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by

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Abstract

With practical applications in mind, a study is made of the representations of the random fields on a sphere based on the theory of representations of the rotation group. The representation of the rotation group is defined in this paper by a class of rotational shift transformations of the random fields generated by a homogeneous random field on the sphere, 'homogeneous' with respect to the rotational motions. First, the spectral decomposition of a homogeneous scalar random field on a sphere is given a simple interpratation: it is a sum of the invariant vectors in the irreducible representation spaces of the rotational shift transformations. Some representations of the random fields are given in terms of the stochastic integrals with respect to a homogeneous random measure on the sphere. A homogeneous l-vector random field with $2l+1$ components in an irreducible space of weight l representation is defined as an invariant tensor under rotational shift transformations. A 'stochastic' spherical harmonics is defined as one of such random fields, which is a stochastic version of the l-vector spherical harmonic. Similarly, a 'stochastic' solid harmonic is defined in terms of stochasic spherical harmonics and generalized spherical Bessel functions. It is expressible as a l-vector Fourier integral over a sphere as well as a tensorial Fourier integral, and satisfies the l-vector Helmholtz equation. Such 'stochastic' harmonic functions can be used effectively in dealing with the scattering problem associated with a random sphere.

1. Introduction

A homogeneous random field in the n-dimensional (n-D) Euclidean space, $X(\boldsymbol{r})$, $\boldsymbol{r}\in R_n$ is characterized by the invariance of its probability distributions (in the strict sense) or of its correlation function (in the wide sense) under spatial translations $\boldsymbol{r}\to\boldsymbol{r}+\boldsymbol{a}$, $\boldsymbol{a}$ denoting an arbitrary vector in R_n: the 1-D homogeneous random field on the time axis therefore corresponds to a stationary process. In what follows we consider a 3-D homogeneous random field with zero mean. It is well known that as a result of the translation invariance the homogeneous

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random field has the spectral representation in the following form[1]:

$$X(\boldsymbol{r}) = \int_{R_3} e^{i\lambda\cdot r} dZ(\lambda) \tag{1}$$

$$<d\overline{Z(\lambda)}dZ(\lambda')> = \delta_{\lambda\lambda'} dF(\lambda) \tag{2}$$

where $\boldsymbol{\lambda}\cdot\boldsymbol{r}$ denotes the inner product, $dZ(\boldsymbol{\lambda})$ the random spectral measure with zero mean, $\langle\ \rangle$ the expectation, the overbar the complex conjugate, $dF(\lambda)$ the spectral measure, and $\delta_{\lambda\lambda'}=1$ $(\boldsymbol{\lambda}=\boldsymbol{\lambda}')$, $=0$ $(\boldsymbol{\lambda}\neq\boldsymbol{\lambda}')$. The spectral representation of the correlation function then becomes

$$R(\boldsymbol{r}) = \langle X(\boldsymbol{r}_1)X(\boldsymbol{r}_2)\rangle = \int_{R_3} e^{i\lambda\cdot r} dF(\boldsymbol{\lambda}) \tag{3}$$

where $\boldsymbol{r}=\boldsymbol{r}_1-\boldsymbol{r}_2$. The orthogonality (2) of the random measure is due to the homogeneity. If the random field has further symmetry, such as the rotational invariance,

$$R(g\boldsymbol{r}) = R(\boldsymbol{r}), \quad dF(g\boldsymbol{\lambda}) = dF(\boldsymbol{\lambda}) \tag{4}$$

where g denotes the arbitrary rotation in 3-D space or the corresponding Euler matrix, then the random field is said to be homogeneous and isotropic. In this case, $R(\boldsymbol{r})$ and $dF(\boldsymbol{\lambda})$ depend only on $r\equiv|\boldsymbol{r}|$ and $\lambda\equiv|\lambda|$, respectively, and the spectral reresentation (3) can be expressed as

$$R(r) = 4\pi\int_0^\infty j_0(\lambda r) dF(\lambda) \tag{5}$$

where $j_0(z) \equiv (\sin z)/z$ denotes the spherical Bessel function of 0-th order. Corresponding to (5), the spectral representation (1) can be cast into the polar from[2-5]: using the polar cordinates $\boldsymbol{r}\equiv(r, \theta, \phi)$,

$$X(\boldsymbol{r}) = \sum_{l=0}^{\infty}\sum_{m=l}^{l}\int_0^\infty J_l^m(\lambda r, \theta, \phi) dZ_l^m(\lambda) \tag{6}$$

$$\begin{aligned}&\langle d\overline{Z_l^m(\lambda)}dZ_{l'}^{m'}(\lambda')\rangle = (4\pi)^2\delta_{ll'}\delta_{mm'}\delta_{\lambda\lambda'}dF_l(\lambda)\\ &\qquad l, l'=0, 1, 2, \ldots, \ |m|\leq l, \ |m'|\leq l'\end{aligned} \tag{7}$$

where $dZ_l^m(\lambda)$ denotes a set of random spectral measures with the spectral measure $dF_l(\lambda)$, and the solid harmonics

$$J_l^m(\lambda r, \theta, \varphi) = j_l(\lambda r) Y_l^m(\theta, \varphi) \tag{8}$$

are given by the product of the spherical Bessel function and the normalized spherical harmonics (A. 9). The correlation function calculated from (6), (7) is readily reduced to (5) using the addition theorems for the spherical harmonics (A. 18) and the sphereical Bessel function.

Now we consider the random field on the sphere S_3 in 3-D space. A random field on the sphere whose probability distribution (correlation function) is invariant under arbitrary rotations is said to be homogeneous in the strict (wide) sense. The spectral representation of the homogeneous random field $X(\boldsymbol{t})$, $\boldsymbol{t} \equiv (1, \theta, \varphi)$ can be given in terms of spherical harmonics and the orthogonal random variables Z_l^m [6];

$$X(\boldsymbol{t}) = \sum_{l=0}^{\infty} \sum_{m=l}^{l} F_l Y_l^m(\theta, \varphi) Z_l^m$$

$$\langle Z_l^m \rangle = 0, \quad \langle \overline{Z_l^m}\, Z_{l'}^{m'} \rangle = \delta_{ll'} \delta_{mm'} \tag{10}$$

where Z_l^m satisfies the orthogonal property (10). The spectral representation for the correlation function can be written using (A. 18);

$$R(\theta) = \langle \overline{X(\boldsymbol{t}_1)} X(\boldsymbol{t}_2) \rangle = \sum_{l=0}^{\infty} |F_l|^2 \sum_{m=-l}^{l} \overline{Y_l^m(\theta_1, \varphi_1)} Y_l^m(\theta_2, \varphi_2) \tag{11}$$

$$= \frac{1}{4\pi} \sum_{l=0}^{\infty} (2l+1) |F_l|^2 P_l(\cos\theta) \tag{12}$$

where $\cos\theta = \boldsymbol{t}_1 \cdot \boldsymbol{t}_2$, θ denoting the angle between the two vectors $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$, (A. 19). We call $|F_l|^2$ the power spectrum and $F_l Z_l^m$ the random spectrum. Particularly, in view of (A. 18) and (A. 22), the 'white' spectrum $|F_l|^2 = 1$ (const.) gives the delta correlation for the white noise on the sphere: $R(\theta) = \delta(\theta)$, $\delta(\theta)$ denoting the delta function on the sphere with the measure $dS = \sin\theta d\theta d\varphi$. It should be noticed that the spectral representation (9) is partially involved in the spherical part of (6). The spectral representation (9) can be obtained either by the direct orthogonal expansion in spherical harmonics or by the Karhunen-Loéve expansion[7] on the sphere for a rotationally invariant correlation function such that $R(\boldsymbol{t}, \boldsymbol{t}') = R(g\boldsymbol{t}, g\boldsymbol{t}')$. In fact, for the eigenvalue integral equation on the sphere,

$$\lambda\varphi(\boldsymbol{t}) = \int_{S_3} R(\boldsymbol{t}, \boldsymbol{t}')\varphi(\boldsymbol{t}') dS', \quad dS = \sin\theta' d\theta' d\varphi' \tag{13}$$

with the kernel $R(\boldsymbol{t}, \boldsymbol{t}')$ commuting with the operator S^g defined by (A. 4), we see that $Y_l^m(\theta, \varphi)$, a vector in the invariant space of S^g, is also an eigenfunction of (13) with the degenerate eigenvalue (multiplicity $2l+1$)

$$\lambda_l = |F_l|^2 \ (\geq 0), \quad l = 0, 1, 2, \ldots, \quad m = -l, \ldots, l \tag{14}$$

Therefore, (11) is none other than the Hilbert-Schmidt expansion of the integral kernel and the spectral representation (9) corresponds to the Karhunen-Loeve expansion.

Noting that in the Euclidean space the translations or the rotations form a group of motions, we consider more generally a 'homogeneous' space on which a group of motions is defined. The theory of special functions on a homogeneous space has been reformulated from a group-theoretic point of view[8-10]. When the motion group is represented by means of a certain transformation group in a function space, the matrix elements of an irreducible representation are regarded as the special functions on the homogeneous space. Then, the irreducible decomposition gives an orthogonal expansion in terms of special functions, and the geometrical property of the special functions known as the addition theorem is simply displayed by the matrix multiplication. We could apply the same line of thought to a random field defined on a homogeneous space. The homogeneity of the random field is defined in regard to its invariance of the probability distributions (strict sense) or the correlation function (wide sense) under the group of motions in the homogeneous space[1, 11-13]. Using this group-theoretic concept, for instance, the spectral representation of a homogeneous random field could be obtained as a decomposition of a vector into irreducible spaces, and a more extended calcus of the random fields could be formulated.

In the present paper we restrict ourselves to the homogeneous random field on a spere where the rotational motions form the rotation group $SO(3)$[9]. For later reference some necessary definitions and formulas concerning the representation of the rotation group are summarized in the appendix. The representation of the rotation group G is made in this paper by introducing a transformation group U^g, $g \in G$, operating on the random variables and by the transformation group D^g operating on the random fields. The group 'representation' is meant to be the group homomorphism $g \rightarrow U^g$, or $g \rightarrow D^g$, and should not be confused with the 'representation' of the random field in what follows. We first show in Sec. 2 that the spectral representation of a scalar homogeneous random field (9) is obtained simply as a result of the irreducible decomposition and that it can be interpreted in a simple manner in terms of the invariant l-vectors, that is, the l-vector is the vectorial quantity which is transformed upon spatial rotation by the matrix of irreducible representation of weight l. In Sec. 3, elementary random variables forming the l-vector basis for the transformation U^g are constructed by means of the stochastic integrals on the sphere. In Sec. 4, the operator D^g is defined, which keeps invariant a homogeneous random field. Some representations of homogeneous random fields are derived from the stochastic integrals, a homogeneous l-vector random field is defined in connection with its transformation property under D^g. Sec. 5 gives a brief discussion on the multiple Wiener integral, Wiener-Hermite expansion and its tensorial property under U^g. As homogeneous l-vector random fields on S_3 we define in Sec. 6 a set of 'stochastic spherical harmonics', in terms of which we can expand any random field on the sphere.

Similarly, in these terms we define a set of 'stochastic solid harmonics' on R_3, which satisfy the l-vector Helmholtz equation. All these are the stochastic analogue to the l-vector spherical functions in the nonrandom case[14]. Therefore, for example, the random wave field scattered from a homogeneous random sphere can be expanded in terms of the stochastic solid harmonics so defined. Such a scattering problem is treated elsewhere[25].

2. Spectral Representation of a Homogeneous Random Field on Sphere

Let $(\Omega, \mathscr{B}, \boldsymbol{P})$ denote the probability space (Ω denotes the sample space, $\mathscr{B}$ the Borel field on Ω and $\boldsymbol{P}$ the probability measure), and let $Y(\omega)$ denote a random variabe ($\mathscr{B}$ measurable function); ω indicates the probability parameter denoting a sample point in Ω which will be often supressed for brevity. Let $L^2(\Omega)$ denote the Hilbert space of random variables such that $\langle |Y|^2\rangle < \infty$ with inner product $(Y_1, Y_2)_\Omega = \langle \overline{Y}_1 Y_2\rangle$, $\langle\ \rangle$denoting the average over Ω. Let a point on the sphere S_3 be denoted by the vector $\equiv(1, \theta, \varphi)$ and $X(\boldsymbol{t})\equiv X(\boldsymbol{t}, \omega)$ be a q. m. (quadratic mean) continuous random field on S_3. In this section, let $L_x^2(\Omega)$ denote the sub-Hilbert space of random variables which is linearly generated from $X(\boldsymbol{t})$ (q. m. limit of linear transformation). The random field on the sphere, $X(\boldsymbol{t})$, can be as well regarded as random field on G by (A. 2);

$$X(\boldsymbol{t})\equiv X(g_t\boldsymbol{e}_0), \quad \boldsymbol{t}\equiv g_t\boldsymbol{e}_0, \ g_t \in G \tag{15}$$

where $\boldsymbol{e}_0$ denotes a unit vector along the polar axis. The scalar field (15) is independent of the third Euler angle φ_2 or of the rotation around $\boldsymbol{t}$. The random field $X(\boldsymbol{t})$ is said to be homogeneous in the wide sense, if the correlation function $R(\boldsymbol{t}_1, \boldsymbol{t}_2)=\langle X(\boldsymbol{t}_1)X(\boldsymbol{t}_2)\rangle$ is invariant under arbitrary rotations;

$$R(\boldsymbol{t}_1, \boldsymbol{t}_2)=R(g\boldsymbol{t}_1, g\boldsymbol{t}_2), \quad \boldsymbol{t}_1, \boldsymbol{t}_2 \in S_3, \ g \in G \tag{16}$$

from which follows that $R(\boldsymbol{t}_1, \boldsymbol{t}_2)$ is a function only of $(\boldsymbol{t}_1 \cdot \boldsymbol{t}_2)=(\boldsymbol{e}_0 \cdot g_1^{-1}g_2\boldsymbol{e}_0)=[g_1^{-1}g_2]_{00}=\cos\theta$, θ being given by (A.19). Hence, we write the correlation function as $R(\theta)\equiv R(\boldsymbol{t}_1, \boldsymbol{t}_2)$.

Let the rotational transformation of the homogeneous random field $X(\boldsymbol{t})$ be as defined by (A. 4);

$$S^g:\ X(\boldsymbol{t}) \to X(g^{-1}\boldsymbol{t}),\quad g \in G \tag{17}$$

Then the transformation S^g on $X(\boldsymbol{t})$ induces a transformation U^g on random variables $Y \in L_x^2(\Omega)$;

$$U^g:\ Y \to U^g Y,\ g \in G \tag{18}$$

which we call the shift transformation. From the invariance (16), it easily follows that the covariance of $Y, Z \in L_x^2(\Omega)$ is invariant under U^g;

$$\langle \overline{U^g Y}\ U^g Z \rangle = \langle \overline{Y} Z \rangle,\quad Y, Z \in L_x^2(\Omega) \tag{19}$$

implying that U^g is a unitary operator in $L_x^2(\Omega)$, and from (A. 5) and (18) follows the group property,

$$U^{g_1} U^{g_2} = U^{g_1 g_2},\ [U^g]^{-1} = U^{g^{-1}},\ U^e = 1 \tag{20}$$

meaning that U^g, $g \in G$, gives the unitary representation of the rotation group in $L_x^2(\Omega)$. It also follows from the q. m. continuity of $X(\boldsymbol{t})$ that U^g is continuous with respect to the parameter g.

As a special case of (18), namely $U^g X(\boldsymbol{t}) = X(g^{-1}\boldsymbol{t})$, the homogeneous scalar random field $X(\boldsymbol{t})$ is expressible as

$$X(\boldsymbol{t}) = X(g\boldsymbol{e}_0) = U^{g^{-1}} X(\boldsymbol{e}_0),\quad \boldsymbol{t} \equiv g\boldsymbol{e}_0 \in S_3,\quad g \in G \tag{21}$$

that is, the value of $X(\boldsymbol{t})$ at $\boldsymbol{t} \equiv g\boldsymbol{e}_0$ is obtained by the transformation $U^{g^{-1}}$ from $X(\boldsymbol{e}_0)$, i. e., the value at the 'north pole' $\boldsymbol{e}_0 \equiv \boldsymbol{e}_z$. Denote by H the subgroup of rotations around the 'polar axis' $\boldsymbol{e}_0$; $h\boldsymbol{e}_0 = \boldsymbol{e}_0$, $H \in H$, and by H_t the subgroup of rotations around the vector $t = g\boldsymbol{e}_0$: $h_t \boldsymbol{t} = \boldsymbol{t}$; $h_t = ghg^{-1} \in H_t$, $h \in H$. Then, for the scalar field $X(\boldsymbol{t})$ we have

$$U^h X(\boldsymbol{t}) = X(\boldsymbol{t}),\ h \in H_t;\quad U^h X(\boldsymbol{e}_0) = X(\boldsymbol{e}_0),\ h \in H \tag{22}$$

To generalize (21) we can construct a random field on G generated from $X(\boldsymbol{t})$ in the following manner. Using a random variable Y in $L_x^2(\Omega)$, we put

$$Y(g) \equiv U^{g^{-1}} Y,\quad g \in G \tag{23}$$

which is a homogeneous random field on G with the correlation function

$$\langle \overline{Y(g_1)}\, Y(g_2) \rangle = \langle \overline{Y} U^{g_1 g_2^{-1}} Y \rangle \tag{24}$$

depending on $g_1 g_2^{-1}$ by virtue of (19) and (20).

By the representation theory of the rotation group, the representation space $L_x^2(\Omega)$ for U^g can be decomposed into the sum of irreducible spaces. Correspondingly, a vector $X(e_0)$ in $L_x^2(\Omega)$ can be decomposed into the vectors of orthogonal irreducible spaces. For convenience we denote by $D_l(\Omega)$ an irreducible space of the weight-l representation for U^g. Denote the canonical basis in $D_l(\Omega)$ fixed at the north pole $e_0 \equiv e_z$ by

$$Z_l^m \equiv Z_l^m(e_0), \quad m=-l, \dots, l \tag{25}$$

which satisfies the orthogonality relation,

$$\langle \overline{Z_l^m} Z_{l'}^{m'} \rangle = \delta_{ll'} \cdot \delta_{mm'}. \tag{26}$$

Since by (22) $X(e_0)$ has only the 0-th canonical components relative to e_0, its irreducible decomposition can be written in terms only of Z_l^0,

$$X(e_0) = \sum_{l=0}^{\infty} \sqrt{\frac{2l+1}{4\pi}} F_l Z_l^0 \tag{27}$$

where the expansion coefficient can be given

$$\sqrt{\frac{2l+1}{4\pi}} F_l = \langle \overline{Z_l^0} X(e_0) \rangle, \quad l=0, 1, 2, \dots \tag{28}$$

By (21) we obtain $X(t)$ from (27) by applying $U^{g^{-1}}$,

$$X(t) = \sum_{l=0}^{\infty} \sqrt{\frac{2l+1}{4\pi}} F_l U^{g^{-1}} Z_l^0 = \sum_{l=0}^{\infty} \sqrt{\frac{2l+1}{4\pi}} F_l Z_l^0(t) \tag{29}$$

where

$$Z_l^m(t) \equiv U^{g^{-1}} Z_l^m, \qquad m=-l, \dots, l, \ t \equiv g e_0 \tag{30}$$

denotes the moving canonical basis in $D_l(\Omega)$ attached to t.

The expansion (29) gives the spectral representation of $X(t)$ in terms of the moving canonical basis in $D_l(\Omega)$. That each term of (29) has only 0-th canonical component implies according to (A.27) that the homogeneous random field $X(t)$ is decomposed into the sum of the isotropic l-vector field in $D_l(\Omega)$, which is a simple geometrical interpretation for the spectral representation.

Now let us represent (29) in terms of the fixed canonical basis at the 'north pole' e_0. By (30) and (A.15) we represent $Z_l^0(t)$ in terms of the fixed canonical basis, using (A.9) and the unitarity of $T_{mn}^l(g)$,

$$Z_l^0(t) \equiv U^{g^{-1}} Z_l^0 = \sum_{m=-l}^{l} T_{om}^l(g^{-1}) Z_l^m$$

$$=\sqrt{\frac{4\pi}{2l+1}}\sum_{m=-l}^{l}Y_l^m(\theta,\varphi)Z_l^m \tag{31}$$

Substituting this into the righthand side of (29) we recover the spectral representation (9) in the original form. Thus we have seen that the random spectrum $F_l Z_l^m$ in (9) can be interpreted simply as the fixed canonical basis (25) in $D_l(\Omega)$. Therefore, the simple spectral representation (29) is a kind of 'coordinate-free' representation while the original one (9) is a 'coordinate-fixed' representation. Likewise, the polar spectral representation (6) for the homogeneous and isotropic random field in R_3 can be concisely expressed in terms of the moving canonical basis in $D_l(\Omega)$. Such group-theoretic or geometric simplification is greatly helpful when we deal with the random fields generated by the original homogeneous random field.

3. Homogeneous Random Measure and Stochastic Integral on the Sphere

Analogous to the 1-D case[15-17] we define the random measure and the stochastic integral over the sphere. Let Δ denote an interval or the joint or the meeting of intervals on the sphere S_3 (more generally, $\Delta\in\mathscr{B}_s$; $\mathscr{B}_s$ denoting the Borel field on S_3 generated by the intervals), and $|\Delta|$ denoting its area (measure): then $|S_3|=4\pi$. Denote by $g\Delta$ ($g\in G$) the rotated interval. Then

$$|g\Delta|=|\Delta|,\ g\in G,\ \Delta\in\mathscr{B}_s \tag{32}$$

Consider the real random measure $B(\Delta)\equiv B(\Delta,\omega)$ on S_3 such that

$$\langle B(\Delta)\rangle=0 \tag{33}$$

$$\langle B(\Delta_1)B(\Delta_2)\rangle=|\Delta_1\Delta_2|,\ \langle B(\Delta)^2\rangle=|\Delta| \tag{34}$$

$$B(\sum_i\Delta_i)=\sum_i B(\Delta_i);\quad \Delta_i\Delta_j=\phi,\ i\neq j \tag{35}$$

ϕ being the null set. Obviously by (32), we have

$$\langle B(g\Delta_1)B(g\Delta_2)\rangle=\langle B(\Delta_1)B(\Delta_2)\rangle=|\Delta_1\Delta_2| \tag{36}$$

implying the wide-sense homogeneity of the random measure. We now assume the random measure to be homogeneous in the strict sense such that for any set of disjoint intervals Δ_i, the multidimensional probability distribution for $B(g\Delta_i)$ is invariant under rotations $g\in G$. Specifically, $B(\Delta)$ is termed the homogeneous Gaussian random measure if $B(\Delta_i)$ obeys the Gaussian distribution.

In what follows, let $\mathscr{B}$ denote the Borel field generated by the random measure $B(\varDelta, \omega)$, so that a $\mathscr{B}$ measurable function $Y(\omega)$ implies a random variable or a nonlinear functional generated by $B(\varDelta, \omega)$. Analogous to (17) and (18), the rotation of the random measure,

$$S^g B(\varDelta, \omega) = B(g^{-1}\varDelta, \omega), \quad g \in G \tag{37}$$

induces the transformation of a random variable, denoted by U^g;

$$U^g: \quad Y(\omega) \rightarrow Y'(\omega) \equiv U^g Y(\omega) \tag{38}$$

which satisfies the same group property (20). We call U^g the shift transformation. The strict-sense homogeneity of the random measure implies that the measure-preserving set transformation $T^g: A \rightarrow A' = T^g A$, is induced on Ω, such that $P(T^g A) = P(A)$, A, $A' \in \mathscr{B}$. For convenience we write this formally as a point transformation on Ω without any essential loss of rigor[18],

$$T^g: \quad \omega \rightarrow \omega' = T^g \ , \ \omega, \omega' \in \Omega, \ g \in G \tag{39}$$

and rewrite (38) in the following manner,

$$U^g Y(\omega) = Y(\omega') \equiv Y(T^g \omega) \tag{40}$$

which is intuitively understandable because the sample point ω can be looked upon as if it were a coodinate parameter. Then the rotational homogeneity of the random measure can be written

$$B(g^{-1}\varDelta, \omega) = B(\varDelta, \ T^g \omega) \tag{41}$$

From this follows the relation, e. g., $B((g_1 g_2)^{-1}\varDelta, \omega) = B(g_2^{-1} g_1^{-1}\varDelta, \omega) = B(g_1^{-1}\varDelta, T^{g_2}\omega)$ $= B(\varDelta, \ T^{g_1} T^{g_2}\omega)$. Hence, we have the group property of T^g analogous to (20);

$$T^{g_1} T^{g_2} = T^{g_1 g_2}, \ [T^g]^{-1} = T^{g^{-1}}, \ T^e = I \tag{42}$$

We call T^g, $g \in G$, the rotational transformation on Ω

Let $L^2(S_3)$ denote the Hilbert space consisting of the (complex-valued) functions $f(\boldsymbol{t})$ on the sphere, and define the inner product and the norm $\|f\|_S$ by

$$(f, g)_S \equiv \int_{S_3} \overline{f(\boldsymbol{t})} g(\boldsymbol{t}) dt\, dS, \quad \|f\|_S^2 \equiv (f, f)_S \tag{43}$$

Denote by

$$I(f) = \int_{S_3} f(\boldsymbol{t}) dB(\boldsymbol{t}) \tag{44}$$

where $dB(\boldsymbol{t}) = B(d\boldsymbol{t})$, the stochastic integral of $f \in L^2(S_3)$ defined with respect to the random measure in the q. m. sense, which satisfies the following properties,

$$I(af+bg)=aI(f)+bI(g) \tag{45}$$

$$\langle I(f)\rangle=0 \tag{46}$$

$$\langle I(f)I(g)\rangle=(f, g)_S, \ \langle \mid I(f) \mid ^2\rangle= \| f \|_S^2 \tag{47}$$

where $f, g \in L^2(S_3)$ and a, b are constants. For an orthonormal system $\varphi_n(t)$ in $L^2(S_3)$ such that

$$(\varphi_n, \varphi_m)_S=\delta_{nm}, \quad n, m=1, 2, \ldots \tag{48}$$

we can form a set of orthogonal random varaibles $B_n\equiv B_n(\omega)$ by

$$B_n\equiv I(\varphi_n)=\int_{S_3}\varphi_n(t)dB(t) \tag{49}$$

$$\langle \overline{B_n}B_m\rangle=\delta_{nm}, \quad n, m=1, 2, \ldots. \tag{50}$$

If $\{\varphi_n\}$ is a complete system in $L^2(S_3)$, $\{B_n\}$ forms a complete orthonormal system in $L_B^2(\Omega)$ ($\subset L^2(\Omega)$), a subspace linearly generated from $B(\Delta)$. When f is expanded as

$$f(t)=\sum_{n=1}^{\infty}f_n\varphi_n(t), \quad f_n\equiv(\varphi_n, f)_S \tag{51}$$

then the stochastic integral (44) is expanded in terms of B_n (in q. m. sense);

$$\int_{S_3}f(t)dB(t)=\sum_{n=1}^{\infty}f_nB_n \tag{52}$$

For a specific example, we define a set of orthogonal random variables by the stochastic integral of spherical harmonics;

$$B_l^m=\int_{S_3}\overline{Y_l^m(\theta, \varphi)}\, dB(t) \tag{53}$$

$$=\sqrt{\frac{2l+1}{4\pi}}\int_{S_3}T_{mo}^l(g_t)dB(g_te_0) \tag{54}$$

$$l=0, 1, 2, \ldots, \quad m=-l, \ldots, l$$

$$\langle \overline{B_l^m}\, B_{l'}^{m'}\rangle=\delta_{ll'}\delta_{mm'} \tag{55}$$

If f is developed in terms of spherical harmonics,

$$f(t)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}f_l^mY_l^m(\theta, \varphi), \quad f_l^m\equiv(Y_l^m, f)_S \tag{56}$$

then we have the expansion

$$\int_{S_3} f(\boldsymbol{t})dB(\boldsymbol{t}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} f_l^m B_l^m \tag{57}$$

which can be regarded as the transformation of the representation for a random variable $I(f)$.

Now, for later reference we apply $U^{g^{-1}}$ or $T^{g^{-1}}$ to (54), using (41), (A.20) and (A.16), to obtain the transformation rule for B_l^m:

$$U^{g^{-1}} B_l^m \equiv B_l^m (T^{g^{-1}}\omega) \tag{58}$$

$$= \sqrt{\frac{2l+1}{4\pi}} \int_{S_3} T^l_{mo}(g\boldsymbol{t})dB(gg_t\boldsymbol{e}_0) = \sqrt{\frac{2l+1}{4\pi}} \int_{S_3} T^l_{mo}(g^{-1}g_t)dB(g_t\boldsymbol{e}_0)$$

$$= \sum_{s=-l}^{l} \overline{T^l_{sm}(g)}\, B_l^s,\ m=-l,\dots,l \tag{59}$$

When compared to (A.15) or (31), (59) shows that B_l^m, $m=-l,\dots,l$, gives a fixed canonical basis in the representation space $D_l(\Omega)$, which is transformed by $U^{g^{-1}}$ as a $\bar{l}$-vector like $\overline{\boldsymbol{e}_{(l)m}}$. Therefore,

$$B_l^m(\boldsymbol{t},\omega) \equiv B_l^m(T^{g^{-1}}\omega),\ \ m=l\dots,l,\ \ \boldsymbol{t}\equiv g\boldsymbol{e}_0 \tag{60}$$

form a moving canonical basis in $D_l(\Omega)$ relative to $\boldsymbol{t}\equiv g\boldsymbol{e}_0$, which is transformed as a $\bar{l}$-vector upon rotation, and satisfies the orthogonality:

$$\langle \overline{B_l^m(\boldsymbol{t})} B_{l'}^{m'}(\boldsymbol{t})\rangle = \langle \overline{B_l^m} B_{l'}^{m'}\rangle = \delta_{ll'}\delta_{mm'} \tag{61}$$

readily following from (55) and (A.13). To generalize this further, we calculate the correlation function,

$$\langle \overline{B_l^m(\boldsymbol{t}_1)} B_{l'}^n(\boldsymbol{t}_2)\rangle = \delta_{ll'}\ \overline{T^l_{mn}(g_2^{-1}g_1)},\ \boldsymbol{t}_1\equiv g_1\boldsymbol{t}_0,\ \boldsymbol{t}_2\equiv g_2\boldsymbol{e}_0 \tag{62}$$

which is analogous to (A.17), and as a special case we have

$$\langle \overline{B_l^0(\boldsymbol{t}_1)} B_{l'}^0(\boldsymbol{t}_2)\rangle = \delta_{ll'}P_l(\cos\theta),\ \ \cos\theta = \boldsymbol{t}_1\cdot\boldsymbol{t}_2 \tag{63}$$

which is analogous to (A.18).

4. Shift Transformation and Homogeneous Random Fields on the Sphere

In what follows we deal with a random field on the sphere $\Psi(\boldsymbol{t},\omega)$ as a ($\mathscr{B}_S \times \mathscr{B}$ measurable) function on $S_3\times\Omega$, or more generally a random field $\Psi(g,\omega)$ on G. For the sake of practical applications we introduce the shift transformation D^g operating on random fields using the convenient notation T^g, instead of U^g.

We define the operator D^g, $g \in G$, by

$$D^g \Psi(\boldsymbol{t}, \omega) = \Psi(g^{-1}\boldsymbol{t}, T^{g^{-1}}\omega) \tag{64}$$

$$D^g \Psi(g_0, \omega) = \Psi(g^{-1}g_0, T^{g^{-1}}\omega) \tag{65}$$

Writing $\boldsymbol{t} \equiv g_0\boldsymbol{e}_0$, (64) is a special case of (65). From (A.5) and (42) it easily follows that D^g gives a representation of the rotation group G:

$$D^{g_1}D^{g_2} = D^{g_1 g_2}, \ [D^g]^{-1} = D^{g^{-1}}, \ D^e = 1 \tag{66}$$

since, for instance, we have $D^{g_1}D^{g_2}\Psi(g, \omega) = D^{g_1}\Psi(g_2^{-1}g, T^{g_2^{-1}}\omega) =$ $\Psi(g_2^{-1}g_1^{-1}g, T^{g_2^{-1}}T^{g_1^{-1}}\omega) = \Psi((g_1g_2)^{-1}g, T^{(g_1g_2)^{-1}}\omega)$. The operator D^g, being a measure transformation on $S_3 \times \Omega$, can be applied to the random measure as well. The shift operator D^g introduced here is analogous to the shift operator operating on stationary processes[19].

The homogeneity of the random measure (41) can be rephrased as the D^g invariance;

$$D^g B(\Delta, \omega) \equiv B(g^{-1}\Delta, T^{g^{-1}}\omega) = B(\Delta, \omega) \tag{67}$$

that is, the homogeneous random measure is D^g invariant. Analogously, if a random field $X(\boldsymbol{t}, \omega)$ generated by $B(\Delta)$ is D^g invariant, that is,

$$D^g X(\boldsymbol{t}, \omega) \equiv X(g^{-1}\boldsymbol{t}, T^{g^{-1}}\omega) = X(\boldsymbol{t}, \omega) \tag{68}$$

then $X(\boldsymbol{t}, \omega)$ is a homogeneous (scalar) random field on S_3. In fact, (68) implies the homogeneity of the random field as well as (41). Therefore, rewriting the relation (68), we have the expression similar to (21);

$$X(\boldsymbol{t}, \omega) = X(\boldsymbol{e}_0, T^{g^{-1}}\omega) \equiv U^{g^{-1}}X(\boldsymbol{e}_0, \omega), \quad \boldsymbol{t} \equiv g\boldsymbol{e}_0 \tag{69}$$

that is, the value at $\boldsymbol{t} \equiv g\boldsymbol{e}_0$ can be obtained by applying $U^{g^{-1}}$ from the value at the north pole, $X(\boldsymbol{e}_0, \omega)$, which is a scalar quantity being invariant upon rotation around $\boldsymbol{e}_0$;

$$X(h^{-1}\boldsymbol{e}_0, \omega) = X(\boldsymbol{e}_0, T^h\omega) = X(\boldsymbol{e}_0, \omega); \ h \in H, \ h\boldsymbol{e}_0 = \boldsymbol{e}_0 \tag{70}$$

More generally, using a random variable $Y(\omega)$ ($\mathscr{B}$ measurable) we make a random field on G,

$$Y(g, \omega) \equiv Y(T^{g^{-1}}\omega), \ g \in G \tag{71}$$

which is easily shown to be D^g invariant and hence a homogeneous random field on G. A random field on S_3, like (69), is a special case of (71) such that $Y(T^h\omega)$

$=Y(\omega),\ \ h \in H.$

Now we derive a homogeneous random field from a stochastic integral,

$$X(\omega)=\int_{S_3} f(t')dB(t', \omega) \tag{72}$$

where $f(t)$ is assumed to be zonal, that is,

$$f(ht)=f(t),\ \ h \in H \tag{73}$$

Putting $X(\omega)\equiv X(e_0, \omega)$, we apply $U^{g^{-1}}$ to (72) to obtain

$$\begin{aligned} X(T^{g^{-1}}\omega)&=\int_{S_3} f(t')dB(t', T^{g^{-1}}\omega) \\ &=\int_{S_3} f(t')dB(gt', \omega)=\int_{S_3} f(g^{-1}t')dB(t', \omega) \end{aligned} \tag{74}$$

in the same manner as (59). The U^h invariance for $X(\omega)$ is readily checked putting $g \to h$ in (74) and using (73). Thus, the stochastic integral

$$X(t, \omega)=\int_{S_3} f(g^{-1}t')dB(t', \omega),\ \ t\equiv ge_0 \tag{75}$$

gives a homogeneous scalar random field, which we call the 'moving average' on the sphere, an analogy of the moving average for a stationary process.

Since $f(t)$ is zonal by (73), it can be expanded in terms of the zonal spherical functions ($m=0$) in the form

$$f(t)=\sum_{l=0}^{\infty}\sqrt{\frac{2l+1}{4\pi}}F_l Y_l^0(\theta, \varphi) \tag{76}$$

which is substituted into (72) to yield

$$X(\omega)=\sum_{l=0}^{\infty}\sqrt{\frac{2l+1}{4\pi}}F_l B_l^0(\omega) \tag{77}$$

Therefore, applying $U^{g^{-1}}$ to (74) and using (59), we obtain

$$X(t, \omega)\equiv X(T^{g^{-1}}\omega)=\sum_{l=0}^{\infty}\sqrt{\frac{2l+1}{4\pi}}F_l B_l^0(T^{g^{-1}}\omega) \tag{78}$$

$$=\sum_{l=0}^{\infty}\sqrt{\frac{2l+1}{4\pi}}F_l \sum_{m=-l}^{l}\overline{T_{mo}^l(g)}\,B_l^m(\omega)$$

$$=\sum_{l=0}^{\infty}F_l \sum_{m=-l}^{l} Y_l^m(\theta, \varphi)\, B_l^m(\omega) \tag{79}$$

which again gives the spectral representation for a homogeneous random field. (78) is in the coordinate-free form (29), while (79) is in the original coordinate

-fixed form (9).

Now we calcuate the correlation function of the moving average (75), using (47), (76), (A.18) and (A.20),

$$R(t_1, t_2) = \int_{S_3} f(g_1^{-1}t) f(g_2^{-1}t) dS = \int_{S_3} f(t) f(g_2^{-1}g_1 t) dS$$

$$= \frac{1}{4\pi} \sum_{l=0}^{\infty} (2l+1) \mid F_l \mid^2 P_l(\cos\theta) \tag{80}$$

which is the spectral representation (12) for the correlation function, easily following from (78) and (63). As mentioned at (12) the 'white' spectrum $\mid F_l \mid^2 = 1$ gives the delta correlation for a white noise on the sphere.

Next, we consider a l-vector random field with components,

$$X_l^m(t, \omega), \quad m = -l, \ldots, l, \quad t \equiv g_t e_0 \tag{81}$$

which are transformed by D^g in the following manner:

$$D^g X_l^m(t, \omega) = \sum_{s=-l}^{l} \overline{T_{sm}^l(g)} X_l^s(t, \omega), \quad m = -l, \ldots, \tag{82}$$

Such a random field is said to be a homogeneous l-vector random field. One of such l-vector random fields can be represented in the following form,

$$X_l^m(t, \omega) = T_{mn}^l(g_t) X_n(t, \omega), \quad m = -l, \ldots, , \tag{83}$$

where $X_n(t,)$, $n = -l, \ldots, l$, represents a D^g-invariant scalar random field of the form (69). In fact, the factor $T_{mn}^l(g_t)$ in (83) is transformed under D^g or S^g like (82) according to (A.16). Since by (A.15) $T_{mn}^l(g_t)$ is the m-th component of $e_{(l)n}(t)$ in the fixed canoical basis, (83) can be expressed in the l-vector notation as

$$X_{(l)n}(t, \omega) = e_{(l)n}(t) X_n(t, \omega) \tag{84}$$

As the l-vector fields, (84) with a different n is linearly independent of (orthogonal to) each other. A homogeneous l-vector random field can be written generally as a linear combination of (84) in n, examples of which will appear in Sec. 6. It is to be noted that the random field of the form (83) or (84) is a rotational counterpart of the 'stochastic' Floquet theorem based on the translational motion[19].

5. Wiener-Hermite Expansion on the Sphere

If $B(\Delta, \omega)$ is a Gaussian random measure we can further define the multiple stochastic inteqral called the multiple Wiener integral in much the same way as the 1-D Euclidean case[15,16,20]. To demonstrate the transformation properties, a convenient definition can be made in terms of Wiener–Hermite differentials[19,21],

$$h_n[dB(t_1),\ldots,dB(t_n)],\ n=0,1,2,\ldots \tag{85}$$

where h_n is defined using n-variate Hermite polynomials[22] (see (95) below). First, a few differentials can be written

$$\begin{aligned} &h_0=1,\ h_1[dB(t)]=dB(t),\\ &h_2[dB(t_1),dB(t_2)]=dB(t_1)dB(t_2)-\delta_{t_1t_2}dt_1\\ &h_3[dB(t_1),dB(t_2),dB(t_3)]=dB(t_1)dB(t_2)dB(t_3)\\ &\qquad-[\delta_{t_1t_2}dt_1dB(t_3)+\delta_{t_2t_3}dt_2dB(t_1)+\delta_{t_3t_1}dB(t_2)] \end{aligned} \tag{86}$$

where $\delta_{tt'}=1,\ t=t'$; $=0,\ t\neq t'$. Omitting details (c.f., references), we give here the formal expression for the n-tuple Wiener integral,

$$I_n(f)=\int_{S_3}\cdots\int_{S_3}f(t_1,\ldots,t_n)h_n[dB(t_1),\ldots,dB(t_n)] \tag{87}$$

where $f(t_1,\ldots,t_n)\in L^2(S_3^n)$ denotes a n-variate function on S_3, $L^2(S_3^n)$ denoting the Hilbert space of n-variate functions with the inner product

$$(f,g)_n=\int_{S_3}\cdots\int_{S_3}f_n(t_1,\ldots,t_n)g(t_1,\ldots,t_n)dt_1\ldots dt_n \tag{88}$$

and the norm $\|f\|_n=[(f,f)_n]^{1/2}$. The multiple Wiener integrals satisfy the following properties;

$$I_n(af+bg)=aI_n(f)+bI_n(g) \tag{89}$$

$$I_n(f)=I_n(\tilde{f}) \tag{90}$$

$$\langle\overline{I_n(f)}I_m(g)\rangle=\delta_{nm}n!\,(\tilde{f},\tilde{g})_n,\ \langle|I_n(f)|^2\rangle=\|f\|_n^2 \tag{91}$$

$$\tilde{f}(t_1,\ldots,t_n)\equiv\frac{1}{n!}\sum_{perm.}f(t_1,\ldots,t_n) \tag{92}$$

where the summation is to be taken over all $n!$ permutations of n variables to symmetrize f. The case of $n=1$ reduces to (44)–(47).

When the n-variate function f_n is expanded in terms of the complete orthonormal system $\varphi_i(t)$.

$$f(t_1,\ldots,t_n)=\sum_{i_1}\cdots\sum_{i_n}f_{i_1\ldots i_n}\varphi_{i_1}(t_1)\ldots\varphi_{i_n}(t_n) \tag{93}$$

it is easily shown by virtue of the tensorial property of n-variate Hermite polynomials that the n-tuple Wiener integral (87) can be transformed into the n

-tuple summation,

$$I_n(f)=\int_{S_3}\cdots\int_{S_3}f(\boldsymbol{t}_1,\ldots,\boldsymbol{t}_n)h_n[dB(\boldsymbol{t}_1),\ldots,dB(\boldsymbol{t}_n)]$$

$$=\sum_{i_1}\cdots\sum_{i_n}f_{i_1\cdots i_n}H_n(B_{i_1},\ldots,B_{i_n}) \tag{94}$$

where

where B_i defined by (49) gives a sequence of independent Gaussian variables and $H_n(B_1,\ldots,B_n)$ denotes the n-variate Hermite polynomial. First few polynomials are

$$\begin{aligned}&H_0=1,\ H_1(B)=B,\quad H_2(B_i,B_j)=B_iB_j-\delta_{ij}\\&H_3(B_i,B_j,B_k)=B_iB_jB_k-\delta_{ij}B_k-\delta_{jk}B_i-\delta_{ki}B_j\end{aligned} \tag{95}$$

It is noted that the n-variate Hermite polynomial is transformed as a tensor of degree n under linear transformations of the variables.

A random variable $X(\omega)$ generated by the Gaussian random measure $B(\varDelta)$ on the sphere ($L^2(\Omega)$ functional) can be developed in terms of the orthogonal functionals,

$$X(\omega)=\sum_{n=0}^{\infty}\int_{S_3}\cdots\int_{S_3}f_n(\boldsymbol{t}_1,\ldots,\boldsymbol{t}_n)h_n[dB(\boldsymbol{t}_1),\ldots,dB(\boldsymbol{t}_n)] \tag{96}$$

$$=\sum_{n=0}^{\infty}\sum_{i_1}\cdots\sum_{i_n}f_{i_1\cdots i_n}H_n(B_{i_1},\ldots,B_{i_n}) \tag{97}$$

which are the Wiener-Hermite (W-H) expansions on the sphere and the discrete representation (97) corresponds to the Cameron-Martin expansion[23].

When spherical harmonics $Y_l^m(\boldsymbol{t})$, $\boldsymbol{t}\equiv(1,\theta,\varphi)$, are used in the expansion (93),

$$f(\boldsymbol{t}_1,\ldots,\boldsymbol{t}_n)=\sum_{l_1m_1}\cdots\sum_{l_nm_n}f_{l_1\cdots l_n}^{m_1\cdots m_n}Y_{l_1}^{m_1}(\boldsymbol{t}_1)\cdots Y_{l_n}^{m_n}(\boldsymbol{t}_n) \tag{98}$$

then (94) becomes

$$I_n(f)=\sum_{l_1m_1}\cdots\sum_{l_nm_n}f_{l_1\cdots l_n}^{m_1\cdots m_n}H_n(B_{l_1}^{m_1},\ldots,B_{l_n}^{m_n}) \tag{99}$$

which by (59) means that, under the rotational transformation $U^{g^{-1}}$ or $T^{g^{-1}}$, (99) is transformed as a $\bar{l}\times\cdots\times\bar{l}$-tensor of degree n in the n-tuple product space $D_l(\Omega)\times\cdots\times D_l(\Omega)$.

We now slightly generalize the equations (72)–(79) in the case of a Gaussian random measure. Letting the Wiener kernel $f_n(t_1, \ldots, t_n)$ be zonal, that is, invariant under rotations, $h \in H$, around the polar axis e_0,

$$f(ht_1, \ldots, ht_n) = f_n(t_1, \ldots, t_n), \quad h \in H, \tag{100}$$

then we easily see that the n-tuple Wiener integral (94), and hence (96), is invariant under U^h (or T^h), $h \in H$. Therefore, applying $U^{g^{-1}}$ to (96), and putting $t \equiv ge_0$, we have

$$X(t, \omega) = X(T^{g^{-1}}\omega)$$

$$= \sum_{n=0}^{\infty} \int_{S_3} \cdots \int_{S_3} f_n(g^{-1}t_1, \ldots, g^{-1}t_n) h_n[dB(t_1), \ldots, dB(t_n)] \tag{101}$$

which is a W–H expansion of a homogeneous random field on the sphere. Similarly, applying $U^{g^{-1}}$ to (97) we obtain a discrete representation of the W–H expansion where B_l^m in (97) is replaced by $B_l^m(t)$ in (60), and the summation is to be taken under the restriction $m_1 + \ldots + m_n = 0$ because of (100).

6. Stochastic Spherical Harmonics and Stochastic Solid Harmonics

Let $Z_l^m \equiv Z_l^m\ (\omega)$ be a fixed canonical basis in $D_l(\Omega)$, and $Z_l^m(t) \equiv Z_l^m(t, \omega)$ be the moving canonical basis relative to $t \equiv ge_0$, such that

$$\langle \overline{Z_l^m} Z_{l'}^{m'} \rangle = \langle \overline{Z_l^m(t)} Z_{l'}^{m'}(t) \rangle = \delta_{ll'} \delta_{mm'} \tag{102}$$

$$Z_l^m(t) \equiv Z_l^m(T^{g^{-1}}\omega) = \sum_{s=-l}^{l} \overline{T_{sm}^l(g)} Z_l^s, \quad t \equiv ge_0 \tag{103}$$

(c. f., (59), (60)). For comparison and reference we quote the similar relations from (A.14) and (A.15) for a l-vector canonical basis $e_{(l)n}$ in D_l and the moving canonical basis $e_{(l)n}(t)$:

$$(e_{(l)m} \cdot e_{(l')m'}) = (e_{(l)m}(t) \cdot e_{(l')m'}(t)) = \delta_{ll'} \delta_{mm'} \tag{104}$$

$$e_{(l)n}(t) \equiv T^l(g) e_{(l)n} = \sum_{s=-l}^{l} T_{sm}^l(g) e_{(l)s}, \quad t \equiv ge_0 \tag{105}$$

It is to be reminded that Z_l^m is transformed as a $\bar{l}$-vector under $U^{g^{-1}}$ or $T^{g^{-1}}$ while $e_{(l)n}$ is transformed as a l-vector under S^g or $T^l(g)$

We define the "stochastic l-vector spherical harmonics" associated with Z_l^m by the fromula;

$$\boldsymbol{P}^{l'}_{(l)n}(\boldsymbol{t};\omega)=\sum_{m=-l'}^{l'}\boldsymbol{P}^{l'm}_{(l)n}(\theta,\varphi)Z^{m}_{l'}(\omega),\quad \boldsymbol{t}\equiv(1,\theta,\varphi) \tag{106}$$

$$=\sqrt{\frac{2l'+1}{4\pi}}\,\boldsymbol{e}_{(l)n}(\boldsymbol{t})Z^{n}_{l'}(\boldsymbol{t}) \tag{107}$$

$$l,\ l'=0,1,2,\dots,\quad L=min(l,l'),\quad n=-L,\dots,L$$

where $\boldsymbol{P}^{l'm}_{(l)n}$ (θ,φ) denotes the l-vector spherical harmonic defined by (A.29), and (106) is rewritten into (107) using (A.29), (103) and (105). Since, by befinition (103), $Z^{n}_{l}(\boldsymbol{t})$ is invariant under $D^{g'}$, (107) is a homogeneous l-vector random field of the form (84). The correlation function can be easily calculated using (62),

$$\langle\overline{\boldsymbol{P}^{l'}_{(l)n}(\boldsymbol{t}_1;\omega)}P^{l'}_{(l)n}(\boldsymbol{t}_2;\omega)\rangle=\frac{2l'+1}{4\pi}\,T^{l'}_{nn}(g_1^{-1}g_2)\overline{\boldsymbol{e}_{(l)n}(\boldsymbol{t}_1)}\boldsymbol{e}_{(l)n}(\boldsymbol{t}_2) \tag{108}$$

which is an isotropic (2-point) tensor field by virtue of (A.28).

Now we introduce the "stochastic solid harmonics" $\boldsymbol{J}^{l}_{(l)n}$ $(r;\omega)$ such that it is a homogeneous l-vector random field on R_3 which satisfies the l-vector Helmholtz equation. We denote the position and the wave vector by $\boldsymbol{r}\equiv$ (r,θ,ω) and $\boldsymbol{k}\equiv$ (k,u,v) in the polar coordinate, respectively, and let them stand for the spherical coordinates (θ,φ) and (u,v) as well. Let us define the stochastic solid harmonic by the integral

$$\boldsymbol{J}^{l'}_{(l)n}(\boldsymbol{r};\omega)\equiv\frac{1}{4\pi i^{l'-l}}\int_{S_3}e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\boldsymbol{P}^{l'}_{(l)n}(\boldsymbol{k};\omega)dS_{\boldsymbol{k}} \tag{109}$$

$$l'=0,1,2,\dots,\quad n=-L,\dots,L,\quad L=min(l,l')$$

That this bears the desired transformation properties under D^{g} easily follows from that of the integrand, or from the following expressions. Substituting (106) into (109) and using (A.35), (A.31), (A.29) and (107), we obtain several expressions for (109);

$$\boldsymbol{J}^{l}_{(l)n}(\boldsymbol{r};\omega)=\sum_{m=-L}^{L}\boldsymbol{J}^{l'm}_{(l)n}(kr,\theta,\varphi)Z^{m}_{l'} \tag{110}$$

$$=\sum_{t=-L}^{L}j^{l'l}_{nt}(kr)\boldsymbol{P}^{l'}_{(l)t}(\boldsymbol{k};\omega) \tag{111}$$

$$=\sqrt{\frac{2l'+1}{4\pi}}\sum_{t=-L}^{L}j^{l'l}_{nt}(kr)\boldsymbol{e}_{(l)t}(\boldsymbol{r})Z^{t}_{l'}(\boldsymbol{t}) \tag{112}$$

Since $Z^{t}_{l}(\boldsymbol{r})$ is D^{g} invariant, (112) is the sum of the functions of the form (84).

The integral representation (109) is a stochastic analogue to (A.35), and (111) is another analogue to (A.31). Furthermore, substituting (112) into the lefthand side of (109), and (107) into the righthand side, we obtain the formula analogous to (A.36),

$$\sum_{t=-L}^{L} j_{nt}^{l'l}(kr)\boldsymbol{e}_{(l)t}(\boldsymbol{r})Z_{l'}^{t}(\boldsymbol{t})$$

$$= \frac{1}{4\pi i^{l'-l}} \int_{S_3} \boldsymbol{e}_{(l)n}(\boldsymbol{k}) Z_{l'}^{n}(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{r}} dS_{\boldsymbol{k}} \tag{113}$$

which is the tensor integral representation, where $\boldsymbol{e}_{(l)n}$ is a canonical l-vector in D_l and $Z_{l'}^{n}$ a canonical $\overline{l'}$-vector in $D_{l'}(\Omega)$ so that $\boldsymbol{e}_{(l)n}(\boldsymbol{k})Z_{l'}^{n}(\boldsymbol{k})$ gives an isotropic $l\times\overline{l'}$-tensor field in $D_l\times D_{l'}(\Omega)$ according to (A.28). It is obvious from (109) or from (110) and (A.33) that $\boldsymbol{J}_{(l)n}^{l'}(\boldsymbol{r}, \omega)$ satisfies the l-vector Helmholtz equation;

$$(\nabla^2+k^2)\boldsymbol{J}_{(l)n}^{l'}(\boldsymbol{r}, \omega)=0 \tag{114}$$

analogous to (A.33). The above mentioned analogies would justify the name of "stochastic l-vector solid harmonics". Similarly, we can define the stochastic solid harmonics $\boldsymbol{H}_{(l)n}^{(1)l'}(\boldsymbol{r};\omega)$ by replacing $\boldsymbol{J}_{(l)n}^{l'}$ by $\boldsymbol{H}_{(l)n}^{(1)l'}$ and $j_{nt}^{l'l}(kr)$ by $h_{nt}^{(1)l'l}(kr)$ in the righthand side of (110)-(112), which represent the stochastic outgoing l-vector wave satisfying the Helmholtz wave eqxation. When Z_l^m is a linear functional of $B(\Delta, \omega)$), then Z_l^m can be replaced by B_l^m. Otherwise, it is to be represented by the W-H expansion.

The "stochastic spherical harmonics" and the "stochastic solid harmonics" as introduced in this section can be effectively used as a powerful tool to formulate the wave scattering theory associated with a random spherical surface[25].

Appendix

Some necessary definitions and notations concerning the representation of the rotation group are briefly summarized for reference and for indicating our choice among various definitions. Several formulas are given in convenient forms for later applications. For details of the theory of the rotation group see Ref. 9, and also the appendix of Ref. 14.

Rotation group A rotation $g\equiv g(\varphi_1, \theta, \varphi_2)$ described by the three Euler angles is defined by the successive rotations in this order: a rotation g_{φ_1} about $\boldsymbol{e}_z$, g_θ about $\boldsymbol{e}_x{}'=g_{\varphi_1}\boldsymbol{e}_x$ and g_{φ_2} about $\boldsymbol{e}_z{}'=g_\theta\boldsymbol{e}_z$, $(\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z)$ denoting the 3-D unit vectors along x, y, z axes, respectively. The rotations g (identity e, inverse g^{-1}) form the rotation group G, and the rotation of a 3-D vector is represented by the Euler matrix $[g]=[g_{\varphi_1}][g_\theta][g_{\varphi_2}]$.

Canonical basis We call the set of vectors $\boldsymbol{e}_m$, $m=-1, 0, 1$, *i. e.*,

$$\boldsymbol{e}_0=\boldsymbol{e}_z, \quad \boldsymbol{e}_{\mp}=[\pm\boldsymbol{e}_x-i\boldsymbol{e}_y]/\sqrt{2} \tag{A.1}$$

the fixed canonical basis relative to $\boldsymbol{e}_z$. The Euler matrix is to be represented in the fixed canonical basis. Let $(\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_\varphi)$ be the unit orthogonal vector basis for the polar coordinates $\boldsymbol{r}\equiv(r, \theta, \varphi)$. The vector $\boldsymbol{t}\equiv\boldsymbol{e}_r$ giving a point a point $\boldsymbol{t}\equiv(1, \theta, \varphi)$ on the sphere can be written

$$\boldsymbol{t}\equiv\boldsymbol{e}_r=g_r\boldsymbol{e}_z, \quad g_r\equiv g(\varphi+\pi/2, \theta, \varphi_2) \tag{A.2}$$

We define the moving canonical basis by the set of vectors, $\boldsymbol{e}_m(\boldsymbol{r})\equiv g_r\boldsymbol{e}_m$, $m=-1, 0, 1$, namely,

$$\boldsymbol{e}_0(\boldsymbol{r})\equiv g_r\boldsymbol{e}_0=\boldsymbol{e}_r, \quad \boldsymbol{e}_{\mp 1}(\boldsymbol{r})=g_r\boldsymbol{e}_+=[\pm\boldsymbol{e}_\varphi+i\boldsymbol{e}_\theta]e^{\pm i\varphi_2}/\sqrt{2} \tag{A.3}$$

which is the canonical basis relative to $\boldsymbol{r}$ or $\boldsymbol{t}$.

Representation of the rotation group For the functions on the sphere $\Psi(\boldsymbol{t})$, $\boldsymbol{t}\in S_3$, or more generally the functions on G, $\Psi(g)$, $g\in G$, we define the transformation S^g by

$$S^g\Psi(g_0)\equiv\Psi(g^{-1}g_0), \quad g, g_0\in G \tag{A.4}$$

which gives the representation of G:

$$S^{g_1}S^{g_2}=S^{g_1g_2}, \quad [S^g]^{-1}=S^{g^{-1}}, \quad S^e=I \tag{A.5}$$

We denote by D_l the $(2l+1)$-D invariant space of the irreducible representation of weight l, of which the matrix of unitary representation is written

$$\boldsymbol{T}^l(g)\equiv[T^l_{mn}(\varphi_1, \theta, \varphi_2)], \quad -l\leq m, n\leq l \tag{A.6}$$

$$T^l_{mn}(\varphi_1, \theta, \varphi_2)\equiv e^{-im\varphi_1}P^l_{mn}(\cos\theta)e^{-in\varphi_2} \tag{A.7}$$

where $T^l_{mn}(e)=\delta_{mn}$ and $T^1_{mn}(g)=g_{mn}$, g_{mn} being the Euler matrix relative to (A.1). The matrix representation is referred to the fixed canonical basis in D_l, which is a set of $(2l+1)$ orthogonal vectors of $(2l+1)$-dimension, $\boldsymbol{e}_{(l)n}$, $n=-l, \ldots, l$: each being the eigenvector with the eigenvalue $e^{-in\varphi_2}$ for the rotation g around $\boldsymbol{e}_z$. In the present paper we deal with the representation of the integral weight l: $l=0, 1, 2, \cdots$. The matrix element (A.7) is called the generalized spherical function of order l[9], and in parcicular for $n=0$, we have

$$T^l_{m0}(\varphi_1, \theta, \varphi_2)=\sqrt{4\pi/(2l+1)}\,i^m Y^m_l(\theta, \varphi_1) \tag{A.8}$$

where $Y_l^m(\theta, \varphi)$ denotes the normalized spherical harmonics

$$Y_l^m(\theta, \varphi) = (-1)^l \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\varphi} \tag{A.9}$$

Vector and tensor $A(2l+1)-D$ vector with components $a_{(l)n}$, $n=-l, \ldots, l$, which is transformed by the matrix $\boldsymbol{T}^l(g)$ upon rotation g as

$$a'_{(l)m} = \sum_{n=-l}^{l} T^l_{mn}(g) a_{(l)n}, \quad m=l, \ldots, l \tag{A.10}$$

is called a l-vector in D_l; hence the ordinary $3-$D vector transformed by g_{mn} is a 1-vector. Similarly, a $(2l+1)-D$ vector transformed by the matrix $\overline{\boldsymbol{T}^l(g)}$ is called a $\bar{l}$-vector in $\overline{D_l}$ (the overbar implying the complex conjugate). Further, we consider a tensorial quantity in a product space, For instance, a tensor with $(2l'+1) \times (2l+1)$ components $a^{(l')n'}_{(l)n}$ which is transformed under rotation g as

$$a^{(l')m'}_{(l)m} = \sum_{n=-l}^{l} \sum_{n'=-l'}^{l'} \overline{T^{l'}_{m'n'}(g)} T^l_{mn}(g) a^{(l')n'}_{(l)n} \tag{A.11}$$

is called for simplicty a $\overline{l'} \times l$-tensor in $\overline{D_{l'}} \times D_l$, where the superscript refers to the component of a $\overline{l'}$-vector in $\overline{D_{l'}}$. The inner product of two l-vectors, $\boldsymbol{a}_{(l)}$ and $\boldsymbol{b}_{(l)}$, as well as the contraction of a tensor, can be defined as

$$(\boldsymbol{a}_{(l)} \cdot \boldsymbol{b}_{(l)}) = \sum_{m=-l}^{l} \overline{a_{(j)m}} b_{(l)m} \tag{A.12}$$

Properties of the matrix The unitarity of the representation matrix,

$$\sum_{s=-l}^{l} \overline{T^l_{sm}(g)}\, T^l_{sn}(g) = \delta_{nm} \tag{A.13}$$

can be interpreted as the orthonormal relation of a set of $(2l+1)$ l-vectors $\boldsymbol{e}_{(l)n}(\boldsymbol{r})$ with respect to the inner product, namely

$$(\boldsymbol{e}_{(l)m}(\boldsymbol{r}) \cdot \boldsymbol{e}_{(l)n}(\boldsymbol{r})) = \delta_{mn} \tag{A.14}$$

where we have put

$$\boldsymbol{e}_{(l)n}(\boldsymbol{r}) \equiv \boldsymbol{T}^l(g_r) \boldsymbol{e}_{(l)n} = \sum_{s=-l}^{l} T^l_{sn}(g_r) \boldsymbol{e}_{(l)s} \qquad n=-l, \ldots, l \tag{A.15}$$

which is obtained from $\boldsymbol{e}_{(l)n}$ by rotation g_r: $T^l_{sn}(g_r)$ in the righthand side giving the $2l+1$ components in the fixed canonical basis of D_l. The set of $(2l+1)$ vectors (A.15) is called the moving canonical basis in D_l relative to $\boldsymbol{r}$ or $\boldsymbol{t}$, which is reduced to (A. 3) for $l=1$. As (A.14) shows, the coordinate-free l-vector notation on the lefthand side of (A.15) makes the vector and tensor formulas considerably simpler than the coordinate-fixed notation on the righhand side.

The multiplicative law of the group representation can be written as

$$T^l_{mn}(g_2g_1)=\sum_{s=-l}^{l}T^l_{ms}(g_2)T^l_{sn}(g_1),\quad T^l_{mn}(g_2^{-1}g_1)=\sum_{s=-l}^{l}\overline{T^l_{sm}(g_2)}T^l_{sn}(g_1) \tag{A.16}$$

The first equality simply shows that $\boldsymbol{e}_{(l)n}(\boldsymbol{r})$ is a l-vector, having the property (A. 10). The second following from the first implies the addition theorem for generalized spherical functions, which can be interpreted as the inner product of two l-vectors:

$$T^l_{mn}(g_2^{-1}g_1)=(\boldsymbol{e}_{(l)m}(\boldsymbol{r}_2)\cdot\boldsymbol{e}_{(l)n}(\boldsymbol{r}_1)) \tag{A.17}$$

where $\boldsymbol{r}_1\equiv g_1\boldsymbol{e}_0$, $\boldsymbol{r}_2\equiv g_2\boldsymbol{e}_0$; (A.17) reduces to (A.14) when $g_1=g_2$. In particular, for $m=n=0$, (A.17) gives the well known addition formula for the zonal spherical function;

$$\begin{aligned}P_l(\cos\theta)&=(\boldsymbol{e}_{(l)0}(\boldsymbol{r}_2)\cdot\boldsymbol{e}_{(l)0}(\boldsymbol{r}_1))\\&=\frac{4\pi}{2l+1}\sum_{m=-l}^{l}\overline{Y_l^m(\theta_1,\varphi_1)}Y_l^m(\theta_2,\varphi_2)\end{aligned} \tag{A.18}$$

$$\cos\theta=\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\cos(\varphi_1-\varphi_2) \tag{A.19}$$

The integration over G has the invariance properties under rotational transformation of the variable:

$$\int_G f(g)dg=\int_G f(gg_0)dg=\int_G f(g_0g)dg=\int_G f(g^{-1})dg \tag{A.20}$$

$$f(g)\equiv f(\varphi_1,\theta,\varphi_2),\quad dg=d\varphi_1\sin\theta d\theta d\varphi_2$$

For the function $f(\boldsymbol{t})$ on S_3, (A.20) imples the integrgl over S_3 multiplied by 2π. The orthogonality and the completeness of the set of generalized spherical functions are written as

$$\int_G\overline{T^{l'}_{m'n'}(g)}T^l_{mn}(g)dg=\delta_{ll'}\delta_{mm'}\delta_{nn'}\frac{8\pi^2}{2l+1} \tag{A.21}$$

$$\frac{1}{8\pi^2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sum_{n=-l}^{l}(2l+1)\overline{T^l_{mn}(g)}T^l_{mn}(g')=\delta(g-g') \tag{A.22}$$

Vector and tensor fields Upon rotation g a l-vector field on R_3 is transformed into a new vector field by the formula,

$$a'_{(l)m}(\boldsymbol{r})=\sum_{n=-l}^{l} T^{l}_{mn}(g)a_{(l)n}(g^{-1}\boldsymbol{r}),\ \boldsymbol{r}\in R_3 \tag{A.23}$$

In a similar manner, a tensor field, e.g., a $\overline{l'}\times l$-tensor field is transformed according to

$$a'^{(l')m'}_{(l)m}(\boldsymbol{r})=\sum_{n=-l}^{l}\sum_{n'=l'}^{l'}\overline{T^{l'}_{m'n'}(g)}\,T^{l}_{mn}(g)\,a^{(l')n'}_{(l)n}(g^{-1}\boldsymbol{r}) \tag{A.24}$$

Isotropic vector field and isotropic tensor field When a vector or a tensor field is invariant under rotations, for instance, when

$$a'_{(l)m}(\boldsymbol{r})=a_{(l)m}(\boldsymbol{r}) \tag{A.25}$$

$$a'^{(l')m'}_{(l)m}(\boldsymbol{r}) \tag{A.26}$$

hold for (A.23) and (A.24), then l-vector field or the $\overline{l'}\times l$-tensor field is said to be isotropic. It is easily shown[14] that, when referred to the moving canonical basis, an isotropic l-vector field has only the 0-th canonical component depending on $r\equiv|\boldsymbol{r}|$, that is

$$a_{(l)m}(\boldsymbol{r})=\delta_{m0}a(r),\quad m=-l,\ldots,l \tag{A.27}$$

and that similarly an isotropic $\overline{l'}\times l$-tensor field has the components only for $m'=m$;

$$a^{(l')m'}_{(l)m}(\boldsymbol{r})=\delta_{mm'}a_m(r),\ \ m=-l,\ldots,l,\ m'=-l',\ldots,l' \tag{A.28}$$

For the isotropic field on the sphere $r=1$, the components are constants.

Vector harmonic functions For reference in the text we summarize the definitions and formulas concerning the l-vector spherical and solid harmonics which are derived from the representation of the rotation group[14].

Let a l-vector function on S_3 having only n-th canonical component be

$$\boldsymbol{P}^{l'm}_{(l)n}(\theta,\varphi)\equiv\sqrt{(2l'+1)/4\pi}\,\overline{T^{l'}_{mn}(g)}\,\boldsymbol{e}_{(l)n}(\boldsymbol{r}) \tag{A.29}$$

$$n=-l,\ldots,l,\quad l'=0,1,2,\ldots,\quad m=-l',\ldots,l'$$

in the coordinate-free notation. The l-vector function (A.29) is called the l-vector spherical harmonic and satisfies the orthogonality relation,

$$\int_{S_3}(\boldsymbol{P}^{l'm'}_{(l)n'}\cdot\boldsymbol{P}^{l''m''}_{(l)n''})ds=\delta_{n'n''}\delta_{l'l''}\delta_{m'm''},\quad dS\equiv\sin\theta d\theta d\varphi \tag{A.30}$$

Let a l-vector functions on R_3 be defined by

$$J_{(l)n}^{l'm}(kr,\theta,\varphi)=\sum_{t=-l}^{l} j_{nt}^{l'l}(kr)P_{(l)t}^{l'm}(\theta,\varphi), \tag{A.31}$$

which we call the l-vector solid harmonic, where $j_{nt}^{l'l}(kr)$ is defined by

$$j_{mn}^{ll'}(kr)\equiv\sum_{L=|l-l'|}^{l+l'} i^{L-l+l'}(-1)^{m+n}(l-ml'm\mid ll'LO)(l-nl'n\mid ll'LO)j_L(kr) \tag{A.32}$$

$j_L(kr)$ being the spherical Bessel funtion and $(l-ml'm \mid ll'LO)$ denoting the Clebsch-Gordan coefficient[24]. $j_{mn}^{ll'}(kr)$ defined by (A.32) is called the generalized spherical Bessel function, having orthogonality with respect to integration, and is derived from the matrix element of the translation group in R_3[14]. The l-vector solid harmonics are shown to satisfy the l-vector Helmholtz equation,

$$(\nabla^2+k^2)J_{(l)n}^{l'm}(kr,\theta,\varphi)=O \tag{A.33}$$

and the orthogonality relation;

$$\frac{2}{\pi}\int_0^\infty\int_{S_3}(J_{(l)n'}^{l'm'}(k'r,\theta,\varphi)\cdot J_{(l)n'}^{l'm'}(k''r,\theta,\varphi))dSr^2dr$$

$$=\delta_{n'n'}\delta_{l'l'}\delta_{m'm'}\frac{\delta(k'-k'')}{k'^2} \tag{A.34}$$

The following vector and tensor integral representations hold for l-vector harmonic functions:

$$J_{(l)n}^{l'm}(kr,\theta,\varphi)=\frac{1}{4\pi i^{l'-l}}\int_{S_3}e^{ik\cdot r}P_{(l)n}^{l'm}(u,v)dS \tag{A.35}$$

$$\sum_{t=-L}^{L} j_{nt}^{ll'}(kr)e_{(l)t}(r)\overline{e_{(l')t}(r)} \qquad (L=min(l,l'))$$

$$=\frac{1}{4\pi i^{l-l'}}\int_{S_3}e_{(l)n}(k)\overline{e_{(l')n}(k)}\,e^{ik\cdot r}dS \tag{A.36}$$

where $k=(k,u,v)$ in the polar coordinates and $dS=\sin u\,du\,dv$. These two are equivalent representations with different interpretations. The first is written as the Fourier transform of the l-vector field over a sphere, while the second gives the Fourier transform of an isotropic $l\times\overline{l'}$-tensor field over the sphere.

Analogous to $j_{mn}^{ll'}$ and $J_{(l)n}^{l'm}$ given in terms of $j_L(kr)$ we can define $h_{mn}^{(1)l'm}$ and the solid harmonics $H_{(l)n}^{(1)l'm}$ in terms of the spherical Hankel function $h_L^{(1)}(kr)$. *The* l-vector solid harmonic $H_{(l)n}^{(1)l'm}$ satisfies the Helmholtz equation (A.33) also and has similar integral representations. The definitions and formulas for l-vector harmonics are reduced to the vector case for $l=1$ and to the familiar scalar case for $l=0$.

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