Canonical Liftings of Level Two of Tetrapods in Characteristic Three

HOSHI, Yuichiro

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By

Yuichiro HOSHI

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京都大学 数理解析研究所

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES

KYOTO UNIVERSITY, Kyoto, Japan

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Abstract. — In the present paper, we give concrete descriptions of the canonical liftings of level two of tetrapods in characteristic three.

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Introduction

In the present paper, we study the theory of *hyperbolically ordinary curves* established in [5]. Let p be an odd prime number, and let k be an algebraically closed field of characteristic p. Let us first recall that we shall say that a hyperbolic curve over k is *hyperbolically ordinary* [cf. [5, Chapter II, Definition 3.3]] if the hyperbolic curve admits a nilpotent [cf. [5, Chapter II, Definition 2.4]] ordinary [cf. [5, Chapter II, Definition 3.1]] indigenous bundle [cf. [5, Chapter I, Definition 2.2]]. It was proved that every hyperbolic curve of type $(0,3)$ over k is hyperbolically ordinary [cf. [5, Chapter II, Proposition 3.7]]. It was also proved that every hyperbolic curve of type $(1,1)$ over k in the case where $p=5$ is hyperbolically ordinary [cf. [6, Chapter IV, §1.3]]. One main result of the theory of hyperbolically ordinary curves is that a "sufficiently general" hyperbolic curve of type (g,r) over k is hyperbolically ordinary [cf. [5, Chapter II, Corollary 3.8]]. The following [weaker version of the] basic question in p-adic Teichmüller theory is discussed in [6, Introduction, §2.1, (1)]:

Is every hyperbolic curve over k hyperbolically ordinary?

The author of the present paper has studied this basic question. In particular, the author of the present paper proved the following two assertions that are closely related to this basic question:

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• If the inclusion

$$(g, r, p) \in \{(0,4,3), (1,1,3), (1,1,7), (2,0,3)\}$$

holds, then every hyperbolic curve of type (g, r) over k is hyperbolically ordinary [cf. [2, Theorem D], [3, Theorem C]].

• If $p = 3$, and $g \leq 5$, then every hyperelliptic hyperbolic curve of type $(g, 0)$ over k is hyperbolically ordinary [cf. [4, Theorem A]].

Write W_∞ for the ring of Witt vectors over k and

$$W \stackrel{\text{def}}{=} W_\infty / p^2 W_\infty$$

for the ring of truncated Witt vectors of length two over k. One significance of the notion of a hyperbolically ordinary curve, which was discussed in the preceding paragraph, is the following interesting consequence of the theory of hyperbolically ordinary curves established in [5]:

(a) A nilpotent ordinary indigenous bundle on a hyperbolic curve over k determines a canonical lifting over W_∞ of the given hyperbolic curve [cf. [5, Chapter III, §3, Canonical Liftings of Curves over Witt Vectors], [5, Chapter IV, §1, The Canonical Frobenius Lifting over the Ordinary Locus]].

Moreover, the theory also concludes that

(b) a nilpotent admissible [cf. [5, Chapter II, Definition 2.4]] indigenous bundle on a hyperbolic curve over k determines a canonical lifting over W of the given hyperbolic curve [cf. [5, Chapter II, §2, The p-Curvature of an Admissible Indigenous Bundle]].

Note that every nilpotent ordinary indigenous bundle is a nilpotent admissible indigenous bundle [cf. [5, Chapter II, Proposition 3.2]]; moreover,

(c) the base-change by the natural surjective homomorphism $W_\infty \twoheadrightarrow W$ of the canonical lifting over W_∞ associated to a nilpotent ordinary indigenous bundle discussed in (a) is no other than the canonical lifting over W associated to the nilpotent ordinary indigenous bundle discussed in (b) [cf. [5, Chapter IV, §1, The Canonical Frobenius Lifting over the Ordinary Locus]].

There are few concrete examples of canonical liftings of hyperbolic curves. *L. R. A. Finotti* established some explicit examples of canonical liftings over W of hyperbolic curves of type $(2, 0)$ over k in the case where $p = 3$ [cf. [1, Theorem 2.8, (3)]]. In the present paper, we discuss the case of *tetrapods in characteristic three*. More precisely, the main purpose of the present paper is to give a complete list of the canonical liftings over W of hyperbolic curves of type $(0, 4)$ over k in the case where $p = 3$.

In the present Introduction, let

$$\lambda \in W$$

be such that both λ and $1 - \lambda$ are invertible in W. Write

$$\overline{\lambda} \in k$$

for the image of $\lambda \in W$ in k and

$$U \stackrel{\text{def}}{=} \operatorname{Spec}\Big(W[t, s_\lambda]/\big(s_\lambda t(1-t)(\lambda - t) - 1\big)\Big)$$

— where t and s_λ are indeterminates. Write, moreover,

• X for the projective smooth curve over W obtained by forming the smooth compactification of the smooth curve U over W and

• $D \subseteq X$ for the closed subscheme of X obtained by forming the disjoint union of the closed subscheme defined by the equality $t^{-1} = 0$ and the closed subscheme defined by the equality $s_\lambda^{-1} = 0$.

In particular, the pair

$$(X, D)$$

forms a hyperbolic curve of type $(0, 4)$ over W, which thus implies that the pair

$$(X_k \stackrel{\text{def}}{=} X \times_W k, D_k \stackrel{\text{def}}{=} D \times_W k)$$

forms a hyperbolic curve of type $(0, 4)$ over k. For each $a \in k$, write

$$[a] \in W$$

for the Teichmüller lift of $a \in k$. Then the complete list of the canonical liftings over W of the hyperbolic curve (X_k, D_k) of type $(0, 4)$ over k in the case where $p = 3$ is given as follows [cf. Theorem 4.4]:

Theorem A. — *Suppose that the equality $p = 3$ holds. Then the following two conditions are equivalent:*

(1) *The hyperbolic curve (X, D) over W is a canonical lifting of the hyperbolic curve (X_k, D_k) over k.*

(2) *One of the following three equalities holds:*

$$\lambda = [\overline{\lambda}] + 3 \cdot \left[\overline{\lambda}^2(\overline{\lambda} + 1)(\overline{\lambda} - 1)\right]^{1/3},$$

$$\lambda = [\overline{\lambda}] - 3 \cdot \left[\overline{\lambda}^3(\overline{\lambda} - 1)\right]^{1/3}, \qquad \lambda = [\overline{\lambda}] - 3 \cdot \left[\overline{\lambda}^2(\overline{\lambda} - 1)\right]^{1/3}.$$

One interesting consequence of Theorem A is as follows [cf. Corollary 4.5]:

Theorem B. — *Suppose that the equality $p = 3$ holds, and that the hyperbolic curve (X, D) over W is a canonical lifting of the hyperbolic curve (X_k, D_k) over k. Then the following three conditions are equivalent:*

(1) *The element $\lambda \in W$ coincides with the Teichmüller lift of some element of k.*

(2) *The equality $\overline{\lambda} = -1$ in k holds.*

(3) *The equality $\lambda = -1$ in W holds.*

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1. Notational Conventions

In the present §1, we introduce some notational conventions applied in the present paper.

1.a. — Throughout the present paper, let p be an odd prime number, and let k be an algebraically closed field of characteristic p. We shall write

$$W \stackrel{\text{def}}{=} W_2(k)$$

for the *ring of truncated Witt vectors of length* 2 over k and

$$p^{*} \stackrel{\text{def}}{=} \frac{p-1}{2}.$$

If a is an element of k, then we shall write

$$[a] \in W$$

for the *Teichmüller lift* of $a \in k$. If x is an element of W, then we shall write

$$\overline{x} \in k$$

for the image of $x \in W$ in k,

$$x^F \in W$$

for the image of $x \in W$ by the unique ring automorphism of W that lifts the absolute Frobenius automorphism of k, and

$$\epsilon(x) \in k$$

for the unique element of k such that

$$x = [\overline{x}] + p[\epsilon(x)].$$

1.b. — Throughout the present paper, let (g, r) be a pair of nonnegative integers such that $2 - 2g - r < 0$, and let

$$(X, D)$$

be a *hyperbolic curve* of type (g, r) over W, i.e., a pair that consists of a projective smooth connected curve X of genus g over W and a [possibly empty] closed subscheme $D \subseteq X$ of X that is étale and of degree r over W. We shall write

$$X^F, \qquad D^F$$

for the respective base-changes of X, D by the unique ring automorphism of W that lifts the absolute Frobenius automorphism of k, and

$$X_k \stackrel{\text{def}}{=} X \times_W k, \qquad D_k \stackrel{\text{def}}{=} D \times_W k, \qquad X_k^F \stackrel{\text{def}}{=} X^F \times_W k, \qquad D_k^F \stackrel{\text{def}}{=} D^F \times_W k$$

for the respective base-changes of X, D, X^F, D^F by the natural quotient ring homomorphism $W \twoheadrightarrow k$. Thus, the pairs

$$(X^F, D^F), \qquad (X_k, D_k), \qquad (X_k^F, D_k^F)$$

are *hyperbolic curves* of type (g, r) over W, k, k, respectively. We shall write

$$X^{\log}, \qquad (X^F)^{\log}, \qquad X_k^{\log}, \qquad (X_k^F)^{\log}$$

for the respective log schemes obtained by equipping X, X^F, X_k, X_k^F with the log structures determined by the divisors [determined by the closed subschemes] $D \subseteq X$, $D^F \subseteq X^F$, $D_k \subseteq X_k$, $D_k^F \subseteq X_k^F$ with normal crossings.

1.c. — We shall define a *tripod* (respectively, *tetrapod*) to be a hyperbolic curve of type $(0,3)$ (respectively, of type $(0,4)$).

1.d. — We shall write

$$\mathcal{O} \stackrel{\mathrm{def}}{=} \mathcal{O}_{X_k}, \qquad \mathcal{O}^F \stackrel{\mathrm{def}}{=} \mathcal{O}_{X_k^F}$$

for the respective *structure sheaves* of the schemes X_k, X_k^F,

$$\omega^{\log}, \qquad (\omega^{\log})^F$$

for the respective *cotangent sheaves* of the log schemes $X_k^{\log}$, $(X_k^F)^{\log}$ over k,

$$\tau^{\log} \stackrel{\mathrm{def}}{=} \mathcal{H}om_{\mathcal{O}}(\omega^{\log}, \mathcal{O}), \qquad (\tau^{\log})^F \stackrel{\mathrm{def}}{=} \mathcal{H}om_{\mathcal{O}^F}((\omega^{\log})^F, \mathcal{O}^F)$$

for the respective *tangent sheaves* of the log schemes $X_k^{\log}$, $(X_k^F)^{\log}$ over k,

$$d\colon \mathcal{O} \longrightarrow \omega^{\log}$$

for the *exterior differentiation operator* with respect to $X_k^{\log}/k$,

$$\Phi\colon X_k \longrightarrow X_k^F$$

for the *relative Frobenius morphism* with respect to X_k/k, and

$$\Phi^{\log}\colon X_k^{\log} \longrightarrow (X_k^F)^{\log}$$

for the *relative Frobenius morphism* with respect to $X_k^{\log}/k$. Observe that if $\mathcal{F}$ is a locally free coherent $\mathcal{O}^F$-module, then one verifies easily that the homomorphism of k-modules

$$d_{\mathcal{F}}\colon \Phi^*\mathcal{F} = \mathcal{O} \otimes_{\Phi^{-1}\mathcal{O}^F} \Phi^{-1}\mathcal{F} \xrightarrow{d \otimes \mathrm{id}} \omega^{\log} \otimes_{\Phi^{-1}\mathcal{O}^F} \Phi^{-1}\mathcal{F} = \omega^{\log} \otimes_{\mathcal{O}} \Phi^*\mathcal{F}$$

is a connection on the $\mathcal{O}$-module $\Phi^*\mathcal{F}$ relative to $X_k^{\log}/k$.

2. Review of Canonical Liftings

In [5], *S. Mochizuki* studied the notion of a *canonical lifting* over W of a hyperbolic curve over k associated to a nilpotent admissible indigenous bundle. In the present §2, let us review some portions of the theory of canonical liftings from the point of view of the present paper.

Let us first recall that it follows immediately from [5, Chapter II, §1, Deformations and FL-Bundles] that there exists a unique extension of $\mathcal{O}$ by $\Phi^*(\tau^{\log})^F$

$$0 \longrightarrow \Phi^*(\tau^{\log})^F \longrightarrow \mathcal{E} \longrightarrow \mathcal{O} \longrightarrow 0$$

that satisfies the following condition:

For each open subscheme $V \subseteq X_k$ of X_k, write

- $\widetilde{V} \subseteq X$ for the open subscheme of X determined by the open subscheme $V \subseteq X_k$ and the natural closed immersion $X_k \hookrightarrow X$ [which is a homeomorphism] and

- $V^{\log}$, $\widetilde{V}^{\log}$ for the respective log schemes obtained by equipping V, $\widetilde{V}$ with the log structures obtained by pulling back the log structures of $X_k^{\log}$, $X^{\log}$ by the natural open immersions $V \hookrightarrow X_k$, $\widetilde{V} \hookrightarrow X$.

Then there exists a(n) [necessarily unique] isomorphism of

- the $\Phi^*(\tau^{\log})^F$-torsor on X_k that assigns, to each open subscheme $V \subseteq X_k$ of X_k, the $\Phi^*(\tau^{\log})^F(V)$-torsor that consists of morphisms $\widetilde{V}^{\log} \to (X^F)^{\log}$ over W that lift the restriction $\Phi^{\log}|_{V^{\log}} \colon V^{\log} \to (X_k^F)^{\log}$ of the relative Frobenius morphism $\Phi^{\log}$ with respect to $X_k^{\log}/k$ with

- the $\Phi^*(\tau^{\log})^F$-torsor on X_k that assigns, to each open subscheme $V \subseteq X_k$ of X_k, the $\Phi^*(\tau^{\log})^F(V)$-torsor that consists of splittings of the extension $0 \to \Phi^*(\tau^{\log})^F|_V \to \mathcal{E}|_V \to \mathcal{O}|_V \to 0$ obtained by restricting the above exact sequence $0 \to \Phi^*(\tau^{\log})^F \to \mathcal{E} \to \mathcal{O} \to 0$ to $V \subseteq X_k$.

Moreover, it follows from the final inclusion of [5, Chapter II, Proposition 1.1] that there exists a unique connection on $\mathcal{E}$ relative to $X_k^{\log}/k$

$$\nabla_{\mathcal{E}} \colon \mathcal{E} \longrightarrow \omega^{\log} \otimes_{\mathcal{O}} \mathcal{E}$$

that fits into an exact sequence of $\mathcal{O}$-modules equipped with connections relative to $X_k^{\log}/k$

$$0 \longrightarrow \left(\Phi^*(\tau^{\log})^F, d_{(\tau^{\log})^F}\right) \longrightarrow (\mathcal{E}, \nabla_{\mathcal{E}}) \longrightarrow (\mathcal{O}, d) \longrightarrow 0$$

[cf. §1.d] whose underlying exact sequence of $\mathcal{O}$-modules is the above exact sequence $0 \to \Phi^*(\tau^{\log})^F \to \mathcal{E} \to \mathcal{O} \to 0$. We shall refer to the pair

$$(\mathcal{E}, \nabla_{\mathcal{E}})$$

as the *FL-bundle* associated to $X^{\log}/W$ [cf. [5, Chapter II, Definition 1.3]].

DEFINITION 2.1 (cf. [5, Chapter II, §2, The p-Curvature of an Admissible Indigenous Bundle]; also [5, Chapter IV, §1, The Canonical Frobenius Lifting over the Ordinary Locus])**.**

(i) We shall say that the hyperbolic curve (X, D) over W is a *canonical lifting* of the hyperbolic curve (X_k, D_k) over k if the projectivization of $(\mathcal{E}, \nabla_{\mathcal{E}})$ is an indigenous bundle [cf. [5, Chapter I, Definition 2.2]] on $X_k^{\log}/k$. Here, let us recall from [5, Chapter II, Proposition 1.4] that, in this situation, the resulting indigenous bundle is necessarily nilpotent admissible [cf. [5, Chapter II, Definition 2.4]].

(ii) Suppose that the hyperbolic curve (X, D) over W is a canonical lifting of the hyperbolic curve (X_k, D_k) over k. Then we shall write

$$X_k^{\mathrm{ord}} \subseteq X_k$$

for the open subscheme of X_k obtained by forming the complement in X_k of the [support of the] supersingular divisor [cf. [5, Chapter II, Proposition 2.6, (3)]] of the nilpotent admissible indigenous bundle obtained by forming the projectivization of $(\mathcal{E}, \nabla_{\mathcal{E}})$,

$$X^{\mathrm{ord}} \subseteq X$$

for the open subscheme of X determined by the open subscheme $X_k^{\mathrm{ord}} \subseteq X_k$ and the natural closed immersion $X_k \hookrightarrow X$ [which is a homeomorphism], and

$$(X_k^{\mathrm{ord}})^{\log}, \qquad (X^{\mathrm{ord}})^{\log}$$

for the respective log schemes obtained by equipping X_k^{ord}, X^{ord} with the log structures obtained by pulling back the log structures of $X_k^{\log}$, $X^{\log}$ by the natural open immersions $X_k^{\mathrm{ord}} \hookrightarrow X_k$, $X^{\mathrm{ord}} \hookrightarrow X$. We shall refer to X_k^{ord}, X^{ord}, $(X_k^{\mathrm{ord}})^{\log}$, $(X^{\mathrm{ord}})^{\log}$ as the *ordinary loci* of X_k, X, $X_k^{\log}$, $X^{\log}$, respectively.

(iii) Suppose that the hyperbolic curve (X, D) over W is a canonical lifting of the hyperbolic curve (X_k, D_k) over k. Thus, the Hodge section [cf. [5, Chapter I, Proposition 2.4]] of the nilpotent admissible indigenous bundle obtained by forming the projectivization of $(\mathcal{E}, \nabla_{\mathcal{E}})$ determines, relative to the unique isomorphism of $\Phi^*(\tau^{\log})^F$-torsors that appears in the characterization of the extension $\mathcal{E}$, a lifting

$$(X^{\mathrm{ord}})^{\log} \longrightarrow (X^F)^{\log}$$

of the restriction $\Phi^{\log}|_{(X_k^{\mathrm{ord}})^{\log}} \colon (X_k^{\mathrm{ord}})^{\log} \to (X_k^F)^{\log}$ of the relative Frobenius morphism $\Phi^{\log}$ with respect to $X_k^{\log}/k$. Then we shall refer to this lifting $(X^{\mathrm{ord}})^{\log} \to (X^F)^{\log}$ of $\Phi^{\log}|_{(X_k^{\mathrm{ord}})^{\log}}$ as the *canonical Frobenius lifting* associated to the canonical lifting (X, D) over W.

PROPOSITION 2.2. — *The following two conditions are equivalent:*

(1) *There exists a reduced closed subscheme $E \subseteq X_k \setminus D_k$ of $X_k \setminus D_k$ of degree $\leq p^{*}(2g-2+r)$* [*cf.* §1.a] *over k that is liftable with respect to $(\mathcal{E}, \nabla_{\mathcal{E}})$* [*cf.* [3, Definition 3.4]], *i.e., such that the base-change $\mathcal{E}|_{\mathcal{O}(-E)} \twoheadrightarrow \mathcal{O}(-E)$ of the surjective homomorphism $\mathcal{E} \twoheadrightarrow \mathcal{O}$ by the natural inclusion $\mathcal{O}(-E) \hookrightarrow \mathcal{O}$ admits a splitting.*

(2) *The hyperbolic curve (X, D) over W is a canonical lifting of the hyperbolic curve (X_k, D_k) over k, i.e., the projectivization of $(\mathcal{E}, \nabla_{\mathcal{E}})$ is a nilpotent admissible indigenous bundle on $X_k^{\log}/k$.*

If, moreover, these two conditions are satisfied, then

(a) *the divisor* [*determined by the closed subscheme*] *$E \subseteq X_k$ of* (1) *is of degree $p^{*}(2g-2+r)$ and coincides with the supersingular divisor of the nilpotent admissible indigenous bundle on $X_k^{\log}/k$ obtained by forming the projectivization of $(\mathcal{E}, \nabla_{\mathcal{E}})$* [*cf.* (2)],

(b) *the splitting of the base-change $\mathcal{E}|_{\mathcal{O}(-E)} \twoheadrightarrow \mathcal{O}(-E)$ of* (1) *is unique,*

(c) *the cokernel of the composite $\mathcal{O}(-E) \to \mathcal{E}$ of the unique* [*cf.* (b)] *splitting of the base-change $\mathcal{E}|_{\mathcal{O}(-E)} \twoheadrightarrow \mathcal{O}(-E)$ of* (1) *with the natural inclusion $\mathcal{E}|_{\mathcal{O}(-E)} \hookrightarrow \mathcal{E}$ is an invertible sheaf on X_k, and*

(d) *the Hodge section of the nilpotent admissible indigenous bundle on $X_k^{\log}/k$ obtained by forming the projectivization of $(\mathcal{E}, \nabla_{\mathcal{E}})$* [*cf.* (2)] *is given by the section determined by the composite $\mathcal{O}(-E) \to \mathcal{E}$ discussed in* (c).

PROOF. — The equivalence (1) $\Leftrightarrow$ (2) follows from [3, Proposition 3.7], together with [3, Proposition 3.2, (ii), (iii)]. Next, to verify the final portion, suppose that conditions (1), (2) are satisfied. Then it follows from [3, Lemma 3.6] and the final portion of [3, Proposition 3.7] that condition (a) is satisfied. Moreover, since the invertible sheaf $\Phi^*(\tau^{\log})^F \otimes_{\mathcal{O}} \mathcal{O}(E)$ on X_k is of degree $-p^*(2g-2+r) < 0$, condition (b) is satisfied. Next, observe that it follows immediately from [3, Lemma 3.6] that condition (c) is satisfied. Finally, it follows immediately from the proof of the implication (2) $\Rightarrow$ (1) of [3, Proposition 3.7] that condition (d) is satisfied. This completes the proof of Proposition 2.2. $\square$

THEOREM 2.3. — *Suppose that $r \geq 3$. Write*

$$\lambda_{r-2} \stackrel{\text{def}}{=} 0 \in W, \quad \lambda_{r-1} \stackrel{\text{def}}{=} 1 \in W.$$

Let

$$\lambda_1, \ldots, \lambda_{r-3} \in W$$

be $r-3$ elements of W such that

$$\#\{\overline{\lambda}_1, \ldots, \overline{\lambda}_{r-3}, \overline{\lambda}_{r-2}, \overline{\lambda}_{r-1}\} = r - 1$$

[*cf.* §1.a]. *Write*

$$U \stackrel{\text{def}}{=} \mathrm{Spec}\Big(W[t, s_\lambda]/\big(s_\lambda(\lambda_1 - t)\cdots(\lambda_{r-1} - t) - 1\big)\Big)$$

$$\hookleftarrow U_k \stackrel{\text{def}}{=} \mathrm{Spec}\Big(k[t, s_\lambda]/\big(s_\lambda(\overline{\lambda}_1 - t)\cdots(\overline{\lambda}_{r-1} - t) - 1\big)\Big)$$

— *where t and s_λ are indeterminates. Suppose that*

- *the projective smooth curve X over W is given by the smooth compactification of the smooth curve U over W, and that*
- *the closed subscheme $D \subseteq X$ of X is given by the closed subscheme of X obtained by forming the disjoint union of the closed subscheme defined by the equality $t^{-1} = 0$ and the closed subscheme defined by the equality $s_\lambda^{-1} = 0$.*

In particular, the pair (X, D) is a hyperbolic curve of type $(0, r)$ over W. Let

$$e_1, \ldots, e_{p^*(r-2)} \in W$$

be $p^*(r-2)$ elements of W such that

$$\#\{\overline{\lambda}_1, \ldots, \overline{\lambda}_{r-3}, \overline{\lambda}_{r-2}, \overline{\lambda}_{r-1}, \overline{e}_1, \ldots, \overline{e}_{p^*(r-2)}\} = r - 1 + p^*(r-2),$$

and let

$$G(t) = \sum_{i=0}^{p^*(r-2)+p} c_i t^i \in W[t]$$

be a polynomial such that $c_{p^*(r-2)+p} \in W$ is invertible. Write

$$F(t, s_e) \stackrel{\text{def}}{=} s_e \cdot G(t) \in W[t, s_e]/\big(s_e(e_1 - t)\cdots(e_{p^*(r-2)} - t) - 1\big),$$

i.e., roughly speaking,

$$\text{“}F(t, s_e) = \frac{G(t)}{\prod_{i=1}^{p^*(r-2)}(e_i - t)}\text{”},$$

and

$$E \subseteq U_k$$

for the closed subscheme of U_k *defined by the polynomial*

$$\prod_{i=1}^{p^*(r-2)} (\overline{e}_i - t) \in k[t, s_\lambda]/\big(s_\lambda(\overline{\lambda}_1 - t) \cdots (\overline{\lambda}_{r-1} - t) - 1\big).$$

Suppose, moreover, that the following three conditions are satisfied:

(1) *The inclusion*

$$F(t, s_e) - t^p \in p \cdot W[t, s_e]/\big(s_e(e_1 - t) \cdots (e_{p^*(r-2)} - t) - 1\big)$$

holds.

(2) *For each* $i \in \{1, \dots, r-1\}$*, the equality*

$$F\big(\lambda_i, \prod_{j=1}^{p^*(r-2)} (e_j - \lambda_i)^{-1}\big) = \lambda_i^F$$

[*cf.* §1.a] *holds.*

(3) *There exists a unit* $u \in W^\times$ *such that the equality*

$$dF(t, s_e) = ups_e^2 \cdot \prod_{i=1}^{r-1} (\lambda_i - t)^{p-1} \cdot dt$$

holds.

Then the following assertions hold:

(i) *The hyperbolic curve* (X, D) *over* W *is a canonical lifting of the hyperbolic curve* (X_k, D_k) *over* k*, i.e., the projectivization of* $(\mathcal{E}, \nabla_{\mathcal{E}})$ *is a nilpotent admissible indigenous bundle on* $X_k^{\log}/k$*.*

(ii) *The supersingular divisor of the nilpotent admissible indigenous bundle on* $X_k^{\log}/k$ *obtained by forming the projectivization of* $(\mathcal{E}, \nabla_{\mathcal{E}})$ [*cf.* (i)] *is given by the divisor* [*determined by the closed subscheme*] $E \subseteq (U_k \subseteq)\ X_k$*.*

(iii) *The canonical Frobenius lifting associated to the canonical lifting* (X, D) *over* W [*cf.* (i)] *is given by the morphism*

$$(X^{\mathrm{ord}})^{\log} \longrightarrow (X^F)^{\log}$$

determined by $t \mapsto F(t, s_e)$ [*cf.* (ii)].

Proof. — For each $i \in \{1, \dots, r-1\}$, write A_{λ_i}, $\overline{A}_{\lambda_i}$ for the respective localizations of $W[t]$, $k[t]$ at the maximal ideals generated by [the images of] p and $\lambda_i - t$. We begin the proof of Theorem 2.3 with the following claim:

> **Claim 2.3.A.** — Let i be an element of $\{1, \dots, r-1\}$. Then [the image in A_{λ_i} of] $\lambda_i^F - F(t, s_e)$ may be written as the product of $(\lambda_i - t)^p$ and a unit of A_{λ_i}.

To verify Claim 2.3.A, let us first observe that it follows immediately from condition (1) that the inclusion

$$F(t, s_e) - \lambda_i^F + (\lambda_i - t)^p \in p \cdot W[t, s_e]/\big(s_e(e_1 - t)\cdots(e_{p^*(r-2)} - t) - 1\big)$$

holds. In particular, there exists a unique element

$$f(t, s_e) \in k[t, s_e]/\big(s_e(\overline{e}_1 - t)\cdots(\overline{e}_{p^*(r-2)} - t) - 1\big)$$

such that the equality

$$F(t, s_e) = \lambda_i^F - (\lambda_i - t)^p + p[f(t, s_e)]$$

holds — where

$$[f(t, s_e)] \in W[t, s_e]/\big(s_e(e_1 - t)\cdots(e_{p^*(r-2)} - t) - 1\big)$$

is a lifting of $f(t, s_e) \in k[t, s_e]/(s_e(\overline{e}_1 - t)\cdots(\overline{e}_{p^*(r-2)} - t) - 1)$. Thus, it follows from condition (2) that

$$\lambda_i^F = F(\lambda_i, \prod_{j=1}^{p^*(r-2)} (e_j - \lambda_i)^{-1}) = \lambda_i^F + p[f(\overline{\lambda}_i, \prod_{j=1}^{p^*(r-2)} (\overline{e}_j - \overline{\lambda}_i)^{-1})]$$

— where $[f(\overline{\lambda}_i, \prod_{j=1}^{p^*(r-2)} (\overline{e}_j - \overline{\lambda}_i)^{-1})] \in W$ is a lifting of $f(\overline{\lambda}_i, \prod_{j=1}^{p^*(r-2)} (\overline{e}_j - \overline{\lambda}_i)^{-1}) \in k$ — which thus implies that

$$f(\overline{\lambda}_i, \prod_{j=1}^{p^*(r-2)} (\overline{e}_j - \overline{\lambda}_i)^{-1}) = 0.$$

Next, observe that it follows from condition (3) that

$$p(\lambda_i - t)^{p-1}dt + p \cdot d[f(t, s_e)] = dF(t, s_e) = ups_e^2 \cdot \prod_{i=1}^{r-1} (\lambda_i - t)^{p-1} \cdot dt,$$

which thus implies that

$$(\overline{\lambda}_i - t)^{p-1}dt + df(t, s_e) - \overline{u}s_e^2 \cdot \prod_{i=1}^{r-1} (\overline{\lambda}_i - t)^{p-1} \cdot dt = 0.$$

In particular, one concludes that [the image in $\overline{A}_{\lambda_i}$ of] df/dt is divisible by $(\overline{\lambda}_i - t)^{p-1}$. In particular, it follows from the equality $f(\overline{\lambda}_i, \prod_{j=1}^{p^*(r-2)} (\overline{e}_j - \overline{\lambda}_i)^{-1}) = 0$ [i.e., verified in the preceding paragraph] that [the image in $\overline{A}_{\lambda_i}$ of] $f(t, s_e)$ is divisible by $(\overline{\lambda}_i - t)^p$. Let $g(t) \in \overline{A}_{\lambda_i}$ be such that

$$f(t, s_e) = (\overline{\lambda}_i - t)^p g(t).$$

Thus, it follows that

$$\lambda_i^F - F(t, s_e) = (\lambda_i - t)^p - p(\lambda_i - t)^p[g(t)] = (\lambda_i - t)^p\big(1 - p[g(t)]\big)$$

— where $[g(t)] \in A_{\lambda_i}$ is a lifting of $g(t) \in \overline{A}_{\lambda_i}$. This completes the proof of Claim 2.3.A.

Next, write $Y \subseteq X$ for the open subscheme of X determined by the open subscheme $X_k \setminus E \subseteq X_k$ and the natural closed immersion $X_k \hookrightarrow X$ [which is a homeomorphism] and $(X_k \setminus E)^{\log}$, $Y^{\log}$ for the respective log schemes obtained by equipping $X_k \setminus E$, Y with the log structures obtained by pulling back the log structures of $X_k^{\log}$, $X^{\log}$ by the natural open immersions $X_k \setminus E \hookrightarrow X_k$, $Y \hookrightarrow X$. Then observe that it follows from condition (1) that the morphism

$$Y \longrightarrow X^F$$

determined by $t \mapsto F(t, s_e)$ is a lifting of the restriction $\Phi|_{X_k \setminus E} \colon X_k \setminus E \to X_k^F$ of the relative Frobenius morphism Φ with respect to X_k/k. Moreover, it follows immediately from Claim 2.3.A, together with the assumption that $c_{p^*(r-2)+p} \in W$ is invertible [cf. the statement of Theorem 2.3], that this lifting $Y \to X^F$ [necessarily uniquely] determines a lifting

$$\Psi^{\log} \colon Y^{\log} \longrightarrow (X^F)^{\log}$$

of the restriction $\Phi^{\log}|_{(X_k \setminus E)^{\log}} \colon (X_k \setminus E)^{\log} \to (X_k^F)^{\log}$ of the relative Frobenius morphism $\Phi^{\log}$ with respect to $X_k^{\log}/k$.

Next, let us observe that one verifies easily from the definition of $F(t, s_e)$ that, for each closed point $x \in X_k$ of X_k, the local height of this lifting $\Psi^{\log}$ at $x \in X_k$ [cf. the discussion following [5, Chapter IV, Definition 4.7]] is ≤ 1 (respectively, is equal to 0) if x is contained in (respectively, is not contained in) the support of E. Write $E_0 \subseteq X_k$ for the [closed subscheme determined by the] divisor on X_k determined by these local heights of this lifting $\Psi^{\log}$. [In particular, it follows that $E_0 \leq E$.] Thus, it follows immediately from a similar argument to the argument applied in the proof of [5, Chapter IV, Proposition 4.8] that the splitting of the restriction $\mathcal{E}|_{X_k \setminus E} \twoheadrightarrow \mathcal{O}|_{X_k \setminus E}$ of the surjective homomorphism $\mathcal{E} \twoheadrightarrow \mathcal{O}$ to the open subscheme $X_k \setminus E \subseteq X_k$ that corresponds, relative to the unique isomorphism of $\Phi^*(\tau^{\log})^F$-torsors that appears in the characterization of the extension $\mathcal{E}$, to this lifting $\Psi^{\log}$ extends to a splitting of the base-change $\mathcal{E}|_{\mathcal{O}(-E_0)} \twoheadrightarrow \mathcal{O}(-E_0)$ of the surjective homomorphism $\mathcal{E} \twoheadrightarrow \mathcal{O}$ by the natural inclusion $\mathcal{O}(-E_0) \hookrightarrow \mathcal{O}$. In particular, since [it is immediate that] the [necessarily reduced] closed subscheme $E \subseteq X_k$ of X_k is of degree $p^*(r-2)$ over k, Theorem 2.3 follows immediately from Proposition 2.2. This completes the proof of Theorem 2.3. □

3. The Case of Tripods in Characteristic Three and Five

In the present §3, we establish concrete descriptions of canonical Frobenius liftings in the case of *tripods in characteristic three and five* [cf. Theorem 3.3 below and Theorem 3.6 below]. In the present §3, write

$$U \stackrel{\text{def}}{=} \operatorname{Spec}\Big(W[t, s_\lambda]/\big(s_\lambda t(1-t) - 1\big)\Big) \hookleftarrow U_k \stackrel{\text{def}}{=} \operatorname{Spec}\Big(k[t, s_\lambda]/\big(s_\lambda t(1-t) - 1\big)\Big)$$

— where t and s_λ are indeterminates. Suppose that

- the projective smooth curve X over W is given by the smooth compactification of the smooth curve U over W, and that
- the closed subscheme $D \subseteq X$ of X is given by the closed subscheme of X obtained by forming the disjoint union of the closed subscheme defined by the equality $t^{-1} = 0$ and the closed subscheme defined by the equality $s_\lambda^{-1} = 0$.

In particular, the hyperbolic curve (X, D) is a *tripod* over W [cf. §1.c].

DEFINITION 3.1. — If $p = 3$, then we shall write

$$F_{\mathrm{tpd}_{/3}}(t, s_e) \stackrel{\mathrm{def}}{=} s_e(4t^4 + 7t^3) \in W[t, s_e]/\big(s_e(t+1) - 1\big),$$

i.e., roughly speaking,

$$\text{“}F_{\mathrm{tpd}_{/3}}(t, s_e) = \frac{4t^4 + 7t^3}{t+1}\text{”}.$$

LEMMA 3.2. — *If $p = 3$, then the following assertions hold:*

(i) *The inclusion*

$$F_{\mathrm{tpd}_{/3}}(t, s_e) - t^3 \in 3 \cdot W[t, s_e]/\big(s_e(t+1) - 1\big)$$

holds.

(ii) *The equality*

$$\big(F_{\mathrm{tpd}_{/3}}(0, 1^{-1}), F_{\mathrm{tpd}_{/3}}(1, 2^{-1})\big) = (0, 1)$$

holds.

(iii) *The equality*

$$dF_{\mathrm{tpd}_{/3}}(t, s_e) = 3s_e^2 t^2 (t-1)^2 dt$$

holds.

PROOF. — These assertions are immediate. □

THEOREM 3.3. — *Suppose that we are in the situation at the beginning of the present* §3*, and that the equality*

$$p = 3$$

holds. Then the following assertions hold:

(i) *The hyperbolic curve (X, D) over W is a canonical lifting of the hyperbolic curve (X_k, D_k) over k.*

(ii) *Write*

$$E \subseteq U_k$$

for the closed subscheme of U_k defined by the polynomial

$$t + 1 \in k[t, s_\lambda]/\big(s_\lambda t(1-t) - 1\big),$$

$Y \subseteq X$ for the open subscheme of X determined by the open subscheme $X_k \setminus E \subseteq X_k$ and the natural closed immersion $X_k \hookrightarrow X$ [which is a homeomorphism], and $Y^{\log}$ for the log scheme obtained by equipping Y with the log structure obtained by pulling back the log structure of $X^{\log}$ by the natural open immersion $Y \hookrightarrow X$. Then the canonical Frobenius lifting associated to the canonical lifting (X, D) over W [cf. (i)*] is given by the morphism*

$$Y^{\log} \longrightarrow (X^F)^{\log}$$

determined by $t \mapsto F_{\mathrm{tpd}_{/3}}(t, s_e)$.

PROOF. — This assertion is a formal consequence of Theorem 2.3 and Lemma 3.2. □

Definition 3.4. — If $p = 5$, then we shall write

$$F_{\mathrm{tpd}_{/5}}(t, s_e) \stackrel{\mathrm{def}}{=} s_e(11t^7 - t^6 + 16t^5) \in W[t, s_e]/\big(s_e(t^2 - t + 1) - 1\big),$$

i.e., roughly speaking,

$$\text{“}F_{\mathrm{tpd}_{/5}}(t, s_e) = \frac{11t^7 - t^6 + 16t^5}{t^2 - t + 1}\text{”}.$$

Lemma 3.5. — *If $p = 5$, then the following assertions hold:*

(i) *The inclusion*

$$F_{\mathrm{tpd}_{/5}}(t, s_e) - t^5 \in 5 \cdot W[t, s_e]/\big(s_e(t^2 - t + 1) - 1\big)$$

holds.

(ii) *The equality*

$$\big(F_{\mathrm{tpd}_{/5}}(0, 1^{-1}), F_{\mathrm{tpd}_{/5}}(1, 1^{-1})\big) = (0, 1)$$

holds.

(iii) *The equality*

$$dF_{\mathrm{tpd}_{/5}}(t, s_e) = 5s_e^2 t^4 (t-1)^4 dt$$

holds.

Proof. — These assertions are immediate. □

Theorem 3.6. — *Suppose that we are in the situation at the beginning of the present §3, and that the equality*

$$p = 5$$

holds. Then the following assertions hold:

(i) *The hyperbolic curve (X, D) over W is a canonical lifting of the hyperbolic curve (X_k, D_k) over k.*

(ii) *Write*

$$E \subseteq U_k$$

for the closed subscheme of U_k defined by the polynomial

$$t^2 - t + 1 \in k[t, s_\lambda]/\big(s_\lambda t(1-t) - 1\big),$$

$Y \subseteq X$ for the open subscheme of X determined by the open subscheme $X_k \setminus E \subseteq X_k$ and the natural closed immersion $X_k \hookrightarrow X$ [which is a homeomorphism], and $Y^{\log}$ for the log scheme obtained by equipping Y with the log structure obtained by pulling back the log structure of $X^{\log}$ by the natural open immersion $Y \hookrightarrow X$. Then the canonical Frobenius lifting associated to the canonical lifting (X, D) over W [cf. (i)*] is given by the morphism*

$$Y^{\log} \longrightarrow (X^F)^{\log}$$

determined by $t \mapsto F_{\mathrm{tpd}_{/5}}(t, s_e)$.

Proof. — This assertion is a formal consequence of Theorem 2.3 and Lemma 3.5. □

4. The Case of Tetrapods in Characteristic Three

In the present §4, we establish concrete descriptions of canonical liftings in the case of *tetrapods in characteristic three* [cf. Theorem 4.4 below]. In the present §4, let λ be an element of W such that both λ and $1-\lambda$ are invertible in W. Write

$$U \stackrel{\text{def}}{=} \mathrm{Spec}\Big(W[t, s_\lambda]/\big(s_\lambda t(1-t)(\lambda - t) - 1\big)\Big)$$

$$\hookleftarrow U_k \stackrel{\text{def}}{=} \mathrm{Spec}\Big(k[t, s_\lambda]/\big(s_\lambda t(1-t)(\overline{\lambda} - t) - 1\big)\Big)$$

— where t and s_λ are indeterminates. Suppose that

- the prime number p is equal to 3, that
- the projective smooth curve X over W is given by the smooth compactification of the smooth curve U over W, and that
- the closed subscheme $D \subseteq X$ of X is given by the closed subscheme of X obtained by forming the disjoint union of the closed subscheme defined by the equality $t^{-1} = 0$ and the closed subscheme defined by the equality $s_\lambda^{-1} = 0$.

In particular, the hyperbolic curve (X, D) is a *tetrapod* over W [cf. §1.c].

Definition 4.1. We shall write

$$F_1(t, s_e) \stackrel{\text{def}}{=} s_e\big((3\lambda+1)t^5 - 3(\lambda+1)t^4 + (-\lambda+3)t^3\big) \in W[t, s_e]/\big(s_e(t^2-\lambda) - 1\big),$$

i.e., roughly speaking,

$$\text{“}F_1(t, s_e) \stackrel{\text{def}}{=} \frac{(3\lambda+1)t^5 - 3(\lambda+1)t^4 + (-\lambda+3)t^3}{t^2 - \lambda}\text{”}.$$

Lemma 4.2. — *The following assertions hold:*

(i) *The inclusion*

$$F_1(t, s_e) - t^3 \in 3 \cdot W[t, s_e]/\big(s_e(t^2-\lambda) - 1\big)$$

holds.

(ii) *The equality*

$$\big(F_1(0, (-\lambda)^{-1}), F_1(1, (1-\lambda)^{-1})\big) = (0, 1)$$

holds.

(iii) *The following two conditions are equivalent:*

- *The equality [i.e., in W]*

$$F_1(\lambda, (\lambda^2-\lambda)^{-1}) = \lambda^F$$

holds.

- *The equality [i.e., in k]*

$$\epsilon(\lambda)^3 = \overline{\lambda}^2(\overline{\lambda}+1)(\overline{\lambda}-1)$$

[*cf.* §1.a] *holds.*

(iv) *The equality*

$$dF_1(t, s_e) = 3s_e^2 t^2 (t-1)^2 (t-\lambda)^2 dt$$

holds.

PROOF. — Assertions (i), (ii), (iv) are immediate. Finally, we verify assertion (iii). Let us first observe that we have equalities

$$F_1(\lambda, (\lambda^2-\lambda)^{-1}) = (\lambda^2-\lambda)^{-1} \cdot (3\lambda^6 - 2\lambda^5 - 4\lambda^4 + 3\lambda^3) = 3\lambda^4 + \lambda^3 - 3\lambda^2.$$

Thus, since $\lambda = [\overline{\lambda}] + 3[\epsilon(\lambda)]$, which thus implies that

$$3\lambda^2 = 3[\overline{\lambda}]^2, \qquad \lambda^3 = [\overline{\lambda}]^3, \qquad 3\lambda^4 = 3[\overline{\lambda}]^4, \qquad \lambda^F = [\overline{\lambda}]^3 + 3[\epsilon(\lambda)]^3,$$

one verifies easily that the two conditions of assertion (iii) are equivalent. This completes the proof of assertion (iii), hence also of Lemma 4.2. □

LEMMA 4.3. — *The following assertions hold:*

(i) *Write* $\eta \overset{\text{def}}{=} 1 - \lambda$. *Then the equality*

$$\epsilon(\eta)^3 = \overline{\eta}^2(\overline{\eta}+1)(\overline{\eta}-1)$$

holds if and only if the equality

$$\epsilon(\lambda)^3 = -\overline{\lambda}^3(\overline{\lambda}-1)$$

holds.

(ii) *Write* $\eta \overset{\text{def}}{=} \lambda/(\lambda-1)$. *Then the equality*

$$\epsilon(\eta)^3 = \overline{\eta}^2(\overline{\eta}+1)(\overline{\eta}-1)$$

holds if and only if the equality

$$\epsilon(\lambda)^3 = -\overline{\lambda}^2(\overline{\lambda}-1)$$

holds.

PROOF. — These assertions follow immediately from the following well-known [cf., e.g., [7, Examples in p.42]] facts:

- For x, $y \in W$, the equalities

$$\epsilon(x+y)^3 = \epsilon(x)^3 + \epsilon(y)^3 - \overline{x^2y} - \overline{xy^2}, \qquad \epsilon(xy) = \epsilon(x)\cdot\overline{y} + \epsilon(y)\cdot\overline{x}$$

hold.

- For $x \in W^\times$, the equality

$$\overline{x}^2 \cdot \epsilon(x^{-1}) = -\epsilon(x)$$

holds. □

Theorem 4.4. — *Suppose that we are in the situation at the beginning of the present §4. Then the hyperbolic curve (X, D) over W is a canonical lifting of the hyperbolic curve (X_k, D_k) over k if and only if one of the following three conditions is satisfied:*

($\dagger_1$) *The equality* $\epsilon(\lambda)^3 = \overline{\lambda}^2(\overline{\lambda}+1)(\overline{\lambda}-1)$ *holds.*

($\dagger_0$) *The equality* $\epsilon(\lambda)^3 = -\overline{\lambda}^3(\overline{\lambda}-1)$ *holds.*

($\dagger_\infty$) *The equality* $\epsilon(\lambda)^3 = -\overline{\lambda}^2(\overline{\lambda}-1)$ *holds.*

Proof. — First, we verify the sufficiency portion. If condition ($\dagger_1$) is satisfied, then it follows from Theorem 2.3 and Lemma 4.2 that

- the hyperbolic curve (X, D) over W is a canonical lifting of the hyperbolic curve (X_k, D_k) over k, and that
- the supersingular divisor of the nilpotent admissible indigenous bundle on $X_k^{\log}/k$ obtained by forming the projectivization of the FL-bundle associated to $X^{\log}/W$ is given by the divisor [determined by the closed subscheme] defined by the equality $t^2 - \overline{\lambda} = 0$.

Next, let $\Box$ be an element of $\{0, \infty\}$. Suppose that condition ($\dagger_\Box$) is satisfied. Write

$$\sigma_0(t, s_\sigma) \overset{\text{def}}{=} 1 - t \in W[t, s_\sigma], \qquad \sigma_\infty(t, s_\sigma) \overset{\text{def}}{=} -s_\sigma t \in W[t, s_\sigma]/(s_\sigma(1-t)-1),$$

$$\overline{\sigma}_0(t, s_\sigma) \overset{\text{def}}{=} 1 - t \in k[t, s_\sigma], \qquad \overline{\sigma}_\infty(t, s_\sigma) \overset{\text{def}}{=} -s_\sigma t \in k[t, s_\sigma]/(s_\sigma(1-t)-1),$$

$$f_0(t) \overset{\text{def}}{=} t^2 + t + \overline{\lambda} \in k[t], \qquad f_\infty(t) \overset{\text{def}}{=} t^2 + \overline{\lambda} t + \overline{\lambda} \in k[t].$$

Then it follows from the conclusion of the preceding paragraph, together with Lemma 4.3, that

- the hyperbolic curve over W

$$\mathrm{Spec}\Big(W[t, s_\lambda]/\big(s_\lambda t(1-t)(\sigma_\Box(\lambda, (1-\lambda)^{-1}) - t) - 1\big)\Big)$$

is a canonical lifting of the hyperbolic curve (X_k, D_k) over k

$$\mathrm{Spec}\Big(k[t, s_\lambda]/\big(s_\lambda t(1-t)(\overline{\sigma}_\Box(\overline{\lambda}, , (1-\overline{\lambda})^{-1}) - t) - 1\big)\Big),$$

and that

- the supersingular divisor of the nilpotent admissible indigenous bundle on $X_k^{\log}/k$ obtained by forming the projectivization of the FL-bundle associated to this canonical lifting over W is given by the divisor [determined by the closed subscheme] defined by the equality $t^2 - \overline{\sigma}_\Box(\overline{\lambda}, (1-\overline{\lambda})^{-1}) = 0$.

Thus, one verifies immediately, by considering the commutative diagram

$$\begin{array}{ccc} U_k = \mathrm{Spec}\Big(k[t, s_\lambda]/\big(s_\lambda t(1-t)(\overline{\lambda} - t) - 1\big)\Big) & \xrightarrow{\sim} & \mathrm{Spec}\Big(k[t, s_\lambda]/\big(s_\lambda t(1-t)(\overline{\sigma}_\Box(\overline{\lambda}, (1-\overline{\lambda})^{-1}) - t) - 1\big)\Big) \\ \downarrow & & \downarrow \\ U = \mathrm{Spec}\Big(W[t, s_\lambda]/\big(s_\lambda t(1-t)(\lambda - t) - 1\big)\Big) & \xrightarrow{\sim} & \mathrm{Spec}\Big(W[t, s_\lambda]/\big(s_\lambda t(1-t)(\sigma_\Box(\lambda, (1-\lambda)^{-1}) - t) - 1\big)\Big) \end{array}$$

— where the upper, lower horizontal arrows are isomorphisms over k, W determined by $\overline{\sigma}_{\Box}(t, s_\sigma)$, $\sigma_{\Box}(t, s_\sigma)$, respectively, and the vertical arrows are the natural closed immersions — that

- the hyperbolic curve (X, D) over W is a canonical lifting of the hyperbolic curve (X_k, D_k) over k, and that
- the supersingular divisor of the nilpotent admissible indigenous bundle on $X_k^{\log}/k$ obtained by forming the projectivization of the FL-bundle associated to $X^{\log}/W$ is given by the divisor [determined by the closed subscheme] defined by the equality $f_{\Box}(t) = 0$.

This completes the proof of the sufficiency portion.

Next, we verify the necessity portion. Suppose that the hyperbolic curve (X, D) over W is a canonical lifting of the hyperbolic curve (X_k, D_k) over k, i.e., that the projectivization $\mathcal{P}_{(X,D)}$ of the FL-bundle associated to $X^{\log}/W$ is a nilpotent admissible indigenous bundle on $X_k^{\log}/k$. Write

$$\lambda_1 \stackrel{\text{def}}{=} [\overline{\lambda}] + 3 \cdot \left[\overline{\lambda}^2(\overline{\lambda} + 1)(\overline{\lambda} - 1)\right]^{1/3},$$

$$\lambda_0 \stackrel{\text{def}}{=} [\overline{\lambda}] - 3 \cdot \left[\overline{\lambda}^3(\overline{\lambda} - 1)\right]^{1/3}, \qquad \lambda_\infty \stackrel{\text{def}}{=} [\overline{\lambda}] - 3 \cdot \left[\overline{\lambda}^2(\overline{\lambda} - 1)\right]^{1/3}.$$

Write, moreover, $\mathcal{P}_1$, $\mathcal{P}_0$, $\mathcal{P}_\infty$ for the nilpotent admissible indigenous bundles on $X_k^{\log}/k$ obtained by forming the projectivizations of the FL-bundles associated to the canonical liftings over W in the case where we take the "λ" to be λ_1, λ_0, λ_∞, respectively [cf. the sufficiency portion already proved]. Thus, it follows from the proof of the sufficiency portion that the supersingular divisors of the nilpotent admissible indigenous bundles $\mathcal{P}_1$, $\mathcal{P}_0$, $\mathcal{P}_\infty$ are given by the divisors [determined by the closed subschemes] defined by $t^2 - \overline{\lambda}$, $f_0(t)$, $f_\infty(t) \in k[t]$, respectively. In particular, since the polynomials $t^2 - \overline{\lambda}$, $f_0(t)$, and $f_\infty(t) \in k[t]$ are distinct, the isomorphism classes of $\mathcal{P}_1$, $\mathcal{P}_0$, $\mathcal{P}_\infty$ determine a subset of cardinality three of the set of isomorphism classes of nilpotent indigenous bundles on $X_k^{\log}/k$. Now let us recall [cf., e.g., [3, Proposition 4.6, (i)]] that the set of isomorphism classes of nilpotent indigenous bundles on $X_k^{\log}/k$ is of cardinality three. In particular, one concludes that there exists an element $\Box$ of $\{0, 1, \infty\}$ such that $\mathcal{P}_{(X,D)}$ is isomorphic to $\mathcal{P}_{\Box}$. On the other hand, let us also recall that it follows immediately from [5, Chapter II, Proposition 1.2] [cf. also [5, Chapter II, Corollary 1.6]] that the isomorphism class of a lifting over W of the hyperbolic curve (X_k, D_k) over k is completely determined by the isomorphism class of the projectivization of the associated FL-bundle. Thus, one concludes that the lifting (X, D) of the hyperbolic curve (X_k, D_k) over k is isomorphic to the canonical lifting over W that satisfies condition $(\dagger_{\Box})$, which thus implies that

$$\lambda \in \left\{\lambda_{\Box}, \frac{1}{\lambda_{\Box}}, 1 - \lambda_{\Box}, \frac{1}{1 - \lambda_{\Box}}, \frac{\lambda_{\Box}}{\lambda_{\Box} - 1}, \frac{\lambda_{\Box} - 1}{\lambda_{\Box}}\right\}.$$

In particular, since [it follows from the definition of $\lambda_{\Box}$ that] the equality $\overline{\lambda} = \overline{\lambda}_{\Box}$ holds, it follows that either

- the equality $\lambda = \lambda_{\Box}$ or
- the equality $\overline{\lambda}_{\Box} = -1$

holds. Thus, to complete the verification of the necessity portion, we may assume without loss of generality that the equalities $\overline{\lambda} = \overline{\lambda}_{\Box} = -1$ hold. Then it follows from the definitions of λ_1, λ_0, λ_∞ that the equalities

$$\{\lambda_1, \lambda_0, \lambda_\infty\} = \{-1, 2, -4\} = \left\{\lambda_\square, \frac{1}{\lambda_\square}, 1-\lambda_\square, \frac{1}{1-\lambda_\square}, \frac{\lambda_\square}{\lambda_\square - 1}, \frac{\lambda_\square - 1}{\lambda_\square}\right\} \qquad (\ni \lambda)$$

hold. In particular, we have an inclusion $\lambda \in \{\lambda_1, \lambda_0, \lambda_\infty\}$, as desired. This completes the proof of the necessity portion, hence also of Theorem 4.4. □

Corollary 4.5. — *Suppose that we are in the situation at the beginning of the present §4. Suppose, moreover, that the hyperbolic curve (X, D) over W is a canonical lifting of the hyperbolic curve (X_k, D_k) over k. Then the following three conditions are equivalent:*

(1) *The element $\lambda \in W$ coincides with the Teichmüller lift of some element of k.*

(2) *The equality $\overline{\lambda} = -1$ in k holds.*

(3) *The equality $\lambda = -1$ in W holds.*

Proof. — This assertion is a formal consequence of Theorem 4.4. □

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(Yuichiro Hoshi) Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, JAPAN
Email address: yuichiro@kurims.kyoto-u.ac.jp