}\frac{x_j}{|x|}\sum_{k=1}^{n}(\partial_k f)(\omega)\left(\frac{\delta_{j,k}}{|x|}-\frac{x_jx_k}{|x|^3}\right)\\
&= \sum_{j=1}^{n}\frac{x_j}{|x|}(\partial_j f)(\omega)\frac{1}{|x|}-\sum_{j=1}^{n}\sum_{k=1}^{n}(\partial_k f)(\omega)\frac{x_j^2x_k}{|x|^4}=0,\\
L_j f(x) &= \left(\partial_j-\frac{x_j}{|x|}\partial_r\right)\left(f\left(\frac{x}{|x|}\right)\right)=\partial_j\left(f\left(\frac{x}{|x|}\right)\right)=\partial_j f(x),
\end{aligned}$$

if we continue to calcurate

$$\begin{aligned}
\partial_j\left(f\left(\frac{x}{|x|}\right)\right) &= \frac{1}{|x|}(\partial_j f)(\omega)-\sum_{k=1}^{n}(\partial_k f)(\omega)\frac{x_jx_k}{|x|^3}\\
&= \frac{1}{|x|}(\partial_j f)(\omega)-\frac{x_j}{|x|^2}(\partial_r f)(\omega)\\
&= \frac{1}{|x|}(L_j f)(\omega)=\left(\frac{1}{|x|}L_j f\right)(w)=(L_j f)(\omega).
\end{aligned}$$

Since we have Leibniz rule

$$\partial_r(fg)=f\partial_r g+g\partial_r f,\qquad L_j(fg)=fL_jg+gL_jf,$$

we estimate for $f=f(r), g=g(\omega)$ and general $h=h(r,\omega)$ as

$$\partial_r(gh)=g\partial_r h,\qquad L_j(fh)=fL_jh. \tag{1.1}$$

We have the pointwise orthogonality between Lf for any f and x as

$$Lf\cdot x=\sum_{j=1}^{n}\left(\partial_j f-\frac{x_j}{|x|}\partial_r f\right)x_j=x\cdot\nabla f-|x|\partial_r f=0.$$

Therefore we have the orthogonality pointwisely between Lf and $\frac{x}{|x|}\partial_r f$ for any f,

$$Lf\cdot\frac{x}{|x|}\partial_r f=\frac{\partial_r f}{|x|}(Lf\cdot x)=0.$$

We use the inner product and the norm for $L^2(\mathbb{R}^n)$,

$$(f|g)=\int_{\mathbb{R}^n}f(x)\overline{g(x)}dx,\qquad \|f\|_2=\|f\|_{L^2}=\sqrt{(f|f)}.$$

We define

$$\|Lf\|_2^2=\sum_{j=1}^{n}\|L_jf\|_2^2.$$

We have the relation $\nabla=\frac{x}{|x|}\partial_r+L$ and, as a corollary, the following orthogonality in the Hilbert space $L^2(\mathbb{R}^n)$.

Proposition 1.

$$\|\nabla f\|_2^2 = \|\partial_r f\|_2^2 + \|Lf\|_2^2. \tag{1.2}$$

Proof. We estimate

$$\begin{aligned}
\|\nabla f\|_2^2 &= \sum_{j=1}^n \left\| \frac{x_j}{|x|}\partial_r f + L_j f \right\|_2^2 \\
&= \sum_{j=1}^n \left\| \frac{x_j}{|x|}\partial_r f \right\|_2^2 + 2\mathrm{Re}\int_{\mathbb{R}^n} \frac{\partial_r f}{|x|} x \cdot \overline{Lf}\, dx + \sum_{j=1}^n \|L_j f\|_2^2 = \|\partial_r f\|_2^2 + \|Lf\|_2^2.
\end{aligned}$$

□

We give the following two results by parts for L_j.

$$\begin{aligned}
(L_j u|v) &= -(u|\partial_j v) + \sum_{k=1}^n \left(u \middle| \partial_k \left(\frac{x_j x_k}{|x|^2} v \right) \right) \\
&= -(u|L_j v) + (n-1)\left(u \middle| \frac{x_j}{|x|^2} v \right),
\end{aligned}$$

and so

$$\sum_{j=1}^n (L_j u|L_j v) = -\sum_{j=1}^n (u|L_j^2 v) + (n-1)\left(u \middle| \frac{1}{|x|^2} x \cdot Lv \right) = -\sum_{j=1}^n (u|L_j^2 v).$$

We consider second order derivative. We introduce the Laplacian on the unit sphere, that is, the Laplace-Beltrami operator.

$$\Delta_{S^{n-1}} := \sum_{1\le j<k\le n} (x_j\partial_k - x_k\partial_j)^2.$$

We have the following relation between the second order of the spherical derivative L and the Laplace-Beltrami operator.

$$\begin{aligned}
&\sum_{j=1}^n (|x|L_j)^2 = |x|^2 \sum_{j=1}^n L_j^2 = |x|^2 \sum_{j=1}^n \left(\partial_j^2 - 2\frac{x_j}{|x|}\partial_j\partial_r + \frac{x_j^2}{|x|^2}\partial_r^2 \right) \\
=&(x_1^2 + \cdots + x_n^2)(\partial_1^2 + \cdots + \partial_n^2) - (x_1^2\partial_1^2 + \cdots + x_n^2\partial_n^2) - 2\sum_{1\le j<k\le n} x_j\partial_j x_k\partial_k \\
=&\sum_{1\le j<k\le n} (x_j^2\partial_k^2 + x_k^2\partial_j^2) - 2\sum_{1\le j<k\le n} x_j\partial_j x_k\partial_k = \Delta_{S^{n-1}},
\end{aligned}$$

where we use (1.1) at the first equality. The following is also well-known

$$\Delta = \partial_r^2 + \frac{n-1}{|x|}\partial_r + \frac{1}{|x|^2}\Delta_{S^{n-1}}.$$

2. Preliminaries part 2

We denote by $L^2_{\text{rad}}(\mathbb{R}^n)$ and $H^1_{\text{rad}}(\mathbb{R}^n)$ the closed subspaces $L^2(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$, respectively, of radial functions:

$$L^2_{\text{rad}}(\mathbb{R}^n) := \{f \in L^2(\mathbb{R}^n);\ \text{There exists } u \in L^2(0,\infty) \text{ such that } f(x) = u(|x|)|x|^{\frac{1-n}{2}} \text{ for almost all } x \in \mathbb{R}^n\backslash\{0\}\},$$

$$H^1_{\text{rad}}(\mathbb{R}^n) := \{f \in (H^1 \cap L^2_{\text{rad}})(\mathbb{R}^n);\ \partial_r f \in L^2_{\text{rad}}(\mathbb{R}^n)\}.$$

For any $f \in L^2(\mathbb{R}^n)$, we denote by Pf its radial average over the unit sphere

$$(Pf)(x) := \frac{1}{\sigma_{n-1}} \int_{S^{n-1}} f(|x|\omega) d\sigma(\omega), \quad x \in \mathbb{R}^n.$$

Then $P : f \mapsto Pf$ induces the orthogonal projection from $L^2(\mathbb{R}^n)$ onto $L^2_{\text{rad}}(\mathbb{R}^n)$ as well as from $H^1(\mathbb{R}^n)$ onto $H^1_{\text{rad}}(\mathbb{R}^n)$ (see Proposition 3 below).

Lemma 2.

$$P^2 = P, \qquad P^* = P.$$

Proof. For any radial function $f \in L^2(\mathbb{R}^n)$, and we write $f(x) = f(|x|)$, we have

$$(Pf)(x) = \frac{1}{\sigma_{n-1}} \int_{S^{n-1}} f(|x|) d\sigma(\omega) = \frac{f(|x|)}{\sigma_{n-1}} \int_{S^{n-1}} d\sigma(\omega) = f(x).$$

Therefore for any f we have the radial Pf and so we have $P(Pf) = Pf$. For any u, v, we calclate

$$\begin{aligned}(Pu|v) &= \int_{\mathbb{R}^n} \frac{1}{\sigma_{n-1}} \int_{S^{n-1}} u(|x|\omega) d\sigma(\omega) \bar{v}(x) dx \\ &= \int_{S^{n-1}} \int_0^\infty \left(\frac{1}{\sigma_{n-1}} \int_{S^{n-1}} u(r\omega) d\sigma(\omega) \right) \bar{v}(r\theta) r^{n-1} dr d\sigma(\theta) \\ &= \int_{S^{n-1}} \int_0^\infty \left(\frac{1}{\sigma_{n-1}} \int_{S^{n-1}} \bar{v}(r\theta) d\sigma(\theta) \right) u(r\omega) r^{n-1} dr d\sigma(\omega) = (u|Pv).\end{aligned}$$

□

We define the orthogonal projection onto the orthogonal complement of these spaces

$$P^\perp = I - P.$$

We have $f = Pf + P^\perp f$.

Proposition 3. *The following relations hold:*

(1) $(P^\perp)^2 = P^\perp, \quad (P^\perp)^* = P^\perp$.
(2) $PP^\perp = P^\perp P = 0$.
(3) $(Pu|P^\perp v) = 0, \quad \|u\|_2^2 = \|Pu\|_2^2 + \|P^\perp u\|_2^2, \quad u, v \in L^2(\mathbb{R}^n)$.
(4) $(\nabla Pu|\nabla P^\perp v) = 0, \quad \|\nabla u\|_2^2 = \|\nabla Pu\|_2^2 + \|\nabla P^\perp u\|_2^2, \quad u, v \in H^1(\mathbb{R}^n)$.

Proof. The all orthogonal projections satisfy the properties (1), (2) and (3).

(1) From Lemma 2,

$$(P^\perp)^2 = I - 2P + P^2 = I - 2P + P = P^\perp,$$

$$(P^\perp u|v) = (u|v) - (Pu|v) = (u|v) - (u|Pv) = (u|P^\perp v).$$

(2) From Lemma 2,

$$PP^{\perp} = P^{\perp}P = P - P^2 = P - P = 0.$$

(3) From (2),

$$(Pu|P^{\perp}v) = (u|PP^{\perp}v) = (u|0) = 0.$$

$$\|u\|^2 = \|Pu + P^{\perp}u\|^2 = \|Pu\|^2 + 2\mathrm{Re}(Pu|P^{\perp}v) + \|P^{\perp}v\|^2 = \|Pu\|^2 + \|P^{\perp}v\|^2.$$

(4) We have

$$\begin{aligned}\partial_r Pu &= \frac{1}{\sigma_{n-1}|x|}\sum_{j=1}^{n} x_j \int_{S^{n-1}} \partial_j(u(|x|\omega))d\sigma(\omega)\\ &= \frac{1}{\sigma_{n-1}|x|}\int_{S^{n-1}}\sum_{j=1}^{n} x_j \frac{x_j}{|x|}(\nabla u)(|x|\omega)\cdot\omega d\sigma(\omega) = P(\nabla u\cdot\omega) = P\partial_r u.\end{aligned}$$

We have seen $L_j v = 0$ for any radial function v, and so $Lv = 0$ and $LP = 0$.

□

By using the orthogonal projection P and $P^{\perp}$, we have the following orthogonal decompositions:

$$L^2(\mathbb{R}^n) = L^2_{\mathrm{rad}}(\mathbb{R}^n) \oplus (L^2_{\mathrm{rad}}(\mathbb{R}^n))^{\perp},$$
$$H^1(\mathbb{R}^n) = H^1_{\mathrm{rad}}(\mathbb{R}^n) \oplus (H^1_{\mathrm{rad}}(\mathbb{R}^n))^{\perp}.$$

We also use the complete orthogonal decomposition

$$L^2(\mathbb{R}^n) = \bigoplus_{k\geq 0} \mathcal{H}^k(\mathbb{R}^n), \tag{2.1}$$

where $\mathcal{H}^k(\mathbb{R}^n)$ is a closed subspace spanned by spherical harmonics of order k multiplied by radial functions. We denote by P_k the associated orthogonal projection. We refer the reader to [4, 5, 6, 24, 26] for details on the decomposition (2.1). Here we notice that

$$P = P_0,$$
$$L^2_{\mathrm{rad}}(\mathbb{R}^n) = \mathcal{H}^0(\mathbb{R}^n).$$

We introduce the eigenvalues of the spherical harmonics

$$\Delta_{S^{n-1}} P_k = -k(k+n-2)P_k, \quad k \geq 0.$$

3. Hardy's inequalities and equalities, Known results.

In this talk, we study the classical Hardy inequality of the form

$$\left(\frac{n-2}{2}\right)^2 \left\|\frac{f}{|x|}\right\|_2^2 \leq \|\nabla f\|_2^2 \tag{3.1}$$

for all $f \in H^1(\mathbb{R}^n)$ with $n \geq 3$. We remark the constant $(n-2/2)^2$ is optimal and is not attained. The follwoing also holds

$$\left(\frac{n-2}{2}\right)^2 \left\|\frac{f}{|x|}\right\|_2^2 \leq \|\partial_r f\|_2^2$$

for all $f \in H^1(\mathbb{R}^n)$ with $n \geq 3$.

We call a kinds of the followings Hardy's equalities. Dolbeault-Volzone (2012) and Bogdan-Dyda-Kim (2016) showed

$$\left(\frac{n-2}{2}\right)^2 \left\|\frac{f}{|x|}\right\|_2^2 = \|\nabla f\|_2^2 - \left\|\nabla f + \frac{n-2}{2}\frac{x}{|x|^2} f\right\|_2^2.$$

M-Ozawa-Wadade (2016) showed

$$\left(\frac{n-2}{2}\right)^2 \left\|\frac{f}{|x|}\right\|_2^2 = \|\partial_r f\|_2^2 - \left\|\partial_r f + \frac{n-2}{2|x|} f\right\|_2^2.$$

Meanwhile, Ekholm-Frank (2006) introduce the fillowing as an improved Hardy inequality

$$\frac{n^2}{4}\left\|\frac{f}{|x|}\right\|_2^2 \leq \|\nabla f\|_2^2 \tag{3.2}$$

for all $f \in C_0^\infty(\mathbb{R}^n)$ with $n \geq 2$ satisfying

$$\int_{S^{n-1}} f(r\omega)d\sigma(\omega) = 0 \tag{3.3}$$

for all $r \geq 0$, where σ is the Lebesgue measure on the unit sphere $S^{n-1} = \{\omega \in \mathbb{R}^n; |\omega| = 1\}$, [7, 12]. In [12], the inequality (3.2) is referred to as an improved Hardy inequality on the basis of the improvement in the coefficient $\frac{n^2}{4}$ on the left hand side of (3.2), which is larger than the corresponding coefficient $\left(\frac{n-2}{4}\right)^2$ on the left hand side of (3.1), as well as of the applicable range of dimensions, in particular, $n = 2$ is now admissible.

4. Hardy's inequalities and equalities, Theorems.

We now state the main results in this paper.

Theorem 4. *Let $n \geq 2$. Then, the following equality*

$$(n-1)\left\|\frac{P^\perp f}{|x|}\right\|_2^2 = \|LP^\perp f\|_2^2 - \sum_{k=2}^{\infty}(k-1)(k+n-1)\left\|\frac{P_k f}{|x|}\right\|_2^2 \tag{4.1}$$

holds for all $f \in H^1(\mathbb{R}^n)$.

Corollary 5. *Let $n \geq 2$. Then the following inequality*

$$(n-1)\left\|\frac{P^\perp f}{|x|}\right\|_2^2 \leq \|LP^\perp f\|_2^2 \tag{4.2}$$

holds for all $f \in H^1(\mathbb{R}^n)$. Equality holds in (4.2) if and only if there exist $\alpha \in \mathbb{C}^n$ and $g, h \in H^1_{\mathrm{rad}}(\mathbb{R}^n)$ such that

$$f(x) = (\alpha \cdot x)g(x) + h(x) \tag{4.3}$$

for almost all $x \in \mathbb{R}^n \backslash \{0\}$, where $\alpha \cdot y = \sum_{j=1}^n \alpha_j y_j$ for $y \in \mathbb{R}^n$. In this case, both sides of (4.2) are given by

$$(n-1)\left\|\frac{P^\perp f}{|x|}\right\|_2^2 = \|LP^\perp f\|_2^2 = \frac{n-1}{n}\sigma_{n-1}|\alpha|^2 \int_0^\infty |u(r)|^2 dr, \tag{4.4}$$

where $u \in L^2(0,\infty)$ satisfies $g(x) = u(|x|)|x|^{-\frac{n-1}{2}}$ for almost all $x \in \mathbb{R}^n \backslash \{0\}$ and $|\alpha|^2 = \sum_{j=1}^n \alpha_j \bar{\alpha}_j$.

Theorem 6. *Let $n \geq 2$. Then, the following equalities*

$$\begin{aligned}
&\left(\frac{n-2}{2}\right)^2 \left\|\frac{f}{|x|}\right\|_2^2 + (n-1)\left\|\frac{P^\perp f}{|x|}\right\|_2^2 \\
&= \left(\frac{n-2}{2}\right)^2 \left\|\frac{Pf}{|x|}\right\|_2^2 + \frac{n^2}{4}\left\|\frac{P^\perp f}{|x|}\right\|_2^2 \qquad (4.5)\\
&= \|\nabla f\|_2^2 - \left\|\left(\partial_r + \frac{n-2}{2|x|}\right) f\right\|_2^2 - \sum_{k=2}^{\infty}(k-1)(k+n-1)\left\|\frac{P_k f}{|x|}\right\|_2^2
\end{aligned}$$

hold for all $f \in H^1(\mathbb{R}^n)$.

Corollary 7. *Let $n \geq 2$. Then, the following inequality*

$$\left(\frac{n-2}{2}\right)^2 \left\|\frac{f}{|x|}\right\|_2^2 + (n-1)\left\|\frac{P^\perp f}{|x|}\right\|_2^2 \leq \|\nabla f\|_2^2. \qquad (4.6)$$

holds for all $f \in H^1(\mathbb{R}^n)$. Equality holds in (4.6) *if and only if $f = 0$.*

Theorem 8. *Let $n \geq 2$. Then, the following equality*

$$\begin{aligned}
\frac{n^2}{4}\left\|\frac{P^\perp f}{|x|}\right\|_2^2 &= \|\nabla P^\perp f\|_2^2 - \left\|\left(\partial_r + \frac{n-2}{2|x|}\right) P^\perp f\right\|_2^2 \qquad (4.7)\\
&\quad - \sum_{k=2}^{\infty}(k-1)(k+n-1)\left\|\frac{P_k f}{|x|}\right\|_2^2
\end{aligned}$$

holds for all $f \in H^1(\mathbb{R}^n)$.

Corollary 9 (Improved Hardy inequality [7, 12])**.** *Let $n \geq 2$. Then, the following inequality*

$$\frac{n^2}{4}\left\|\frac{P^\perp f}{|x|}\right\|_2^2 \leq \|\nabla P^\perp f\|_2^2 \qquad (4.8)$$

holds for all $f \in H^1(\mathbb{R}^n)$. Equality holds in (4.8) *if and only if $f \in H^1_{\mathrm{rad}}(\mathbb{R}^n)$. In this case, both sides of* (4.8) *vanish.*

The inequality (4.6) improved the Hardy inequality (3.1) in the sense that (4.6) reveals a novel term $(n-1)\left\|\frac{P^\perp f}{|x|}\right\|_2^2$ on the left hand side and that the improvement in (3.2) in two space dimensions arises as a result of the existence of the novel term $\left\|\frac{P^\perp f}{|x|}\right\|_2^2$ when $n = 2$. Moreover, (4.6) clarifies the contribution by the orthogonal component to $H^1_{\mathrm{rad}}(\mathbb{R}^n)$ with coefficient $n-1$, which together with the standard coefficient $\left(\frac{n-2}{2}\right)^2$ yields the improved coefficient $\frac{n^2}{4}$ in (3.2) on the basis of the simple identity $\left(\frac{n-2}{2}\right)^2 + (n-1) = \frac{n^2}{4}$.

In Section 2, we prove a density lemma, which enables us to prove the main theorems for functions in $C_0^\infty(\mathbb{R}^n \backslash \{0\})$. In Section 3, we prove the main results stated above. Furthermore, we also include a justification of the observation that the constant $\frac{n^2}{4}$ in the improved Hardy inequality (4.8) is best possible and thus we establish the nonexistence of nontrivial extremizers for the improved Hardy inequality.

5. Proofs of the main theorems

The following proposition is important in the proofs of the main theorems:

Proposition 10. *$P^\perp(C_0^\infty(\mathbb{R}^n\backslash\{0\}))$ is dense in $P^\perp(H^1(\mathbb{R}^n)) = (H^1_{\mathrm{rad}}(\mathbb{R}^n))^\perp$ if $n \geq 2$.*

In this section, we prove Theorems 4, 6, 8 and their corollaries. By a density argument based on Proposition 10, it suffices to prove the theorems for functions in $C_0^\infty(\mathbb{R}^n\backslash\{0\})$. In the proofs below, all functions are supposedly elements of $C_0^\infty(\mathbb{R}^n\backslash\{0\})$.

Proof of Theorem 4. Let $f \in C_0^\infty(\mathbb{R}^n\backslash\{0\})$. By Propositions 3 and **??**, the first term on the right hand side of (4.1) is represented as

$$\begin{aligned}
\|LP^\perp f\|_2^2 &= \sum_{j=1}^n \|L_j P^\perp f\|_2^2 \\
&= -\sum_{j=1}^n (L_j^2 P^\perp f | P^\perp f) \\
&= -\sum_{j=1}^n \sum_{k=0}^\infty (L_j^2 P^\perp P_k f | P^\perp f) \\
&= -\sum_{j=1}^n \sum_{k=1}^\infty (L_j^2 P^\perp P_k f | P^\perp f) \\
&= -\sum_{k=1}^\infty (|x|^{-2} \Delta_{S^{n-1}} P_k P^\perp f | P^\perp f)
\end{aligned}$$

and hence

$$\begin{aligned}
\|LP^\perp f\|_2^2 &= -\sum_{k=1}^\infty \int_0^\infty (\Delta_{S^{n-1}} P_k P^\perp f)(r(\cdot) | P^\perp f(r(\cdot)))_{L^2(S^{n-1})} r^{n-3} dr \\
&= \sum_{k=1}^\infty k(k+n-2) \int_0^\infty (P_k P^\perp f)(r(\cdot) | P^\perp f(r(\cdot)))_{L^2(S^{n-1})} r^{n-3} dr \\
&= \sum_{k=1}^\infty k(k+n-2) \left\| \frac{1}{|x|} P_k P^\perp f \right\|_2^2 \\
&= (n-1) \sum_{k=1}^\infty \left\| \frac{1}{|x|} P_k P^\perp f \right\|_2^2 + \sum_{k=1}^\infty (k(k+n-2) - (n-1)) \left\| \frac{1}{|x|} P_k P^\perp f \right\|_2^2 \\
&= (n-1) \sum_{k=0}^\infty \left\| P_k \left(\frac{P^\perp f}{|x|} \right) \right\|_2^2 + \sum_{k=2}^\infty (k-1)(k+n-1) \left\| \frac{1}{|x|} P_k P^\perp f \right\|_2^2 \\
&= (n-1) \left\| \frac{P^\perp f}{|x|} \right\|_2^2 + \sum_{k=2}^\infty (k-1)(k+n-1) \left\| \frac{P_k f}{|x|} \right\|_2^2,
\end{aligned}$$

where we have used relations

$$
\begin{aligned}
P_0 P^{\perp} &= P^{\perp} P_0 = 0, \\
P_k P^{\perp} &= P_k(I - P_0) = P_k \quad \text{for } k \geq 1, \\
P_k\left(\frac{g}{|x|}\right) &= \frac{1}{|x|} P_k g \quad \text{for } k \geq 0.
\end{aligned}
$$

This completes the proof. □

Proof of Corollary 5. The inequality (4.2) is a direct consequence of (4.1). The equality in (4.2) holds if and only if $\frac{P_k f}{|x|} = 0$ for all nonnegative integers k with $k \geq 2$, namely, $\frac{f}{|x|} \in \mathcal{H}_0 \oplus \mathcal{H}_1$. This proves (4.3). Then we take $g, h \in H^1_{\mathrm{rad}}(\mathbb{R}^n)$ as in (4.3). In this case, we have

$$
\begin{aligned}
(P^{\perp} f)(x) &= (\alpha \cdot x) g(x), \\
(L P^{\perp} f)(x) &= \left(\alpha - \frac{x}{|x|}\left(\alpha \cdot \frac{x}{|x|}\right)\right) g(x)
\end{aligned}
$$

for almost all $x \in \mathbb{R}^n \backslash \{0\}$. Since g is radial and $\frac{P^{\perp} f}{|x|} \in \mathcal{H}_1$, a new function $u \in L^2(0, \infty)$ is defined to satisfy $g(x) = u(|x|)|x|^{-\frac{n-1}{2}}$ for almost all $x \in \mathbb{R}^n \backslash \{0\}$. We evaluate two integrals in (4.2) as

$$
\begin{aligned}
\left\|\frac{P^{\perp} f}{|x|}\right\|_2^2 &= \int_0^{\infty} \int_{S^{n-1}} |\alpha \cdot \omega|^2 |g(r\omega)|^2 d\sigma(\omega) r^{n-1} dr \\
&= \int_{S^{n-1}} |\alpha \cdot \omega|^2 d\sigma(\omega) \int_0^{\infty} |u(r)|^2 dr \\
&= \frac{\sigma_{n-1}}{n} |\alpha|^2 \int_0^{\infty} |u(r)|^2 dr,
\end{aligned}
$$

$$
\begin{aligned}
\|L P^{\perp} f\|_2^2 &= \int_0^{\infty} \int_{S^{n-1}} |\alpha - \omega(\alpha \cdot \omega)|^2 d\sigma(\omega) |u(r)|^2 dr \\
&= \int_{S^{n-1}} (|\alpha|^2 - |\alpha \cdot \omega|^2) d\sigma(\omega) \int_0^{\infty} |u(r)|^2 dr \\
&= \left(1 - \frac{1}{n}\right) \sigma_{n-1} |\alpha|^2 \int_0^{\infty} |u(r)|^2 dr,
\end{aligned}
$$

where we have used

$$
\begin{aligned}
\int_{S^{n-1}} |\alpha \cdot \omega|^2 d\sigma(\omega) &= \sum_{j=1}^{n} \sum_{k=1}^{n} \alpha_j \bar{\alpha}_k \int_{S^{n-1}} \omega_j \omega_k d\sigma(\omega) \\
&= \sum_{j=1}^{n} |\alpha_j|^2 \int_{S^{n-1}} \omega_j^2 d\sigma(\omega) \\
&= \sum_{j=1}^{n} |\alpha_j|^2 \frac{1}{n} \sigma_{n-1} = \frac{1}{n} \sigma_{n-1} |\alpha|^2.
\end{aligned}
$$

This proves (4.4). □

Proof of Theorem 6. The equality (4.5) follows from (**??**), (1.2), (4.1), and $\|LP^{\perp}f\|_2 = \|Lf\|_2$. □

Proof of Corollary 7. The equality in (4.6) holds if and only if (4.3) and

$$|x|^{1-\frac{n}{2}}\partial_r(|x|^{\frac{n}{2}-1}f) = \left(\partial_r + \frac{n-2}{2|x|}\right) f = 0.$$

Then f is written as $f(x) = |x|^{1-\frac{n}{2}}\psi\left(\frac{x}{|x|}\right)$ for some function $\psi : S^{n-1} \to \mathbb{C}$, which together with (4.3) implies that $f(x) = |x|^{1-\frac{n}{2}}\left(\alpha \cdot \frac{x}{|x|}\right)$ for some $\alpha \in \mathbb{C}^n$. In this case, $\frac{f}{|x|} \in L^2(\mathbb{R}^n)$ if and only if $\alpha = 0$, which means $f = 0$. □

Proof of Theorem 8. The equality (4.7) follows by substituting f by $P^{\perp}f$ in (4.5). □

Proof of Corollary 9. The equality (4.8) follows if and only if $P^{\perp}f = 0$, which means $f \in H^1_{\mathrm{rad}}(\mathbb{R}^n)$. □

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