Probability Distribution of Maxima of Random Surface

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by

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Abstract

The joint probability distributions are obtained of the extremum value, principal curvatures and the angle of their orientation of a random surface described by a homogeneous Gaussian random field. The probability density for the extrema is calculated as a marginal distribution, and is cast into a Gram-Charlier series in a form convenient for numerical calculation. Probability densities are numerically calculated for some isotropic spectral densities for an application in the microwave backscattering from ocean waves.

Key Words: random surface, extremum, curvature, probability density, Gram-Charlier series

1. Introduction

With a practical application in mind, the probability distributions are obtained for the values of extrema of a Gaussian random surface. We can extend Rice's theory for the maxima of a stationary Gaussian process [1] to the case of a 2-dimensional (2-D) random surface described by a homogeneous Gaussian random field, and obtain a joint probability distribution for 4 variables; extremum, two principal curvatures and the angle of their orientations at the extremum point. The probability density for extrema, i.e., minima, maxima or saddle points, can be given as a marginal distribution by integrating 3 other variables. For numerical calculation, it is conveniently expressed as a Gram-Charlier series in terms of its moments given by 3-D integrals. The present work was motivated by the problem of radio-wave scattering from a random surface. It is closely associated with the design of a satellite altimeter using backscattered microwave from ocean waves owing to the fact that the scattering takes place mostly at points of extremum, and the scattering cross-section is inversely proportional to the Gaussian curvature. However, our results could also be applied to a problem concerning image processing.

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2. Probability Distribution for Partial Derivatives

We summarize well known formulas for the sake of definition. Let a random surface $z = f(x, y)$ be given by a homogeneous Gaussian random field on a 2-D space, with mean 0 and the correlation function:

$$R(\boldsymbol{r}) = \langle f(\boldsymbol{r}) f(0) \rangle = \int_{-\infty}^{\infty} e^{i\boldsymbol{\lambda}\cdot\boldsymbol{r}} S(\boldsymbol{\lambda})\, d\boldsymbol{\lambda} \tag{1}$$

where $\boldsymbol{r} = (x, y)$, $\boldsymbol{\lambda} = (\lambda, \mu)$, $d\boldsymbol{\lambda} = d\lambda d\mu$, and the spectral density $S(\boldsymbol{\lambda})$ is unchanged against inversion: $S(\lambda, \mu) = S(-\lambda, -\mu)$. We assume $f(\boldsymbol{r})$ is twice q.m. differentiable w.r.t. $\boldsymbol{r}$. Let the partial derivatives of $f(x, y)$ be denoted by $u \equiv f_x$, $v \equiv f_y$, $r \equiv f_{xx}$, $s \equiv f_{xy}$, $t \equiv f_{yy}$. where the subscripts indicate differentiation. Let the 6×6 covariance matrix of the partial derivatives (z, r, t, s, u, v) at a fixed point be written in the form:

$$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{R}_1 & 0 \\ 0 & \boldsymbol{R}_2 \end{bmatrix}$$

$$\equiv \left[\begin{array}{cccc|cc} R & -R_1 & -R_2 & -R_3 & & \\ -R_1 & R_{11} & R_{33} & R_{13} & & 0 \\ -R_2 & R_{33} & R_{22} & R_{23} & & \\ -R_3 & R_{13} & R_{23} & R_{33} & & \\ \hline & & 0 & & R_1 & R_3 \\ & & & & R_3 & R_2 \end{array}\right] \tag{2}$$

$$\langle z^2 \rangle = R \equiv \int S(\boldsymbol{\lambda})\, d\boldsymbol{\lambda} \tag{3}$$

$$\langle u^2 \rangle = -\langle zr \rangle = R_1 \equiv \int \lambda^2 S(\boldsymbol{\lambda})\, d\boldsymbol{\lambda} \tag{4}$$

$$\langle v^2 \rangle = -\langle zt \rangle = R_2 \equiv \int \mu^2 S(\boldsymbol{\lambda})\, d\boldsymbol{\lambda} \tag{5}$$

$$\langle uv \rangle = -\langle zs \rangle = R_3 \equiv \int \lambda\mu S(\boldsymbol{\lambda})\, d\boldsymbol{\lambda} \tag{6}$$

$$\langle s^2 \rangle = \langle rt \rangle = R_{33} \equiv \int \lambda^2 \mu^2 S(\boldsymbol{\lambda})\, d\boldsymbol{\lambda} \tag{7}$$

$$\langle r^2 \rangle = R_{11} \equiv \int \lambda^4 S(\boldsymbol{\lambda})\, d\boldsymbol{\lambda} \tag{8}$$

$$\langle rs\rangle = R_{13} \equiv \int \lambda^3 \mu S(\lambda)\,d\lambda \tag{9}$$

$$\langle t^2\rangle = R_{22} \equiv \int \mu^4 S(\lambda)\,d\lambda \tag{10}$$

$$\langle ts\rangle = R_{23} \equiv \int \lambda\mu^3 S(\lambda)\,d\lambda \tag{11}$$

In the isotropic case where $S(\lambda) = S(\Lambda)$, $\Lambda \equiv \sqrt{\lambda^2 + \mu^2}$, there are further relations: $R_1 = R_2$, $R_{11} = R_{22}$, $R_3 = R_{13} = R_{23} = 0$.

Let the inverse matrix of $\boldsymbol{R}$ be denoted by

$$\boldsymbol{R}^{-1} = \begin{bmatrix} \boldsymbol{R}_1^{-1} & 0 \\ 0 & \boldsymbol{R}_2^{-1} \end{bmatrix} \tag{12}$$

$$\boldsymbol{R}_1^{-1} = \begin{bmatrix} A & A_1 & A_2 & A_3 \\ A_1 & A_{11} & A_{12} & A_{13} \\ A_2 & A_{12} & A_{22} & A_{23} \\ A_3 & A_{13} & A_{23} & A_{33} \end{bmatrix} \tag{13}$$

$$\boldsymbol{R}_2^{-1} = \frac{1}{|D_2|}\begin{bmatrix} R_2 & -R_3 \\ -R_3 & R_1 \end{bmatrix} \tag{14}$$

$$D_1 \equiv \det \boldsymbol{R}_1,\quad D_2 \equiv \det \boldsymbol{R}_2 = R_1 R_2 - R_3^2 \tag{15}$$

The 6-D Gaussian density function for (z, r, t, s, u, v) can be written

$$p(z, r, t, s, u, v) = p_1(z, r, t, s)p_2(u, v) \tag{16}$$

$$p_1(z, r, t, s) \equiv \frac{1}{(2\pi)^2|D_1|^{1/2}} \exp\left\{-\frac{1}{2}\boldsymbol{z}^t \boldsymbol{R}_1^{-1}\boldsymbol{z}\right\} \tag{17}$$

$$p_2(u, v) \equiv \frac{1}{2\pi|D_2|^{1/2}} \exp\left\{-\frac{1}{2}\boldsymbol{u}^t \boldsymbol{R}_2^{-1}\boldsymbol{u}\right\} \tag{18}$$

where the exponents in (17) and (18) give the quadratic forms of $\boldsymbol{z} = (z, t, r, s)$ and $\boldsymbol{u} = (u, v)$ w.r.t. $\boldsymbol{R}_1^{-1}$ and $\boldsymbol{R}_2^{-1}$, respectively.

$$\boldsymbol{z}^t \boldsymbol{R}_1^{-1}\boldsymbol{z} \equiv Az^2 + 2(A_1 r + A_2 t + A_3 s)z + A_{11}r^2 + A_{22}t^2 + A_{33}s^2 + 2(A_{12}rt + A_{13}rs + A_{23}ts) \tag{19}$$

$$\boldsymbol{u}^t \boldsymbol{R}_2^{-1}\boldsymbol{u} \equiv (R_2 u^2 - 2R_3 uv + R_1 v^2)/(R_1 R_2 - R_3^2) \tag{20}$$

3. Probability Distribution for Extrema and Curvatures

Extrema and Curvature of Surface At a point of extremum of the surface we have

$$u=0, \qquad v=0 \tag{21}$$

Let ρ be the radius of curvature of the curve formed on the intersection of the surface $z=f(x, y)$ and a vertical plane with an angle θ with x axis. When θ is so chosen as to satisfy

$$\tan 2\theta=\frac{2s}{r-t}, \left(-\frac{\pi}{4}\leq\theta\leq\frac{\pi}{4}\right) \tag{22}$$

then ρ takes a maximum or minimum value, which is a root of the equation,

$$\frac{1}{\rho^2}-(r+t)\frac{1}{\rho}+(rt-s^2)=0 \tag{23}$$

Let $1/\rho_1\equiv r_0$, $1/\rho_2\equiv t_0$ denote the maximum and minimum curvature, which are, respectively, given by

$$\frac{1}{\rho_1}\equiv r_0=\frac{r+t}{2}+\sqrt{\left(\frac{r-t}{2}\right)^2+s^2} \tag{24}$$

$$\frac{1}{\rho_2}\equiv t_0=\frac{r+t}{2}-\sqrt{\left(\frac{r-t}{2}\right)^2+s^2} \tag{25}$$

Obviously, the principal curvatures r_0 and t_0 satisfy the relation,

$$r_0+t_0=r+t \tag{26}$$

$$r_0t_0=rt-s^2 \tag{27}$$

where (26) gives the mean curvature and (27) the Gaussian total curvature.

If we write (r, t, s) in terms of (r_0, t_0, θ), we have

$$r=r_0\cos^2\theta+t_0\sin^2\theta \tag{28}$$

$$t=r_0\sin^2\theta+t_0\cos^2\theta \tag{29}$$

$$s=(r_0-t_0)\sin\theta\cos\theta \tag{30}$$

By the coordinate rotation $(x, y)\rightarrow(x', y')$

$$\begin{aligned} x&=x'\cos\theta-y'\sin\theta\\ y&=x'\sin\theta+y'\cos\theta \end{aligned} \tag{31}$$

we obtain the transformation $(u, v)\rightarrow(u', v')$, in the same form as (31), and the transformation

$(r, t, s) \to (r', t', s')$:

$$r = r'\cos^2\theta + t'\sin^2\theta - s'\sin 2\theta$$
$$t = r'\sin^2\theta + t'\cos^2\theta + s'\sin 2\theta \qquad (32)$$
$$s = \frac{1}{2}(r' - t')\sin 2\theta + s'\cos 2\theta$$

Particularly, when θ is chosen as in (22), we have

$$r' = r_0,\ t' = t_0,\ s' = 0 \qquad (33)$$

and the relations (28)–(30). We denote the parameter regions for maxima, minima and saddle points by R's as defined by the following inequalities:

$$R_{\min}: r_0 \ > 0\ t_0 > 0;\ rt - s^2 > 0,\ r > 0$$
$$R_{\max}: r_0 \ < 0\ t_0 < 0;\ rt - s^2 > 0,\ r < 0 \qquad (34)$$
$$R_{\text{sad}}: r_0 t_0 < 0 \qquad ;\ rt - s^2 < 0,$$

Probability Distribution for Extrema and Curvatures By the coordinate transformation $(r, t, s, u, v) \to (r_0, t_0, \theta, u', v')$ according to (31)–(30), the volume elements are transformed as follows:

$$\mathrm{d}r\mathrm{d}t\mathrm{d}s = \left|\frac{\partial(r, t, s)}{\partial(r_0, t_0, \theta)}\right| \mathrm{d}r_0\mathrm{d}t_0\mathrm{d}\theta$$
$$= |r_0 - t_0|(1 + \sin^2 2\theta)\,\mathrm{d}r_0\mathrm{d}t_0\mathrm{d}\theta \qquad (35)$$
$$\mathrm{d}u\mathrm{d}v \ = \mathrm{d}u'\mathrm{d}v'$$

so that we can rewrite the probability distribution in the following manner:

$$p_1(z, r, t, s)\,p_2(u, v)\,\mathrm{d}r\mathrm{d}t\mathrm{d}s\mathrm{d}u\mathrm{d}v$$
$$= p_1(z, r_0\cos^2\theta + t_0\sin^2\theta, r_0\sin^2\theta + t_0\cos^2\theta, (r_0 - t_0)\sin\theta\cos\theta)$$
$$\times\ p_2(u'\cos\theta - v'\sin\theta, u'\sin\theta + v'\cos\theta)|r_0 - t_0|(1 + \sin^2 2\theta)\,\mathrm{d}r_0\mathrm{d}t_0\mathrm{d}\theta\mathrm{d}u'\mathrm{d}v'$$
$$\equiv p_3(z, r_0, t_0, \theta; u', v')\,\mathrm{d}r_0\mathrm{d}t_0\mathrm{d}\theta\mathrm{d}u'\mathrm{d}v' \qquad (36)$$

With this form of probability distribution, we can apply Rice's method for the maxima of a stationary process [1] to the case of a homogeneous random surface. With the angle θ fixed, we seek an extremum along the curve on the surface by which the principal curvature is defined. The value of $u'(v')$ at the distance $\mathrm{d}x'(\mathrm{d}y')$ from the extremum point where $u' = v' = 0$ in a fixed direction $\theta(\theta + \pi/2)$ is given by $u' = -r_0\mathrm{d}x'$ $(v' = -t_0\mathrm{d}y')$. Therefore, the probability that the random field has an extremum within a small volume $\mathrm{d}z\mathrm{d}x\mathrm{d}y$ at the point (x, y) with the parameters (r_0, t_0, θ) in an infinitesimal region $\mathrm{d}r_0\mathrm{d}t_0\mathrm{d}\theta$ is given by

$$dz\int_{|r_0|dx'} du' \int_{|t_0|y'} dv' p_3(z, r_0, t_0, \theta; u', v') = dz dx dy |r_0 t_0| p_3(z, r_0, t_0, \theta; 0, 0) \quad (37)$$

times $dr_0 dt_0 d\theta$, where $dx'dy' = dxdy \equiv d\boldsymbol{r}$, and $r_0 t_0 = rt - s^2$ is the Gaussian curvature by (27). Obviously the distribution is uniform with respect to the spacial coordinate (x, y).

From (37) we can derive various probability distributions by integrating it over an appropriate parameter region. For instance, we can obtain a joint distribution of maxima and Gaussian curvature $g \equiv r_0 t_0$.

Similarly, the probability that the random field has a min, max or saddle point in the volume $dxdydz$ is obtained by integrating (37) over the region of (r_0, t_0, θ) given by (34):

$$P_{\min}(z) dz d\boldsymbol{r} = dz d\boldsymbol{r} \int_0^\infty dr_0 \int_0^\infty dt_0 \int_{-\pi/4}^{\pi/4} d\theta |r_0 t_0| p_3(z, r_0, t_0, \theta; 0, 0) \quad (38)$$

$$= dz d\boldsymbol{r} \iint_{rt > s^2, r > 0} dr dt ds |rt - s^2| p(z, r, t, s; 0, 0) \quad (39)$$

$$P_{\max}(z) dz d\boldsymbol{r} = dz d\boldsymbol{r} \int_{-\infty}^0 dr_0 \int_{-\infty}^0 dt_0 \int_{-\pi/4}^{\pi/4} d\theta |r_0 t_0| p_3(z, r_0, t_0, \theta; 0, 0) \quad (40)$$

$$= dz d\boldsymbol{r} \iint_{rt > s^2, r < 0} dr dt ds |rt - s^2| p(z, r, t, s; 0, 0) \quad (41)$$

By (17) and (36) we have $p_3(z, -r_0, -t_0, \theta: 0, 0) = p_3(-z, r_0, t_0, 0, 0)$, so that we obtain

$$P_{\max}(z) dz d\boldsymbol{r} = P_{\min}(-z) dz d\boldsymbol{r} \quad (42)$$

The equality (42) is due to the symmetry of Gaussian distribution.

$$P_{\text{sad}}(z) dz d\boldsymbol{r}$$

$$= dz d\boldsymbol{r} \left[\int_{-\infty}^0 dr_0 \int_0^\infty dt_0 + \int_0^\infty dr_0 \int_{-\infty}^0 dt_0 \right] \int_{-\pi/4}^{\pi/4} d\theta |r_0 t_0| p_3(z, r_0, t_0, \theta; 0, 0) \quad (43)$$

$$= dz d\boldsymbol{r} \iint_{rt < s^2} dr dt ds |rt - s^2| p(z, r, t, s; 0, 0) \quad (44)$$

4. Moments and Gram-Charlier Expansion

Gram-Charlier Expansion Numerical integration of (38)–(44) can be performed since the integrand is positive and well-behaved. In this section, instead of integrating for every value of z, we express $P(z)$ as a Gram-Charlier series in terms of its moments, which we write

$$M_n \equiv \frac{1}{M}\int_{-\infty}^{\infty} z^n P(z)\,\mathrm{d}z, \qquad n = 1, 2, 3, \cdots$$

$$M \equiv \int_{-\infty}^{\infty} P(z)\,\mathrm{d}z \tag{45}$$

where M denotes the normalizing constant. We write the Gram-Charlier expansion as:

$$P(z) = MG(z-\mu;\sigma^2)\left[1 + \sum_{n=3}^{\infty} \frac{b_n}{n!\,\sigma^{2n}} h_n(z-\mu;\sigma^2)\right] \tag{46}$$

$$G(z-\mu;\sigma^2) \equiv \frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{(z-\mu)^2}{2\sigma^2}\right] \tag{47}$$

$$h_n(z,\sigma^2) \equiv \sigma^{2n} G(z;\sigma^2)^{-1}\left(-\frac{\mathrm{d}}{\mathrm{d}z}\right)^n G(z;\sigma^2) \tag{48}$$

$$\exp\left[\lambda z - \frac{\lambda^2}{2}\sigma^2\right] = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} h_n(z;\sigma^2) \tag{49}$$

where $G(z-\mu, \sigma^2)$ denotes the Gaussian distribution with mean $\mu \equiv M_1$ and the variance $\sigma \equiv M_2 - M_1^2$, and (48) gives n-th order Hermite polynomial and (49) the generating function of Hermite polynomial.

The coefficient b_n, which can be obtained using the orthogonality of h_n,

$$b_n \equiv \frac{1}{M}\int_{-\infty}^{\infty} h_n(z-\mu;\sigma^2)P(z)\,\mathrm{d}z, \qquad n = 0, 1, 2, \cdots \tag{50}$$

can be given in terms of the moments up to n-th order, where, $b_0 = 1$, $b_1 = b_2 = 0$. For instance,

$$b_3 = M_3 - 3M_1M_2 + 2M_1^3 \tag{51}$$

$$b_4 = M_4 - 4M_1M_3 - 3M_2^2 + 12M_1^2M_2 - 6M_1^4 \tag{52}$$

Moment Generating Function We calculate the moment M_n by means of the moment generating function,

$$\varphi(v) = \langle e^{vz}\rangle_z = \frac{1}{M}\int_{-\infty}^{\infty} e^{vz} P_{\min}(z)\,\mathrm{d}z \tag{53}$$

$$= \sum_{n=0}^{\infty} \frac{v^n}{n!} M_n \quad (M_0 = 1) \tag{54}$$

From (16)–(18) and (39) the probability density can be written as

$$P(z) = \int_R |rt - s^2| p_1(z, r, t, s) p_2(0, 0)\,\mathrm{d}r\mathrm{d}t\mathrm{d}s \tag{55}$$

$$= \frac{1}{N}\int_R |rt - s^2| \exp\left[-\frac{A}{2}(z^2 - 2Lz + Q)\right] drdtds \tag{56}$$

$$L = -\frac{1}{A}(A_1 r + A_2 t + A_3 s) \tag{57}$$

$$Q = \frac{1}{A}[A_{11}r^2 + A_{22}t^2 + A_{33}s^2 + 2(A_{12}rt + A_{13}rs + A_{23}ts)] \tag{58}$$

$$N = (2\pi)^3 |D_1 D_2|^{1/2} \tag{59}$$

where L and Q denotes a linear and a quadratic form of r, t, s, respectively, and the integration domain R denotes $R_{\min}, R_{\max}, R_{\text{sad}}$ defined by (34), according as P represents $P_{\min}, P_{\max}, P_{\text{sad}}$, respectively. Upon substituting (56) into (53), we make use of the formula;

$$\Phi(v, L) \equiv \int_{-\infty}^{\infty} e^{vz} e^{-\frac{A}{2}(z^2 - 2Lz)} dz = \sqrt{\frac{2\pi}{A}} e^{\frac{A}{2}(L + \frac{L}{A})^2} = \sqrt{\frac{2\pi}{A}} e^{\frac{A}{2}L^2} e^{vL + \frac{v^2}{2A}} \tag{60}$$

$$= \sqrt{\frac{2\pi}{A}} e^{\frac{A}{2}L^2} \sum_{n=0}^{\infty} \frac{v^n}{n!} h_n\left(L; -\frac{1}{A}\right) \tag{61}$$

where the righthand is obtained using the generating function of the Hermite polynomial (49). Then we can calculate the moment generating function $\varphi(v)$ expanded in powers of v;

$$\varphi(v) = \frac{1}{M}\int_{-\infty}^{\infty} e^{vz} P(z) dz$$

$$= \frac{1}{MN}\int_R \Phi(v, L) |rs - t^2| e^{-\frac{A}{2}Q} drdtds \tag{62}$$

$$= \frac{1}{MN}\sqrt{\frac{2\pi}{A}} \sum_{n=0}^{\infty} \frac{v^n}{n!} \int_R |rt - s^2| h_n\left(L, -\frac{1}{A}\right) e^{-\frac{A}{2}(Q - L^2)} drdtds \tag{63}$$

Comparing this with (54) we get the moment M_n as the coefficient of $v^n/n!$:

$$M = \frac{1}{N}\sqrt{\frac{2\pi}{A}} \int_R |rt - s^2| e^{-\frac{A}{2}(Q - L^2)} drdtds \tag{64}$$

$$M_n = \frac{1}{MN}\sqrt{\frac{2\pi}{A}} \int_R |rt - s^2| h_n\left(L, -\frac{1}{A}\right) e^{-\frac{A}{2}(Q - L^2)} drdtds, \qquad n = 0, 1, 2, \cdots \tag{54}$$

Some low order moments are

$$M_1 = \frac{1}{MN}\sqrt{\frac{2\pi}{A}}\int_R |rt - s^2| L e^{-\frac{A}{2}(Q - L^2)} \mathrm{d}r \mathrm{d}t \mathrm{d}s \tag{66}$$

$$M_2 = \frac{1}{MN}\sqrt{\frac{2\pi}{A}}\int_R |rt - s^2| \left(L^2 + \frac{1}{A}\right) e^{-\frac{A}{2}(Q - L^2)} \mathrm{d}r \mathrm{d}t \mathrm{d}s \tag{67}$$

$$M_3 = \frac{1}{MN}\sqrt{\frac{2\pi}{A}}\int_R |rt - s^2| \left(L^3 + \frac{3}{A}L\right) e^{-\frac{A}{2}(Q - L^2)} \mathrm{d}r \mathrm{d}t \mathrm{d}s \tag{68}$$

$$M_4 = \frac{1}{MN}\sqrt{\frac{2\pi}{A}}\int_R |rt - s^2| \left(L^4 + \frac{6}{A}L^2 + \frac{3}{A^2}\right) e^{-\frac{A}{2}(Q - L^2)} \mathrm{d}r \mathrm{d}t \mathrm{d}s \tag{69}$$

where L and Q are given by (57) and (58), respectively. That the quadratic form in the exponent of the integrand is positive;

$$Q' \equiv Q - L^2 > 0 \tag{70}$$

is obvious if we put $z = L$ in the equation,

$$z^2 - 2Lz + Q \geq 0 \tag{71}$$

which is the quandratic form with respect to the positive definite matrix $\boldsymbol{R}^{-1}$. Since the integrand of (65) is a well behaved function, even the 3D integration over the domain $R_{\min}$, $R_{\max}$ or R_{sad} could be numerically performed if necessary.

Method of Integration for Moments The 3D integration of (39), (41), (44) or (65) is not an easy task. Here, using the transformation of variables (28)–(30), we bring the 3D integration w.r.t. (r_0, t_0, θ) to the sum of 1D integrations after integrating termwise over (r_0, t_0).

First we deal with $P_{\min}(z)$ and then $P_{\max}(z)$ is given by $P_{\min}(-z)$. The integral for moments (65) is written as

$$M_n = \iiint_{\substack{rt > s^2 \\ r > 0}} |rt - s^2| \psi_n(r, t, s) \mathrm{d}r \mathrm{d}t \mathrm{d}s \tag{72}$$

$$\psi_n(r, t, s) \equiv \frac{1}{MN}\sqrt{\frac{2\pi}{A}} h_n\left(L, -\frac{1}{A}\right) e^{-\frac{A}{2}(Q - L^2)} \tag{73}$$

By the transformatin (28)–(30) and (35) we have

$$M_n = \int_{-\pi/4}^{\pi/4} (1 + \sin^2\theta) \int_0^\infty \int_0^\infty r_0 t_0 |r_0 - t_0| \widehat{\psi_n}(r_0, t_0, \theta) \mathrm{d}r_0 \mathrm{d}t_0 \tag{74}$$

$$\widehat{\psi_n}(r_0, t_0, \theta) \equiv \psi_n(r, t, \theta) \tag{75}$$

where the integration w.r.t. (r_0, t_0) is written

$$\iint_{r_0>t_0} r_0t_0(r_0-t_0)\widehat{\psi_n}(r_0, t_0, \theta)\,dr_0dt_0 + \iint_{t_0>r_0} r_0t_0(t_0-r_0)\widehat{\psi_n}(r_0, t_0, \theta)\,dr_0dt_0$$

$$= \iint_{r_0>t_0} r_0t_0(r_0-t_0)[\widehat{\psi_n}(r_0, t_0, \theta) + \widehat{\psi_n}(t_0, r_0, \theta)]\,dr_0dt_0 \tag{76}$$

$$= \iint_{r_0>t_0} r_0t_0(r_0-t_0)\left[\widehat{\psi_n}(r_0, t_0, \theta) + \widehat{\psi_n}\left(r_0, t_0, \theta+\frac{\pi}{2}\right)\right]dr_0dt_0 \tag{77}$$

Here (77) is obtained by interchanging the variables r_0, t_0 in $\widehat{\psi_n}$, which amounts to the transformations $\theta \to \theta + \pi/2$, $\cos\theta \to -\sin\theta$, $\sin\theta \to \cos\theta$. This implies that the region for integration is expandable to $-\pi/4 \le \theta \le 3\pi/4$, or to $0 \le \theta \le \pi$. Thus, (74) is further rewritten as

$$M_n = \int_0^{\pi}(1+\sin^2 2\theta)\,d\theta \iint_{r_0>t_0} r_0t_0(r_0+t_0)\widehat{\psi_n}(r_0, t_0, \theta)\,dr_0dt_0 \tag{78}$$

$$= \int_0^{\pi}(1+\sin^2 2\theta)\,d\theta \int_0^{\infty}\int_0^{\infty} s_0 + t_0)s_0t_0\widehat{\psi_n}(s_0+t_0, t_0, \theta)\,ds_0dt_0 \tag{79}$$

where we have made the change of variable $(r_0, t_0) \to (s_0, t_0)$:

$$s_0 \equiv r_0 - t_0,\quad r_0t_0 = (s_0+t_0)t_0,\quad dr_0dt_0 = ds_0dt_0 \tag{80}$$

In this manner the linear and quadratic forms L and Q w.r.t. (r, s, t) are cast into the linear and quadratic forms w.r.t. (t_0, s_0) as follows:

$$L \equiv L(t_0, s_0) = -\frac{1}{A}(\alpha t_0 + \beta s_0) \tag{81}$$

$$Q \equiv Q(t_0, s_0) = \frac{1}{A}(\gamma t_0^2 + 2\delta t_0 s_0 + \varepsilon s_0^2) \tag{82}$$

$$\alpha = A_1 + A_2 \tag{83}$$

$$\beta = A_1\cos^2\theta + A_2\sin^2\theta + A_2\sin\theta\cos\theta \tag{84}$$

$$\gamma = A_{11} + A_{22} + A_{12} \tag{85}$$

$$\delta = A_{11}\cos^2\theta + A_{22}\sin^2\theta + A_{12} + (A_{13} + A_{23})\sin\theta\cos\theta \tag{86}$$

$$\varepsilon = A_{11}\cos^4\theta + A_{22}\sin^4\theta + (2A_{12} + A_{33})\sin^2\theta\cos^2\theta + 2A_{13}\sin\theta\cos^3\theta + 2A_{23}\sin^3\theta\cos\theta \tag{87}$$

Thus, the quadratic form in the exponent of (73) is written

$$A(Q - L^2) = \xi t_0^2 + 2\eta t_0 s_0 + \zeta s_0^2 \tag{88}$$

$$\xi \equiv \gamma - \frac{\alpha^2}{A}, \ \eta \equiv \delta - \frac{\alpha\beta}{A}, \ \zeta \equiv \varepsilon - \frac{\beta^2}{A} \tag{89}$$

We note that $h_n(L, -1/A)$ is a polynomial in L and accordingly that it is a polynomial in (t_0, s_0). Using the formula

$$\int_0^\infty dx \int_0^\infty dy\, x^n y^m e^{-(x^2+2axy+y^2)} = \frac{1}{4}\sum_{r=0}^{\infty} \frac{(-2a)^r}{r!} \Gamma\left(\frac{n+r+1}{2}\right)\Gamma\left(\frac{m+r+1}{2}\right) \tag{90}$$

$$\Gamma(n+1) = n!$$

$$\Gamma\left(n + \frac{1}{2}\right) = (2n-1)(2n-3)\cdots 3 \cdot 1 \frac{\sqrt{\pi}}{2^n}$$

which may be rewritten as

$$\int_0^\infty \int_0^\infty t_0^n s_0^m e^{-\frac{1}{2}[\xi t_0^2 + 2\eta t_0 s_0 + \zeta s_0^2]} dt_0 ds_0$$

$$= \frac{1}{4}\left[\frac{2}{\xi}\right]^{\frac{n+1}{2}}\left[\frac{2}{\zeta}\right]^{\frac{m+1}{2}} \sum_{r=0}^{\infty} \frac{1}{r!}\left[\frac{-2\eta}{\sqrt{\xi\zeta}}\right]^r \Gamma\left(\frac{n+r+1}{2}\right)\Gamma\left(\frac{m+r+1}{2}\right) \tag{91}$$

we can perform the integration (79) termwise w.r.t. (t_0, s_0) after expanding

$$(s_0 + t_0)s_0 t_0 h_n\left(L, -\frac{1}{A}\right) = \sum_k \sum_m a_{km} t_0^k s_0^m \tag{92}$$

in terms of powers of t_0, s_0, and using (91). In this way the integral (79) is reduced to the sum of integrals over θ, which can be numerically computed.

In a similar manner we can calculate the moments for $P_{\text{sad}}(z)$. We first write

$$M_n = \iiint_{rt<s^2} |rt - s^2|\psi_n(r, t, s) dr dt ds$$

$$= \int_{-\pi/4}^{\pi/4} (1 + \sin^2 2\theta) d\theta \left[\int_0^\infty dr_0 \int_{-\infty}^0 dt_0 + \int_{-\infty}^0 dr_0 \int_0^\infty dt_0\right]$$

$$\times |r_0 t_0 (r_0 - t_0)| \widehat{\psi_n}(r_0, t_0, \theta)$$

$$= \int_{-\pi/4}^{\pi/4} (1 + \sin^2 2\theta) d\theta \int_0^\infty \int_0^\infty r_0 t_0 (r_0 + t_0) [\psi_n(-r_0, t_0, \theta)$$

$$+ \widehat{\psi_n}(r_0, -t_0, \theta)] dr_0 dt_0 \tag{93}$$

where

$$\widehat{\psi}_n(r_0, t_0, \theta) = \frac{1}{MN}\sqrt{\frac{2\pi}{A}}\psi_n\left(L, -\frac{1}{A}\right)e^{-\frac{A}{2}(Q-L^2)} \tag{94}$$

$$\begin{aligned} L \equiv L(r_0, t_0) &= -\frac{1}{A}[\alpha t_0 + \beta(r_0 - t_0)] \\ &= -\frac{1}{A}[(\alpha - \beta)t_0 + \beta r_0] \end{aligned} \tag{95}$$

$$\begin{aligned} Q \equiv Q(r_0, t_0) &= \frac{1}{A}[\gamma t_0^2 + 2\delta t_0(r_0 - t_0) + \varepsilon(r_0 - t_0)^2] \\ &= \frac{1}{A}[(\gamma - 2\delta + \varepsilon)t_0^2 + 2(\delta - \varepsilon)r_0 t_0 + \varepsilon r_0^2] \end{aligned} \tag{96}$$

Therefore, substituting these into (93), we can perform the integration over (r_0, t_0) to have the sum of 1D integrals w.r.t. θ.

5. Spectral Model for Ocean Waves

In this section we give some examples of spectral density and its related quantities. Such a spectrum is intended to be a model spectrum for possible ocean waves, because we are interested in the probability distribution for the extrema and curvatures of the ocean waves which undergo irregular undulating motion with a certain average wave length and wave direction.

For the purpose of application we are taking up four types of spectrum; Gaussian and rational spectral types combined with isotropic and anisotropic ones. The Gaussian spectrum has a rapidly decreasing spectral tail with fewer short wave components, which physically means that the wave has few ripples and a very smooth surface. A rational spectrum, on the other hand, has a longer spectral tail and more short wave components, which implies a rougher surface with more ripples on the wave. The correlation length or the spectral width is described by a parameter ℓ. In order to model a spectrum for ocean waves, $S(\boldsymbol{\lambda})$ should have a spectral peak at the spatial frequency $\lambda \equiv |\boldsymbol{\lambda}| = \Lambda_p$ corresponding to the average ocean wave lenght, which in turn is closely related to ℓ. An isotropic spectrum $S(\Lambda)$, $\Lambda \equiv \sqrt{\lambda^2 + \mu^2}$ implies the omnidirectional waves under windless conditins. An anisotropic spectrum $S(\boldsymbol{\lambda}) \equiv S(\Lambda, \varphi)$ with the spectral peak in the direction $\varphi = \varphi_0$ implies the directional waves related to the wind direction. In what follows we will list four types of spectral model together with brief comments, and tabulate their parameters, correlation function and several spectral moments which equal the covariances between the partical derivatives of the random surface.

5.1 Isotopic Gaussian Spectrum-Omni-directional Wave Model Spectral Density

$$S(\Lambda) = \frac{R}{2\pi K_0} e^{-\ell^2 \Lambda^2}, \quad \Lambda = \sqrt{\lambda^2 + \mu^2} \quad \text{(Gaussian)} \tag{97}$$

$$S(\Lambda) = \frac{R}{2\pi K_m} \Lambda^{2m} e^{-\ell^2 \Lambda^2}, \quad m = 1, 2, \cdots \quad \text{(Omnidirectional Wave Model)} \tag{98}$$

where K_m denotes the normalizing constant given by

$$K_m \equiv \int_0^\infty \Lambda^{2m+1} e^{-\ell^2 \Lambda^2} \mathrm{d}\Lambda = \frac{m!}{2\ell^{2(m+1)}}, \qquad m = 0, 1, 2, \cdots \tag{99}$$

(97) shows a simple isotropic Gaussian spectrum, whereas (98) gives a spectrum for an omnidirectional wave model.

Spectral Peak Frequency

$$\Lambda_p = \frac{\sqrt{m}}{\ell}, \quad m = 0, 1, 2, \cdots, \ \ell = \frac{\sqrt{m}}{\Lambda_p} \tag{100}$$

Correlation Function

$$R(r) = \int_0^\infty \int_0^{2\pi} e^{i\boldsymbol{\lambda}\cdot\boldsymbol{r}} S(\boldsymbol{\lambda}) \mathrm{d}\boldsymbol{\lambda}, \qquad \mathrm{d}\boldsymbol{\lambda} \equiv \Lambda \mathrm{d}\Lambda \mathrm{d}\varphi \tag{101}$$

$$= \frac{R}{K_m} \int_0^\infty J_0(\Lambda r) \Lambda^{2m+1} e^{-\ell^2 \Lambda^2} \mathrm{d}\Lambda \tag{102}$$

$$= \frac{R}{K_m} \frac{m!}{2\ell^{2(m+1)}} {}_1F_1\left(m+1;\, 1;\, -\frac{r^2}{4\ell^2}\right) \tag{103}$$

$$= R \frac{1}{m!} \sum_{n=0}^{\infty} \frac{(m+n)!}{(n!)^2} \left(-\frac{r^2}{4\ell^2}\right)^n, \qquad m = 0, 1, 2, \cdots, \tag{104}$$

$$\equiv \begin{cases} R \exp\left(-\dfrac{r^2}{4\ell^2}\right) & (m = 0,\ \text{Gauss}) \quad (105) \\[2ex] R\left(1 - \dfrac{r^2}{4\ell^2}\right) \exp\left(-\dfrac{r^2}{4\ell^2}\right) & (m = 1) \quad (106) \end{cases}$$

$$\left| \begin{array}{ll} R\left[1-\frac{r^2}{2\ell^2}+\frac{1}{2}\left(\frac{r^2}{4\ell^2}\right)^2\right]\exp\left(-\frac{r^2}{4\ell^2}\right) & (m=2) \quad (107) \\ R\left[1-\frac{3r^2}{4\ell^2}+\frac{3}{2}\left(\frac{r^2}{4\ell^2}\right)^2-\frac{1}{6}\left(\frac{r^2}{4\ell^2}\right)^3\right]\exp\left(-\frac{r^2}{4\ell^2}\right) & (m=3) \quad (108) \end{array}\right.$$

where ${}_1F_1(\alpha;\gamma;z)$ denotes the confluent hypergeometric function:

$$\int_0^r J_{2n}(\Lambda r)\Lambda^{2m+1}e^{-\ell^2\Lambda^2}\mathrm{d}\Lambda$$

$$=\frac{\Gamma(m+n+1)r^{2n}}{2^{2n+1}\ell^{2(m+n+1)}\Gamma(2n+1)}\,{}_1F_1(m+n+1;2n+1,-r^2/4\ell^2) \tag{109}$$

$${}_1F_1(\alpha;\gamma;z)=\frac{\Gamma(\gamma)}{\Gamma(\alpha)}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n)}{\Gamma(\gamma+n)}\frac{z^n}{n!} \tag{110}$$

Larger m gives more oscillatory correlation function.

Covariance Matrix

$$R\equiv R(0)=\langle z^2\rangle=\int S(\lambda)\mathrm{d}\lambda=2\pi\int_0^{\infty}S(\Lambda)\Lambda\mathrm{d}\Lambda \tag{111}$$

$$R_1=R_2=\int\lambda^2S(\lambda)\mathrm{d}\lambda=\pi\int_0^{\infty}\Lambda^3S(\Lambda)\mathrm{d}\Lambda$$

$$=\frac{R}{2K_m}\int_0^{\infty}\Lambda^{2m+3}e^{-\ell^2\Lambda^2}\mathrm{d}\Lambda \quad (R(r)r\to 0,\ m\to m+1) \tag{112}$$

$$=\frac{R}{2}\frac{K_{m+1}}{K_m}=R\frac{m+1}{2\ell^2} \tag{113}$$

$$R_3=0 \tag{114}$$

$$R_{11}=R_{22}=\int\lambda^4S(\lambda)\mathrm{d}\lambda=\frac{R}{2\pi K_m}\int_0^{2\pi}\cos^4\varphi\mathrm{d}\varphi\int_0^{\infty}\Lambda^{2m+5}e^{-\ell^2\Lambda^2}\mathrm{d}\Lambda \tag{115}$$

$$=R\frac{3}{8}\frac{K_{m+2}}{K_m}=R\frac{3(m+2)(m+1)}{8\ell^4} \tag{116}$$

$$R_{33}=\int\lambda^2\mu^2S(\lambda)\mathrm{d}\lambda=\frac{R}{K_m}\int_0^{2\pi}\cos^2\varphi\sin^2\mathrm{d}\varphi\int_0^{\infty}\Lambda^{2m+5}e^{-\ell^2\Lambda^2}\mathrm{d}\Lambda \tag{117}$$

$$= R\frac{1}{8}\frac{K_{m+2}}{K_m} = R\frac{(m+2)(m+1)}{8\ell^4} \tag{118}$$

$$R_{13} = R_{23} = 0 \tag{119}$$

5.2 Anisotropic Gaussian Spectrum-Directional Wave Model Spectral Density

$$S(\lambda) = S(\Lambda, \varphi) = \frac{R}{K_m L_n}\cos^{2n}(\varphi - \varphi_0)\Lambda^{2m}e^{-\ell^2\Lambda^2}, \quad m = 0, 1, 2, \cdots, n = 0, 1, 2, \cdots \tag{120}$$

where K_m and L_n are given by

$$K_m \equiv \int_0^\infty \Lambda^{2m}e^{-\ell^2\Lambda^2}\mathrm{d}\Lambda = \frac{m!}{2\ell^{2(m+1)}} \tag{121}$$

$$L_n \equiv \int_0^{2\pi}\cos^{2n}\varphi\mathrm{d}\varphi = 2\pi\frac{(2n-1)!!}{(2n)!!} \tag{122}$$

The spectrum with $n = 0$ corresponds to isotropic omnidirectional waves, and $n \geq 1$ gives anisotropic directional waves in the direction $\varphi = \varphi_0$. Larger n implies the stronger directivity of waves, and therefore, $n = 1$ corresponds to the weakest directivity.

Correlation Function

$$R(r, \theta) = \int_0^\infty\int_0^{2\pi} e^{i\Lambda r\cos(\theta-\varphi)}S(\Lambda, \varphi)\Lambda\mathrm{d}\Lambda\mathrm{d}\varphi \tag{123}$$

In what follows we gives the quantities for the weakest directional wave model $n = 1$ which can be easily calculated.

Spectral Density $(n = 1)$

$$S(\Lambda, \varphi) = \frac{R}{\pi K_m}\cos^2(\varphi - \varphi_0)\Lambda^{2m}e^{-\ell^2\Lambda^2} \tag{124}$$

$$R = \int_0^\infty\int_0^{2\pi} S(\lambda, \varphi)\Lambda\mathrm{d}\Lambda\mathrm{d}\varphi \tag{125}$$

Correlation Function $(n = 1)$

$$R(r, \theta) = \frac{R}{\pi K_m} \int_0^\infty \int_0^{2\pi} e^{i\Lambda r\cos(\theta-\varphi)} \cos^2(\varphi - \varphi_0) \mathrm{d}\varphi_0 \Lambda^{2m+1} e^{-\ell^2\Lambda^2} \mathrm{d}\Lambda \tag{126}$$

$$= \frac{2R}{K_m} \int_0^\infty [J_0(\Lambda r) - J_2(\Lambda)\cos 2(\theta - \varphi_0)] \Lambda^{2m+1} e^{-\ell^2\Lambda^2} \mathrm{d}\Lambda \tag{127}$$

which can be written in terms of the hypergeometric function, but is expressible as a sum in terms of $r^n e^{-r^2/4\ell^2}$ when m is small. The first term in (127) is the isotropic part and the second the directional part.

Covariance Matrix $(n = 1)$

$$R_1 = R\frac{K_{m+1}}{K_m} \frac{1}{\pi} \int_0^{2\pi} \cos^2\varphi \cos^2(\varphi - \varphi_0) \mathrm{d}\varphi = R\frac{m+1}{4\ell^2}(1 + \cos^2\varphi_0) \tag{128}$$

$$R_2 = R\frac{K_{m+1}}{K_m} \frac{1}{\pi} \int_0^{2\pi} \sin^2\varphi \cos^2(\varphi - \varphi_0) \mathrm{d}\varphi = R\frac{m+1}{4\ell^2}(1 + \sin^2\varphi_0) \tag{129}$$

$$R_3 = R\frac{K_{m+1}}{K_m} \frac{1}{\pi} \int_0^{2\pi} \cos\varphi \sin\varphi \cos^2(\varphi - \varphi_0) \mathrm{d}\varphi = R\frac{m+1}{4\ell^2}\sin 2\varphi_0 \tag{130}$$

$$R_{11} = R\frac{K_{m+2}}{K_m} \frac{1}{\pi} \int_0^{2\pi} \cos^4\varphi \cos^2(\varphi - \varphi_0) \mathrm{d}\varphi$$

$$= R\frac{(m+2)(m+1)}{\ell^4}\left(\frac{5}{8}\cos^2\varphi_0 + \frac{1}{8}\sin^2\varphi_0\right) \tag{131}$$

$$R_{22} = R\frac{K_{m+2}}{K_m} \frac{1}{\pi} \int_0^{2\pi} \sin^4\varphi \cos^2(\varphi - \varphi_0) \mathrm{d}\varphi$$

$$= R\frac{(m+2)(m+1)}{\ell^4}\left(\frac{1}{8}\cos^2\varphi_0 + \frac{5}{8}\sin^2\varphi_0\right) \tag{132}$$

$$R_{33} = R\frac{K_{m+2}}{K_m} \frac{1}{\pi} \int_0^{2\pi} \cos^2\varphi \sin^2\varphi \cos^2(\varphi - \varphi_0) \mathrm{d}\varphi = R\frac{(m+2)(m+1)}{8\ell^4} \tag{133}$$

$$R_{13} = R\frac{K_{m+2}}{K_m} \frac{1}{\pi} \int_0^{2\pi} \cos^3\varphi \sin\varphi \cos^2(\varphi - \varphi_0) \mathrm{d}\varphi = R\frac{(m+2)(m+1)}{8\ell^4}\sin 2\varphi_0 \tag{134}$$

$$R_{23} = R\frac{K_{m+2}}{K_m}\frac{1}{\pi}\int_0^{2\pi}\cos\varphi\sin^3\varphi\cos^2(\varphi-\varphi_0)\,\mathrm{d}\varphi = R_{13} \tag{135}$$

5.3 Rational Isotropic Spectrum-Omni-directional Wave Model Spectral Density

$$S(\Lambda) = \frac{R}{2\pi K_{0\ell}}\frac{1}{(\Lambda^2+\kappa^2)^{\ell+1}}, \qquad \ell = 3, 4, \cdots \tag{136}$$

$$S(\Lambda) = \frac{R}{2\pi K_{m\ell}}\frac{\Lambda^{2m}}{(\Lambda^2+\kappa^2)^{\ell+1}}, \qquad \ell \geq m+3 \tag{137}$$

where the normalizing constant $K_{m\ell}$ is given by

$$K_{m\ell} = \int_0^\infty \frac{\Lambda^{2m+1}\,\mathrm{d}\Lambda}{(\Lambda^2+\kappa^2)} = \frac{1}{2\kappa^{2(\ell-m)}}\frac{m!(\ell-m-1)!}{\ell!} \tag{138}$$

Spectral Peak Frequency

$$\Lambda_p = \kappa\sqrt{\frac{m}{\ell-m+1}} \tag{139}$$

The condition for (137) is due to the condition that $f(\boldsymbol{r})$ be differentiable up to the second order partial derivatives.

Correlation Function

$$R(r) = \frac{R}{K_{m\ell}}\int_0^\infty \frac{\Lambda^{2m+1}J_0(\Lambda r)}{(\Lambda^2+\kappa^2)^{\ell+1}}\mathrm{d}\Lambda, \qquad R(0) = R,\ \ell \geq m+1 \tag{140}$$

which is expressible in terms of the hypergeometric function ${}_1F_2$ but we omit the details. When $m = 0$ we can calculate as follows:

$$R(r) = \frac{R}{K_{0\ell}}\int_0^\infty \frac{J_0(\Lambda r)\Lambda\mathrm{d}\Lambda}{(\Lambda^2+\kappa^2)^{\ell+1}} \tag{141}$$

$$= \frac{R}{2^{\ell-1}(\ell-1)!}(\kappa r)^\ell K_\ell(\kappa r), \qquad \ell = 1, 2, \cdots \tag{142}$$

$$\simeq R[1-(\ell-2)!\,2^{\ell-3}(\kappa r)^2] \qquad r \simeq 0 \tag{143}$$

$$\sim R\sqrt{\frac{\pi}{2}}\frac{1}{2^{\ell-1}(\ell-1)!}(\kappa r)^{\ell-\frac{1}{2}}e^{-\kappa r} \qquad r \longrightarrow \infty \tag{144}$$

Covariance Matrix

$$R_1 = R_2 = R\frac{1}{2}\frac{K_{m+1,\ell}}{K_{m\ell}} \tag{145}$$

$$R_3 = 0 \tag{146}$$

$$R_{11} = R_{22} = R\frac{3}{8}\frac{K_{m+2,\ell}}{K_{m\ell}} \tag{147}$$

$$R_{33} = R\frac{1}{8}\frac{K_{m+2,\ell}}{K_{m\ell}} \tag{148}$$

$$R_{13} = R_{23} = 0 \tag{149}$$

where

$$\frac{K_{m+1,\ell}}{K_{m\ell}} = \kappa^2\frac{m+1}{\ell-m-1} \tag{150}$$

$$\frac{K_{m+2,\ell}}{K_{m\ell}} = \kappa^4\frac{(m+2)(m+1)}{(\ell-m-1)(\ell-m-2)} \tag{151}$$

5.4 Anisotropic Rational Spectrum-Directional Wave Model Spectral Density

$$S(\Lambda, \varphi) = \frac{R}{K_{m\ell}L_n}\cos^{2n}(\varphi-\varphi_0)\frac{\Lambda^{2m}}{(\Lambda^2+\kappa^2)^{\ell+1}}, \qquad \ell \geq m+1,\ m,\ n = 0, 1, 2, \cdots \tag{152}$$

Covariance Matrix

$$R_1 = R\frac{K_{m+1,\ell}}{K_{m\ell}}\frac{1}{4}(1+2\cos^2\varphi_0) \tag{153}$$

$$R_2 = R\frac{K_{m+1,\ell}}{K_{m\ell}}\frac{1}{4}(1+2\sin^2\varphi_0) \tag{154}$$

$$R_3 = R\frac{K_{m+1,\ell}}{K_{m\ell}}\frac{1}{4}\sin 2\varphi_0 \tag{155}$$

$$R_{11} = R\frac{K_{m+2,\ell}}{K_{m\ell}}\left(\frac{5}{8}\cos^2\varphi_0 + \frac{1}{8}\sin^2\varphi_0\right) \tag{156}$$

$$R_{22} = R\frac{K_{m+2,\ell}}{K_{m\ell}}\left(\frac{1}{8}\cos^2\varphi_0 + \frac{5}{8}\sin^2\varphi_0\right) \tag{157}$$

$$R_{33} = R\frac{1}{8}\frac{K_{m+2,\ell}}{K_{m\ell}} \tag{158}$$

$$R_{13} = R_{23} = R\frac{1}{8}\frac{K_{m+2,\ell}}{K_{m\ell}}\sin 2\varphi_0 \tag{159}$$

6. Example of Probability Distribution for Extrema

Lastly, we show some examples of the probability distributions $P_{\min}$, $P_{\max}$ and P_{sad} calculated for the isotropic Gaussian and rational spectral densities. The Gaussian spectrum (98) and the rational spectrum (137), together with the correlation functions, are shown in Figs. 1–4 for $m = 1, 2, 3, 10$, with $R = 1$ and with the spectral peak position normalized as $\Lambda_p = 1$. The parameter m specifies the spectral form of the ocean waves; a larger m implies a sharper spectral peak and more wavy correlation function (ordinary Gaussian spectrum with $m = 0$ is shown only for comparison). We have shown the rational spectrum only for the case $\ell = m + 3$.

Figs. 5–8 show the probability distributions $P_{\min}(z)$ and $P_{\text{sad}}(z)$ corresponding to the Gaussian and rational spectra shown in Figs. 1 and 3. $P_{\max}(z)$ is shown only for $m = 0$ in

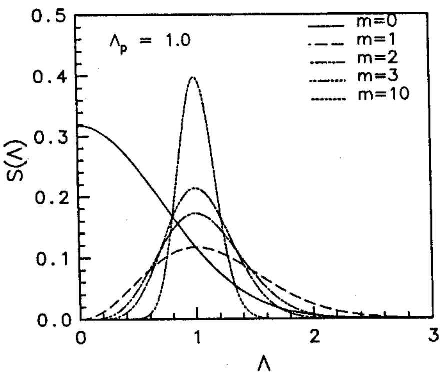

Fig. 1 Gaussian Model Spectrum. $\Lambda = \sqrt{\lambda^2 + \mu^2}$, $\ell = 1$ $(m = 0)$, $\ell = \sqrt{m}(m = 1, 2, \cdots)$. Peak spatial frequency normalized as $\Lambda_p = \sqrt{m}/\ell = 1$. $R = 1$.

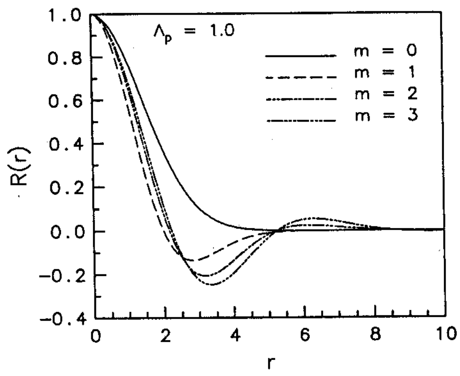

Fig. 2 Correlation Function for Gaussian Model Spectrum. $r = \sqrt{x^2 + y^2}, R = 1$.

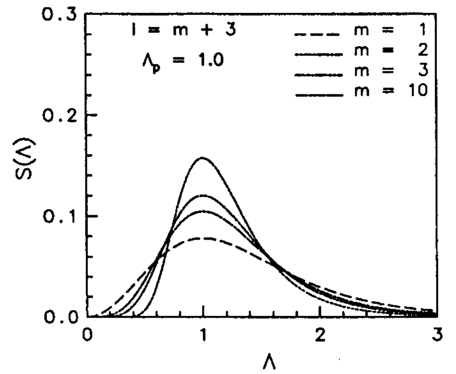

Fig. 3 Rational Model Spectrum. $\Lambda=\sqrt{\lambda^2+\mu^2}$, $\ell=m+3$ ($m=1, 2,\cdots$). Peak spatial frequency normalized as $\Lambda_p=\kappa\sqrt{m/(\ell-m-1)}=1$. $R=1$.

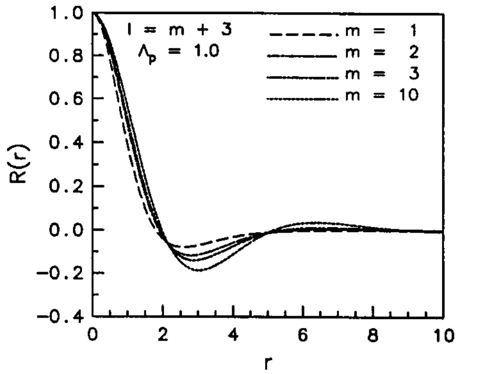

Fig. 4 Correlation Function for Rational Model Spectrum. $r=\sqrt{x^2+y^2}$, $R=1$.

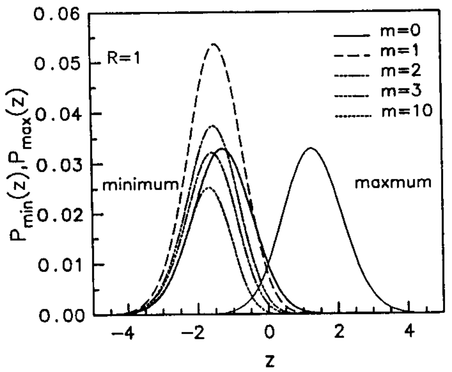

Fig. 5 Probability distributions $P_{min}(z)$ and $P_{max}(z)$ (Gaussian Spectrum). $R=1$. As $P_{max}(z)=P_{min}(-z)$, $P_{max}(z)$ is shown only for $m=0$.

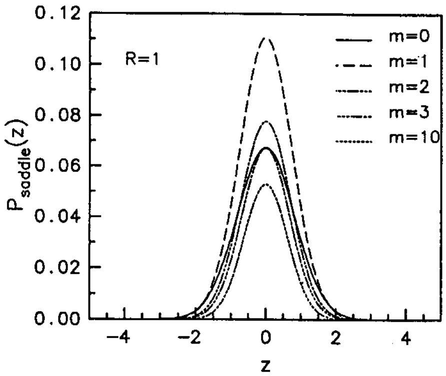

Fig. 6 Probability Distribution $P_{sad}(z)$ (Gaussian Spectrum). $R=1$.

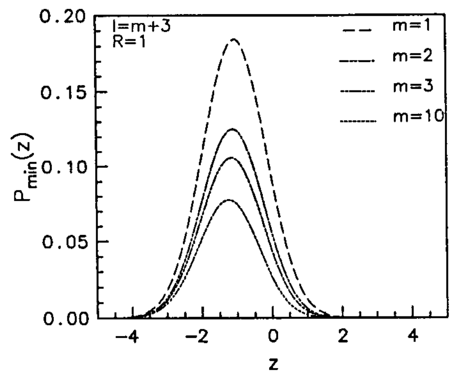

Fig. 7 Probability Distribution $P_{min}(z)$ (Rational Spectrum). $\ell=m+3$, $R=1$. $P_{max}(z)=P_{min}(-z)$.

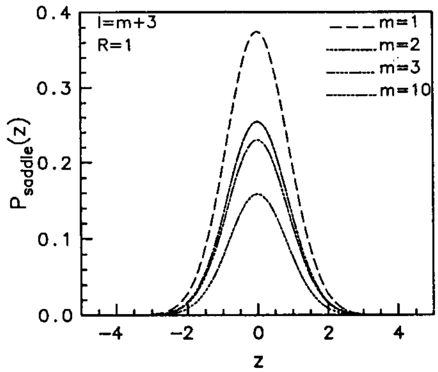

Fig. 8 Probability Distribution $P_{sad}(z)$ (Rational Spectrum). $\ell=m+3$, $R=1$.

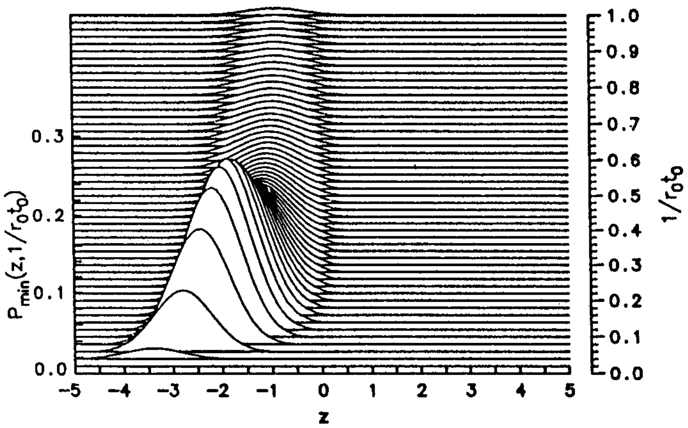

Fig. 9 Joint Probability Distribution $P_{min}(z, 1/r_0t_0)$ (Gaussian Spectrum) $R=1$, $m=1$. $r_0t_0=rt-s^2$: Gaussian curvature of minimum point.

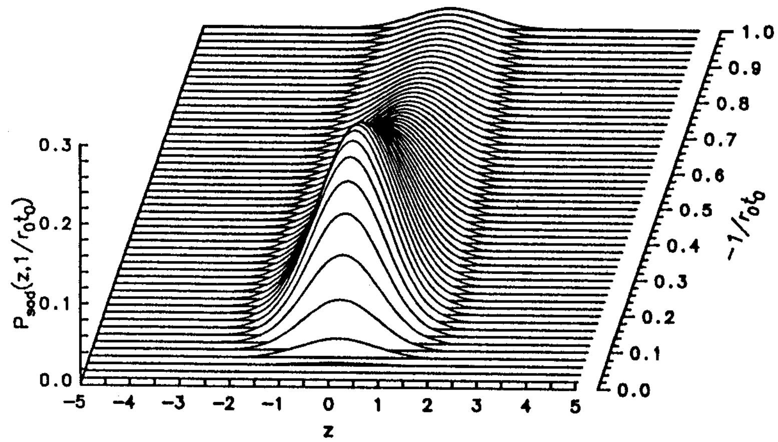

Fig. 10 Joint Probability Distribution $P_{sad}(z, 1/r_0t_0)$ (Gaussian Spectrum) $R=1$, $m=1$. $r_0t_0=rt-s^2$: Gaussian curvature of saddle point.

Fig. 5, since $P_{max}(z)=P_{min}(-z)$. Figs. 9 and 10 show the examples of the joint distribution $P_{min}(z, g)$ and $P_{sad}(z, g)$ for z and Gaussian radius of curvature $g \equiv 1/r_0t_0$, which are calculated for the Gaussian spectrum in Fig. 1 with $m=1$. Similar but broader distributins are obtained for the rational spectrum, but are omitted here.

References

[1] S. O. Rice: Mathematical analysis of random noise, *BSTJ* **24** (1945).